CHAPTER 2

PHYSICS AND MATHEMATICS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics. In the present course we shall constantly be using the techniques of algebra, trigonometry and geometry as well as vector algebra, differential calculus and integral calculus. In this chapter we shall discuss the latter three topics. Errors in measurement and the concept of significant digits are also introduced.

2.1 VECTORS AND SCALARS

Certain physical quantities are completely described by a numerical value alone (with units specified) and are added according to the ordinary rules of algebra. As an example the mass of a system is described by saying that it is 5 kg. If two bodies one having a mass of 5 kg and other having a mass of 2 kg are added together to make a composite system, the total mass of the system becomes 5 kg + 2 kg = 7 kg. Such quantities are called *scalars*.

The complete description of certain physical quantities requires a numerical value (with units specified) as well as a direction in space. Velocity of a particle is an example of this kind. The magnitude of velocity is represented by a number such as 5 m/s and tells us how fast a particle is moving. But the description of velocity becomes complete only when the direction of velocity is also specified. We can represent this velocity by drawing a line parallel to the velocity and putting an arrow showing the direction of velocity. We can decide beforehand a particular length to represent 1 m/s and the length of the line representing a velocity of 5 m/s may be taken as 5 times this unit

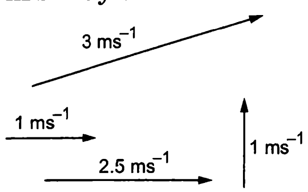


Figure 2.1

length. Figure (2.1) shows representations of several velocities in this scheme. The front end (carrying the arrow) is called the head and the rear end is called the tail.

Further, if a particle is given two velocities simultaneously its resultant velocity is different from the two velocities and is obtained by using a special rule. Suppose a small ball is moving inside a long tube at a speed 3 m/s and the tube itself is moving in the room at a speed 4 m/s along a direction perpendicular to its length. In which direction and how fast is the ball moving as seen from the room ?

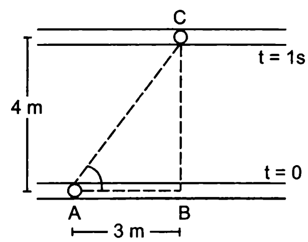


Figure 2.2

Figure (2.2) shows the positions of the tube and the ball at t = 0 and t = 1 s. Simple geometry shows that the ball has moved 5 m in a direction $\theta = 53°$ from the tube. So the resultant velocity of the ball is 5 m/s along this direction. The general rule for finding the resultant of two velocities may be stated as follows.

Draw a line AB representing the first velocity with B as the head. Draw another line BC representing the second velocity with its tail B coinciding with the head of the first line. The line AC with A as the tail and C as the head represents the resultant velocity. Figure (2.3) shows the construction.

The resultant is also called the sum of the two velocities. We have added the two velocities AB and BC and have obtained the sum AC. This rule of addition is called the "triangle rule of addition".

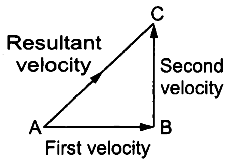


Figure 2.3

The physical quantities which have magnitude and direction and which can be added according to the triangle rule, are called *vector quantities*. Other examples of vector quantities are force, linear momentum, electric field, magnetic field etc.

The vectors are denoted by putting an arrow over the symbols representing them. Thus, we write $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. Sometimes a vector is represented by a single letter such as $\vec{v}$, $\vec{F}$ etc. Quite often in printed books the vectors are represented by bold face letters like **AB**, **BC**, **v**, **f** etc.

If a physical quantity has magnitude as well as direction but does not add up according to the triangle rule, it will not be called a vector quantity. Electric current in a wire has both magnitude and direction but there is no meaning of triangle rule there. Thus, electric current is not a vector quantity.

2.2 EQUALITY OF VECTORS

Two vectors (representing two values of the same physical quantity) are called equal if their magnitudes and directions are same. Thus, a parallel translation of a vector does not bring about any change in it.

2.3 ADDITION OF VECTORS

The triangle rule of vector addition is already described above. If $\vec{a}$ and $\vec{b}$ are the two vectors to be added, a diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$. The vector joining the tail of $\vec{a}$ with the head of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$. Figure (2.4a) shows the construction. The same rule

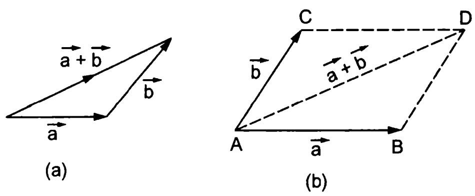


Figure 2.4

may be stated in a slightly different way. We draw the vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding (figure 2.4b). Taking these two as the adjacent sides we complete the parallelogram. The diagonal through the common tails gives the sum of the two vectors. Thus, in figure, (2.4b) $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$.

Suppose the magnitude of $\vec{a} = a$ and that of $\vec{b} = b$. What is the magnitude of $\vec{a} + \vec{b}$ and what is its direction ? Suppose the angle between $\vec{a}$ and $\vec{b}$ is θ. It is easy to see from figure (2.5) that

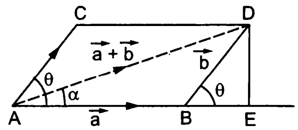


Figure 2.5

$$AD^2 = (AB + BE)^2 + (DE)^2$$
$$= (a + b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 + 2ab\cos\theta + b^2.$$

Thus, the magnitude of $\vec{a} + \vec{b}$ is

$$\sqrt{a^2 + b^2 + 2ab\cos\theta}. \qquad \ldots (2.1)$$

Its angle with $\vec{a}$ is α where

$$\tan\alpha = \frac{DE}{AE} = \frac{b\sin\theta}{a + b\cos\theta}. \qquad \ldots (2.2)$$

Example 2.1

Two vectors having equal magnitudes A make an angle θ with each other. Find the magnitude and direction of the resultant.

Solution : The magnitude of the resultant will be

$$B = \sqrt{A^2 + A^2 + 2AA\cos\theta}$$
$$= \sqrt{2A^2(1+\cos\theta)} = \sqrt{4A^2\cos^2\frac{\theta}{2}}$$
$$= 2A\cos\frac{\theta}{2}.$$

The resultant will make an angle α with the first vector where

$$\tan\alpha = \frac{A\sin\theta}{A + A\cos\theta} = \frac{2A\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2A\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

or, $\alpha = \dfrac{\theta}{2}$

Thus, the resultant of two equal vectors bisects the angle between them.

2.4 MULTIPLICATION OF A VECTOR BY A NUMBER

Suppose $\vec{a}$ is a vector of magnitude a and k is a number. We define the vector $\vec{b} = k\vec{a}$ as a vector of magnitude $|ka|$. If k is positive the direction of the vector $\vec{b} = k\vec{a}$ is same as that of $\vec{a}$. If k is negative, the direction of $\vec{b}$ is opposite to $\vec{a}$. In particular, multiplication by (–1) just inverts the direction of the vector. The vectors $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

If $\vec{a}$ is a vector of magnitude a and $\vec{u}$ is a vector of unit magnitude in the direction of $\vec{a}$, we can write $\vec{a} = a\vec{u}$.

2.5 SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$, invert the direction of $\vec{b}$ and add to $\vec{a}$. Figure (2.6) shows the process.

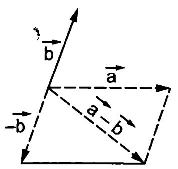


Figure 2.6

Example 2.2

Two vectors of equal magnitude 5 unit *have an angle* 60° *between them. Find the magnitude of* (a) *the sum of the vectors and* (b) *the difference of the vectors.*

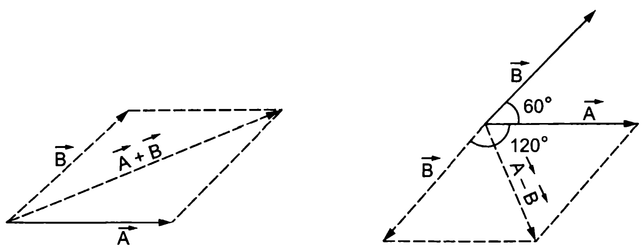


Figure 2.7

Solution : Figure (2.7) shows the construction of the sum $\vec{A} + \vec{B}$ and the difference $\vec{A} - \vec{B}$.

(a) $\vec{A} + \vec{B}$ is the sum of $\vec{A}$ and $\vec{B}$. Both have a magnitude of 5 unit and the angle between them is 60°. Thus, the magnitude of the sum is

$$|\vec{A} + \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 60^\circ}$$
$$= 2 \times 5 \cos 30^\circ = 5\sqrt{3} \text{ unit.}$$

(b) $\vec{A} - \vec{B}$ is the sum of $\vec{A}$ and $(-\vec{B})$. As shown in the figure, the angle between $\vec{A}$ and $(-\vec{B})$ is 120°. The magnitudes of both $\vec{A}$ and $(-\vec{B})$ is 5 unit. So,

$$|\vec{A} - \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 120^\circ}$$
$$= 2 \times 5 \cos 60^\circ = 5 \text{ unit.}$$

2.6 RESOLUTION OF VECTORS

Figure (2.8) shows a vector $\vec{a} = \overrightarrow{OA}$ in the X-Y plane drawn from the origin O. The vector makes an angle α with the X-axis and β with the Y-axis. Draw perpendiculars AB and AC from A to the X and Y axes respectively. The length OB is scalled the projection of $\overrightarrow{OA}$ on X-axis. Similarly OC is the projection of $\overrightarrow{OA}$ on Y-axis. According to the rules of vector addition

$$\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}.$$

Thus, we have resolved the vector $\vec{a}$ into two parts, one along OX and the other along OY. The magnitude of the part along OX is $OB = a\cos\alpha$ and the magnitude of the part along OY is $OC = a\cos\beta$. If $\vec{i}$ and $\vec{j}$ denote vectors of unit magnitude along OX and OY respectively, we get

$$\overrightarrow{OB} = a\cos\alpha\,\vec{i} \text{ and } \overrightarrow{OC} = a\cos\beta\,\vec{j}$$

so that $$\vec{a} = a\cos\alpha\,\vec{i} + a\cos\beta\,\vec{j}.$$

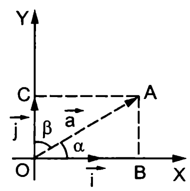


Figure 2.8

If the vector $\vec{a}$ is not in the X-Y plane, it may have nonzero projections along X,Y,Z axes and we can resolve it into three parts i.e., along the X, Y and Z axes. If α, β, γ be the angles made by the vector $\vec{a}$ with the three axes respectively, we get

$$\vec{a} = a\cos\alpha\,\vec{i} + a\cos\beta\,\vec{j} + a\cos\gamma\,\vec{k} \qquad \ldots (2.3)$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along X, Y and Z axes respectively. The magnitude ($a\cos\alpha$) is called the component of $\vec{a}$ along X-axis, ($a\cos\beta$) is called the component along Y-axis and ($a\cos\gamma$) is called the component along Z-axis. In general, the component of a vector $\vec{a}$ along a direction making an angle θ with it is a cosθ (figure 2.9) which is the projection of $\vec{a}$ along the given direction.

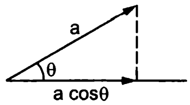


Figure 2.9

Equation (2.3) shows that any vector can be expressed as a linear combination of the three unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$.

Example 2.3

A force of 10·5 N *acts on a particle along a direction making an angle of* 37° *with the vertical. Find the component of the force in the vertical direction.*

Solution : The component of the force in the vertical direction will be

$$F_{\perp} = F\cos\theta = (10{\cdot}5\text{ N})(\cos 37°)$$

$$= (10{\cdot}5\text{ N})\frac{4}{5} = 8{\cdot}40\text{ N}.$$

We can easily add two or more vectors if we know their components along the rectangular coordinate axes. Let us have

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

and $$\vec{c} = c_x\vec{i} + c_y\vec{j} + c_z\vec{k}$$

then

$$\vec{a} + \vec{b} + \vec{c} = (a_x + b_x + c_x)\vec{i} + (a_y + b_y + c_y)\vec{j} + (a_z + b_z + c_z)\vec{k}.$$

If all the vectors are in the X-Y plane then all the z components are zero and the resultant is simply

$$\vec{a} + \vec{b} + \vec{c} = (a_x + b_x + c_x)\vec{i} + (a_y + b_y + c_y)\vec{j}.$$

This is the sum of two mutually perpendicular vectors of magnitude $(a_x + b_x + c_x)$ and $(a_y + b_y + c_y)$. The resultant can easily be found to have a magnitude

$$\sqrt{(a_x + b_x + c_x)^2 + (a_y + b_y + c_y)^2}$$

making an angle α with the X-axis where

$$\tan\alpha = \frac{a_y + b_y + c_y}{a_x + b_x + c_x}.$$

2.7 DOT PRODUCT OR SCALAR PROUDCT OF TWO VECTORS

The dot product (also called scalar product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a}\cdot\vec{b} = ab\cos\theta \qquad \ldots (2.4)$$

where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the angle between them. The dot product between two mutually perpendicular vectors is zero as cos90° = 0.

