CHAPTER 15

WAVE MOTION AND WAVES ON A STRING

15.1 WAVE MOTION

When a particle moves through space, it carries kinetic energy with itself. Wherever the particle goes, the energy goes with it. The energy is associated with the particle and is transported from one region of the space to the other together with the particle just like we ride a car and are taken from Lucknow to Varanasi with the car.

There is another way to transport energy from one part of space to the other without any bulk motion of material together with it. Sound is transmitted in air in this manner. When you say "Hello" to your friend, no material particle is ejected from your lips and falls on your friend's ear. You create some disturbance in the part of the air close to your lips. Energy is transferred to these air particles either by pushing them ahead or pulling them back. The density of the air in this part temporarily increases or decreases. These disturbed particles exert force on the next layer of air, transferring the disturbance to that layer. In this way, the disturbance proceeds in air and finally the air near the ear of the listener gets disturbed.

The disturbance produced in the air near the speaker travels in air, the air itself does not move. The air that is near the speaker at the time of uttering a word remains all the time near the speaker even when the message reaches the listener. This type of motion of energy is called a *wave motion.*

To give another example of propagation of energy without bulk motion of matter, suppose many persons are standing in a queue to buy cinema tickets from the ticket counter. It is not yet time, the counter is closed and the persons are getting annoyed. The last person in the queue is somewhat unruly, he leans forward pushing the man in front of him and then stands straight. The second last person, getting the jerk from behind, is forced to lean forward and push the man in front. This second last person manages to stand straight again but the third last person temporarily loses balance and leans forward. The jerk thus travels down the queue and finally the person at the front of the queue feels it. With the jerk, travels the energy down the queue from one end to another though the last person and the first person are still in their previous positions.

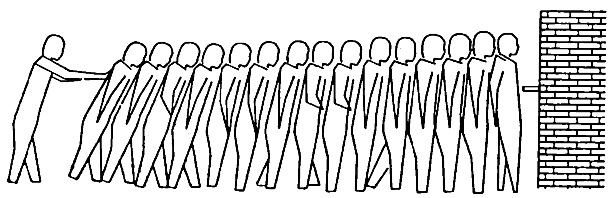


Figure 15.1

The world is full of examples of wave motion. When raindrops hit the surface of calm water, circular waves can be seen travelling on the surface. Any particle of water is only locally displaced for a short time but the disturbance spreads and the particles farther and farther get disturbed when the wave reaches them. Another common example of wave motion is the wave associated with light. One speciality about this wave is that it does not require any material medium for its propagation. The waves requiring a medium are called *mechanical waves* and those which do not require a medium are called *nonmechanical waves.*

In the present chapter, we shall study the waves on a stretched string, a mechanical wave in one dimension.

15.2 WAVE PULSE ON A STRING

Let us consider a long string with one end fixed to a wall and the other held by a person. The person pulls on the string keeping it tight. Suppose the person snaps his hand a little up and down producing a bump in the string near his hand (Figure 15.2). The operation takes a very small time say one tenth of a second after which the person stands still holding the string tight in his hand. What happens as time passes ?

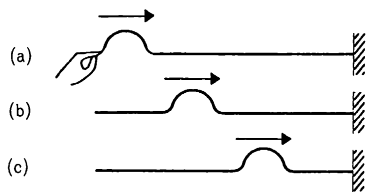


Figure 15.2

Experiments show that if the vertical displacement given is small, the disturbance travels down the string with constant speed. Figure (15.2) also shows the status of the string at successive instants. As time passes, the "bump" travels on the string towards right. For an elastic and homogeneous string, the bump moves with constant speed to cover equal distances in equal time. Also, the shape of the bump is not altered as it moves, provided the bump is small. Notice that no part of the string moves from left to right. The person is holding the left end tight and the string cannot slip from his hand. The part of the string, where the bump is present at an instant, is in up-down motion. As time passes, this part again regains its normal position. The person does some work on the part close to his hand giving some energy to that part. This disturbed part exerts elastic force on the part to the right and transfers the energy, the bump thus moves on to the right. In this way, different parts of the string are successively disturbed, transmitting the energy from left to right.

When a disturbance is localised only to a small part of space at a time, we say that a *wave pulse* is passing through that part of the space. This happens when the source producing the disturbance (hand in this case) is active only for a short time. If the source is active for some extended time repeating its motion several times, we get a *wave train* or a *wave packet.* For example, if the person in figure (15.2) decides to vibrate his hand up and down 10 times and then stop, a wave train consisting of 10 loops will proceed on the string.

Equation of a Travelling Wave

Suppose, in the example of figure (15.2), the man starts snapping his hand at $t = 0$ and finishes his job at $t = \Delta t$. The vertical displacement y of the left end of the string is a function of time. It is zero for $t < 0$, has non-zero value for $0 < t < \Delta t$ and is again zero for $t > \Delta t$. Let us represent this function by $f(t)$. Take the left end of the string as the origin and take the X-axis along the string towards right. The function $f(t)$ represents the displacement y of the particle at $x = 0$ as a function of time

$$y(x = 0, t) = f(t).$$

The disturbance travels on the string towards right with a constant speed v. Thus, the displacement, produced at the left end at time t, reaches the point x at time $t + x/v$. Similarly, the displacement of the particle at point x at time t was originated at the left end at the time $t - x/v$. But the displacement of the left end at time $t - x/v$ is $f(t - x/v)$. Hence,

$$y(x, t) = y(x = 0, t - x/v)$$
$$= f(t - x/v).$$

The displacement of the particle at x at time t i.e., $y(x, t)$ is generally abbreviated as y and the wave equation is written as

$$y = f(t - x/v). \qquad \dots (15.1)$$

Equation (15.1) represents a wave travelling in the positive x-direction with a constant speed v. Such a wave is called a *travelling wave* or a *progressive wave.* The function f is arbitrary and depends on how the source moves. The time t and the position x must appear in the wave equation in the combination $t - x/v$ only. For example,

$$y = A \sin \frac{(t - x/v)}{T}, \quad y = A e^{-\frac{(t - x/v)}{T}}$$

etc. are valid wave equations. They represent waves travelling in positive x-direction with constant speed. The equation $y = A \sin \frac{(x^2 - v^2 t^2)}{L^2}$ does not represent a wave travelling in x-direction with a constant speed.

If a wave travels in negative x-direction with speed v, its general equation may be written as

$$y = f(t + x/v). \qquad \dots (15.2)$$

The wave travelling in positive x-direction (equation 15.1) can also be written as

$$y = f\left(\frac{vt - x}{v}\right)$$

or,
$$y = g(x - vt), \qquad \dots (15.3)$$

where g is some other function having the following meaning. If we put $t = 0$ in equation (15.3), we get the displacement of various particles at $t = 0$ i.e.,

$$y(x, t = 0) = g(x).$$

Thus, $g(x)$ represents the shape of the string at $t = 0$. If the displacement of the different particles at $t = 0$ is represented by the function $g(x)$, the displacement of the particle at x at time t will be $y = g(x - vt)$. Similarly, if the wave is travelling along the negative x-direction and the displacement of different particles at $t = 0$ is $g(x)$, the displacement of the particle at x at time t will be

$$y = g(x + vt). \quad \dots (15.4)$$

Thus, the function f in equation (15.1) and (15.2) represents the displacement of the point $x = 0$ as time passes and g in (15.3) and (15.4) represents the displacement at $t = 0$ of different particles.

Example 15.1

A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given as

$$y = y_0 e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2}$$

where $y_0 = 4$ mm, $T = 1{\cdot}0$ s *and* $\lambda = 4$ cm. *(a) Find the velocity of the wave. (b) Find the function f(t) giving the displacement of the particle at x = 0. (c) Find the function g(x) giving the shape of the string at t = 0. (d) Plot the shape g(x) of the string at t = 0. (e) Plot the shape of the string at t = 5 s.*

Solution : (a) The wave equation may be written as

$$y = y_0 e^{-\frac{1}{T^2}\left(t - \frac{x}{\lambda/T}\right)^2}.$$

Comparing with the general equation $y = f(t - x/v)$, we see that

$$v = \frac{\lambda}{T} = \frac{4 \text{ cm}}{1{\cdot}0 \text{ s}} = 4 \text{ cm/s}.$$

(b) putting $x = 0$ in the given equation,

$$f(t) = y_0 e^{-(t/T)^2}. \quad \dots \text{(i)}$$

(c) putting $t = 0$ in the given equation

$$g(x) = y_0 e^{-(x/\lambda)^2}. \quad \dots \text{(ii)}$$

(d)

fig-15.3 (a)

(e)

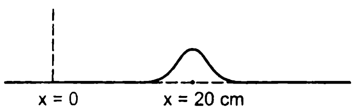


fig-15.3 (b)

15.3 SINE WAVE TRAVELLING ON A STRING

What happens if the person holding the string in figure (15.2) keeps waving his hand up and down continuously. He keeps doing work on the string and the energy is continuously supplied to the string. Any part of the string continues to vibrate up and down once the first disturbance has reached it. It receives energy from the left, transmits it to the right and the process continues till the person is not tired. The nature of vibration of any particle is similar to that of the left end, the only difference being that the motion is repeated after a time delay of x/v.

A very important special case arises when the person vibrates the left end $x = 0$ in a simple harmonic motion. The equation of motion of this end may then be written as

$$f(t) = A \sin \omega t, \quad \dots (15.5)$$

where A represents the amplitude and ω the angular frequency. The time period of oscillation is $T = 2\pi/\omega$ and the frequency of oscillation is $\nu = 1/T = \omega/2\pi$. The wave produced by such a vibrating source is called a *sine wave* or *sinusoidal wave.*

Since the displacement of the particle at $x = 0$ is given by (15.5), the displacement of the particle at x at time t will be

$$y = f(t - x/v)$$

or, $$y = A \sin \omega(t - x/v). \quad \dots (15.6)$$

This follows from the fact that the wave moves along the string with a constant speed v and the displacement of the particle at x at time t was originated at $x = 0$ at time $t - x/v$.

The velocity of the particle at x at time t is given by

$$\frac{\partial y}{\partial t} = A \omega \cos \omega(t - x/v). \quad \dots (15.7)$$

The symbol $\frac{\partial}{\partial t}$ is used in place of $\frac{d}{dt}$ to indicate that while differentiating with respect to t, we should treat x as constant. It is the same particle whose displacement should be considered as a function of time.

This velocity is totally different from the wave velocity v. The wave moves on the string at a constant velocity v along the x-axis, but the particle moves up and down with velocity $\frac{\partial y}{\partial t}$ which changes with x and t according to (15.7).

Figure (15.4) shows the shape of the string as time passes. Each particle of the string vibrates in simple harmonic motion with the same amplitude A and frequency ν. The phases of the vibrations are, however, different. When a particle P (figure 15.4) reaches its extreme position in upward direction, the particle Q little to its right, is still coming up and the particle R little to its left, has already crossed that phase and is going down. The phase difference is larger if the particles are separated farther.

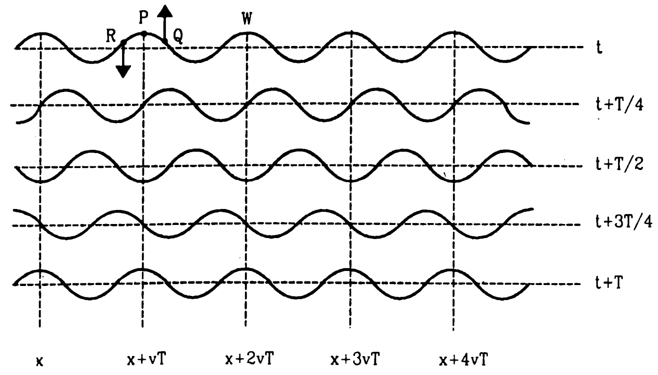


Figure 15.4

Each particle copies the motion of another particle at its left with a time delay of x/v, where x is the separation between the two particles. For the particles P and W, shown in figure (15.4), the separation is $\Delta x = vT$ and the particle W copies the motion of P after a time delay of $\Delta x/v = T$. But the motion of any particle at any instant is identical in all respects to its motion a time period T later. So, a delay of one time period is equivalent to no delay and hence, the particles P and W vibrate in the same phase. They reach their extreme positions together, they cross their mean positions together, their displacements are identical and their velocities are identical at any instant. Same is true for any pair of particles separated by a distance vT. This separation is called the *wavelength* of the wave and is denoted by the Greek letter λ. Thus, $\lambda = vT$.

The above relation can easily be derived mathematically. Suppose, the particles at x and $x + L$ vibrate in the same phase. By equation (15.6) and (15.7),

$$A \sin\left[\omega\left(t - \frac{x}{v}\right)\right] = A \sin\left[\omega\left(t - \frac{x+L}{v}\right)\right]$$

and $$A\,\omega \cos\left[\omega\left(t - \frac{x}{v}\right)\right] = A\,\omega \cos\left[\omega\left(t - \frac{x+L}{v}\right)\right].$$

This gives

$$\omega\left(t - \frac{x}{v}\right) = \omega\left(t - \frac{x+L}{v}\right) + 2\,n\,\pi,$$

where n is an integer.

or, $$0 = -\frac{\omega L}{v} + 2\,n\,\pi$$

or, $$L = \frac{v}{\omega} 2\,n\,\pi.$$

The minimum separation between the particles vibrating in same phase is obtained by putting $n = 1$ in the above equation. Thus, the wavelength is

$$\lambda = \frac{v}{\omega} 2\pi = vT. \qquad \ldots (15.8)$$

Also, $$v = \lambda/T = \nu\lambda, \qquad \ldots (15.9)$$

where $\nu = 1/T$ is the frequency of the wave.

This represents an important relation between the three characteristic parameters of a sine wave namely, the wave velocity, the frequency and the wavelength.

The quantity $2\pi/\lambda$ is called the *wave number* and is generally denoted by the letter k.

Thus, $$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{v} = \frac{\omega}{v}.$$

The segment, where the disturbance is positive, is called a *crest* of the wave and the segment, where the disturbance is negative, is called a *trough*. The separation between consecutive crests or between consecutive troughs is equal to the wavelength.