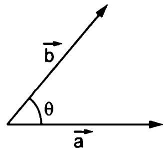


Figure 2.10

The dot product is commutative and distributive.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$

Example 2.4

The work done by a force $\vec{F}$ during a displacement $\vec{r}$ is given by $\vec{F}\cdot\vec{r}$. Suppose a force of 12 N *acts on a particle in vertically upward direction and the particle is displaced through* 2·0 m *in vertically downward direction. Find the work done by the force during this displacement.*

Solution : The angle between the force $\vec{F}$ and the displacement $\vec{r}$ is 180°. Thus, the work done is

$$W = \vec{F}\cdot\vec{r}$$

$$= Fr\cos\theta$$

$$= (12\text{ N})(2{\cdot}0\text{ m})(\cos 180°)$$

$$= -24\text{ N–m} = -24\text{ J}.$$

Dot Product of Two Vectors in terms of the Components along the Coordinate Axes

Consider two vectors $\vec{a}$ and $\vec{b}$ represented in terms of the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ along the coordinate axes as

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

and $$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}.$$

Then

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})\cdot(b_x\vec{i} + b_y\vec{j} + b_z\vec{k})\\ &= a_x b_x\,\vec{i}\cdot\vec{i} + a_x b_y\,\vec{i}\cdot\vec{j} + a_x b_z\,\vec{i}\cdot\vec{k}\\ &\quad + a_y b_x\,\vec{j}\cdot\vec{i} + a_y b_y\,\vec{j}\cdot\vec{j} + a_y b_z\,\vec{j}\cdot\vec{k}\\ &\quad + a_z b_x\,\vec{k}\cdot\vec{i} + a_z b_y\,\vec{k}\cdot\vec{j} + a_z b_z\,\vec{k}\cdot\vec{k} \qquad \ldots (i)\end{aligned}$$

Since, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are mutually orthogonal, we have $\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = \vec{k} \cdot \vec{j} = 0$.

Also, $\vec{i} \cdot \vec{i} = 1 \times 1 \cos 0 = 1$.

Similarly, $\vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$.

Using these relations in equation (i) we get

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.$$

2.8 CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ is itself a vector. The magnitude of this vector is

$$|\vec{a} \times \vec{b}| = ab \sin\theta \qquad \text{... (2.5)}$$

where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the smaller angle between the two. When two vectors are drawn with both the tails coinciding, two angles are formed between them (figure 2.11). One of the angles is smaller than 180°

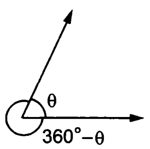


Figure 2.11

and the other is greater than 180° unless both are equal to 180°. The angle θ used in equation (2.5) is the smaller one. If both the angles are equal to 180°, $\sin\theta = \sin 180° = 0$ and hence $|\vec{a} \times \vec{b}| = 0$. Similarly if $\theta = 0$, $\sin\theta = 0$ and $|\vec{a} \times \vec{b}| = 0$. The cross product of two parallel vectors is zero.

The direction of $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Thus, it is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$. To determine the direction of arrow on this perpendicular several rules are in use. In order to avoid confusion we here describe just one rule.

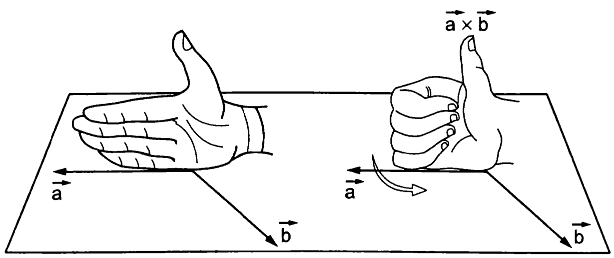


Figure 2.12

Draw the two vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding (figure 2.12). Now place your stretched right palm perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that the fingers are along the vector $\vec{a}$ and when the fingers are closed they go towards $\vec{b}$. The direction of the thumb gives the direction of arrow to be put on the vector $\vec{a} \times \vec{b}$.

This is known as the *right hand thumb rule*. The left handers should be more careful in using this rule as it must be practiced with right hand only.

Note that this rule makes the cross product noncommutative. In fact

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

The cross product follows the distributive law

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

It does not follow the associative law

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

When we choose a coordinate system any two perpendicular lines may be chosen as X and Y axes. However, once X and Y axes are chosen, there are two possible choices of Z-axis. The Z-axis must be perpendicular to the X-Y plane. But the positive direction of Z-axis may be defined in two ways. We choose the positive direction of Z-axis in such a way that

$$\vec{i} \times \vec{j} = \vec{k}.$$

Such a coordinate system is called a *right handed system*. In such a system

$$\vec{j} \times \vec{k} = \vec{i} \quad \text{and} \quad \vec{k} \times \vec{i} = \vec{j}.$$

Of course $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$.

Example 2.5

The vector $\vec{A}$ *has a magnitude of* 5 unit, $\vec{B}$ *has a magnitude of* 6 unit *and the cross product of* $\vec{A}$ *and* $\vec{B}$ *has a magnitude of* 15 unit. *Find the angle between* $\vec{A}$ *and* $\vec{B}$.

Solution : If the angle between $\vec{A}$ and $\vec{B}$ is θ, the cross product will have a magnitude

$$|\vec{A} \times \vec{B}| = AB \sin\theta$$

or, $$15 = 5 \times 6 \sin\theta$$

or, $$\sin\theta = \frac{1}{2}.$$

Thus, $$\theta = 30° \text{ or, } 150°.$$

Cross Product of Two Vectors in terms of the Components along the Coordinate Axes

Let $$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

and $$\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}.$$

Then $\vec{a} \times \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \times (b_x \vec{i} + b_y \vec{j} + b_z \vec{k})$

$$
\begin{aligned}
&= a_x b_x \vec{i} \times \vec{i} + a_x b_y \vec{i} \times \vec{j} + a_x b_z \vec{i} \times \vec{k} \\
&\quad + a_y b_x \vec{j} \times \vec{i} + a_y b_y \vec{j} \times \vec{j} + a_y b_z \vec{j} \times \vec{k} \\
&\quad + a_z b_x \vec{k} \times \vec{i} + a_z b_y \vec{k} \times \vec{j} + a_z b_z \vec{k} \times \vec{k} \\
&= a_x b_y \vec{k} + a_x b_z (-\vec{j}) + a_y b_x (-\vec{k}) + a_y b_z (\vec{i}) \\
&\quad + a_z b_x (\vec{j}) + a_z b_y (-\vec{i}) \\
&= (a_y b_z - a_z b_y) \vec{i} + (a_z b_x - a_x b_z) \vec{j} \\
&\quad + (a_x b_y - a_y b_x) \vec{k}.
\end{aligned}
$$

Zero Vector

If we add two vectors $\vec{A}$ and $\vec{B}$, we get a vector. Suppose the vectors $\vec{A}$ and $\vec{B}$ have equal magnitudes but opposite directions. What is the vector $\vec{A} + \vec{B}$? The magnitude of this vector will be zero. For mathematical consistency it is convenient to have a vector of zero magnitude although it has a little significance in physics. This vector is called zero vector. The direction of a zero vector is indeterminate. We can write this vector as $\vec{0}$. The concept of zero vector is also helpful when we consider vector product of parallel vectors. If $\vec{A} \parallel \vec{B}$, the vector $\vec{A} \times \vec{B}$ is zero vector. For any vector $\vec{A}$,

$$\vec{A} + \vec{0} = \vec{A}$$

$$\vec{A} \times \vec{0} = \vec{0}$$

and for any number λ,

$$\lambda \vec{0} = \vec{0}.$$

2.9 DIFFERENTIAL CALCULUS : $\frac{dy}{dx}$ AS RATE MEASURER

Consider two quantities y and x interrelated in such a way that for each value of x there is one and only one value of y. Figure (2.13) represents the graph

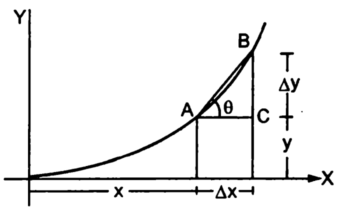


Figure 2.13

of y versus x. The value of y at a particular x is obtained by the height of the ordinate at that x. Let x be changed by a small amount Δx, and the corresponding change in y be Δy. We can define the "rate of change" of y with respect to x in the following way. When x changes by Δx, y changes by Δy so that the rate of change seems to be equal to $\frac{\Delta y}{\Delta x}$. If A be the point (x, y) and B be the point $(x + \Delta x, y + \Delta y)$, the rate $\frac{\Delta y}{\Delta x}$ equals the slope of the line AB. We have

$$\frac{\Delta y}{\Delta x} = \frac{BC}{AC} = \tan\theta.$$

However, this cannot be the precise definition of the rate. Because the rate also varies between the points A and B. The curve is steeper at B than at A. Thus, to know the rate of change of y at a particular value of x, say at A, we have to take Δx very small. However small we take Δx, as long as it is not zero the rate may vary within that small part of the curve. However, if we go on drawing the point B closer to A and everytime calculate $\frac{\Delta y}{\Delta x} = \tan\theta$, we shall see that as Δx is made smaller and smaller the slope $\tan\theta$ of the line AB approaches the slope of the tangent at A. This slope of the tangent at A thus gives the rate of change of y with respect to x at A. This rate is denoted by $\frac{dy}{dx}$. Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

For small changes Δx we can approximately write

$$\Delta y = \frac{dy}{dx} \Delta x.$$

Note that if the function y increases with an increase in x at a point, $\frac{dy}{dx}$ is positive there, because both Δy and Δx are positive. If the function y decreases with an increase in x, Δy is negative when Δx is positive. Then $\frac{\Delta y}{\Delta x}$ and hence $\frac{dy}{dx}$ is negative.

Example 2.6

From the curve given in figure (2.14) find $\frac{dy}{dx}$ at x = 2, 6 and 10.

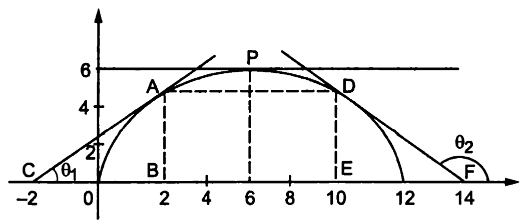


Figure 2.14

Solution : The tangent to the curve at x = 2 is AC. Its slope is $\tan\theta_1 = \frac{AB}{BC} = \frac{5}{4}$.

Thus, $\frac{dy}{dx} = \frac{5}{4}$ at $x = 2$.

The tangent to the curve at $x = 6$ is parallel to the X-axis.

Thus, $\frac{dy}{dx} = \tan\theta = 0$ at $x = 6$.

The tangent to the curve at $x = 10$ is DF. Its slope is

$$\tan\theta_2 = \frac{DE}{EF} = -\frac{5}{4}\cdot$$

Thus, $\frac{dy}{dx} = -\frac{5}{4}$ at $x = 10\cdot$

If we are given the graph of y versus x, we can find $\frac{dy}{dx}$ at any point of the curve by drawing the tangent at that point and finding its slope. Even if the graph is not drawn and the algebraic relation between y and x is given in the form of an equation, we can find $\frac{dy}{dx}$ algebraically. Let us take an example.

The area A of a square of length L is $A = L^2$.

If we change L to $L + \Delta L$, the area will change from A to $A + \Delta A$ (figure 2.15).