Alternative Forms of Wave Equation

We have written the wave equation of a wave travelling in x-direction as

$$y = A \sin \omega(t - x/v).$$

This can also be written in several other forms such as,

$$y = A \sin(\omega t - kx) \qquad \ldots (15.10)$$

$$= A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \ldots (15.11)$$

$$= A \sin k(vt - x). \qquad \ldots (15.12)$$

Also, it should be noted that we have made our particular choice of $t = 0$ in writing equation (15.5) from which the wave equation is deduced. The origin of time is chosen at an instant when the left end $x = 0$ is crossing its mean position $y = 0$ and is going up. For a general choice of the origin of time, we will have to add a phase constant so that the equation will be

$$y = A \sin[\omega(t - x/v) + \phi]. \qquad \ldots (15.13)$$

The constant ϕ will be $\pi/2$ if we choose $t = 0$ at an instant when the left end reaches its extreme position $y = A$. The equation will then be

$$y = A \cos \omega(t - x/v). \qquad \ldots (15.14)$$

If $t = 0$ is taken at the instant when the left end is crossing the mean position from upward to downward direction, ϕ will be π and the equation will be

$$y = A \sin \omega\left(\frac{x}{v} - t\right)$$

or, $$y = A \sin(kx - \omega t). \qquad \ldots (15.15)$$

Example 15.2

Consider the wave $y = (5\text{ mm}) \sin[(1\text{ cm}^{-1})x - (60\text{ s}^{-1})t]$. Find (a) the amplitude (b) the wave number, (c) the *wavelength, (d) the frequency, (e) the time period and (f) the wave velocity.*

Solution : Comparing the given equation with equation (15.15), we find

(a) amplitude $A = 5$ mm

(b) wave number $k = 1 \text{ cm}^{-1}$

(c) wavelength $\lambda = \frac{2\pi}{k} = 2\pi$ cm

(d) frequency $\nu = \frac{\omega}{2\pi} = \frac{60}{2\pi}$ Hz $= \frac{30}{\pi}$ Hz

(e) time period $T = \frac{1}{\nu} = \frac{\pi}{30}$ s

(f) wave velocity $v = \nu\lambda = 60$ cm/s.

15.4 VELOCITY OF A WAVE ON A STRING

The velocity of a wave travelling on a string depends on the elastic and the inertia properties of the string. When a part of the string gets disturbed, it exerts an extra force on the neighbouring part because of the elastic property. The neighbouring part responds to this force and the response depends on the inertia property. The elastic force in the string is measured by its tension F and the inertia by its mass per unit length. We have used the symbol F for tension and not T in order to avoid confusion with the time period.

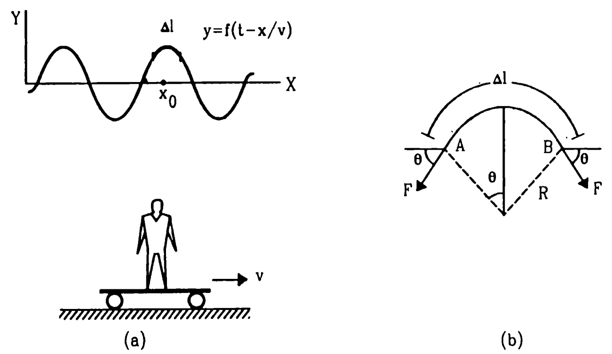


Figure 15.5

Suppose a wave $y = f\left(t - \frac{x}{v}\right)$ is travelling on the string in the positive x-direction with a speed v. Let us choose an observer who is riding on a car that moves along the x-direction with the same velocity v (figure 15.5). Looking from this frame, the pattern of the string is at rest but the entire string is moving towards the negative x-direction with a speed v. If a crest is opposite to the observer at any instant, it will always remain opposite to him with the same shape while the string will pass through this crest in opposite direction like a snake.

Consider a small element AB of the string of length Δl at the highest point of a crest. Any small curve may be approximated by a circular arc. Suppose the small element Δl forms an arc of radius R. The particles of the string in this element go in this circle with a speed v as the string slides through this part. The general situation is shown in figure (15.5a) and the expanded view of the part near Δl is shown in figure (15.5b).

We assume that the displacements are small so that the tension in the string does not appreciably change because of the disturbance. The element AB is pulled by the parts of the string to its right and to its left. Resultant force on this element is in the downward direction as shown in figure (15.5b) and its magnitude is

$$F_r = F\sin\theta + F\sin\theta = 2F\sin\theta.$$

As Δl is taken small, θ will be small and

$$\sin\theta \approx \frac{\Delta l/2}{R}$$

so that the resultant force on Δl is

$$F_r = 2F\left(\frac{\Delta l/2}{R}\right) = F\Delta l/R.$$

If μ be the mass per unit length of the string, the element AB has a mass $\Delta m = \Delta l\,\mu$. Its downward acceleration is

$$a = \frac{F_r}{\Delta m} = \frac{F\Delta l/R}{\mu\,\Delta l} = \frac{F}{\mu R}.$$

But the element is moving in a circle of radius R with a constant speed v. Its acceleration is, therefore, $a = \frac{v^2}{R}$. The above equation becomes

$$\frac{v^2}{R} = \frac{F}{\mu R}$$

or, $$v = \sqrt{F/\mu}. \qquad \dots (15.16)$$

The velocity of the wave on a string thus depends only on the tension F and the linear mass density μ. We have used the approximation that the tension F remains almost unchanged as the part of the string vibrates up and down. This approximation is valid only for small amplitudes because as the string vibrates, the lengths of its parts change during the course of vibration and hence, the tension changes.

Example 15.3

Figure (15.6) shows a string of linear mass density 1·0 g/cm on which a wave pulse is travelling. Find the

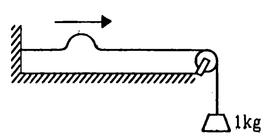


Figure 15.6

time taken by the pulse in travelling through a distance of 50 cm *on the string. Take* $g = 10 \text{ m/s}^2$.

Solution : The tension in the string is $F = mg = 10$ N. The mass per unit length is $\mu = 1{\cdot}0$ g/cm $= 0{\cdot}1$ kg/m. The wave velocity is, therefore, $v = \sqrt{F/\mu} = \sqrt{\dfrac{10\text{ N}}{0{\cdot}1\text{ kg/m}}} = 10$ m/s.

The time taken by the pulse in travelling through 50 cm is, therefore, 0·05 s.

15.5 POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

When a travelling wave is established on a string, energy is transmitted along the direction of propagation of the wave. Consider again a sine wave travelling along a stretched string in x-direction. The equation for the displacement in y-direction is

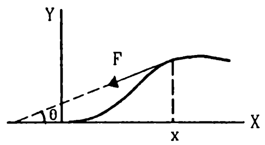


Figure 15.7

$$y = A \sin \omega(t - x/v). \qquad \text{... (i)}$$

Figure (15.7) shows a portion of the string at a time t to the right of position x. The string on the left of the point x exerts a force F on this part. The direction of this force is along the tangent to the string at position x. The component of the force along the Y-axis is

$$F_y = -F\sin\theta \approx -F\tan\theta = -F\frac{\partial y}{\partial x}.$$

The power delivered by the force F to the string on the right of position x is, therefore,

$$P = \left(-F\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial t}.$$

By (i), it is

$$-F\left[\left(-\frac{\omega}{v}\right)A\cos\omega(t - x/v)\right][\omega A\cos\omega(t - x/v)]$$

$$= \frac{\omega^2 A^2 F}{v}\cos^2\omega\,(t - x/v).$$

This is the rate at which energy is being transmitted from left to right across the point at x. The $\cos^2$ term oscillates between 0 and 1 during a cycle and its average value is 1/2. The average power transmitted across any point is, therefore,

$$P_{av} = \frac{1}{2}\frac{\omega^2 A^2 F}{v} = 2\pi^2 \mu\, v A^2 \nu^2. \qquad \text{... (15.17)}$$

The power transmitted along the string is proportional to the square of the amplitude and square of the frequency of the wave.

Example 15.4

The average power transmitted through a given point on a string supporting a sine wave is 0·20 W *when the amplitude of the wave is* 2·0 mm. *What power will be transmitted through this point if the amplitude is increased to* 3·0 mm.

Solution : Other things remaining the same, the power transmitted is proportional to the square of the amplitude.

Thus,

$$\frac{P_2}{P_1} = \frac{A_2^2}{A_1^2}$$

or, $$\frac{P_2}{0{\cdot}20\text{ W}} = \frac{9}{4} = 2{\cdot}25$$

or, $$P_2 = 2{\cdot}25 \times 0{\cdot}20\text{W} = 0{\cdot}45\text{ W}.$$

15.6 INTERFERENCE AND THE PRINCIPLE OF SUPERPOSITION

So far we have considered a single wave passing on a string. Suppose two persons are holding the string at the two ends and snap their hands to start a wave pulse each. One pulse starts from the left end and travels on the string towards right, the other starts at the right end and travels towards left. The pulses travel at same speed although their shapes depend on how the persons snap their hands. Figure (15.8) shows the shape of the string as time passes.

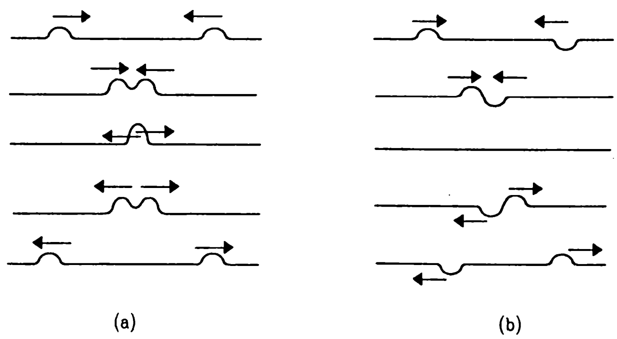


Figure 15.8

The pulses travel towards each other, overlap and recede from each other. The remarkable thing is that the shapes of the pulses, as they emerge after the overlap, are identical to their original shapes. Each pulse has passed the overlap region so smoothly as if the other pulse was not at all there. After the encounter, each pulse looks just as it looked before and each pulse travels just as it did before. The waves can pass through each other freely without being modified.

This is a unique property of the waves. The particles cannot pass through each other, they collide and their course of motion changes. How do we determine the shape of the string at the time when the pulses actually overlap? The mechanism to know the resultant displacement of a particle which is acted upon by two or more waves simultaneously is very simple. The displacement of the particle is equal to the sum of the displacements the waves would have individually produced. If the first wave alone is travelling, let us say it displaces the particle by 0·2 cm upward and if the second wave alone is travelling, suppose the displacement of this same particle is 0·4 cm upward at that instant. The displacement of the particle at that instant will be 0·6 cm upward if both the waves pass through that particle simultaneously. The displacement of the particles, if the first wave alone were travelling, may be written as

$$y_1 = f_1(t - x/v)$$

and the displacement if the second wave alone were travelling may be written as

$$y_2 = f_2(t + x/v).$$

If both the waves are travelling on the string, the displacement of its different particles will be given by

$$y = y_1 + y_2 = f_1(t - x/v) + f_2(t + x/v).$$

The two individual displacements may be in opposite directions. The magnitude of the resulting displacement may be smaller than the magnitudes of the individual displacements.

If two wave pulses, approaching each other, are identical in shape except that one is inverted with respect to the other, at some instant the displacement of all the particles will be zero. However, the velocities of the particles will not be zero as the waves will emerge in the two directions shortly. Such a situation is shown in figure (15.8b). We see that there is an instant when the string is straight every where. But soon the wave pulses emerge which move away from each other

Suppose one person snaps the end up and down whereas the other person snaps his end sideways. The displacements produced are at right angles to each other as indicated in figure (15.9). When the two waves overlap, the resultant displacement of any particle is the vector sum of the two individual displacements.

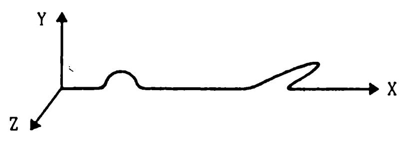


Figure 15.9

The above observations about the overlap of the waves may be summarised in the following statement which is known as the *principle of superposition.*

When two or more waves simultaneously pass through a point, the disturbance at the point is given by the sum of the disturbances each wave would produce in absence of the other wave(s).

In general, the principle of superposition is valid for small disturbances only. If the string is stretched too far, the individual displacements do not add to give the resultant displacement. Such waves are called *nonlinear waves.* In this course, we shall only be talking about linear waves which obey the superposition principle.

When two or more waves pass through the same region simultaneously we say that the waves interfere or the *interference of waves* takes place. The principle of superposition says that the phenomenon of wave interference is remarkably simple. Each wave makes its own contribution to the disturbance no matter what the other waves are doing.

15.7 INTERFERENCE OF WAVES GOING IN SAME DIRECTION

Suppose two identical sources send sinusoidal waves of same angular frequency ω in positive x-direction. Also, the wave velocity and hence, the wave number k is same for the two waves. One source may be started a little later than the other or the two sources may be situated at different points. The two waves arriving at a point then differ in phase. Let the amplitudes of the two waves be A_1 and A_2 and the two waves differ in phase by an angle δ. Their equations may be written as

$$y_1 = A_1 \sin(kx - \omega t)$$

and

$$y_2 = A_2 \sin(kx - \omega t + \delta).$$

According to the principle of superposition, the resultant wave is represented by

$$\begin{aligned} y = y_1 + y_2 &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \delta) \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t) \cos\delta \\ &\qquad + A_2 \cos(kx - \omega t)\sin\delta \\ &= \sin(kx - \omega t)\,(A_1 + A_2\cos\delta) + \cos(kx - \omega t)\,(A_2\sin\delta). \end{aligned}$$

We can evaluate it using the method described in Chapter-12 to combine two simple harmonic motions.