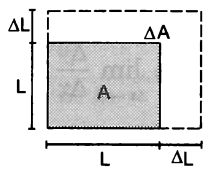


Figure 2.15

$$A + \Delta A = (L + \Delta L)^2$$
$$= L^2 + 2L\,\Delta L + (\Delta L)^2$$

or, $$\Delta A = 2L(\Delta L) + (\Delta L)^2$$

or, $$\frac{\Delta A}{\Delta L} = 2L + \Delta L.$$

Now if ΔL is made smaller and smaller, $2L + \Delta L$ will approach $2L$.

Thus, $$\frac{dA}{dL} = \lim_{\Delta L \to 0} \frac{\Delta A}{\Delta L} = 2L.$$

Table (2.1) gives the formulae for $\frac{dy}{dx}$ for some of the important functions. $\frac{dy}{dx}$ is called the differential coefficient or derivative of y with respect to x.

Table 2.1 : $\frac{dy}{dx}$ *for some common functions*

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	nx^{n-1}	$\sec x$	$\sec x \tan x$
$\sin x$	$\cos x$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cos x$	$-\sin x$	$\ln x$	$\frac{1}{x}$
$\tan x$	$\sec^2 x$	e^x	e^x
$\cot x$	$-\operatorname{cosec}^2 x$		

Besides, there are certain rules for finding the derivatives of composite functions.

(a) $\frac{d}{dx}(cy) = c\,\frac{dy}{dx}$ (c is a constant)

(b) $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

(c) $\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$

(d) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$

(e) $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

With these rules and table 2.1 derivatives of almost all the functions of practical interest may be evaluated.

Example 2.7

Find $\frac{dy}{dx}$ *if* $y = e^x \sin x$.

Solution : $y = e^x \sin x$.

So $$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x) = e^x \frac{d}{dx}(\sin x) + \sin x\,\frac{d}{dx}(e^x)$$
$$= e^x \cos x + e^x \sin x = e^x(\cos x + \sin x).$$

2.10 MAXIMA AND MINIMA

Suppose a quantity y depends on another quantity x in a manner shown in figure (2.16). It becomes maximum at x_1 and minimum at x_2.

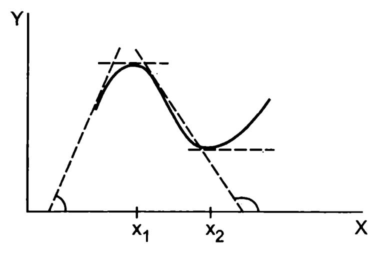


Figure 2.16

At these points the tangent to the curve is parallel to the X-axis and hence its slope is $\tan\theta = 0$. But the slope of the curve y-x equals the rate of change $\frac{dy}{dx}\cdot$ Thus, at a maximum or a minimum,

$$\frac{dy}{dx} = 0.$$

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0 \text{ at a maximum.}$$

The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}\cdot$ Thus, the condition of a maximum is

$$\left.\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{d^2y}{dx^2} &< 0\end{aligned}\right\} \text{— maximum.} \qquad \dots (2.6)$$

Similarly, at a minimum the slope changes from negative to positive. The slope increases at such a point and hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$. The condition of a minimum is

$$\left.\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{d^2y}{dx^2} &> 0\end{aligned}\right\} \text{— minimum.} \qquad \dots (2.7)$$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

Example 2.8

The height reached in time t by a particle thrown upward with a speed u is given by

$$h = ut - \frac{1}{2}gt^2$$

where $g = 9{\cdot}8\ \mathrm{m/s^2}$ is a constant. Find the time taken in reaching the maximum height.

Solution : The height h is a function of time. Thus, h will be maximum when $\frac{dh}{dt} = 0$. We have,

$$h = ut - \frac{1}{2}gt^2$$

or, $$\frac{dh}{dt} = \frac{d}{dt}(ut) - \frac{d}{dt}\left(\frac{1}{2}gt^2\right)$$

$$= u\frac{dt}{dt} - \frac{1}{2}g\frac{d}{dt}(t^2)$$

$$= u - \frac{1}{2}g(2t) = u - gt.$$

For maximum h,

$$\frac{dh}{dt} = 0$$

or, $$u - gt = 0 \quad \text{or,} \quad t = \frac{u}{g}\cdot$$

2.11 INTEGRAL CALCULUS

Let PQ be a curve representing the relation between two quantities x and y (figure 2.17). The point P corresponds to $x = a$ and Q corresponds to $x = b$. Draw perpendiculars from P and Q on the X-axis so as to cut it at A and B respectively. We are interested in finding the area $PABQ$. Let us denote the value of y at x by the symbol $y = f(x)$.

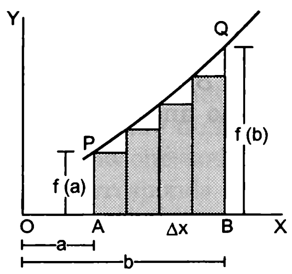


Figure 2.17

Let us divide the length AB in N equal elements each of length $\Delta x = \frac{b-a}{N}\cdot$ From the ends of each small length we draw lines parallel to the Y-axis. From the points where these lines cut the given curve, we draw short lines parallel to the X-axis. This constructs the rectangular bars shown shaded in the figure. The sum of the areas of these N rectangular bars is

$$I' = f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\Delta x)\,\Delta x + \dots$$
$$\dots + f[a + (N-1)\,\Delta x]\,\Delta x\,.$$

This may be written as

$$I' = \sum_{i=1}^{N} f(x_i)\Delta x \qquad \dots (2.8)$$

where x_i takes the values $a, a+\Delta x, a+2\Delta x, .., b-\Delta x$.

This area differs slightly from the area $PABQ$. This difference is the sum of the small triangles formed just under the curve. Now the important point is the following. As we increase the number of intervals N, the vertices of the bars touch the curve PQ at more points and the total area of the small triangles decreases. As N tends to infinity (Δx tends to zero because $\Delta x = \frac{b-a}{N}$) the vertices of the bars touch the curve at infinite number of points and the total area of the triangles tends to zero. In such a limit the sum (2.8) becomes the area I of $PABQ$. Thus, we may write,

$$I = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x.$$

The limit is taken as Δx tends to zero or as N tends to infinity. In mathematics this quantity is denoted as

$$I = \int_a^b f(x)\, dx$$

and is read as the integral of $f(x)$ with respect to x within the limits $x = a$ to $x = b$. Here a is called the lower limit and b the upper limit of integration. The integral is the sum of a large number of terms of the type $f(x)\,\Delta x$ with x continuously varying from a to b and the number of terms tending to infinity.

Let us use the above method to find the area of a trapezium. Let us suppose the line PQ is represented by the equation $y = x$.

The points A and B on the X-axis represent $x = a$ and $x = b$. We have to find the area of the trapezium $PABQ$.

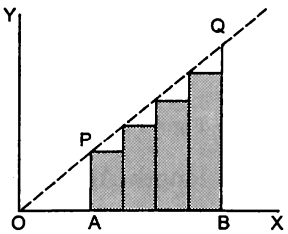


Figure 2.18

Let us divide the length AB in N equal intervals. The length of each interval is $\Delta x = \frac{b-a}{N}$. The height of the first shaded bar is $y = x = a$, of the second bar is $y = x = a + \Delta x$, that of the third bar is $y = x = a + 2\,\Delta x$ etc. The height of the Nth bar is $y = x = a + (N-1)\Delta x$. The width of each bar is Δx, so that the total area of all the bars is

$$I' = a\Delta x + (a + \Delta x)\,\Delta x + (a + 2\Delta x)\,\Delta x + \ldots$$
$$\ldots + [a + (N-1)\Delta x]\Delta x$$
$$= \big[a + (a + \Delta x) + (a + 2\Delta x) + \ldots$$
$$\ldots + \{a + (N-1)\Delta x\}\big]\Delta x \qquad \ldots (2.9)$$

This sum can be written as

$$I' = \sum_{i=1}^{N} x_i\,\Delta x$$

where $\Delta x = \frac{b-a}{N}$ and $x_i = a,\ a + \Delta x, \ldots\ b - \Delta x.$

As $\Delta x \to 0$ the total area of the bars becomes the area of the shaded part $PABQ$.

Thus, the required area is

$$I = \lim_{\Delta x \to 0} \sum_{i=1}^{N} x_i\,\Delta x$$
$$= \int_a^b x dx. \qquad \ldots \text{(i)}$$

Now the terms making the series in the square bracket in equation (2.9) are in arithmetic progression so that this series may be summed up using the formula $S = \frac{n}{2}(a + l)$. Equation (2.9) thus becomes

$$I' = \frac{N}{2}[a + \{a + (N-1)\Delta x\}]\Delta x$$
$$= \frac{N\Delta x}{2}\,[2a + N\Delta x - \Delta x]$$
$$= \frac{b-a}{2}\,[2a + b - a - \Delta x]$$
$$= \frac{b-a}{2}\,[a + b - \Delta x].$$

Thus, the area $PABQ$ is

$$I = \lim_{\Delta x \to 0}\left[\frac{b-a}{2}\right][a + b - \Delta x]$$
$$= \frac{b-a}{2}\,(a + b)$$
$$= \frac{1}{2}\,(b^2 - a^2). \qquad \ldots \text{(ii)}$$

Thus, from (i) and (ii)

$$\int_a^b x\, dx = \frac{1}{2}\,(b^2 - a^2).$$

In mathematics, special methods have been developed to find the integration of various functions $f(x)$. A very useful method is as follows. Suppose we wish to find

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{N}$; $x_i = a,\ a + \Delta x\,, \ldots\ b - \Delta x.$

Now look for a function $F(x)$ such that the derivative of $F(x)$ is $f(x)$ that is, $\frac{dF(x)}{dx} = f(x)$. If you can find such a function $F(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)\,;$$

$F(b) - F(a)$ is also written as $[F(x)]_a^b$.

$F(x)$ is called the indefinite integration or the antiderivative of $f(x)$. We also write $\int f(x)\,dx = F(x)$. This may be treated as another way of writing $\frac{dF(x)}{dx} = f(x)$.

For example, $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2}\frac{d}{dx}(x^2) = \frac{1}{2}\cdot 2x = x$.

$$\text{Thus,} \quad \int_a^b x\,dx = \left[\frac{1}{2}x^2\right]_a^b$$

$$= \left(\frac{1}{2}b^2\right) - \left(\frac{1}{2}a^2\right)$$

$$= \frac{1}{2}(b^2 - a^2)$$

as deduced above.

Table (2.2) lists some important integration formulae. Many of them are essentially same as those given in table (2.1).

Table 2.2 : *Integration Formulae*

$f(x)$	$F(x) = \int f(x)\,dx$	$f(x)$	$F(x) = \int f(x)\,dx$
$\sin x$	$-\cos x$	$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$	$\frac{1}{x}$	$\ln x$
$\sec^2 x$	$\tan x$	$\frac{1}{x^2 + a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\operatorname{cosec}^2 x$	$-\cot x$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\frac{x}{a}$
$\sec x \tan x$	$\sec x$		
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$		

Some useful rules for integration are as follows:

(a) $\int c\,f(x)\,dx = c\int f(x)\,dx$ where c is a constant

(b) Let $\int f(x)\,dx = F(x)$

then $\int f(cx)\,dx = \frac{1}{c}F(cx)$.

(c) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Example 2.9

Evaluate $\int_3^6 (2x^2 + 3x + 5)\,dx$.

Solution : $\int (2x^2 + 3x + 5)\,dx$

$$= \int 2x^2 dx + \int 3x\,dx + \int 5\,dx$$

$$= 2\int x^2 dx + 3\int x\,dx + 5\int x^0 dx$$

$$= 2\frac{x^3}{3} + 3\frac{x^2}{2} + 5\frac{x^1}{1}$$

$$= \frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x.$$

$$\text{Thus,} \quad \int_3^6 (2x^2 + 3x + 5)\,dx = \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x\right]_3^6$$

$$= \frac{2}{3}(216 - 27) + \frac{3}{2}(36 - 9) + 5(6 - 3)$$

$$= 126 + 40{\cdot}5 + 15 = 181{\cdot}5.$$

2.12 SIGNIFICANT DIGITS

When a measurement is made, a numerical value is read generally from some calibrated scale. To measure the length of a body we can place a metre scale in contact with the body. One end of the body may be made to coincide with the zero of the metre scale and the reading just in front of the other end is noted from the scale. When an electric current is measured with an ammeter the reading of the pointer on the graduation of the ammeter is noted. The value noted down includes all the digits that can be directly read from the scale and one doubtful digit at the end. The doubtful digit corresponds to the eye estimation within the smallest subdivision of the scale. This smallest subdivision is known as the *least count* of the instrument. In a metre scale, the major graduations are at an interval of one centimetre and ten subdivisions are made between two consecutive major graduations. Thus, the smallest subdivision measures a millimetre. If one end of the object coincides with the zero of the metre scale, the other end may fall between 10·4 cm and 10·5 cm mark of the scale (figure 2.19). We can estimate the distance between the 10·4 cm mark and the edge of the body as follows.