If we write

$$A_1 + A_2 \cos\delta = A \cos \varepsilon \quad \text{... (i)}$$

and

$$A_2 \sin\delta = A \sin \varepsilon, \quad \text{... (ii)}$$

we get

$$\begin{aligned} y &= A\,[\sin(kx - \omega t) \cos \varepsilon + \cos(kx - \omega t) \sin \varepsilon] \\ &= A \sin(kx - \omega t + \varepsilon). \end{aligned}$$

Thus, the resultant is indeed a sine wave of amplitude A with a phase difference ε with the first wave. By (i) and (ii),

$$A^2 = A^2 \cos^2\varepsilon + A^2 \sin^2\varepsilon$$
$$= (A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta$$

or, $$A = \sqrt{A_1^2 + A_2^2 + 2 A_1 A_2 \cos\delta} \,. \quad \ldots \text{(15.18)}$$

Also, $$\tan\varepsilon = \frac{A\sin\varepsilon}{A\cos\varepsilon} = \frac{A_2 \sin\delta}{A_1 + A_2\cos\delta} \,. \quad \ldots \text{(15.19)}$$

As discussed in Chapter-12, these relations may be remembered by using a geometrical model. We draw a vector of length A_1 to represent $y_1 = A_1 \sin(kx - \omega t)$ and another vector of length A_2 at an angle δ with the first one to represent $y_2 = A_2 \sin(kx - \omega t + \delta)$. The resultant of the two vectors then represents the resultant wave $y = A \sin(kx - \omega t + \varepsilon)$. Figure (15.10) shows the construction.

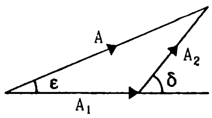


Figure 15.10

Constructive and Destructive Interference

We see from equation (15.18) that the resultant amplitude A is maximum when $\cos\delta = +1$, or $\delta = 2n\pi$ and is minimum when $\cos\delta = -1$, or $\delta = (2n+1)\pi$, where n is an integer. In the first case, the amplitude is $A_1 + A_2$ and in the second case, it is $|A_1 - A_2|$. The two cases are called *constructive* and *destructive* interferences respectively. The conditions may be written as,

$$\left.\begin{array}{l} \text{constructive interference} : \delta = 2n\pi \\ \text{destructive interference} \;\; : \delta = (2n+1)\pi \end{array}\right| \quad \ldots \text{(15.20)}$$

Example 15.5

Two waves are simultaneously passing through a string. The equations of the waves are given by

$$y_1 = A_1 \sin k(x - vt)$$

and $$y_2 = A_2 \sin k(x - vt + x_0),$$

where the wave number $k = 6{\cdot}28 \text{ cm}^{-1}$ *and* $x_0 = 1{\cdot}50$ cm. *The amplitudes are* $A_1 = 5{\cdot}0$ mm *and* $A_2 = 4{\cdot}0$ mm. *Find the phase difference between the waves and the amplitude of the resulting wave.*

Solution : The phase of the first wave is $k(x - vt)$ and of the second is $k(x - vt + x_0)$.

The phase difference is, therefore,

$$\delta = k x_0 = (6{\cdot}28 \text{ cm}^{-1})(1{\cdot}50 \text{ cm}) = 2\pi \times 1{\cdot}5 = 3\pi.$$

The waves satisfy the condition of destructive interference. The amplitude of the resulting wave is given by

$$|A_1 - A_2| = 5{\cdot}0 \text{ mm} - 4{\cdot}0 \text{ mm} = 1{\cdot}0 \text{ mm}.$$

15.8 REFLECTION AND TRANSMISSION OF WAVES

In figure (15.2), a wave pulse was generated at the left end which travelled on the string towards right. When the pulse reaches a particular element, the forces on the element from the left part of the string and from the right part act in such a way that the element is disturbed according to the shape of the pulse.

The situation is different when the pulse reaches the right end which is clamped at the wall. The element at the right end exerts a force on the clamp and the clamp exerts equal and opposite force on the element. The element at the right end is thus acted upon by the force from the string left to it and by the force from the clamp. As this end remains fixed, the two forces are opposite to each other. The force from the left part of the string transmits the forward wave pulse and hence, the force exerted by the clamp sends a return pulse on the string whose shape is similar to the original pulse but is inverted. The original pulse tries to pull the element at the fixed end up and the return pulse sent by the clamp tries to pull it down. The resultant displacement is zero. Thus, the wave is reflected from the fixed end and the reflected wave is inverted with respect to the original wave. The shape of the string at any time, while the pulse is being reflected, can be found by adding an inverted image pulse to the incident pulse (figure 15.11).

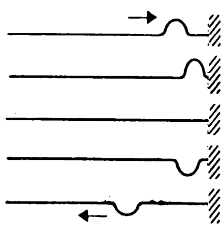


Figure 15.11

Let us now suppose that the right end of the string is attached to a light frictionless ring which can freely move on a vertical rod. A wave pulse is sent on the string from left (Figure 15.12). When the wave reaches the right end, the element at this end is acted on by the force from the left to go up. However, there is no corresponding restoring force from the right as the rod does not exert a vertical force on the ring. As a result, the right end is displaced in upward direction more than the height of the pulse i.e., it overshoots the normal maximum displacement. The lack of restoring force from right can be equivalently described in the following way. An extra force acts from right which sends a wave from right to left with its shape identical to the original one. The element at the end is acted upon by both the incident and the reflected wave and the displacements add. Thus, a wave is reflected by the free end without inversion.

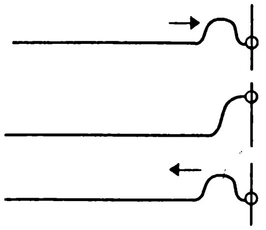


Figure 15.12

Quite often, the end point is neither completely fixed nor completly free to move. As an example, consider a light string attached to a heavier string as shown in figure (15.13). If a wave pulse is produced on the light string moving towards the junction, a part of the wave is reflected and a part is transmitted on the heavier string. The reflected wave is inverted with respect to the original one (figure 15.13a).

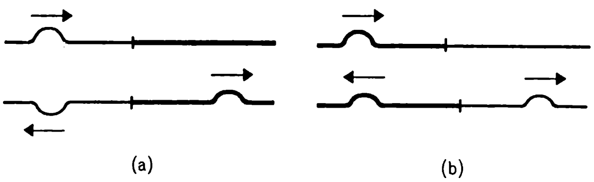


Figure 15.13

On the other hand, if the wave is produced on the heavier string, which moves towards the junction, a part will be reflected and a part transmitted, no inversion of wave shape will take place (figure 15.13b).

The rule about the inversion at reflection may be stated in terms of the wave velocity. The wave velocity is smaller for the heavier string ($v = \sqrt{F/\mu}$) and larger for the lighter string. The above observation may be stated as follows.

If a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

15.9 STANDING WAVES

Suppose two sine waves of equal amplitude and frequency propagate on a long string in opposite directions. The equations of the two waves are given by

$$y_1 = A\sin(\omega t - kx)$$

and $$y_2 = A\sin(\omega t + kx + \delta).$$

These waves interfere to produce what we call *standing waves.* To understand these waves, let us discuss the special case when $\delta = 0$.

The resultant displacements of the particles of the string are given by the principle of superposition as

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\,[\sin(\omega t - kx) + \sin(\omega t + kx)] \\ &= 2A\sin\omega t\cos kx \end{aligned}$$

or, $$y = (2A\cos kx)\sin\omega t. \qquad \text{... (15.21)}$$

This equation can be interpreted as follows. Each particle of the string vibrates in a simple harmonic motion with an amplitude $|2A\cos kx|$. The amplitudes are not equal for all the particles. In particular, there are points where the amplitude $|2A\cos kx| = 0$. This will be the case when

$$\cos kx = 0$$

or, $$kx = \left(n + \frac{1}{2}\right)\pi$$

or, $$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2},$$

where n is an integer.

For these particles, $\cos kx = 0$ and by equation (15.21) the displacement y is zero all the time. Although these points are not physically clamped, they remain fixed as the two waves pass them simultaneously. Such points are known as *nodes.*

For the points where $|\cos kx| = 1$, the amplitude is maximum. Such points are known as *antinodes.*

We also see from equation (15.21) that at a time when $\sin\omega t = 1$, all the particles for which $\cos kx$ is positive reach their positive maximum displacement. At this particular instant, all the particles for which $\cos kx$ is negative, reach their negative maximum displacement. At a time when $\sin\omega t = 0$, all the particles cross their mean positions. Figure (15.14a) shows the change in the shape of the string as time passes. Figure (15.14b) shows the external appearance of the vibrating string. This type of wave is called a *standing wave* or a *stationary wave.* The particles at nodes do not move at all and the particles at the antinodes move with maximum amplitude.

It is clear that the separation between consecutive nodes or consecutive antinodes is $\lambda/2$. As the particles at the nodes do not move at all, energy cannot be transmitted across them. The main differences between a standing wave and a travelling wave are summarised below.

1. In a travelling wave, the disturbance produced in a region propagates with a definite velocity but in a standing wave, it is confined to the region where it is produced.

2. In a travelling wave, the motion of all the particles are similar in nature. In a standing wave, different particles move with different amplitudes.

3. In a standing wave, the particles at nodes always remain in rest. In travelling waves, there is no particle which always remains in rest.

4. In a standing wave, all the particles cross their mean positions together. In a travelling wave, there is no instant when all the particles are at the mean positions together.

5. In a standing wave, all the particles between two successive nodes reach their extreme positions together, thus moving in phase. In a travelling wave, the phases of nearby particles are always different.

6. In a travelling wave, energy is transmitted from one region of space to other but in a standing wave, the energy of one region is always confined in that region.

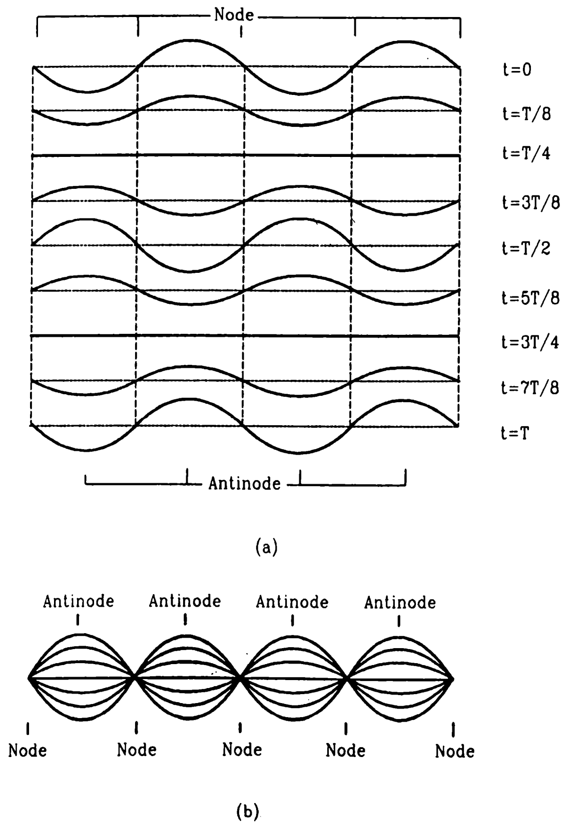


Figure 15.14

Example 15.6

Two travelling waves of equal amplitudes and equal frequencies move in opposite directions along a string. They interfere to produce a standing wave having the equation

$$y = A \cos kx \sin \omega t$$

in which $A = 1{\cdot}0$ mm, $k = 1{\cdot}57\ \mathrm{cm}^{-1}$ *and* $\omega = 78{\cdot}5\ \mathrm{s}^{-1}$. *(a) Find the velocity of the component travelling waves. (b) Find the node closest to the origin in the region* $x > 0$. *(c) Find the antinode closest to the origin in the region* $x > 0$. *(d) Find the amplitude of the particle at* $x = 2{\cdot}33$ cm.

Solution : (a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2}\sin(\omega t - kx) \quad \text{and}$$

$$y_2 = \frac{A}{2}\sin(\omega t + kx).$$

The wave velocity (magnitude) of either of the waves is

$$v = \frac{\omega}{k} = \frac{78{\cdot}5\ \mathrm{s}^{-1}}{1{\cdot}57\ \mathrm{cm}^{-1}} = 50\ \mathrm{cm/s}.$$

(b) For a node, $\cos kx = 0$.

The smallest positive x satisfying this relation is given by

$$kx = \frac{\pi}{2}$$

or, $$x = \frac{\pi}{2k} = \frac{3{\cdot}14}{2 \times 1{\cdot}57\ \mathrm{cm}^{-1}} = 1\ \mathrm{cm}.$$

(c) For an antinode, $|\cos kx| = 1$.

The smallest positive x satisfying this relation is given by

$$kx = \pi$$

or, $$x = \frac{\pi}{k} = 2\ \mathrm{cm}.$$

(d) The amplitude of vibration of the particle at x is given by $|A \cos kx|$. For the given point,

$$kx = (1{\cdot}57\ \mathrm{cm}^{-1})(2{\cdot}33\ \mathrm{cm}) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}\cdot$$

Thus, the amplitude will be

$$(1{\cdot}0\ \mathrm{mm})\ |\cos(\pi + \pi/6)| = \frac{\sqrt{3}}{2}\ \mathrm{mm} = 0{\cdot}86\ \mathrm{mm}.$$

15.10 STANDING WAVES ON A STRING FIXED AT BOTH ENDS (QUALITATIVE DISCUSSION)

Consider a string of length L fixed at one end to a wall and the other end tied to a tuning fork which vibrates longitudinally with a small amplitude (figure 15.15). The fork produces sine waves of amplitude A which travel on the string towards the fixed end and get reflected from this end. The reflected waves which travel towards the fork are inverted in shape because they are reflected from a fixed end. These waves are again reflected from the fork. As the fork is heavy and vibrates longitudinally with a small amplitude, it acts like a fixed end and the waves reflected here are again inverted in shape. Therefore, the wave produced directly by the fork at this instant and the twice reflected wave have same shape, except that the twice reflected wave has already travelled a length $2L$.

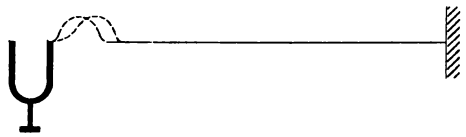


Figure 15.15

Suppose the length of the string is such that $2L = \lambda$. The two waves interfere constructively and the resultant wave that proceeds towards right has an amplitude $2A$. This wave of amplitude $2A$ is again reflected by the wall and then by the fork. This twice reflected wave again interferes constructively with the oncoming new wave and a wave of amplitude $3A$ is produced. Thus, as time passes, the amplitude keeps on increasing. The string gets energy from the vibrations of the fork and the amplitude builds up. Same arguments hold if $2L$ is any integral multiple of λ that is $L = n\lambda/2$ where n is an integer.