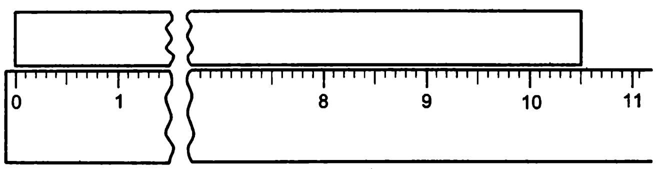


Figure 2.19

We mentally divide the 1 mm division in 10 equal parts and guess on which part is the edge falling. We may note down the reading as 10·46 cm. The digits 1, 0 and 4 are certain but 6 is doubtful. All these digits are called *significant digits*. We say that the length is measured up to four significant digits. The rightmost or the doubtful digit is called the *least significant digit* and the leftmost digit is called the *most significant digit*.

There may be some confusion if there are zeroes at the right end of the number. For example, if a measurement is quoted as 600 mm and we know nothing about the least count of the scale we cannot be sure whether the last zeros are significant or not. If the scale had marking only at each metre then the edge must be between the marks 0 m and 1 m and the digit 6 is obtained only through the eye estimation. Thus, 6 is the doubtful digit and the zeros after that are insignificant. But if the scale had markings at centimetres, the number read is 60 and these two digits are significant, the last zero is insignificant. If the scale used had markings at millimetres, all the three digits 6, 0, 0 are significant. To avoid confusion one may report only the significant digits and the magnitude may be correctly described by proper powers of 10. For example, if only 6 is significant in 600 mm we may write it as 6×10^2 mm. If 6 and the first zero are significant we may write it as $6{\cdot}0 \times 10^2$ mm and if all the three digits are significant we may write it as $6{\cdot}00 \times 10^2$ mm.

If the integer part is zero, any number of continuous zeros just after the decimal part is insignificant. Thus, the number of significant digits in 0·0023 is two and in 1·0023 is five.

2.13 SIGNIFICANT DIGITS IN CALCULATIONS

When two or more numbers are added, subtracted, multiplied or divided, how to decide about the number of significant digits in the answer? For example, suppose the mass of a body A is measured to be 12·0 kg and of another body B to be 7·0 kg. What is the ratio of the mass of A to the mass of B? Arithmetic will give this ratio as

$$\frac{12{\cdot}0}{7{\cdot}0} = 1{\cdot}714285\ldots$$

However, all the digits of this answer cannot be significant. The zero of 12·0 is a doubtful digit and the zero of 7·0 is also doubtful. The quotient cannot have so many reliable digits. The rules for deciding the number of significant digits in an arithmetic calculation are listed below.

1. In a multiplication or division of two or more quantities, the number of significant digits in the answer is equal to the number of significant digits in the quantity which has the minimum number of significant digits. Thus, $\frac{12{\cdot}0}{7{\cdot}0}$ will have two significant digits only.

The insignificant digits are dropped from the result if they appear after the decimal point. They are replaced by zeros if they appear to the left of the decimal point. The least significant digit is rounded according to the rules given below.

If the digit next to the one rounded is more than 5, the digit to be rounded is increased by 1. If the digit next to the one rounded is less than 5, the digit to be rounded is left unchanged. If the digit next to the one rounded is 5, then the digit to be rounded is increased by 1 if it is odd and is left unchanged if it is even.

2. For addition or subtraction write the numbers one below the other with all the decimal points in one line. Now locate the first column from left that has a doubtful digit. All digits right to this column are dropped from all the numbers and rounding is done to this column. The addition or subtraction is now performed to get the answer.

Example 2.10

Round off the following numbers to three significant digits (a) 15462, *(b)* 14·745, *(c)* 14·750 *and (d)* $14{\cdot}650 \times 10^{12}$.

Solution : (a) The third significant digit is 4. This digit is to be rounded. The digit next to it is 6 which is greater than 5. The third digit should, therefore, be increased by 1. The digits to be dropped should be replaced by zeros because they appear to the left of the decimal. Thus, 15462 becomes 15500 on rounding to three significant digits.

(b) The third significant digit in 14·745 is 7. The number next to it is less than 5. So 14·745 becomes 14·7 on rounding to three significant digits.

(c) 14·750 will become 14·8 because the digit to be rounded is odd and the digit next to it is 5.

(d) $14{\cdot}650 \times 10^{12}$ will become $14{\cdot}6 \times 10^{12}$ because the digit to be rounded is even and the digit next to it is 5.

Example 2.11

Evaluate $\frac{25{\cdot}2 \times 1374}{33{\cdot}3}$. *All the digits in this expression are significant.*

Solution : We have $\frac{25{\cdot}2 \times 1374}{33{\cdot}3} = 1039{\cdot}7838\ldots$

Out of the three numbers given in the expression 25·2 and 33·3 have 3 significant digits and 1374 has four. The answer should have three significant digits. Rounding 1039·7838... to three significant digits, it becomes 1040. Thus, we write

$$\frac{25{\cdot}2 \times 1374}{33{\cdot}3} = 1040.$$

Example 2.12

Evaluate $24{\cdot}36 + 0{\cdot}0623 + 256{\cdot}2.$

***Solution* :**

$$\begin{array}{r} 24{\cdot}36 \\ 0{\cdot}0623 \\ \underline{256{\cdot}2} \end{array}$$

Now the first column where a doubtful digit occurs is the one just next to the decimal point (256·2). All digits right to this column must be dropped after proper rounding. The table is rewritten and added below

$$\begin{array}{r} 24{\cdot}4 \\ 0{\cdot}1 \\ \underline{256{\cdot}2} \\ 280{\cdot}7 \end{array}$$

The sum is 280·7.

2.14 ERRORS IN MEASUREMENT

While doing an experiment several errors can enter into the results. Errors may be due to faulty equipment, carelessness of the experimenter or random causes. The first two types of errors can be removed after detecting their cause but the random errors still remain. No specific cause can be assigned to such errors.

When an experiment is repeated many times, the random errors are sometimes positive and sometimes negative. Thus, the average of a large number of the results of repeated experiments is close to the true value. However, there is still some uncertainty about the truth of this average. The uncertainty is estimated by calculating the standard deviation described below.

Let x_1, x_2, x_3, $\ldots x_N$ are the results of an experiment repeated N times. The standard deviation σ is defined as

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

where $\bar{x} = \frac{1}{N}\sum_i x_i$ is the average of all the values of x.

The best value of x derived from these experiments is $\bar{x}$ and the uncertainty is of the order of $\pm\sigma$. In fact $\bar{x} \pm 1{\cdot}96\,\sigma$ is quite often taken as the interval in which the true value should lie. It can be shown that there is a 95% chance that the true value lies within $\bar{x} \pm 1{\cdot}96\,\sigma$.

If one wishes to be more sure, one can use the interval $\bar{x} \pm 3\,\sigma$ as the interval which will contain the true value. The chances that the true value will be within $\bar{x} \pm 3\,\sigma$ is more that 99%.

All this is true if the number of observations N is large. In practice if N is greater than 8, the results are reasonably correct.

Example 2.13

The focal length of a concave mirror obtained by a student in repeated experiments are given below. Find the average focal length with uncertainty in $\pm\,\sigma$ *limit.*

No. of observation	focal length in cm
1	25·4
2	25·2
3	25·6
4	25·1
5	25·3
6	25·2
7	25·5
8	25·4
9	25·3
10	25·7

Solution : The average focal length $\bar{f} = \frac{1}{10}\sum_{i=1}^{10} f_i$

$$= 25{\cdot}37 \approx 25{\cdot}4.$$

The calculation of σ is shown in the table below:

i	f_i cm	$f_i - \bar{f}$ cm	$(f_i - \bar{f})^2$ cm^2	$\Sigma(f_i - \bar{f})^2$ cm^2
1	25·4	0·0	0·00	
2	25·2	− 0·2	0·04	
3	25·6	0·2	0·04	
4	25·1	− 0·3	0·09	
5	25·3	− 0·1	0·01	0·33
6	25·2	− 0·2	0·04	
7	25·5	0·1	0·01	
8	25·4	0·0	0·00	
9	25·3	− 0·1	0·01	
10	25·7	0·3	0·09	

$$\sigma = \sqrt{\frac{1}{10}\sum_i (f_i - \bar{f})^2} = \sqrt{0{\cdot}033\ \text{cm}^2} = 0{\cdot}18\ \text{cm}$$

$$\cong 0{\cdot}2\ \text{cm}.$$

Thus, the focal length is likely to be within (25·4 ± 0·2 cm) and we write

$$f = (25{\cdot}4 \pm 0{\cdot}2)\ \text{cm}.$$

Worked Out Examples

1. *A vector has component along the X-axis equal to* 25 unit *and along the Y-axis equal to* 60 unit. *Find the magnitude and direction of the vector.*

Solution : The given vector is the resultant of two perpendicular vectors, one along the X-axis of magnitude 25 unit and the other along the Y-axis of magnitude 60 units. The resultant has a magnitude A given by

$$A = \sqrt{(25)^2 + (60)^2 + 2 \times 25 \times 60 \cos 90°}$$
$$= \sqrt{(25)^2 + (60)^2} = 65.$$

The angle α between this vector and the X-axis is given by

$$\tan\alpha = \frac{60}{25}.$$

2. *Find the resultant of the three vectors shown in figure (2-W1).*

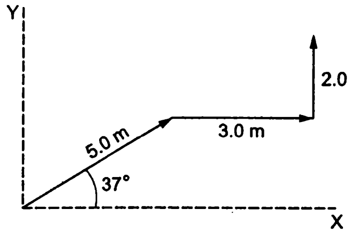


Figure 2-W1

Solution : Take the axes as shown in the figure.

The x-component of the 5·0 m vector $= 5{\cdot}0 \text{ m} \cos 37°$
$= 4{\cdot}0$ m,

the x-component of the 3·0 m vector = 3·0 m

and the x-component of the 2·0 m vector $= 2{\cdot}0 \text{ m} \cos 90°$
$= 0$.

Hence, the x-component of the resultant
= 4·0 m + 3·0 m + 0 = 7·0 m.

The y-component of the 5·0 m vector $= 5{\cdot}0 \text{ m} \sin 37°$
$= 3{\cdot}0$ m,

the y-component of the 3·0 m vector = 0
and the y-component of the 2·0 m vector = 2·0 m.

Hence, the y-component of the resultant
= 3·0 m + 0 + 2·0 m = 5·0 m.

The magnitude of the resultant vector

$$= \sqrt{(7{\cdot}0 \text{ m})^2 + (5{\cdot}0 \text{ m})^2}$$
$$= 8{\cdot}6 \text{ m}.$$

If the angle made by the resultant with the X-axis is θ, then

$$\tan\theta = \frac{y\text{-component}}{x\text{-component}} = \frac{5{\cdot}0}{7{\cdot}0} \quad \text{or,} \quad \theta = 35{\cdot}5°.$$

3. *The sum of the three vectors shown in figure (2-W2) is zero. Find the magnitudes of the vectors* $\vec{OB}$ *and* $\vec{OC}$.

Solution : Take the axes as shown in the figure

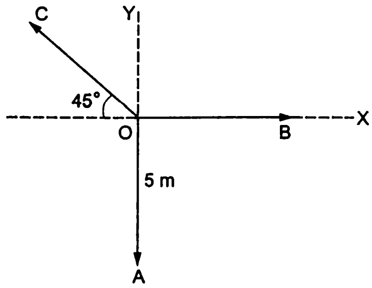


Figure 2-W2

The x-component of $\vec{OA} = (OA)\cos 90° = 0$.