In the above discussion, we have neglected any loss of energy due to air viscosity or due to lack of flexibility of string etc. In actual practice, energy is lost by several processes and the loss increases as the amplitude of vibration increases. Ultimately, a balance is reached when the rate of energy received from the fork equals the rate of energy lost due to various damping processes. In the steady state, waves of constant amplitude are present on the string from left to right as well as from right to left. These waves, propagating in opposite directions, produce standing waves on the string. Nodes and antinodes are formed and the amplitudes of vibration are large at antinodes. We say that the string is in resonance with the fork. The condition, $L = n\lambda/2$, for such a resonance may be stated in a different way. We have from equation (15.9)

$$v = \nu\lambda$$

or, $$\lambda = v/\nu.$$

The condition for resonance is, therefore,

$$L = n\frac{\lambda}{2}$$

or, $$L = \frac{n\,v}{2\,\nu}$$

or, $$\nu = \frac{n\,v}{2L} = \frac{n}{2L}\sqrt{F/\mu}. \qquad \ldots (15.22)$$

The lowest frequency with which a standing wave can be set up in a string fixed at both the ends is thus

$$\nu_0 = \frac{1}{2L}\sqrt{F/\mu}. \qquad \ldots (15.23)$$

This is called the *fundamental frequency* of the string. The other possible frequencies of standing waves are integral multiples of the fundamental frequency. The frequencies given by equation (15.22) are called the *natural frequencies, normal frequencies or resonant frequencies.*

Example 15.7

A 50 cm *long wire of mass* 20 g *supports a mass of* 1·6 kg *as shown in figure (15.16). Find the fundamental frequency of the portion of the string between the wall and the pulley. Take* $g = 10\ \mathrm{m/s^2}$.

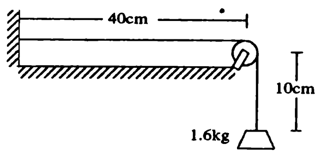


Figure 15.16

Solution **:** The tension in the string is $F = (1{\cdot}6\ \mathrm{kg})(10\ \mathrm{m/s^2})$
$= 16\ \mathrm{N}.$

The linear mass density is $\mu = \dfrac{20\ \mathrm{g}}{50\ \mathrm{cm}} = 0{\cdot}04\ \mathrm{kg/m}.$

The fundamental frequency is

$$\nu_0 = \frac{1}{2L}\sqrt{F/\mu}$$

$$= \frac{1}{2\times(0{\cdot}4\ \mathrm{m})}\sqrt{\frac{16\ \mathrm{N}}{0{\cdot}04\ \mathrm{kg/m}}} = 25\ \mathrm{Hz}.$$

What happens if the resonance condition (15.23) is not met. The phase difference between the twice reflected wave and the new wave is not an integral multiple of 2π.

In fact, the phase difference with the new wave then depends on the number of reflections suffered by the orignal wave and hence, depends on time. At certain time instants, the amplitude is enhanced and at some other time instants, the amplitude is decreased. Thus, the average amplitude does not increase by interference and the vibrations are small. The string absorbs only a little amount of energy from the source.

15.11 ANALYTIC TREATMENT OF VIBRATION OF A STRING FIXED AT BOTH ENDS

Suppose a string of length L is kept fixed at the ends $x = 0$ and $x = L$ and sine waves are produced on it. For certain wave frequencies, standing waves are set up in the string. Due to the multiple reflection at the ends and damping effects, waves going in the positive x-direction interfere to give a resultant wave

$$y_1 = A\sin(kx - \omega t).$$

Similarly, the waves going in the negative x-direction interfere to give the resultant wave

$$y_2 = A\sin(kx + \omega t + \delta).$$

The resultant displacement of the particle of the string at position x and at time t is given by the principle of superposition as

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx + \omega t + \delta)]$$

$$= 2A\sin\left(kx + \frac{\delta}{2}\right)\cos\left(\omega t + \frac{\delta}{2}\right). \quad \text{... (i)}$$

If standing waves are formed, the ends $x = 0$ and $x = L$ must be nodes because they are kept fixed. Thus, we have the boundary conditions

$$y = 0 \text{ at } x = 0 \text{ for all } t$$

and $$y = 0 \text{ at } x = L \text{ for all } t.$$

The first boundary condition is satisfied by equation (i) if $\sin\frac{\delta}{2} = 0$

or, $$\delta = 0.$$

Equation (i) then becomes

$$y = 2A\sin kx\cos\omega t \quad \text{... (15.24)}$$

The second boundary condition will be satisfied if

$$\sin kL = 0$$

or, $kL = n\pi$ where $n = 1,2,3,4,5, \dots$

or, $$\frac{2\pi L}{\lambda} = n\pi$$

or, $$L = \frac{n\lambda}{2}. \quad \text{... (15.25)}$$

If the length of the string is an integral multiple of $\lambda/2$, standing waves are produced. Again writing $\lambda = vT = \frac{v}{\nu}$, equation (15.25) becomes

$$\nu = \frac{nv}{2L} = \frac{n}{2L}\sqrt{F/\mu}$$

which is same as equation (15.22).

The lowest possible frequency is

$$\nu_0 = \frac{v}{2L} = \frac{1}{2L}\sqrt{F/\mu}. \quad \text{... (15.26)}$$

This is the fundamental frequency of the string.

The other natural frequencies with which standing waves can be formed on the string are

$\nu_1 = 2\nu_0 = \frac{2}{2L}\sqrt{F/\mu}$ 1st overtone or 2nd harmonic

$\nu_2 = 3\nu_0 = \frac{3}{2L}\sqrt{F/\mu}$ 2nd overtone or 3rd harmonic

$\nu_3 = 4\nu_0 = \frac{4}{2L}\sqrt{F/\mu}$ 3rd overtone or 4th harmonic

etc. In general, any integral multiple of the fundamental frequency is an allowed frequency. These higher frequencies are called *overtones*. Thus, $\nu_1 = 2\nu_0$ is the first overtone, $\nu_2 = 3\nu_0$ is the second overtone etc. An integral multiple of a frequency is called its *harmonic*. Thus, *for a string fixed at both the ends, all the overtones are harmonics of the fundamental frequency and all the harmonics of the fundamental frequency are overtones.*

This property is unique to the string and makes it so valuable in musical instruments such as violin, guitar, sitar, santoor, sarod etc.

Normal Modes of Vibration

When a string vibrates according to equation (15.24) with some natural frequency, it is said to vibrate in a *normal mode*. For the nth normal mode $k = \frac{n\pi}{L}$ and the equation for the displacement is, from equation (15.24),

$$y = 2A\sin\frac{n\pi x}{L}\cos\omega t. \quad \text{... (15.27)}$$

For fundamental mode, $n = 1$ and the equation of the standing wave is, from (15.27),

$$y = 2A\sin\frac{\pi x}{L}\cos\omega t.$$

The amplitude of vibration of the particle at x is $2A\sin(\pi x/L)$ which is zero at $x = 0$ and at $x = L$. It is maximum at $x = L/2$ where $\sin(\pi x/L) = 1$. Thus, we have nodes at the ends and just one antinode at the middle point of the string.

In the first overtone, also known as the second harmonic, the constant n is equal to 2 and equation(15.27) becomes

$$y = 2A\sin\frac{2\pi x}{L}\cos\omega t.$$

The amplitude $2A\sin\frac{2\pi x}{L}$ is zero at $x = 0$, $L/2$ and L and is maximum at $L/4$ and $3L/4$. The middle point of the string is also a node and is not displaced during the vibration. The points $x = L/4$ and $x = 3L/4$ are the antinodes.

In the second overtone, $n = 3$ and equation(15.27) becomes

$$y = 2A\sin\frac{3\pi x}{L}\cos\omega t.$$

The nodes are at $x = 0$, $L/3$, $2L/3$ and L where $\sin \frac{3\pi x}{L} = 0$. There are two nodes in between the ends. Antinodes occur midway between the nodes, i.e., at $x = L/6$, $L/2$ and $5L/6$.

Similarly, in the nth overtone, there are n nodes between the ends and $n+1$ antinodes midway between the nodes. The shape of the string as it vibrates in a normal mode is shown in figure (15.17) for some of the normal modes.

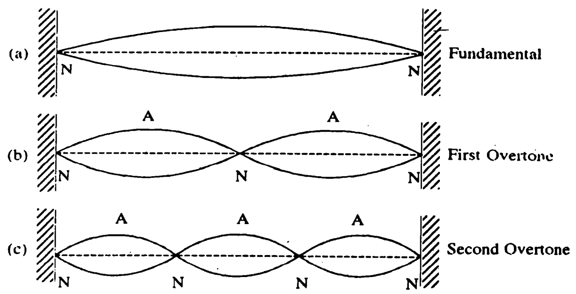


Figure 15.17

When the string of a musical instrument such as a sitar is plucked aside at some point, its shape does not correspond to any of the normal modes discussed above. In fact, the shape of the string is a combination of several normal modes and thus, a combination of frequencies are emitted.

15.12 VIBRATION OF A STRING FIXED AT ONE END

Standing waves can be produced on a string which is fixed at one end and whose other end is free to move in a transverse direction. Such a free end can be nearly achieved by connecting the string to a very light thread.

If the vibrations are produced by a source of "correct" frequency, standing waves are produced. If the end $x = 0$ is fixed and $x = L$ is free, the equation is again given by (15.24)

$$y = 2A \sin kx \cos \omega t$$

with the boundary condition that $x = L$ is an antinode. The boundary condition that $x = 0$ is a node is automatically satisfied by the above equation. For $x = L$ to be an antinode,

$$\sin kL = \pm 1$$

or, $$kL = \left(n + \frac{1}{2}\right)\pi$$

or, $$\frac{2\pi L}{\lambda} = \left(n + \frac{1}{2}\right)\pi$$

or, $$\frac{2L\nu}{v} = n + \frac{1}{2}$$

or, $$\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L} = \frac{n + \frac{1}{2}}{2L}\sqrt{F/\mu}. \quad \text{... (15.28)}$$

These are the normal frequencies of vibration. The fundamental frequency is obtained when $n = 0$, i.e.,

$$\nu_0 = v/4L$$

The overtone frequencies are

$$\nu_1 = \frac{3v}{4L} = 3\nu_0$$

$$\nu_2 = \frac{5v}{4L} = 5\nu_0$$

$$\nu_3 = \frac{7v}{4L} = 7\nu_0 \quad \text{etc.}$$

We see that all the harmonics of the fundamental are not the allowed frequencies for the standing waves. Only the odd harmonics are the overtones. Figure (15.18) shows shapes of the string for some of the normal modes.

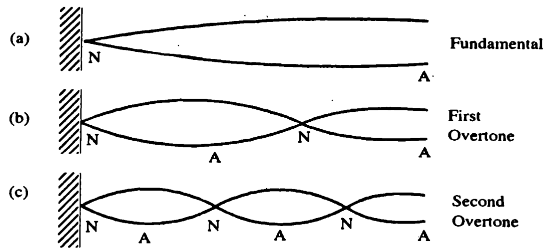


Figure 15.18

15.13 LAWS OF TRANSVERSE VIBRATIONS OF A STRING : SONOMETER

The fundamental frequency of vibration of a string fixed at both ends is given by equation (15.26). From this equation, one can immediately write the following statements known as "Laws of transverse vibrations of a string".

(a) Law of length – *The fundamental frequency of vibration of a string (fixed at both ends) is inversely proportional to the length of the string provided its tension and its mass per unit length remain the same.*

$$\nu \propto 1/L \text{ if } F \text{ and } \mu \text{ are constants.}$$

(b) Law of tension – *The fundamental frequency of a string is proportional to the square root of its tension provided its length and the mass per unit length remain the same.*

$$\nu \propto \sqrt{F} \text{ if } L \text{ and } \mu \text{ are constants.}$$

(c) Law of mass – *The fundamental frequency of a string is inversely proportional to the square root of the linear mass density, i.e., mass per unit length provided the length and the tension remain the same.*

$$\nu \propto \frac{1}{\sqrt{\mu}} \text{ if } L \text{ and } F \text{ are constants.}$$

These laws may be experimentally studied with an apparatus called *sonometer.*

A typical design of a sonometer is shown in figure (15.19). One has a wooden box, also called the sound box, on which two bridges A and B are fixed at the ends. A metal wire C is welded with the bridges and is kept tight. This wire C is called the auxiliary wire. Another wire D, called the experimental wire is fixed at one end to the bridge A and passes over the second bridge B to hold a hanger H on which suitable weights can be put. Two small movable bridges C_1 and C_2 may slide under the auxiliary wire and another two movable bridges D_1 and D_2 may slide under the experimental wire.

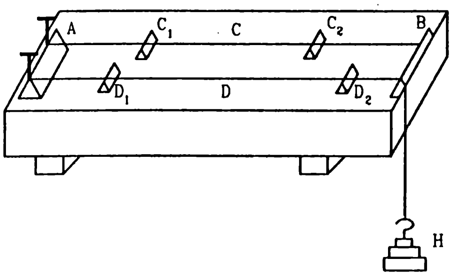


Figure 15.19

The portion of the wire between the movable bridges forms the "string" fixed at both ends. By sliding these bridges, the length of the wire may be changed. The tension of the experimental wire D may be changed by changing the weights on the hanger. One can remove the experimental wire itself and put another wire in its place thereby changing the mass per unit length.

The waves can be produced on the wire by vibrating a tuning fork (by holding its stem and gently hitting a prong on a rubber pad) and pressing its stem on the platform of the sound box of the sonometer. The simple harmonic disturbance is transmitted to the wire through the bridges. The frequency of vibration is same as that of the tuning fork. If this frequency happens to be equal to one of the natural frequencies of the wire, standing waves with large amplitudes are set up on it. The tuning fork is then said to be in "resonance" or in "unison" with the wire.