The x-component of $\vec{OB} = (OB)\cos 0° = OB$.

The x-component of $\vec{OC} = (OC)\cos 135° = -\frac{1}{\sqrt{2}} OC$.

Hence, the x-component of the resultant

$$= OB - \frac{1}{\sqrt{2}} OC. \quad \text{... (i)}$$

It is given that the resultant is zero and hence its x-component is also zero. From (i),

$$OB = \frac{1}{\sqrt{2}} OC. \quad \text{... (ii)}$$

The y-component of $\vec{OA} = OA \cos 180° = -OA$.

The y-component of $\vec{OB} = OB \cos 90° = 0$.

The y-component of $\vec{OC} = OC \cos 45° = \frac{1}{\sqrt{2}} OC$.

Hence, the y-component of the resultant

$$= \frac{1}{\sqrt{2}} OC - OA \quad \text{... (iii)}$$

As the resultant is zero, so is its y-component. From (iii),

$$\frac{1}{\sqrt{2}} OC = OA, \quad \text{or,} \quad OC = \sqrt{2}\, OA = 5\sqrt{2} \text{ m}.$$

From (ii), $OB = \frac{1}{\sqrt{2}} OC = 5$ m.

4. *The magnitudes of vectors* $\vec{OA}$, $\vec{OB}$ *and* $\vec{OC}$ *in figure (2-W3) are equal. Find the direction of* $\vec{OA} + \vec{OB} - \vec{OC}$.

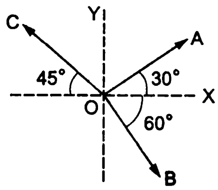


Figure 2-W3

Solution : Let $OA = OB = OC = F$.

x-component of $\vec{OA} = F\cos 30° = F\frac{\sqrt{3}}{2}$.

x-component of $\vec{OB} = F\cos 60° = \frac{F}{2}$.

x-component of $\vec{OC} = F\cos 135° = -\frac{F}{\sqrt{2}}$.

x-component of $\vec{OA} + \vec{OB} - \vec{OC}$

$$= \left(\frac{F\sqrt{3}}{2}\right) + \left(\frac{F}{2}\right) - \left(-\frac{F}{\sqrt{2}}\right)$$
$$= \frac{F}{2}(\sqrt{3} + 1 + \sqrt{2}).$$

y-component of $\vec{OA} = F\cos 60° = \frac{F}{2}$.

y-component of $\vec{OB} = F\cos 150° = -\frac{F\sqrt{3}}{2}$.

y-component of $\vec{OC} = F\cos 45° = \frac{F}{\sqrt{2}}$.

y-component of $\vec{OA} + \vec{OB} - \vec{OC}$

$$= \left(\frac{F}{2}\right) + \left(-\frac{F\sqrt{3}}{2}\right) - \left(\frac{F}{\sqrt{2}}\right)$$
$$= \frac{F}{2}(1 - \sqrt{3} - \sqrt{2}).$$

Angle of $\vec{OA} + \vec{OB} - \vec{OC}$ with the X-axis

$$= \tan^{-1}\frac{\frac{F}{2}(1 - \sqrt{3} - \sqrt{2})}{\frac{F}{2}(1 + \sqrt{3} + \sqrt{2})} = \tan^{-1}\frac{(1 - \sqrt{3} - \sqrt{2})}{(1 + \sqrt{3} + \sqrt{2})}.$$

5. *Find the resultant of the three vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ shown in figure (2-W4). Radius of the circle is R.*

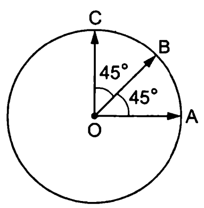


Figure 2-W4

Solution : $OA = OC$.

$\vec{OA} + \vec{OC}$ is along $\vec{OB}$ (bisector) and its magnitude is $2R\cos 45° = R\sqrt{2}$.

$(\vec{OA} + \vec{OC}) + \vec{OB}$ is along $\vec{OB}$ and its magnitude is $R\sqrt{2} + R = R(1 + \sqrt{2})$.

6. *The resultant of vectors $\vec{OA}$ and $\vec{OB}$ is perpendicular to $\vec{OA}$ (figure 2-W5). Find the angle AOB.*

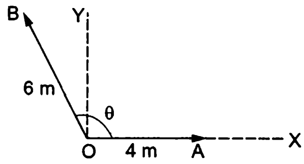


Figure 2-W5

Solution : Take the dotted lines as X, Y axes.

x-component of $\vec{OA} = 4$ m, x-component of $\vec{OB} = 6\text{ m}\cos\theta$.

x-component of the resultant $= (4 + 6\cos\theta)$ m.

But it is given that the resultant is along Y-axis. Thus, the x-component of the resultant = 0

$4 + 6\cos\theta = 0$ or, $\cos\theta = -2/3$.

7. *Write the unit vector in the direction of $\vec{A} = 5\vec{i} + \vec{j} - 2\vec{k}$.*

Solution : $|\vec{A}| = \sqrt{5^2 + 1^2 + (-2)^2} = \sqrt{30}$.

The required unit vector is $\frac{\vec{A}}{|\vec{A}|}$

$$= \frac{5}{\sqrt{30}}\vec{i} + \frac{1}{\sqrt{30}}\vec{j} - \frac{2}{\sqrt{30}}\vec{k}.$$

8. *If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ show that $\vec{a} \perp \vec{b}$.*

Solution : We have $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$
$$= a^2 + b^2 + 2\vec{a}\cdot\vec{b}.$$

Similarly,

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$
$$= a^2 + b^2 - 2\vec{a}\cdot\vec{b}.$$

If $\quad |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$,

$$a^2 + b^2 + 2\vec{a}\cdot\vec{b} = a^2 + b^2 - 2\vec{a}\cdot\vec{b}$$

or, $\quad \vec{a}\cdot\vec{b} = 0$

or, $\quad \vec{a} \perp \vec{b}$.

9. *If $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and $\vec{b} = 4\vec{i} + 3\vec{j} + 2\vec{k}$, find the angle between $\vec{a}$ and $\vec{b}$.*

Solution : We have $\quad \vec{a}\cdot\vec{b} = ab\cos\theta$

or, $\quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Now $\quad \vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

$$= 2 \times 4 + 3 \times 3 + 4 \times 2 = 25.$$

Also $\quad a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$$= \sqrt{4 + 9 + 16} = \sqrt{29}$$

and $\quad b = \sqrt{b_x^2 + b_y^2 + b_z^2} = \sqrt{16 + 9 + 4} = \sqrt{29}.$

Thus, $\cos\theta = \frac{25}{29}$

or, $\theta = \cos^{-1}\left(\frac{25}{29}\right)$.

10. *If* $\vec{A} = 2\vec{i} - 3\vec{j} + 7\vec{k}$, $\vec{B} = \vec{i} + 2\vec{k}$ *and* $\vec{C} = \vec{j} - \vec{k}$ *find* $\vec{A} \cdot (\vec{B} \times \vec{C})$.

Solution : $\vec{B} \times \vec{C} = (\vec{i} + 2\vec{k}) \times (\vec{j} - \vec{k})$

$$= \vec{i} \times (\vec{j} - \vec{k}) + 2\vec{k} \times (\vec{j} - \vec{k})$$

$$= \vec{i} \times \vec{j} - \vec{i} \times \vec{k} + 2\vec{k} \times \vec{j} - 2\vec{k} \times \vec{k}$$

$$= \vec{k} + \vec{j} - 2\vec{i} - 0 = -2\vec{i} + \vec{j} + \vec{k}.$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2\vec{i} - 3\vec{j} + 7\vec{k}) \cdot (-2\vec{i} + \vec{j} + \vec{k})$$

$$= (2)(-2) + (-3)(1) + (7)(1)$$

$$= 0.$$

11. *The volume of a sphere is given by*

$$V = \frac{4}{3}\pi R^3$$

where R is the radius of the sphere. (a) Find the rate of change of volume with respect to R. (b) Find the change in volume of the sphere as the radius is increased from 20·0 cm *to* 20·1 cm. *Assume that the rate does not appreciably change between R* = 20·0 cm *to R* = 20·1 cm.

Solution : (a) $V = \frac{4}{3}\pi R^3$

or, $\frac{dV}{dR} = \frac{4}{3}\pi \frac{d}{dr}(R)^3 = \frac{4}{3}\pi \cdot 3R^2 = 4\pi R^2.$

(b) At R = 20 cm, the rate of change of volume with the radius is

$$\frac{dV}{dR} = 4\pi R^2 = 4\pi\,(400\text{ cm}^2)$$

$$= 1600\,\pi\text{ cm}^2.$$

The change in volume as the radius changes from 20·0 cm to 20·1 cm is

$$\Delta V = \frac{dV}{dR}\Delta R$$

$$= (1600\,\pi\text{ cm}^2)(0{\cdot}1\text{ cm})$$

$$= 160\,\pi\text{ cm}^3.$$

12. *Find the derivative of the following functions with respect to x. (a)* $y = x^2 \sin x$, *(b)* $y = \frac{\sin x}{x}$ *and (c)* $y = \sin(x^2)$.

Solution :

(a) $y = x^2 \sin x$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

$$= x^2 \cos x + (\sin x)(2x)$$

$$= x(2\sin x + x\cos x).$$

(b) $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\sin x) - \sin x\left(\frac{dx}{dx}\right)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}.$$

(c) $\frac{dy}{dx} = \frac{d}{dx^2}(\sin x^2) \cdot \frac{d(x^2)}{dx}$

$$= \cos x^2 (2x)$$

$$= 2x\cos x^2.$$

13. *Find the maximum or minimum values of the function* $y = x + \frac{1}{x}$ *for* $x > 0$.

Solution : $y = x + \frac{1}{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1})$$

$$= 1 + (-x^{-2})$$

$$= 1 - \frac{1}{x^2}.$$

For y to be maximum or minimum,

$$\frac{dy}{dx} = 0$$

or, $1 - \frac{1}{x^2} = 0$

Thus, $x = 1$ or -1.

For $x > 0$ the only possible maximum or minimum is at $x = 1$. At $x = 1$, $y = x + \frac{1}{x} = 2$.

Near $x = 0$, $y = x + \frac{1}{x}$ is very large because of the term $\frac{1}{x}$. For very large x, again y is very large because of the term x. Thus $x = 1$ must correspond to a minimum. Thus, y has only a minimum for $x > 0$. This minimum occurs at $x = 1$ and the minimum value of y is $y = 2$.

14. *Figure (2-W6) shows the curve* $y = x^2$. *Find the area of the shaded part between x* = 0 *and x* = 6.

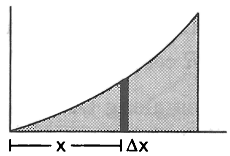


Figure 2-W6

Solution : The area can be divided into strips by drawing ordinates between $x = 0$ and $x = 6$ at a regular interval of dx. Consider the strip between the ordinates at x and $x + dx$. The height of this strip is $y = x^2$. The area of this strip is $dA = y\,dx = x^2 dx$.

The total area of the shaded part is obtained by summing up these strip-areas with x varying from 0 to 6. Thus

$$A = \int_0^6 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^6 = \frac{216 - 0}{3} = 72.$$

15. *Evaluate* $\int_0^t A \sin \omega t\,dt$ *where* A *and* ω *are constants.*

Solution :

$$\int_0^t A \sin \omega t\,dt$$

$$= A\left[\frac{-\cos \omega t}{\omega}\right]_0^t = \frac{A}{\omega}(1 - \cos \omega t).$$

16. *The velocity* v *and displacement* x *of a particle executing simple harmonic motion are related as*

$$v\frac{dv}{dx} = -\omega^2 x.$$

At $x = 0$, $v = v_0$. *Find the velocity* v *when the displacement becomes* x.