How can one identify whether the tuning fork is in resonance with the wire or not? A simple method is to place a small piece of paper (called a paper rider) at the middle point of the wire between the movable bridges. When vibrations in the wire are induced by putting the tuning fork in contact with the board, the paper-piece also vibrates. If the tuning fork is in resonance with the fundamental mode of vibration of the wire, the paper-piece is at the antinode. Because of the large amplitude of the wire there, it violently shakes and quite often jumps off the wire. Thus, the resonance can be detected just by visible inspection.

The paper-piece is also at an antinode if the wire is vibrating in its 3rd harmonic, although the amplitude will not be as large as it would be in the fundamental mode. The paper-piece may shake but not that violently.

Another good method to detect the resonance is based on the interference of sound waves of different frequencies. The tuning fork is sounded by gently hitting a prong on a rubber pad and the wire is plucked by hand. The resultant sound shows a periodic increase and decrease in intensity if the frequency of the fork is close (but not exactly equal) to one of the natural frequencies of the wire. This periodic variation in intensity is called *beats* that we shall study in the next chapter. The length is then only slightly varied till the beats disappear and that ensures resonance.

Law of Length

To study the law of length, only the experimental wire is needed. The wire is put under a tension by placing suitable weights (say 3 to 4 kg) on the hanger.

A tuning fork is vibrated and the length of the wire is adjusted by moving the movable bridges such that the fork is in resonance with the fundamental mode of vibration of the wire. The frequency ν of the tuning fork and the length l of the wire resonating with it are noted. The experiment is repeated with different tuning forks and the product νl is evaluated for each fork which should be a constant by the law of length.

Law of Tension

To study the law of tension, one may proceed as follows. A particular length of the experimental wire is selected by keeping the movable bridges D_1, D_2 fixed. The auxiliary wire is plucked. The vibration is transmitted to the experimental wire through the sound box. By adjusting the movable bridges C_1 and C_2, the fundamental frequency of the auxiliary wire is made equal to the fundamental frequency of the experimental wire by testing that the two wires resonate with each other. The tension in the experimental wire is changed and the length of the auxiliary wire is again adjusted to resonate with it. The experiment is repeated several times with different tensions and the corresponding lengths of the auxiliary wire are noted. Suppose l' represents the length of the auxiliary wire resonating with the fixed length of the experimental wire when the tension in it is T. Also suppose ν is the frequency of vibration of the wires in their fundamental modes in this situation. Then,

$$\nu \propto \frac{1}{l'} \text{ according to the law of length}$$

and $\nu \propto \sqrt{T}$ according to the law of tension.

Hence, $l' \propto 1/\sqrt{T}$.

The product $l'\sqrt{T}$ may be evaluated from the experiments which should be a constant.

Why do we have to use the auxiliary wire in the above scheme and not a tuning fork ? That is because, to adjust for the resonance, the variable quantity should be continuously changeable. As the length of the experimental wire is kept fixed and its frequency is to be compared as a function of tension, we need a source whose frequency can be continuously changed. Choosing different tuning forks to change the frequency will not work as the forks are available for descrete frequencies only.

Law of Mass

To study the law of mass, the length and the tension are to be kept constant and the mass per unit length is to be changed. Again, the auxiliary wire is used to resonate with the fixed length of the experimental wire as was suggested during the study of the law of tension. A fixed length of the experimental wire is chosen between the bridges D_1 and D_2 and a fixed tension is applied to it. The auxiliary wire is given a tension by hanging a certain load and its length is adjusted so that it resonates with the experimental wire. The experiment is repeated with different experimental wires keeping equal lengths between the movable bridges and applying equal tension. Each time the length l' of the auxiliary wire is adjusted to bring it in resonance with the experimental wire. The mass per unit length of each experimental wire is obtained by weighing a known length of the wire. We have

$\nu \propto 1/l'$ according to the law of length

and $\nu \propto 1/\sqrt{\mu}$ according to the law of mass.

Thus, $l' \propto \sqrt{\mu}$.

The law of mass is thus studied by obtaining $\frac{l'}{\sqrt{\mu}}$ each time which should be a constant.

Example 15.8

In a sonometer experiment, resonance is obtained when the experimental wire has a length of 21 cm *between the bridges and the vibrations are excited by a tuning fork of frequency* 256 Hz. *If a tuning fork of frequency* 384 Hz *is used, what should be the length of the experimental wire to get the resonance?*

Solution : By the law of length, $l_1 \nu_1 = l_2 \nu_2$

or, $$l_2 = \frac{\nu_1}{\nu_2} l_1 = \frac{256}{384} \times 21 \text{ cm} = 14 \text{ cm}.$$

15.14 TRANSVERSE AND LONGITUDINAL WAVES

The wave on a string is caused by the displacements of the particles of the string. These displacements are in a direction perpendicular to the direction of propagation of the wave. If the disturbance produced in a wave has a direction perpendicular to the direction of propagation of the wave, the wave is called a *transverse wave.* The wave on a string is a transverse wave. Another example of transverse wave is the light wave. It is the electric field which changes its value with space and time and the changes are propagated in space. The direction of the electric field is perpendicular to the direction of propagation of light when light travels in free space.

Sound waves are not transverse. The particles of the medium are pushed and pulled along the direction of propagation of sound. We shall study in some detail the mechanism of sound waves in the next chapter. If the disturbance produced as the wave passes is along the direction of the wave propagation, the wave is called a *longitudinal wave.* Sound waves are longitudinal.

All the waves cannot be characterised as either longitudinal or transverse. A very common example of a wave that is neither longitudinal nor transverse is a wave on the surface of water. When water in a steady lake is disturbed by shaking a finger in it, waves are produced on the water surface. The water particles move in elliptic or circular path as the wave passes them. The elliptic motion has components both along and perpendicular to the direction of propagation of the wave.

15.15 POLARIZATION OF WAVES

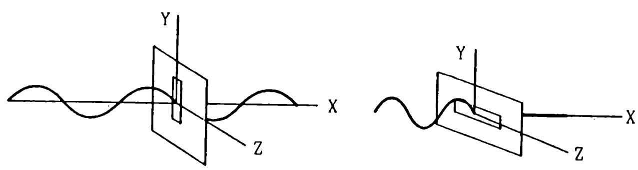


Figure 15.20

Suppose a stretched string goes through a slit made in a cardboard which is placed perpendicular to the string (figure 15.20). If we take the X-axis along the string, the cardboard will be in Y-Z plane. Suppose the particles of the string are displaced in y-direction as the wave passes. If the slit in the cardboard is also along the Y-axis, the part of the string in the slit can vibrate freely in the slit and the wave will pass through the slit. However, if the cardboard is rotated by 90° in its plane, the slit will point along the Z-axis. As the wave arrives at the slit, the part of the string in it tries to move along the Y-axis but the contact force by the cardboard does not allow it. The wave is not able to pass through the slit. If the slit is inclined to the Y-axis at some other angle, only a part of the wave is transmitted and in the transmitted wave the disturbance is produced parallel to the slit. Figure (15.21) suggests the same arrangement with two chairs.

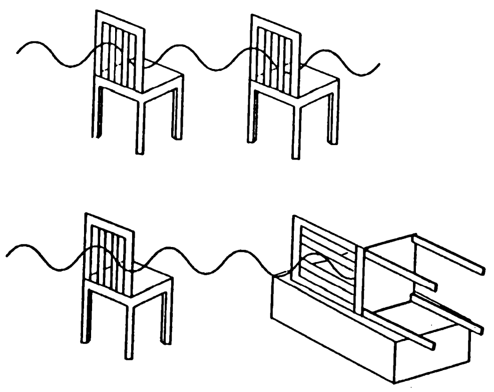


Figure 15.21

If the disturbance produced is always along a fixed direction, we say that the wave is linearly polarized in that direction. The waves considered in this chapter are linearly polarized in y-direction. Similarly, if a wave produces displacement along the z-direction, its equation is given by $z = A \sin \omega(t - x/v)$ and it is a linearly polarized wave, polarized in z-direction. Linearly polarized waves are also called *plane polarized.*

If each particle of the string moves in a small circle as the wave passes through it, the wave is called *circularly polarized.* If each particle goes in ellipse, the wave is called *elliptically polarized.*

Finally, if the particles are randomly displaced in the plane perpendicular to the direction of propagation, the wave is called *unpolarized.*

A circularly polarized or unpolarized wave passing through a slit does not show change in intensity as the slit is rotated in its plane. But the transmitted wave becomes linearly polarized in the direction parallel to the slit.

Worked Out Examples

1. *The displacement of a particle of a string carrying a travelling wave is given by*

$$y = (3{\cdot}0 \text{ cm}) \sin 6{\cdot}28(0{\cdot}50x - 50\, t),$$

where x is in centimeter and t in second. Find (a) the amplitude, (b) the wavelength, (c) the frequency and (d) the speed of the wave.

Solution **:** Comparing with the standard wave equation

$$y = A\sin(kx - \omega t)$$

$$= A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

we see that,

amplitude $= A = 3{\cdot}0$ cm,

wavelength $= \lambda = \dfrac{1}{0{\cdot}50}$ cm $= 2{\cdot}0$ cm,

and the frequency $= \nu = \dfrac{1}{T} = 50$ Hz.

The speed of the wave is $v = \nu\lambda$

$$= (50 \text{ s}^{-1})(2{\cdot}0 \text{ cm})$$

$$= 100 \text{ cm/s}.$$

2. *The equation for a wave travelling in x-direction on a string is*

$$y = (3{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})\, x - (314 \text{ s}^{-1})t].$$

(a) Find the maximum velocity of a particle of the string.

(b) Find the acceleration of a particle at $x = 6{\cdot}0$ cm at time $t = 0{\cdot}11$ s.

Solution **:**

(a) The velocity of the particle at x at time t is

$$v = \frac{\partial y}{\partial t} = (3{\cdot}0 \text{ cm})(-314 \text{ s}^{-1}) \cos[(3{\cdot}14 \text{ cm}^{-1})\, x - (314 \text{ s}^{-1})t]$$

$$= (-9{\cdot}4 \text{ m/s}) \cos[(3{\cdot}14 \text{ cm}^{-1})\, x - (314 \text{ s}^{-1})t].$$

The maximum velocity of a particle will be

$$v = 9{\cdot}4 \text{ m/s}.$$

(b) The acceleration of the particle at x at time t is

$$a = \frac{\partial v}{\partial t} = -(9{\cdot}4 \text{ m/s})(314 \text{ s}^{-1}) \sin[(3{\cdot}14 \text{ cm}^{-1})\, x - (314 \text{ s}^{-1})t]$$

$$= -(2952 \text{ m/s}^2) \sin[(3{\cdot}14 \text{ cm}^{-1})\, x - (314 \text{ s}^{-1})t].$$

The acceleration of the particle at $x = 6{\cdot}0$ cm at time $t = 0{\cdot}11$ s is $a = -(2952 \text{ m/s}^2) \sin[6\pi - 11\pi] = 0.$

3. *A long string having a cross-sectional area* $0{\cdot}80\ \text{mm}^2$ *and density* $12{\cdot}5\ \text{g/cm}^3$, *is subjected to a tension of* 64 N *along the X-axis. One end of this string is attached to a vibrator moving in transverse direction at a frequency of* 20 Hz. *At* $t = 0$, *the source is at a maximum displacement* $y = 1{\cdot}0$ cm. *(a) Find the speed of the wave travelling on the string. (b) Write the equation for the wave. (c) What is the displacement of the particle of the string at* $x = 50$ cm *at time* $t = 0{\cdot}05$ s? *(d) What is the velocity of this particle at this instant?*

Solution :

(a) The mass of 1 m long part of the string is

$$m = (0{\cdot}80\ \text{mm}^2) \times (1\ \text{m}) \times (12{\cdot}5\ \text{g/cm}^3)$$
$$= (0{\cdot}80 \times 10^{-6}\ \text{m}^3) \times (12{\cdot}5 \times 10^3\ \text{kg/m}^3)$$
$$= 0{\cdot}01\ \text{kg}.$$

The linear mass density is $\mu = 0{\cdot}01$ kg/m. The wave speed is $v = \sqrt{F/\mu}$

$$= \sqrt{\frac{64\ \text{N}}{0{\cdot}01\ \text{kg/m}}} = 80\ \text{m/s}.$$

(b) The amplitude of the source is $A = 1{\cdot}0$ cm and the frequency is $\nu = 20$ Hz. The angular frequency is $\omega = 2\pi\nu = 40\pi\text{s}^{-1}$. Also at $t = 0$, the displacement is equal to its amplitude i.e., at $t = 0, x = A$. The equation of motion of the source is, therefore,

$$y = (1{\cdot}0\ \text{cm}) \cos[(40\pi\ \text{s}^{-1})\, t]. \qquad \text{... (i)}$$

The equation of the wave travelling on the string along the positive X-axis is obtained by replacing t with $t - x/v$ in equation (i). It is, therefore,

$$y = (1{\cdot}0\ \text{cm}) \cos\left[(40\pi\text{s}^{-1})\left(t - \frac{x}{v}\right)\right]$$
$$= (1{\cdot}0\ \text{cm}) \cos\left[(40\pi\text{s}^{-1})t - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)x\right], \qquad \text{... (ii)}$$

where the value of v has been put from part (a).

(c) The displacement of the particle at $x = 50$ cm at time $t = 0{\cdot}05$ s is by equation (ii),

$$y = (1{\cdot}0\ \text{cm}) \cos[(40\pi\ \text{s}^{-1})(0{\cdot}05\ \text{s}) - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)(0{\cdot}5\ \text{m})]$$
$$= (1{\cdot}0\ \text{cm}) \cos\left[2\pi - \frac{\pi}{4}\right]$$
$$= \frac{1{\cdot}0\ \text{cm}}{\sqrt{2}} = 0{\cdot}71\ \text{cm}.$$

(d) The velocity of the particle at position x at time t is, by equation (ii),

$$v = \frac{\partial y}{\partial t} = -(1{\cdot}0\ \text{cm})(40\pi\ \text{s}^{-1}) \sin\left[(40\pi\ \text{s}^{-1})\, t - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)x\right].$$

Putting the values of x and t,

$$v = -(40\pi\ \text{cm/s}) \sin\left(2\pi - \frac{\pi}{4}\right)$$
$$= \frac{40\pi}{\sqrt{2}}\ \text{cm/s} \approx 89\ \text{cm/s}.$$

4. *The speed of a transverse wave, going on a wire having a length* 50 cm *and mass* 5·0 g, *is* 80 m/s. *The area of cross-section of the wire is* $1{\cdot}0\ \text{mm}^2$ *and its Young's modulus is* $16 \times 10^{11}\ \text{N/m}^2$. *Find the extension of the wire over its natural length.*

Solution : The linear mass density is

$$\mu = \frac{5 \times 10^{-3}\ \text{kg}}{50 \times 10^{-2}\ \text{m}} = 1{\cdot}0 \times 10^{-2}\ \frac{\text{kg}}{\text{m}}.$$

The wave speed is $v = \sqrt{F/\mu}$.