Solution : We have

$$v\frac{dv}{dx} = -\omega^2 x$$

or, $$v\,dv = -\omega^2 x\,dx$$

or, $$\int_{v_0}^{v} v\,dv = \int_0^x -\omega^2 x\,dx \qquad \ldots \text{(i)}$$

When summation is made on $-\omega^2 x\,dx$ the quantity to be varied is x. When summation is made on $v\,dv$ the quantity to be varied is v. As x varies from 0 to x the velocity varies from v_0 to v. Therefore, on the left the limits of integration are from v_0 to v and on the right they are from 0 to x. Simiplifying (i),

$$\left[\frac{1}{2}v^2\right]_{v_0}^{v} = -\omega^2\left[\frac{x^2}{2}\right]_0^x$$

or, $$\frac{1}{2}(v^2 - v_0^2) = -\omega^2\frac{x^2}{2}$$

or, $$v^2 = v_0^2 - \omega^2 x^2$$

or, $$v = \sqrt{v_0^2 - \omega^2 x^2}.$$

17. *The charge flown through a circuit in the time interval between* t *and* $t + dt$ *is given by* $dq = e^{-t/\tau}\,dt$, *where* τ *is a constant. Find the total charge flown through the circuit between* $t = 0$ *to* $t = \tau$.

Solution : The total charge flown is the sum of all the dq's for t varying from $t = 0$ to $t = t_0$. Thus, the total charge flown is

$$Q = \int_0^\tau e^{-t/\tau}\,dt$$

$$= \left[\frac{e^{-t/\tau}}{-1/\tau}\right]_0^\tau = \tau\left(1 - \frac{1}{e}\right).$$

18. *Evaluate* $(21{\cdot}6002 + 234 + 2732{\cdot}10) \times 13$.

Solution :

$$\begin{array}{r|cr} 21.6002 & & 22 \\ 234 & \Rightarrow & 234 \\ \underline{2732{\cdot}10} & & \underline{2732} \\ & & 2988 \end{array}$$

The three numbers are arranged with their decimal points aligned (shown on the left part above). The column just left to the decimals has 4 as the doubtful digit. Thus, all the numbers are rounded to this column. The rounded numbers are shown on the right part above. The required expression is $2988 \times 13 = 38844$. As 13 has only two significant digits the product should be rounded off after two significant digits. Thus the result is 39000.

□

QUESTIONS FOR SHORT ANSWER

1. Is a vector necessarily changed if it is rotated through an angle ?
2. Is it possible to add two vectors of unequal magnitudes and get zero ? Is it possible to add three vectors of equal magnitudes and get zero ?
3. Does the phrase "direction of zero vector" have physical significance ? Discuss in terms of velocity, force etc.
4. Can you add three unit vectors to get a unit vector ? Does your answer change if two unit vectors are along the coordinate axes ?

5. Can we have physical quantities having magnitude and direction which are not vectors ?

6. Which of the following two statements is more appropriate ?

(a) Two forces are added using triangle rule because force is a vector quantity.

(b) Force is a vector quantity because two forces are added using triangle rule.

7. Can you add two vectors representing physical quantities having different dimensions? Can you multiply two vectors representing physical quantities having different dimensions?

8. Can a vector have zero component along a line and still have nonzero magnitude?

9. Let ε_1 and ε_2 be the angles made by $\vec{A}$ and $-\vec{A}$ with the positive X-axis. Show that $\tan\varepsilon_1 = \tan\varepsilon_2$. Thus, giving $\tan\varepsilon$ does not uniquely determine the direction of $\vec{A}$.

10. Is the vector sum of the unit vectors $\vec{i}$ and $\vec{j}$ a unit vector ? If no, can you multiply this sum by a scalar number to get a unit vector?

11. Let $\vec{A} = 3\vec{i} + 4\vec{j}$. Write four vector $\vec{B}$ such that $\vec{A} \neq \vec{B}$ but $A = B$.

12. Can you have $\vec{A} \times \vec{B} = \vec{A} \cdot \vec{B}$ with $A \neq 0$ and $B \neq 0$? What if one of the two vectors is zero?

13. If $\vec{A} \times \vec{B} = 0$, can you say that (a) $\vec{A} = \vec{B}$, (b) $\vec{A} \neq \vec{B}$?

14. Let $\vec{A} = 5\vec{i} - 4\vec{j}$ and $\vec{B} = -7{\cdot}5\vec{i} + 6\vec{j}$. Do we have $\vec{B} = k\vec{A}$? Can we say $\frac{\vec{B}}{\vec{A}} = k$?

OBJECTIVE I

1. A vector is not changed if
(a) it is rotated through an arbitrary angle
(b) it is multiplied by an arbitrary scalar
(c) it is cross multiplied by a unit vector
(d) it is slid parallel to itself.

2. Which of the sets given below may represent the magnitudes of three vectors adding to zero?
(a) 2, 4, 8 (b) 4, 8, 16 (c) 1, 2, 1 (d) 0·5, 1, 2.

3. The resultant of $\vec{A}$ and $\vec{B}$ makes an angle α with $\vec{A}$ and β with $\vec{B}$,
(a) $\alpha < \beta$ (b) $\alpha < \beta$ if $A < B$
(c) $\alpha < \beta$ if $A > B$ (d) $\alpha < \beta$ if $A = B$.

4. The component of a vector is
(a) always less than its magnitude
(b) always greater than its magnitude
(c) always equal to its magnitude
(d) none of these.

5. A vector $\vec{A}$ points vertically upward and $\vec{B}$ points towards north. The vector product $\vec{A} \times \vec{B}$ is
(a) along west (b) along east
(c) zero (d) vertically downward.

6. The radius of a circle is stated as 2·12 cm. Its area should be written as
(a) 14 cm^2 (b) 14·1 cm^2 (c) 14·11 cm^2 (d) 14·1124 cm^2.

OBJECTIVE II

1. A situation may be described by using different sets of coordinate axes having different orientations. Which of the following do not depend on the orientation of the axes ?
(a) the value of a scalar (b) component of a vector
(c) a vector (d) the magnitude of a vector.

2. Let $\vec{C} = \vec{A} + \vec{B}$.
(a) $|\vec{C}|$ is always greater than $|\vec{A}|$
(b) It is possible to have $|\vec{C}| < |\vec{A}|$ and $|\vec{C}| < |\vec{B}|$
(c) C is always equal to $A + B$
(d) C is never equal to $A + B$.

3. Let the angle between two nonzero vectors $\vec{A}$ and $\vec{B}$ be 120° and its resultant be $\vec{C}$.
(a) C must be equal to $|A - B|$
(b) C must be less than $|A - B|$
(c) C must be greater than $|A - B|$
(d) C may be equal to $|A - B|$.

4. The x-component of the resultant of several vectors
(a) is equal to the sum of the x-components of the vectors
(b) may be smaller than the sum of the magnitudes of the vectors
(c) may be greater than the sum of the magnitudes of the vectors
(d) may be equal to the sum of the magnitudes of the vectors.

5. The magnitude of the vector product of two vectors $|\vec{A}|$ and $|\vec{B}|$ may be
(a) greater than AB (b) equal to AB
(c) less than AB (d) equal to zero.

EXERCISES

1. A vector $\vec{A}$ makes an angle of 20° and $\vec{B}$ makes an angle of 110° with the X-axis. The magnitudes of these vectors are 3 m and 4 m respectively. Find the resultant.

2. Let $\vec{A}$ and $\vec{B}$ be the two vectors of magnitude 10 unit each. If they are inclined to the X-axis at angles 30° and 60° respectively, find the resultant.

3. Add vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ each having magnitude of 100 unit and inclined to the X-axis at angles 45°, 135° and 315° respectively.

4. Let $\vec{a} = 4\vec{i} + 3\vec{j}$ and $\vec{b} = 3\vec{i} + 4\vec{j}$. (a) Find the magnitudes of (a) $\vec{a}$, (b) $\vec{b}$, (c) $\vec{a} + \vec{b}$ and (d) $\vec{a} - \vec{b}$.

5. Refer to figure (2-E1). Find (a) the magnitude, (b) x and y components and (c) the angle with the X-axis of the resultant of $\vec{OA}$, $\vec{BC}$ and $\vec{DE}$.

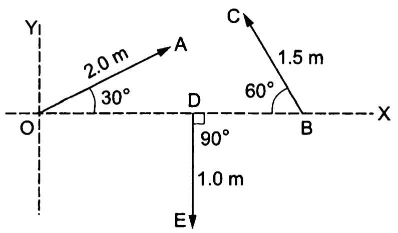


Figure 2-E1

6. Two vectors have magnitudes 3 unit and 4 unit respectively. What should be the angle between them if the magnitude of the resultant is (a) 1 unit, (b) 5 unit and (c) 7 unit.

7. A spy report about a suspected car reads as follows. "The car moved 2·00 km towards east, made a perpendicular left turn, ran for 500 m, made a perpendicular right turn, ran for 4·00 km and stopped". Find the displacement of the car.

8. A carrom board (4 ft × 4 ft square) has the queen at the centre. The queen, hit by the striker moves to the front edge, rebounds and goes in the hole behind the striking line. Find the magnitude of displacement of the queen (a) from the centre to the front edge, (b) from the front edge to the hole and (c) from the centre to the hole.

9. A mosquito net over a 7 ft × 4 ft bed is 3 ft high. The net has a hole at one corner of the bed through which a mosquito enters the net. It flies and sits at the diagonally opposite upper corner of the net. (a) Find the magnitude of the displacement of the mosquito. (b) Taking the hole as the origin, the length of the bed as the X-axis, its width as the Y-axis, and vertically up as the Z-axis, write the components of the displacement vector.

10. Suppose $\vec{a}$ is a vector of magnitude 4·5 unit due north. What is the vector (a) $3\vec{a}$, (b) $-4\vec{a}$?

11. Two vectors have magnitudes 2 m and 3 m. The angle between them is 60°. Find (a) the scalar product of the two vectors, (b) the magnitude of their vector product.

12. Let $A_1 A_2 A_3 A_4 A_5 A_6 A_1$ be a regular hexagon. Write the x-components of the vectors represented by the six sides taken in order. Use the fact that the resultant of these six vectors is zero, to prove that
$\cos 0 + \cos\pi/3 + \cos 2\pi/3 + \cos 3\pi/3 + \cos 4\pi/3 + \cos 5\pi/3 = 0$.
Use the known cosine values to verify the result.

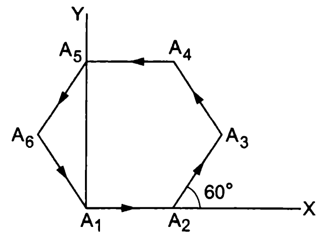


Figure 2-E2

13. Let $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} + 5\vec{k}$. Find the angle between them.

14. Prove that $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$.

15. If $\vec{A} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and $\vec{B} = 4\vec{i} + 3\vec{j} + 2\vec{k}$, find $\vec{A} \times \vec{B}$.

16. If $\vec{A}$, $\vec{B}$, $\vec{C}$ are mutually perpendicular, show that $\vec{C} \times (\vec{A} \times \vec{B}) = 0$. Is the converse true?

17. A particle moves on a given straight line with a constant speed v. At a certain time it is at a point P on its straight line path. O is a fixed point. Show that $\vec{OP} \times \vec{v}$ is independent of the position P.

18. The force on a charged particle due to electric and magnetic fields is given by $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$. Suppose $\vec{E}$ is along the X-axis and $\vec{B}$ along the Y-axis. In what direction and with what minimum speed v should a positively charged particle be sent so that the net force on it is zero?

19. Give an example for which $\vec{A} \cdot \vec{B} = \vec{C} \cdot \vec{B}$ but $\vec{A} \neq \vec{C}$.

20. Draw a graph from the following data. Draw tangents at x = 2, 4, 6 and 8. Find the slopes of these tangents. Verify that the curve drawn is $y = 2x^2$ and the slope of tangent is $\tan\theta = \frac{dy}{dx} = 4x$.

x	1	2	3	4	5	6	7	8	9	10
y	2	8	18	32	50	72	98	128	162	200

21. A curve is represented by $y = \sin x$. If x is changed from $\frac{\pi}{3}$ to $\frac{\pi}{3} + \frac{\pi}{100}$, find approximately the change in y.

22. The electric current in a charging R–C circuit is given by $i = i_0 e^{-t/RC}$ where i_0, R and C are constant parameters of the circuit and t is time. Find the rate of change of current at (a) $t = 0$, (b) $t = RC$, (c) $t = 10\,RC$.