Thus, the tension is $F = \mu v^2$

$$= \left(1{\cdot}0 \times 10^{-2}\ \frac{\text{kg}}{\text{m}}\right) \times 6400\ \frac{\text{m}^2}{\text{s}^2} = 64\ \text{N}.$$

The Young's modulus is given by

$$Y = \frac{F/A}{\Delta L/L}.$$

The extension is, therefore,

$$\Delta L = \frac{FL}{AY}$$
$$= \frac{(64\ \text{N})(0{\cdot}50\ \text{m})}{(1{\cdot}0 \times 10^{-6}\text{m}^2) \times (16 \times 10^{11}\ \text{N/m}^2)} = 0{\cdot}02\ \text{mm}.$$

5. *A uniform rope of length* 12 m *and mass* 6 kg *hangs vertically from a rigid support. A block of mass* 2 kg *is attached to the free end of the rope. A transverse pulse of wavelength* 0·06 m *is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope?*

Solution : As the rope is heavy, its tension will be different at different points. The tension at the free end will be (2 kg)g and that at the upper end it will be (8 kg)g.

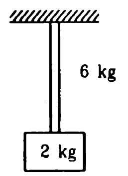


Figure 15-W1

We have, $\qquad v = \nu\lambda$

or, $\qquad \sqrt{F/\mu} = \nu\lambda$

or, $\qquad \sqrt{F}/\lambda = \nu\sqrt{\mu}. \qquad$ ... (i)

The frequency of the wave pulse will be the same everywhere on the rope as it depends only on the frequency of the source. The mass per unit length is also the same throughout the rope as it is uniform. Thus, by (i), $\dfrac{\sqrt{F}}{\lambda}$ is constant.

Hence, $$\frac{\sqrt{(2\ \text{kg})g}}{0{\cdot}06\ \text{m}} = \frac{\sqrt{(8\ \text{kg})g}}{\lambda_1},$$

where λ_1 is the wavelength at the top of the rope. This gives $\lambda_1 = 0{\cdot}12$ m.

6. *Two waves passing through a region are represented by*

$$y = (1{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})x - (157 \text{ s}^{-1})t]$$

and $$y = (1{\cdot}5 \text{ cm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x - (314 \text{ s}^{-1})t].$$

Find the displacement of the particle at $x = 4{\cdot}5$ cm *at time* $t = 5{\cdot}0$ ms.

Solution : According to the principle of superposition, each wave produces its disturbance independent of the other and the resultant disturbance is equal to the vector sum of the individual disturbances. The displacements of the particle at $x = 4{\cdot}5$ cm at time $t = 5{\cdot}0$ ms due to the two waves are,

$$y_1 = (1{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})(4{\cdot}5 \text{ cm}) - (157 \text{ s}^{-1})(5{\cdot}0 \times 10^{-3} \text{ s})]$$

$$= (1{\cdot}0 \text{ cm}) \sin\left[4{\cdot}5\pi - \frac{\pi}{4}\right]$$

$$= (1{\cdot}0 \text{ cm}) \sin[4\pi + \pi/4] = \frac{1{\cdot}0 \text{ cm}}{\sqrt{2}}$$

and

$$y_2 = (1{\cdot}5 \text{ cm}) \sin[(1{\cdot}57 \text{ cm}^{-1})(4{\cdot}5 \text{ cm}) - (314 \text{ s}^{-1})(5{\cdot}0 \times 10^{-3} \text{ s})]$$

$$= (1{\cdot}5 \text{ cm}) \sin\left[2{\cdot}25\pi - \frac{\pi}{2}\right]$$

$$= (1{\cdot}5 \text{ cm}) \sin[2\pi - \pi/4]$$

$$= -(1{\cdot}5 \text{ cm}) \sin\frac{\pi}{4} = -\frac{1{\cdot}5 \text{ cm}}{\sqrt{2}}.$$

The net displacement is

$$y = y_1 + y_2 = \frac{-0{\cdot}5 \text{ cm}}{\sqrt{2}} = -0{\cdot}35 \text{ cm}.$$

7. *The vibrations of a string fixed at both ends are described by the equation*

$$y = (5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x] \sin[(314 \text{ s}^{-1})t].$$

(a) What is the maximum displacement of the particle at $x = 5{\cdot}66$ cm ? *(b) What are the wavelengths and the wave speeds of the two transverse waves that combine to give the above vibration ? (c) What is the velocity of the particle at* $x = 5{\cdot}66$ cm *at time* $t = 2{\cdot}00$ s ? *(d) If the length of the string is* 10·0 cm, *locate the nodes and the antinodes. How many loops are formed in the vibration ?*

Solution :

(a) The amplitude of the vibration of the particle at position x is

$$A = |\,(5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x]\,|.$$

For $x = 5{\cdot}66$ cm,

$$A = \left|(5{\cdot}00 \text{ mm}) \sin\left[\frac{\pi}{2} \times 5{\cdot}66\right]\right|$$

$$= \left|(5{\cdot}00 \text{ mm}) \sin\left(2{\cdot}5\pi + \frac{\pi}{3}\right)\right|$$

$$= \left|(5{\cdot}00 \text{ mm}) \cos\frac{\pi}{3}\right| = 2{\cdot}50 \text{ mm}.$$

(b) From the given equation, the wave number $k = 1{\cdot}57$ cm^{-1} and the angular frequency $\omega = 314$ s^{-1}. Thus, the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2 \times 3{\cdot}14}{1{\cdot}57 \text{ cm}^{-1}} = 4{\cdot}00 \text{ cm}$$

and the frequency is $\nu = \frac{\omega}{2\pi} = \frac{314 \text{ s}^{-1}}{2 \times 3{\cdot}14} = 50 \text{ s}^{-1}$.

The wave speed is $v = \nu\lambda = (50 \text{ s}^{-1})(4{\cdot}00 \text{ cm}) = 2{\cdot}00$ m/s.

(c) The velocity of the particle at position x at time t is given by

$$v = \frac{\partial y}{\partial t} = (5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x] [314 \text{ s}^{-1} \cos(314 \text{ s}^{-1})t]$$

$$= (157 \text{ cm/s}) \sin(1{\cdot}57 \text{ cm}^{-1})x \cos(314 \text{ s}^{-1})t.$$

Putting $x = 5{\cdot}66$ cm and $t = 2{\cdot}00$ s, the velocity of this particle at the given instant is

$$(157 \text{ cm/s}) \sin\left(\frac{5\pi}{2} + \frac{\pi}{3}\right) \cos(200\,\pi)$$

$$= (157 \text{ cm/s}) \times \cos\frac{\pi}{3} \times 1 = 78{\cdot}5 \text{ cm/s}.$$

(d) the nodes occur where the amplitude is zero i.e.,

$$\sin(1{\cdot}57 \text{ cm}^{-1})x = 0.$$

or, $$\left(\frac{\pi}{2} \text{ cm}^{-1}\right)x = n\pi,$$

where n is an integer.

Thus, $$x = 2\,n \text{ cm}.$$

The nodes, therefore, occur at $x = 0$, 2 cm, 4 cm, 6 cm, 8 cm and 10 cm. Antinodes occur in between them i.e., at $x = 1$ cm, 3 cm, 5 cm, 7 cm and 9 cm. The string vibrates in 5 loops.

8. *A guitar string is* 90 cm *long and has a fundamental frequency of* 124 Hz. *Where should it be pressed to produce a fundamental frequency of* 186 Hz ?

Solution : The fundamental frequency of a string fixed at both ends is given by

$$\nu = \frac{1}{2L}\sqrt{\frac{F}{\mu}}.$$

As F and μ are fixed, $\frac{\nu_1}{\nu_2} = \frac{L_2}{L_1}$

or, $$L_2 = \frac{\nu_1}{\nu_2} L_1 = \frac{124 \text{ Hz}}{186 \text{ Hz}} (90 \text{ cm}) = 60 \text{ cm}.$$

Thus, the string should be pressed at 60 cm from an end.

9. *A sonometer wire has a total length of* 1 m *between the fixed ends. Where should the two bridges be placed below the wire so that the three segments of the wire have their fundamental frequencies in the ratio* 1 : 2 : 3 ?

Solution : Suppose the lengths of the three segments are L_1, L_2 and L_3 respectively. The fundamental frequencies are

$$\nu_1 = \frac{1}{2L_1}\sqrt{F/\mu}$$

$$\nu_2 = \frac{1}{2L_2}\sqrt{F/\mu}$$

$$\nu_3 = \frac{1}{2L_3}\sqrt{F/\mu}$$

so that $\nu_1 L_1 = \nu_2 L_2 = \nu_3 L_3$. ... (i)

As $\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3$, we have

$\nu_2 = 2\nu_1$ and $\nu_3 = 3\nu_1$ so that by (i)

$$L_2 = \frac{\nu_1}{\nu_2} L_1 = \frac{L_1}{2}$$

and $$L_3 = \frac{\nu_1}{\nu_3} L_1 = \frac{L_1}{3}.$$

As $L_1 + L_2 + L_3 = 1$ m,

we get $$L_1\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1 \text{ m}$$

or, $$L_1 = \frac{6}{11} \text{ m}.$$

Thus, $$L_2 = \frac{L_1}{2} = \frac{3}{11} \text{ m}$$

and $$L_3 = \frac{L_1}{3} = \frac{2}{11} \text{ m}.$$

One bridge should be placed at $\frac{6}{11}$ m from one end and the other should be placed at $\frac{2}{11}$ m from the other end.

10. *A wire having a linear mass density* $5{\cdot}0 \times 10^{-3}$ kg/m *is stretched between two rigid supports with a tension of* 450 N. *The wire resonates at a frequency of* 420 Hz. *The next higher frequency at which the same wire resonates is* 490 Hz. *Find the length of the wire.*

Solution : Suppose the wire vibrates at 420 Hz in its nth harmonic and at 490 Hz in its $(n+1)$th harmonic.

$$420 \text{ s}^{-1} = \frac{n}{2L}\sqrt{F/\mu} \quad \text{... (i)}$$

and $$490 \text{ s}^{-1} = \frac{(n+1)}{2L}\sqrt{F/\mu}. \quad \text{... (ii)}$$

This gives $\frac{490}{420} = \frac{n+1}{n}$

or, $n = 6$.

Putting the value in (i),

$$420 \text{ s}^{-1} = \frac{6}{2L}\sqrt{\frac{450 \text{ N}}{5{\cdot}0 \times 10^{-3} \text{ kg/m}}} = \frac{900}{L} \text{ m/s}$$

or, $$L = \frac{900}{420} \text{ m} = 2{\cdot}1 \text{ m}.$$

□

QUESTIONS FOR SHORT ANSWER

1. You are walking along a seashore and a mild wind is blowing. Is the motion of air a wave motion ?
2. The radio and TV programmes, telecast at the studio, reach our antenna by wave motion. Is it a mechanical wave or nonmechanical ?
3. A wave is represented by an equation $y = c_1 \sin(c_2 x + c_3 t)$. In which direction is the wave going ? Assume that c_1, c_2 and c_3 are all positive.
4. Show that the particle speed can never be equal to the wave speed in a sine wave if the amplitude is less than wavelength divided by 2π.
5. Two wave pulses identical in shape but inverted with respect to each other are produced at the two ends of a stretched string. At an instant when the pulses reach the middle, the string becomes completely straight. What happens to the energy of the two pulses ?
6. Show that for a wave travelling on a string
$$\frac{y_{max}}{v_{max}} = \frac{v_{max}}{a_{max}},$$
where the symbols have usual meanings. Can we use componendo and dividendo taught in algebra to write
$$\frac{y_{max} + v_{max}}{y_{max} - v_{max}} = \frac{v_{max} + a_{max}}{v_{max} - a_{max}} \ ?$$
7. What is the smallest positive phase constant which is equivalent to $7{\cdot}5\pi$?
8. A string clamped at both ends vibrates in its fundamental mode. Is there any position (except the ends) on the string which can be touched without disturbing the motion ? What if the string vibrates in its first overtone ?

OBJECTIVE I

1. A sine wave is travelling in a medium. The minimum distance between the two particles, always having same speed, is
(a) $\lambda/4$ (b) $\lambda/3$ (c) $\lambda/2$ (d) λ.

2. A sine wave is travelling in a medium. A particular particle has zero displacement at a certain instant. The particle closest to it having zero displacement is at a distance
(a) $\lambda/4$ (b) $\lambda/3$ (c) $\lambda/2$ (d) λ.

3. Which of the following equations represents a wave travelling along Y-axis ?
(a) $x = A \sin(ky - \omega t)$ (b) $y = A \sin(kx - \omega t)$
(c) $y = A \sin ky \cos \omega t$ (d) $y = A \cos ky \sin \omega t$.

4. The equation $y = A \sin^2(kx - \omega t)$ represents a wave motion with
(a) amplitude A, frequency $\omega/2\pi$
(b) amplitude $A/2$, frequency ω/π
(c) amplitude $2A$, frequency $\omega/4\pi$
(d) does not represent a wave motion.

5. Which of the following is a mechanical wave ?
(a) Radio waves. (b) X-rays.
(c) Light waves. (d) Sound waves.

6. A cork floating in a calm pond executes simple harmonic motion of frequency ν when a wave generated by a boat passes by it. The frequency of the wave is
(a) ν (b) $\nu/2$ (c) 2ν (d) $\sqrt{2}\nu$.

7. Two strings A and B, made of same material, are stretched by same tension. The radius of string A is double of the radius of B. A transverse wave travels on A with speed v_A and on B with speed v_B. The ratio v_A/v_B is
(a) 1/2 (b) 2 (c) 1/4 (d) 4.

8. Both the strings, shown in figure (15-Q1), are made of same material and have same cross-section. The pulleys are light. The wave speed of a transverse wave in the string AB is v_1 and in CD it is v_2. Then v_1/v_2 is
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $1/\sqrt{2}$.

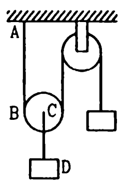


Figure 15-Q1

9. Velocity of sound in air is 332 m/s. Its velocity in vacuum will be
(a) > 332 m/s (b) = 332 m/s
(c) < 332 m/s (d) meaningless.

10. A wave pulse, travelling on a two-piece string, gets partially reflected and partially transmitted at the junction. The reflected wave is inverted in shape as compared to the incident one. If the incident wave has wavelength λ and the transmitted wave λ',
(a) $\lambda' > \lambda$ (b) $\lambda' = \lambda$ (c) $\lambda' < \lambda$
(d) nothing can be said about the relation of λ and λ'.

11. Two waves represented by $y = a \sin(\omega t - kx)$ and $y = a \cos(\omega t - kx)$ are superposed. The resultant wave will have an amplitude
(a) a (b) $\sqrt{2}a$ (c) $2a$ (d) 0.

12. Two wires A and B, having identical geometrical construction, are stretched from their natural length by small but equal amount. The Young's modulus of the wires are Y_A and Y_B whereas the densities are ρ_A and ρ_B. It is given that $Y_A > Y_B$ and $\rho_A > \rho_B$. A transverse signal started at one end takes a time t_1 to reach the other end for A and t_2 for B.
(a) $t_1 < t_2$. (b) $t_1 = t_2$. (c) $t_1 > t_2$.
(d) The information is insufficient to find the relation between t_1 and t_2.

13. Consider two waves passing through the same string. Principle of superposition for displacement says that the net displacement of a particle on the string is sum of the displacements produced by the two waves individually. Suppose we state similar principles for the net velocity of the particle and the net kinetic energy of the particle. Such a principle will be valid for
(a) both the velocity and the kinetic energy
(b) the velocity but not for the kinetic energy
(c) the kinetic energy but not for the velocity
(d) neither the velocity nor the kinetic energy.

14. Two wave pulses travel in opposite directions on a string and approach each other. The shape of one pulse is inverted with respect to the other.
(a) The pulses will collide with each other and vanish after collision.
(b) The pulses will reflect from each other i.e., the pulse going towards right will finally move towards left and vice versa.
(c) The pulses will pass through each other but their shapes will be modified.
(d) The pulses will pass through each other without any change in their shapes.

15. Two periodic waves of amplitudes A_1 and A_2 pass through a region. If $A_1 > A_2$, the difference in the maximum and minimum resultant amplitude possible is
(a) $2A_1$ (b) $2A_2$ (c) $A_1 + A_2$ (d) $A_1 - A_2$.

16. Two waves of equal amplitude A, and equal frequency travel in the same direction in a medium. The amplitude of the resultant wave is
(a) 0 (b) A (c) $2A$ (d) between 0 and $2A$.

17. Two sine waves travel in the same direction in a medium. The amplitude of each wave is A and the phase difference between the two waves is 120°. The resultant amplitude will be
(a) A (b) $2A$ (c) $4A$ (d) $\sqrt{2}A$.

18. The fundamental frequency of a string is proportional to
(a) inverse of its length (b) the diameter
(c) the tension (d) the density.

19. A tuning fork of frequency 480 Hz is used to vibrate a sonometer wire having natural frequency 240 Hz. The wire will vibrate with a frequency of
(a) 240 Hz (b) 480 Hz
(c) 720 Hz (d) will not vibrate.

20. A tuning fork of frequency 480 Hz is used to vibrate a sonometer wire having natural frequency 410 Hz. The wire will vibrate with a frequency
(a) 410 Hz (b) 480 Hz (c) 820 Hz (d) 960 Hz.

21. A sonometer wire of length l vibrates in fundamental mode when excited by a tuning fork of frequency 416 Hz. If the length is doubled keeping other things same, the string will
(a) vibrate with a frequency of 416 Hz
(b) vibrate with a frequency of 208 Hz
(c) vibrate with a frequency of 832 Hz
(d) stop vibrating.

22. A sonometer wire supports a 4 kg load and vibrates in fundamental mode with a tuning fork of frequency 416 Hz. The length of the wire between the bridges is now doubled. In order to maintain fundamental mode, the load should be changed to
(a) 1 kg (b) 2 kg (c) 8 kg (d) 16 kg.

OBJECTIVE II

1. A mechanical wave propagates in a medium along the X-axis. The particles of the medium
(a) must move on the X-axis
(b) must move on the Y-axis
(c) may move on the X-axis
(d) may move on the Y-axis.

2. A transverse wave travels along the Z-axis. The particles of the medium must move
(a) along the Z-axis (b) along the X-axis
(c) along the Y-axis (d) in the X-Y plane.

3. Longitudinal waves cannot
(a) have a unique wavelength (b) transmit energy
(c) have a unique wave velocity (d) be polarized.

4. A wave going in a solid
(a) must be longitudinal (b) may be longitudinal
(c) must be transverse (d) may be transverse.

5. A wave moving in a gas
(a) must be longitudinal (b) may be longitudinal
(c) must be transverse (d) may be transverse.

6. Two particles A and B have a phase difference of π when a sine wave passes through the region.
(a) A oscillates at half the frequency of B.
(b) A and B move in opposite directions.
(c) A and B must be separated by half of the wavelength.
(d) The displacements at A and B have equal magnitudes.

7. A wave is represented by the equation
$$y = (0{\cdot}001\ \text{mm})\sin[(50\ \text{s}^{-1})t + (2{\cdot}0\ \text{m}^{-1})x].$$
(a) The wave velocity = 100 m/s.
(b) The wavelength = 2·0 m.
(c) The frequency = $25/\pi$ Hz.
(d) The amplitude = 0·001 mm.

8. A standing wave is produced on a string clamped at one end and free at the other. The length of the string
(a) must be an integral multiple of $\lambda/4$
(b) must be an integral multiple of $\lambda/2$
(c) must be an integral multiple of λ
(d) may be an integral multiple of $\lambda/2$.

9. Mark out the correct options.
(a) The energy of any small part of a string remains constant in a travelling wave.
(b) The energy of any small part of a string remains constant in a standing wave.
(c) The energies of all the small parts of equal length are equal in a travelling wave.
(d) The energies of all the small parts of equal length are equal in a standing wave.

10. In a stationary wave,
(a) all the particles of the medium vibrate in phase
(b) all the antinodes vibrate in phase
(c) the alternate antinodes vibrate in phase
(d) all the particles between consecutive nodes vibrate in phase.

EXERCISES

1. A wave pulse passing on a string with a speed of 40 cm/s in the negative x-direction has its maximum at $x = 0$ at $t = 0$. Where will this maximum be located at $t = 5$ s ?

2. The equation of a wave travelling on a string stretched along the X-axis is given by
$$y = A e^{-\left(\frac{x}{a} + \frac{t}{T}\right)^2}.$$
(a) Write the dimensions of A, a and T. (b) Find the wave speed. (c) In which direction is the wave travelling ? (d) Where is the maximum of the pulse located at $t = T$? At $t = 2T$?

3. Figure (15-E1) shows a wave pulse at $t = 0$. The pulse moves to the right with a speed of 10 cm/s. Sketch the shape of the string at $t = 1$ s, 2 s and 3 s.

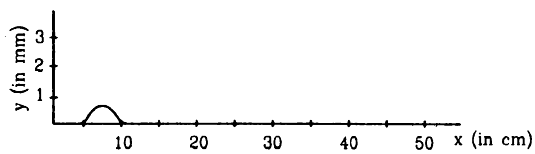


Figure 15-E1

4. A pulse travelling on a string is represented by the function

$$y = \frac{a^3}{(x - vt)^2 + a^2},$$

where $a = 5$ mm and $v = 20$ cm/s. Sketch the shape of the string at $t = 0$, 1 s and 2 s. Take $x = 0$ in the middle of the string.

5. The displacement of the particle at $x = 0$ of a stretched string carrying a wave in the positive x-direction is given by $f(t) = A \sin(t/T)$. The wave speed is v. Write the wave equation.

6. A wave pulse is travelling on a string with a speed v towards the positive X-axis. The shape of the string at $t = 0$ is given by $g(x) = A \sin(x/a)$, where A and a are constants.
(a) What are the dimensions of A and a? (b) Write the equation of the wave for a general time t, if the wave speed is v.

7. A wave propagates on a string in the positive x-direction at a velocity v. The shape of the string at $t = t_0$ is given by $g(x, t_0) = A \sin(x/a)$. Write the wave equation for a general time t.

8. The equation of a wave travelling on a string is

$$y = (0{\cdot}10 \text{ mm}) \sin[(31{\cdot}4 \text{ m}^{-1})x + (314 \text{ s}^{-1})t].$$

(a) In which direction does the wave travel? (b) Find the wave speed, the wavelength and the frequency of the wave. (c) What is the maximum displacement and the maximum speed of a portion of the string?

9. A wave travels along the positive x-direction with a speed of 20 m/s. The amplitude of the wave is 0·20 cm and the wavelength 2·0 cm. (a) Write a suitable wave equation which describes this wave. (b) What is the displacement and velocity of the particle at $x = 2{\cdot}0$ cm at time $t = 0$ according to the wave equation written? Can you get different values of this quantity if the wave equation is written in a different fashion?

10. A wave is described by the equation

$$y = (1{\cdot}0 \text{ mm}) \sin \pi \left(\frac{x}{2{\cdot}0 \text{ cm}} - \frac{t}{0{\cdot}01 \text{ s}} \right).$$

(a) Find the time period and the wavelength. (b) Write the equation for the velocity of the particles. Find the speed of the particle at $x = 1{\cdot}0$ cm at time $t = 0{\cdot}01$ s. (c) What are the speeds of the particles at $x = 3{\cdot}0$ cm, 5·0 cm and 7·0 cm at $t = 0{\cdot}01$ s? (d) What are the speeds of the particles at $x = 1{\cdot}0$ cm at $t = 0{\cdot}011$, 0·012, and 0·013 s?

11. A particle on a stretched string supporting a travelling wave, takes 5·0 ms to move from its mean position to the extreme position. The distance between two consecutive particles, which are at their mean positions, is 2·0 cm. Find the frequency, the wavelength and the wave speed.

12. Figure (15-E2) shows a plot of the transverse displacements of the particles of a string at $t = 0$ through which a travelling wave is passing in the positive x-direction. The wave speed is 20 cm/s. Find (a) the amplitude, (b) the wavelength, (c) the wave number and (d) the frequency of the wave.

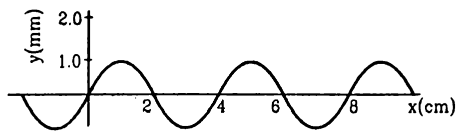


Figure 15-E2

13. A wave travelling on a string at a speed of 10 m/s causes each particle of the string to oscillate with a time period of 20 ms. (a) What is the wavelength of the wave? (b) If the displacement of a particle is 1·5 mm at a certain instant, what will be the displacement of a particle 10 cm away from it at the same instant?

14. A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8 N, what would be the speed of a transverse wave passing on it?

15. A string of length 20 cm and linear mass density 0·40 g/cm is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in figure (15-E3), which travels towards the other end. (a) When will the string have the shape shown in the figure again? (b) Sketch the shape of the string at a time half of that found in part (a).

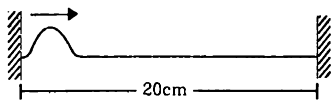


Figure 15-E3

16. A string of linear mass density 0·5 g/cm and a total length 30 cm is tied to a fixed wall at one end and to a frictionless ring at the other end (figure 15-E4). The ring can move on a vertical rod. A wave pulse is produced on the string which moves towards the ring at a speed of 20 cm/s. The pulse is symmetric about its maximum which is located at a distance of 20 cm from the end joined to the ring. (a) Assuming that the wave is reflected from the ends without loss of energy, find the time taken by the string to regain its shape. (b) The shape of the string changes periodically with time. Find this time period. (c) What is the tension in the string?

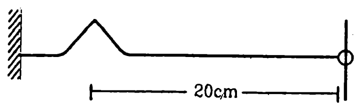


Figure 15-E4

17. Two wires of different densities but same area of cross-section are soldered together at one end and are stretched to a tension T. The velocity of a transverse

wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.

18. A transverse wave described by

$$y = (0{\cdot}02\ \text{m})\sin[(1{\cdot}0\ \text{m}^{-1})x + (30\ \text{s}^{-1})t]$$

propagates on a stretched string having a linear mass density of $1{\cdot}2 \times 10^{-4}$ kg/m. Find the tension in the string.

19. A travelling wave is produced on a long horizontal string by vibrating an end up and down sinusoidally. The amplitude of vibration is 1·0 cm and the displacement becomes zero 200 times per second. The linear mass density of the string is 0·10 kg/m and it is kept under a tension of 90 N. (a) Find the speed and the wavelength of the wave. (b) Assume that the wave moves in the positive x-direction and at $t = 0$, the end $x = 0$ is at its positive extreme position. Write the wave equation. (c) Find the velocity and acceleration of the particle at $x = 50$ cm at time $t = 10$ ms.

20. A string of length 40 cm and weighing 10 g is attached to a spring at one end and to a fixed wall at the other end. The spring has a spring constant of 160 N/m and is stretched by 1·0 cm. If a wave pulse is produced on the string near the wall, how much time will it take to reach the spring?

21. Two blocks each having a mass of 3·2 kg are connected by a wire CD and the system is suspended from the ceiling by another wire AB (figure 15-E5). The linear mass density of the wire AB is 10 g/m and that of CD is 8 g/m. Find the speed of a transverse wave pulse produced in AB and in CD.