23. The electric current in a discharging R–C circuit is given by $i = i_0 e^{-t/RC}$ where i_0, R and C are constant parameters and t is time. Let $i_0 = 2{\cdot}00$ A, $R = 6{\cdot}00 \times 10^5\ \Omega$

and $C = 0{\cdot}500\ \mu F$. (a) Find the current at $t = 0{\cdot}3$ s. (b) Find the rate of change of current at $t = 0{\cdot}3$ s. (c) Find approximately the current at $t = 0{\cdot}31$ s.

24. Find the area bounded under the curve $y = 3x^2 + 6x + 7$ and the X-axis with the ordinates at $x = 5$ and $x = 10$.

25. Find the area enclosed by the curve $y = \sin x$ and the X-axis between $x = 0$ and $x = \pi$.

26. Find the area bounded by the curve $y = e^{-x}$, the X-axis and the Y-axis.

27. A rod of length L is placed along the X-axis between $x = 0$ and $x = L$. The linear density (mass/length) ρ of the rod varies with the distance x from the origin as $\rho = a + bx$. (a) Find the SI units of a and b. (b) Find the mass of the rod in terms of a, b and L.

28. The momentum p of a particle changes with time t according to the relation $\frac{dp}{dt} = (10\text{ N}) + (2\text{ N/s})t$. If the momentum is zero at $t = 0$, what will the momentum be at $t = 10$ s?

29. The changes in a function y and the independent variable x are related as $\frac{dy}{dx} = x^2$. Find y as a function of x.

30. Write the number of significant digits in (a) 1001, (b) 100·1, (c) 100·10, (d) 0·001001.

31. A metre scale is graduated at every millimetre. How many significant digits will be there in a length measurement with this scale?

32. Round the following numbers to 2 significant digits. (a) 3472, (b) 84·16, (c) 2·55 and (d) 28·5.

33. The length and the radius of a cylinder measured with a slide callipers are found to be 4·54 cm and 1·75 cm respectively. Calculate the volume of the cylinder.

34. The thickness of a glass plate is measured to be 2·17 mm, 2·17 mm and 2·18 mm at three different places. Find the average thickness of the plate from this data.

35. The length of the string of a simple pendulum is measured with a metre scale to be 90·0 cm. The radius of the bob plus the length of the hook is calculated to be 2·13 cm using measurements with a slide callipers. What is the effective length of the pendulum? (The effective length is defined as the distance between the point of suspension and the centre of the bob.)

□

ANSWERS

OBJECTIVE I

1. (d) 2. (c) 3. (c) 4. (d) 5. (a) 6. (b)

OBJECTIVE II

1. (a), (c), (d) 2. (b) 3. (c) 4. (a), (b), (d)
5. (b), (c), (d)

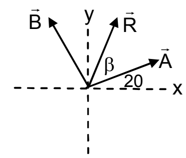
EXERCISES

1. 5 m at 73° with X-axis
2. 20 cos15° unit at 45° with X-axis
3. 100 unit at 45° with X-axis
4. (a) 5 (b) 5 (c) $7\sqrt{2}$ (d) $\sqrt{2}$
5. (a) 1·6 m (b) 0·98 m and 1·3 m respectively (c) $\tan^{-1}(1{\cdot}32)$
6. (a) 180° (b) 90° (c) 0
7. 6·02 km, $\tan^{-1}\frac{1}{12}$.
8. (a) $\frac{2}{3}\sqrt{10}$ ft (b) $\frac{4}{3}\sqrt{10}$ ft (c) $2\sqrt{2}$ ft
9. (a) $\sqrt{74}$ ft (b) 7 ft, 4 ft, 3 ft
10. (a) 13·5 unit due north (b) 18 unit due south
11. (a) 3 m^2 (b) $3\sqrt{3}$ m^2
13. $\cos^{-1}(38/\sqrt{1450})$
15. $-6\vec{i} + 12\vec{j} - 6\vec{k}$
16. no
18. along Z-axis with speed E/B
21. 0·0157
22. (a) $\frac{-i_0}{RC}$ (b) $\frac{-i_0}{RCe}$ (c) $\frac{-i_0}{RCe^{10}}$
23. (a) $\frac{2{\cdot}00}{e}$ A (b) $\frac{-20}{3e}$ A/s (c) $\frac{5{\cdot}8}{3e}$ A
24. 1135
25. 2
26. 1
27. (a) kg/m, kg/m^2 (b) $aL + bL^2/2$
28. 200 kg–m/s
29. $y = \frac{x^3}{3} + C$
30. (a) 4 (b) 4 (c) 5 (d) 4
31. 1, 2, 3 or 4
32. (a) 3500 (b) 84 (c) 2·6 (d) 28
33. 43·7 cm^3
34. 2·17 mm
35. 92·1 cm

□

SOLUTIONS TO CONCEPTS
CHAPTER – 2

1. As shown in the figure,



The angle between $\vec{A}$ and $\vec{B}$ = 110° – 20° = 90°

$|\vec{A}|$ = 3 and $|\vec{B}|$ = 4m

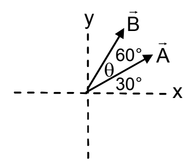
Resultant R = $\sqrt{A^2 + B^2 + 2AB\cos\theta}$ = 5 m

Let β be the angle between $\vec{R}$ and $\vec{A}$

$\beta = \tan^{-1}\left(\dfrac{4\sin 90°}{3 + 4\cos 90°}\right) = \tan^{-1}(4/3) = 53°$

∴ Resultant vector makes angle (53° + 20°) = 73° with x-axis.

2. Angle between $\vec{A}$ and $\vec{B}$ is θ = 60° – 30° =30°



$|\vec{A}|$ and $|\vec{B}|$ = 10 unit

R = $\sqrt{10^2 + 10^2 + 2.10.10.\cos 30°}$ = 19.3

β be the angle between $\vec{R}$ and $\vec{A}$

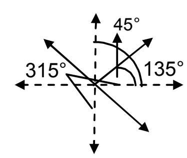
$\beta = \tan^{-1}\left(\dfrac{10\sin 30°}{10 + 10\cos 30°}\right) = \tan^{-1}\left(\dfrac{1}{2+\sqrt{3}}\right) = \tan^{-1}(0.26795) = 15°$

∴ Resultant makes 15° + 30° = 45° angle with x-axis.

3. x component of $\vec{A}$ = 100 cos 45° = $100/\sqrt{2}$ unit

x component of $\vec{B}$ = 100 cos 135° = $100/\sqrt{2}$

x component of $\vec{C}$ = 100 cos 315° = $100/\sqrt{2}$



Resultant x component = $100/\sqrt{2}$ – $100/\sqrt{2}$ + $100/\sqrt{2}$ = $100/\sqrt{2}$

y component of $\vec{A}$ = 100 sin 45° = $100/\sqrt{2}$ unit

y component of $\vec{B}$ = 100 sin 135° = $100/\sqrt{2}$

y component of $\vec{C}$ = 100 sin 315° = $-100/\sqrt{2}$

Resultant y component = $100/\sqrt{2}$ + $100/\sqrt{2}$ – $100/\sqrt{2}$ = $100/\sqrt{2}$

Resultant = 100

Tan α = $\dfrac{\text{y component}}{\text{x component}}$ = 1

⇒ α = $\tan^{-1}$ (1) = 45°

The resultant is 100 unit at 45° with x-axis.

4. $\vec{a} = 4\vec{i} + 3\vec{j}$, $\vec{b} = 3\vec{i} + 4\vec{j}$

a) $|\vec{a}| = \sqrt{4^2 + 3^2}$ = 5

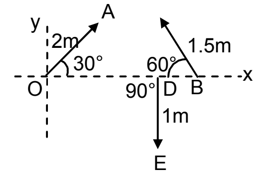
b) $|\vec{b}| = \sqrt{9 + 16}$ = 5

c) $|\vec{a} + \vec{b}| = |7\vec{i} + 7\vec{j}| = 7\sqrt{2}$

d) $\vec{a} - \vec{b} = (-3 + 4)\hat{i} + (-4 + 3)\hat{j} = \hat{i} - \hat{j}$

$|\vec{a} - \vec{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

5. x component of $\overrightarrow{OA}$ = 2cos30° = $\sqrt{3}$



x component of $\overrightarrow{BC}$ = 1.5 cos 120° = –0.75

x component of $\overrightarrow{DE}$ = 1 cos 270° = 0

y component of $\overrightarrow{OA}$ = 2 sin 30° = 1

y component of $\overrightarrow{BC}$ = 1.5 sin 120° = 1.3

y component of $\overrightarrow{DE}$ = 1 sin 270° = –1

R_x = x component of resultant = $\sqrt{3} - 0.75 + 0$ = 0.98 m

R_y = resultant y component = 1 + 1.3 – 1 = 1.3 m

So, R = Resultant = 1.6 m

If it makes and angle α with positive x-axis

Tan α = $\dfrac{\text{y component}}{\text{x component}}$ = 1.32

$\Rightarrow \alpha = \tan^{-1} 1.32$

6. $|\vec{a}|$ = 3m $|\vec{b}|$ = 4

a) If R = 1 unit $\Rightarrow \sqrt{3^2 + 4^2 + 2.3.4.\cos\theta}$ = 1

θ = 180°

b) $\sqrt{3^2 + 4^2 + 2.3.4.\cos\theta}$ = 5

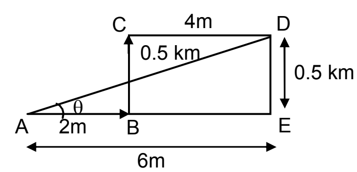
θ = 90°

c) $\sqrt{3^2 + 4^2 + 2.3.4.\cos\theta}$ = 7

θ = 0°

Angle between them is 0°.

7. $\overrightarrow{AD} = 2\hat{i} + 0.5\hat{J} + 4\hat{K} = 6\hat{i} + 0.5\hat{j}$



AD = $\sqrt{AE^2 + DE^2}$ = 6.02 KM

Tan θ = DE / AE = 1/12

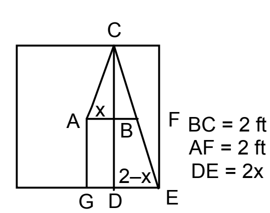
$\theta = \tan^{-1}$ (1/12)

The displacement of the car is 6.02 km along the distance $\tan^{-1}$ (1/12) with positive x-axis.

8. In ΔABC, tanθ = x/2 and in ΔDCE, tanθ = (2 – x)/4 tan θ = (x/2) = (2 – x)/4 = 4x

$\Rightarrow$ 4 – 2x = 4x

$\Rightarrow$ 6x = 4 $\Rightarrow$ x = 2/3 ft



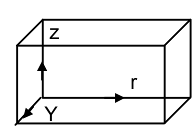
a) In ΔABC, AC = $\sqrt{AB^2 + BC^2}$ = $\dfrac{2}{3}\sqrt{10}$ ft

b) In ΔCDE, DE = 1 – (2/3) = 4/3 ft

CD = 4 ft. So, CE = $\sqrt{CD^2 + DE^2}$ = $\dfrac{4}{3}\sqrt{10}$ ft

c) In ΔAGE, AE = $\sqrt{AG^2 + GE^2}$ = $2\sqrt{2}$ ft.

9. Here the displacement vector $\vec{r} = 7\hat{i} + 4\hat{j} + 3\hat{k}$



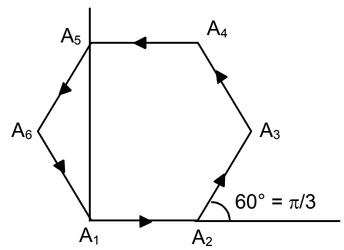
a) magnitude of displacement = $\sqrt{74}$ ft

b) the components of the displacement vector are 7 ft, 4 ft and 3 ft.

10. $\vec{a}$ is a vector of magnitude 4.5 unit due north.
   a) $3|\vec{a}| = 3 \times 4.5 = 13.5$
      $3\vec{a}$ is along north having magnitude 13.5 units.
   b) $-4|\vec{a}| = -4 \times 1.5 = -6$ unit
      $-4\vec{a}$ is a vector of magnitude 6 unit due south.

11. $|\vec{a}| = 2$ m, $|\vec{b}| = 3$ m
   angle between them $\theta = 60°$
   a) $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos 60° = 2 \times 3 \times 1/2 = 3$ m^2
   b) $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin 60° = 2 \times 3 \times \sqrt{3/2} = 3\sqrt{3}$ m^2.

12. We know that according to polygon law of vector addition, the resultant of these six vectors is zero.



   Here A = B = C = D = E = F (magnitude)
   So, Rx = A cosθ + A cos π/3 + A cos 2π/3 + A cos 3π/3 + A cos 4π/4 + A cos 5π/5 = 0
   [As resultant is zero. X component of resultant $R_x = 0$]
   = cos θ + cos π/3 + cos 2π/3 + cos 3π/3 + cos 4π/3 + cos 5π/3 = 0
   Note : Similarly it can be proved that,
      sin θ + sin π/3 + sin 2π/3 + sin 3π/3 + sin 4π/3 + sin 5π/3 = 0

13. $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$; $\vec{b} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

   $\vec{a} \cdot \vec{b} = ab\cos\theta \Rightarrow \theta = \cos^{-1}\dfrac{\vec{a} \cdot \vec{b}}{ab}$

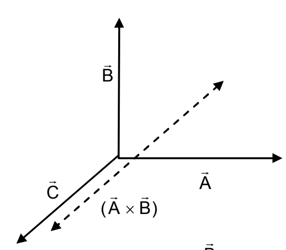
   $\Rightarrow \cos^{-1}\dfrac{2\times3+3\times4+4\times5}{\sqrt{2^2+3^2+4^2}\sqrt{3^2+4^2+5^2}} = \cos^{-1}\left(\dfrac{38}{\sqrt{1450}}\right)$

14. $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ (claim)
   As, $\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$
   AB sin θ $\hat{n}$ is a vector which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, this implies that it is also perpendicular to $\vec{A}$. As dot product of two perpendicular vector is zero.
   Thus $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$.

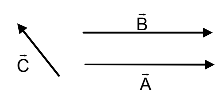
15. $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{B} = 4\hat{i} + 3\hat{j} + 2\hat{k}$

   $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{vmatrix} \Rightarrow \hat{i}(6-12) - \hat{j}(4-16) + \hat{k}(6-12) = -6\hat{i} + 12\hat{j} - 6\hat{k}$.

16. Given that $\vec{A}$, $\vec{B}$ and $\vec{C}$ are mutually perpendicular
   $\vec{A} \times \vec{B}$ is a vector which direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.
   Also $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$
   ∴ Angle between $\vec{C}$ and $\vec{A} \times \vec{B}$ is 0° or 180° (fig.1)



   So, $\vec{C} \times (\vec{A} \times \vec{B}) = 0$
   The converse is not true.
   For example, if two of the vector are parallel, (fig.2), then also

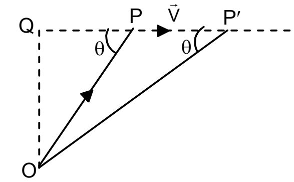


   $\vec{C} \times (\vec{A} \times \vec{B}) = 0$
   So, they need not be mutually perpendicular.

17. The particle moves on the straight line PP' at speed v.

From the figure,

$\overrightarrow{OP} \times v = (OP)v\sin\theta\hat{n}$ = v(OP) sin θ $\hat{n}$ = v(OQ) $\hat{n}$



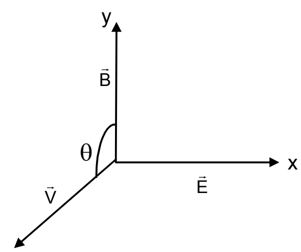
It can be seen from the figure, OQ = OP sin θ = OP' sin θ'

So, whatever may be the position of the particle, the magnitude and direction of $\overrightarrow{OP} \times \vec{v}$ remain constant.

$\therefore$ $\overrightarrow{OP} \times \vec{v}$ is independent of the position P.

18. Give $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) = 0$

$\Rightarrow \vec{E} = -(\vec{v} \times \vec{B})$



So, the direction of $\vec{v} \times \vec{B}$ should be opposite to the direction of $\vec{E}$. Hence, $\vec{v}$ should be in the positive yz-plane.

Again, E = vB sin θ ⇒ v = $\dfrac{E}{B\sin\theta}$

For v to be minimum, θ = 90° and so v_{min} = F/B

So, the particle must be projected at a minimum speed of E/B along +ve z-axis (θ = 90°) as shown in the figure, so that the force is zero.

19. For example, as shown in the figure,

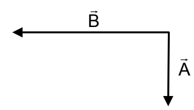
$\vec{A} \perp \vec{B}$ $\quad$ $\vec{B}$ along west

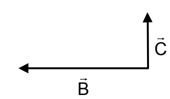
$\vec{B} \perp \vec{C}$ $\quad$ $\vec{A}$ along south

$\quad$ $\vec{C}$ along north

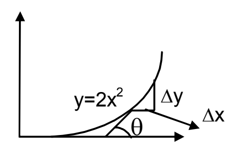
$\vec{A} \cdot \vec{B} = 0$ $\quad$ $\therefore$ $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{C}$

$\vec{B} \cdot \vec{C} = 0$ $\quad$ But $\vec{B} \neq \vec{C}$





20. The graph y = $2x^2$ should be drawn by the student on a graph paper for exact results.



To find slope at any point, draw a tangent at the point and extend the line to meet x-axis. Then find tan θ as shown in the figure.

It can be checked that,

Slope = tan θ = $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x^2)$ = 4x

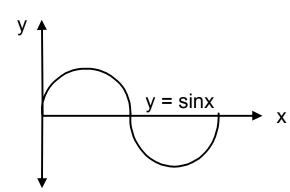
Where x = the x-coordinate of the point where the slope is to be measured.

21. y = sinx

So, y + Δy = sin (x + Δx)

Δy = sin (x + Δx) – sin x

= $\left(\dfrac{\pi}{3} + \dfrac{\pi}{100}\right) - \sin\dfrac{\pi}{3}$ = 0.0157.



22. Given that, i = $i_0 e^{-t/RC}$

$\therefore$ Rate of change of current = $\dfrac{di}{dt} = \dfrac{d}{dt} i_0 e^{-i/RC} = i_0 \dfrac{d}{dt} e^{-t/RC}$ = $\dfrac{-i_0}{RC} \times e^{-t/RC}$

When $\quad$ a) t = 0, $\dfrac{di}{dt} = \dfrac{-i}{RC}$

b) when t = RC, $\dfrac{di}{dt} = \dfrac{-i}{RCe}$

c) when t = 10 RC, $\dfrac{di}{dt} = \dfrac{-i_0}{RCe^{10}}$

23. Equation i = $i_0 e^{-t/RC}$

i_0 = 2A, R = 6 × 10^{-5} Ω, C = 0.0500 × 10^{-6} F = 5 × 10^{-7} F

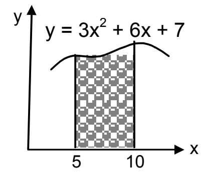
a) i = $2 \times e^{\left(\frac{-0.3}{6\times 0^3 \times 5\times 10^{-7}}\right)} = 2\times e^{\left(\frac{-0.3}{0.3}\right)} = \frac{2}{e}\text{amp}$.

b) $\frac{di}{dt} = \frac{-i_0}{RC}e^{-t/RC}$ when t = 0.3 sec ⇒ $\frac{di}{dt} = -\frac{2}{0.30}e^{(-0.3/0.3)} = \frac{-20}{3e}\text{Amp/sec}$

c) At t = 0.31 sec, i = $2e^{(-0.3/0.3)} = \frac{5.8}{3e}\text{Amp}$.

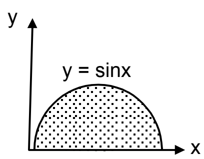
24. y = $3x^2$ + 6x + 7

∴ Area bounded by the curve, x axis with coordinates with x = 5 and x = 10 is given by,



Area = $\int_0^y dy = \int_5^{10}(3x^2 + 6x + 7)dx = 3\left.\frac{x^3}{3}\right]_5^{10} + 5\left.\frac{x^2}{3}\right]_5^{10} + 7x]_5^{10}$ = 1135 sq.units.

25. Area = $\int_0^y dy = \int_0^{\pi} \sin x dx = -[\cos x]_0^{\pi}$ = 2

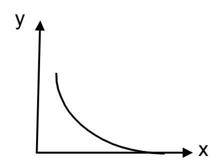


26. The given function is y = e^{-x}

When x = 0, y = e^{-0} = 1

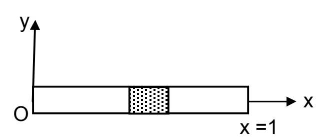
x increases, y value deceases and only at x = ∞, y = 0.

So, the required area can be found out by integrating the function from 0 to ∞.



So, Area = $\int_0^{\infty} e^{-x}dx = -[e^{-x}]_0^{\infty} = 1$.

27. $\rho = \frac{\text{mass}}{\text{length}} = a + bx$



a) S.I. unit of ‘a’ = kg/m and SI unit of ‘b’ = kg/m^2 (from principle of homogeneity of dimensions)

b) Let us consider a small element of length ‘dx’ at a distance x from the origin as shown in the figure.

∴ dm = mass of the element = ρ dx = (a + bx) dx

So, mass of the rod = m = $\int dm = \int_0^L (a + bx)dx = \left[ax + \frac{bx^2}{2}\right]_0^L = aL + \frac{bL^2}{2}$

28. $\frac{dp}{dt}$ = (10 N) + (2 N/S)t

momentum is zero at t = 0

∴ momentum at t = 10 sec will be

dp = [(10 N) + 2Ns t]dt

$\int_0^p dp = \int_0^{10} 10dt + \int_0^{10}(2tdt) = 10t]_0^{10} + 2\left.\frac{t^2}{2}\right]_0^{10}$ = 200 kg m/s.

29. The change in a function of y and the independent variable x are related as $\frac{dy}{dx} = x^2$.

$\Rightarrow$ dy = x^2 dx

Taking integration of both sides,

$\int dy = \int x^2 dx \Rightarrow y = \frac{x^3}{3} + c$

$\therefore$ y as a function of x is represented by $y = \frac{x^3}{3} + c$.

30. The number significant digits
    a) 1001 No.of significant digits = 4
    b) 100.1 No.of significant digits = 4
    c) 100.10 No.of significant digits = 5
    d) 0.001001 No.of significant digits = 4

31. The metre scale is graduated at every millimeter.

    1 m = 100 mm

    The minimum no.of significant digit may be 1 (e.g. for measurements like 5 mm, 7 mm etc) and the maximum no.of significant digits may be 4 (e.g.1000 mm)

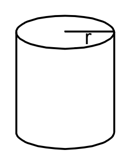
    So, the no.of significant digits may be 1, 2, 3 or 4.

32. a) In the value 3472, after the digit 4, 7 is present. Its value is greater than 5.
    So, the next two digits are neglected and the value of 4 is increased by 1.
    $\therefore$ value becomes 3500
    b) value = 84
    c) 2.6
    d) value is 28.

33. Given that, for the cylinder

    Length = l = 4.54 cm, radius = r = 1.75 cm

    Volume = $\pi r^2 l = \pi \times (4.54) \times (1.75)^2$



    Since, the minimum no.of significant digits on a particular term is 3, the result should have 3 significant digits and others rounded off.

    So, volume $V = \pi r^2 l = (3.14) \times (1.75) \times (1.75) \times (4.54) = 43.6577$ cm^3

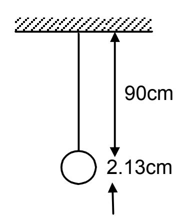
    Since, it is to be rounded off to 3 significant digits, V = 43.7 cm^3.

34. We know that,

    Average thickness = $\frac{2.17 + 2.17 + 2.18}{3}$ = 2.1733 mm

    Rounding off to 3 significant digits, average thickness = 2.17 mm.

35. As shown in the figure,



    Actual effective length = (90.0 + 2.13) cm

    But, in the measurement 90.0 cm, the no. of significant digits is only 2.

    So, the addition must be done by considering only 2 significant digits of each measurement.

    So, effective length = 90.0 + 2.1 = 92.1 cm.

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