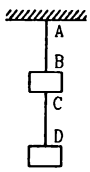


Figure 15-E5

22. In the arrangement shown in figure (15-E6), the string has a mass of 4·5 g. How much time will it take for a transverse disturbance produced at the floor to reach the pulley? Take $g = 10\ \text{m/s}^2$.

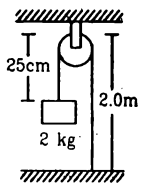


Figure 15-E6

23. A 4·0 kg block is suspended from the ceiling of an elevator through a string having a linear mass density of $19{\cdot}2 \times 10^{-3}$ kg/m. Find the speed (with respect to the string) with which a wave pulse can proceed on the string if the elevator accelerates up at the rate of $2{\cdot}0\ \text{m/s}^2$. Take $g = 10\ \text{m/s}^2$.

24. A heavy ball is suspended from the ceiling of a motor car through a light string. A transverse pulse travels at a speed of 60 cm/s on the string when the car is at rest and 62 cm/s when the car accelerates on a horizontal road. Find the acceleration of the car. Take $g = 10\ \text{m/s}^2$.

25. A circular loop of string rotates about its axis on a frictionless horizontal plane at a uniform rate so that the tangential speed of any particle of the string is v. If a small transverse disturbance is produced at a point of the loop, with what speed (relative to the string) will this disturbance travel on the string?

26. A heavy but uniform rope of length L is suspended from a ceiling. (a) Write the velocity of a transverse wave travelling on the string as a function of the distance from the lower end. (b) If the rope is given a sudden sideways jerk at the bottom, how long will it take for the pulse to reach the ceiling? (c) A particle is dropped from the ceiling at the instant the bottom end is given the jerk. Where will the particle meet the pulse?

27. Two long strings A and B, each having linear mass density $1{\cdot}2 \times 10^{-2}$ kg/m, are stretched by different tensions 4·8 N and 7·5 N respectively and are kept parallel to each other with their left ends at $x = 0$. Wave pulses are produced on the strings at the left ends at $t = 0$ on string A and at $t = 20$ ms on string B. When and where will the pulse on B overtake that on A?

28. A transverse wave of amplitude 0·50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m/s, what average power is the source transmitting to the wire?

29. A 200 Hz wave with amplitude 1 mm travels on a long string of linear mass density 6 g/m kept under a tension of 60 N. (a) Find the average power transmitted across a given point on the string. (b) Find the total energy associated with the wave in a 2·0 m long portion of the sring.

30. A tuning fork of frequency 440 Hz is attached to a long string of linear mass density 0·01 kg/m kept under a tension of 49 N. The fork produces transverse waves of amplitude 0·50 mm on the string. (a) Find the wave speed and the wavelength of the waves. (b) Find the maximum speed and acceleration of a particle of the string. (c) At what average rate is the tuning fork transmitting energy to the string?

31. Two waves, travelling in the same direction through the same region, have equal frequencies, wavelengths and amplitudes. If the amplitude of each wave is 4 mm and the phase difference between the waves is 90°, what is the resultant amplitude?

32. Figure (15-E7) shows two wave pulses at $t = 0$ travelling on a string in opposite directions with the same

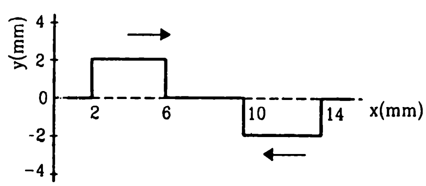


Figure 15-E7

wave speed 50 cm/s. Sketch the shape of the string at $t = 4$ ms, 6 ms, 8 ms, and 12 ms.

33. Two waves, each having a frequency of 100 Hz and a wavelength of 2·0 cm, are travelling in the same direction on a string. What is the phase difference between the waves (a) if the second wave was produced 0·015 s later than the first one at the same place, (b) if the two waves were produced at the same instant but the first one was produced a distance 4·0 cm behind the second one? (c) If each of the waves has an amplitude of 2·0 mm, what would be the amplitudes of the resultant waves in part (a) and (b)?

34. If the speed of a transverse wave on a stretched string of length 1 m is 60 m/s, what is the fundamental frequency of vibration?

35. A wire of length 2·00 m is stretched to a tension of 160 N. If the fundamental frequency of vibration is 100 Hz, find its linear mass density.

36. A steel wire of mass 4·0 g and length 80 cm is fixed at the two ends. The tension in the wire is 50 N. Find the frequency and wavelength of the fourth harmonic of the fundamental.

37. A piano wire weighing 6·00 g and having a length of 90·0 cm emits a fundamental frequency corresponding to the "Middle C" ($\nu = 261{\cdot}63$ Hz). Find the tension in the wire.

38. A sonometer wire having a length of 1·50 m between the bridges vibrates in its second harmonic in resonance with a tuning fork of frequency 256 Hz. What is the speed of the transverse wave on the wire?

39. The length of the wire shown in figure (15-E8) between the pulleys is 1·5 m and its mass is 12·0 g. Find the frequency of vibration with which the wire vibrates in two loops leaving the middle point of the wire between the pulleys at rest.

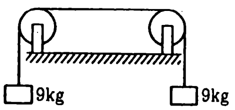


Figure 15-E8

40. A one metre long stretched string having a mass of 40 g is attached to a tuning fork. The fork vibrates at 128 Hz in a direction perpendicular to the string. What should be the tension in the string if it is to vibrate in four loops?

41. A wire, fixed at both ends is seen to vibrate at a resonant frequency of 240 Hz and also at 320 Hz. (a) What could be the maximum value of the fundamental frequency? (b) If transverse waves can travel on this string at a speed of 40 m/s, what is its length?

42. A string, fixed at both ends, vibrates in a resonant mode with a separation of 2·0 cm between the consecutive nodes. For the next higher resonant frequency, this separation is reduced to 1·6 cm. Find the length of the string.

43. A 660 Hz tuning fork sets up vibration in a string clamped at both ends. The wave speed for a transverse wave on this string is 220 m/s and the string vibrates in three loops. (a) Find the length of the string. (b) If the maximum amplitude of a particle is 0·5 cm, write a suitable equation describing the motion.

44. A particular guitar wire is 30·0 cm long and vibrates at a frequency of 196 Hz when no finger is placed on it. The next higher notes on the scale are 220 Hz, 247 Hz, 262 Hz and 294 Hz. How far from the end of the string must the finger be placed to play these notes?

45. A steel wire fixed at both ends has a fundamental frequency of 200 Hz. A person can hear sound of maximum frequency 14 kHz. What is the highest harmonic that can be played on this string which is audible to the person?

46. Three resonant frequencies of a string are 90, 150 and 210 Hz. (a) Find the highest possible fundamental frequency of vibration of this string. (b) Which harmonics of the fundamental are the given frequencies? (c) Which overtones are these frequencies. (d) If the length of the string is 80 cm, what would be the speed of a transverse wave on this string?

47. Two wires are kept tight between the same pair of supports. The tensions in the wires are in the ratio 2 : 1, the radii are in the ratio 3 : 1 and the densities are in the ratio 1 : 2. Find the ratio of their fundamental frequencies.

48. A uniform horizontal rod of length 40 cm and mass 1·2 kg is supported by two identical wires as shown in figure (15-E9). Where should a mass of 4·8 kg be placed on the rod so that the same tuning fork may excite the wire on left into its fundamental vibrations and that on right into its first overtone? Take $g = 10 \text{ m/s}^2$.

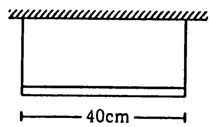


Figure 15-E9

49. Figure (15-E10) shows an aluminium wire of length 60 cm joined to a steel wire of length 80 cm and stretched between two fixed supports. The tension produced is 40 N. The cross-sectional area of the steel wire is 1·0 mm^2 and that of the aluminium wire is 3·0 mm^2. What could be the minimum frequency of a tuning fork which can produce standing waves in the system with the joint as a node? The density of aluminium is 2·6 g/cm^3 and that of steel is 7·8 g/cm^3.

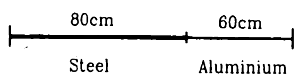


Figure 15-E10

50. **A string of length L fixed at both ends vibrates in its fundamental mode at a frequency ν and a maximum amplitude A. (a) Find the wavelength and the wave number k. (b) Take the origin at one end of the string and the X-axis along the string. Take the Y-axis along**

the direction of the displacement. Take $t = 0$ at the instant when the middle point of the string passes through its mean position and is going towards the positive y-direction. Write the equation describing the standing wave.

51. A 2 m long string fixed at both ends is set into vibrations in its first overtone. The wave speed on the string is 200 m/s and the amplitude is 0·5 cm. (a) Find the wavelength and the frequency. (b) Write the equation giving the displacement of different points as a function of time. Choose the X-axis along the string with the origin at one end and $t = 0$ at the instant when the point $x = 50$ cm has reached its maximum displacement.

52. The equation for the vibration of a string, fixed at both ends vibrating in its third harmonic, is given by

$$y = (0{\cdot}4\ \text{cm})\sin[(0{\cdot}314\ \text{cm}^{-1})\,x]\cos[(600\pi\ \text{s}^{-1})t].$$

(a) What is the frequency of vibration? (b) What are the positions of the nodes? (c) What is the length of the string? (d) What is the wavelength and the speed of two travelling waves that can interfere to give this vibration?

53. The equation of a standing wave, produced on a string fixed at both ends, is

$$y = (0{\cdot}4\ \text{cm})\sin[(0{\cdot}314\ \text{cm}^{-1})\,x]\cos[(600\pi\ \text{s}^{-1})t].$$

What could be the smallest length of the string?

54. A 40 cm wire having a mass of 3·2 g is stretched between two fixed supports 40·05 cm apart. In its fundamental mode, the wire vibrates at 220 Hz. If the area of cross-section of the wire is 1·0 mm^2, find its Young's modulus.

55. Figure (15-E11) shows a string stretched by a block going over a pulley. The string vibrates in its tenth harmonic in unison with a particular tuning fork. When a beaker containing water is brought under the block so that the block is completely dipped into the beaker, the string vibrates in its eleventh harmonic. Find the density of the material of the block.

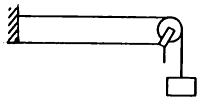


Figure 15-E11

56. A 2·00 m long rope, having a mass of 80 g, is fixed at one end and is tied to a light string at the other end. The tension in the string is 256 N. (a) Find the frequencies of the fundamental and the first two overtones. (b) Find the wavelength in the fundamental and the first two overtones.

57. A heavy string is tied at one end to a movable support and to a light thread at the other end as shown in figure (15-E12). The thread goes over a fixed pulley and supports a weight to produce a tension. The lowest frequency with which the heavy string resonates is 120 Hz. If the movable support is pushed to the right by 10 cm so that the joint is placed on the pulley, what will be the minimum frequency at which the heavy string can resonate?

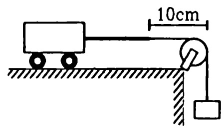


Figure 15-E12

□

ANSWERS

OBJECTIVE I

1. (c)	2. (c)	3. (a)	4. (b)	5. (d)	6. (a)
7. (a)	8. (d)	9. (d)	10. (c)	11. (b)	12. (d)
13. (b)	14. (d)	15. (b)	16. (d)	17. (a)	18. (a)
19. (b)	20. (b)	21. (a)	22. (d)		

OBJECTIVE II

1. (c), (d)	2. (d)	3. (d)
4. (b), (d)	5. (a)	6. (b), (d)
7. (c), (d)	8. (a)	9. (b)
10. (c), (d)		

EXERCISES

1. At $x = -2$ m

2. (a) L, L, T (b) a/T
   (c) negative x-direction (d) $x = -a$ and $x = -2a$

5. $f(x, t) = A\sin\left(\dfrac{t}{T} - \dfrac{x}{vT}\right)$

6. (a) L, L (b) $f(x, t) = A\sin\dfrac{x - vt}{a}$

7. $f(x, t) = A\sin\dfrac{x - v(t - t_0)}{a}$

8. (a) negative x-direction (b) 10 m/s, 20 cm, 50 Hz
   (c) 0·10 mm, 3·14 cm/s

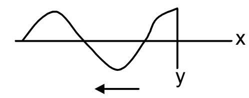
9. (a) $y = (0{\cdot}20\ \text{cm})\sin[(\pi\ \text{cm}^{-1})\,x - (2\pi \times 10^3\ \text{s}^{-1})t]$
   (b) zero, 4π m/s

10. (a) 20 ms, 4·0 cm (b) zero
(c) zero (d) 9·7 cm/s, 18 cm/s, 25 cm/s
11. 50 Hz, 4·0 cm, 2·0 m/s
12. (a) 1·0 mm (b) 4 cm (c) 1·6 cm^{-1} (d) 5 Hz
13. (a) 20 cm (b) − 1·5 mm
14. 32 m/s
15. (a) 0·02 s
16. (a) 2 s (b) 3 s (c) 2×10^{-3} N
17. 0·25
18. 0·108 N
19. (a) 30 m/s, 30 cm
(b) $y = (1\cdot0 \text{ cm}) \cos 2\pi \left[\frac{x}{30 \text{ cm}} - \frac{t}{0\cdot01 \text{ s}}\right]$
(c) − 5·4 m/s, 2·0 km/s^2
20. 0·05 s
21. 79 m/s and 63 m/s
22. 0·02 s
23. 50 m/s
24. 3·7 m/s^2
25. v
26. (a) $\sqrt{gx}$ (b) $\sqrt{4L/g}$
(c) at a distance $\frac{L}{3}$ from the bottom
27. at t = 100 ms at x = 2·0 m
28. 49 mW
29. (a) 0·47 W (b) 9·4 mJ
30. (a) 70 m/s, 16 cm (b) 1·4 m/s, 3·8 km/s (c) 0·67 W
31. $4\sqrt{2}$ mm
33. (a) 3π (b) 4π (c) zero, 4·0 mm
34. 30 Hz
35. 1·00 g/m
36. 250 Hz, 40 cm
37. 1480 N
38. 384 m/s
39. 70 Hz
40. 164 N
41. (a) 80 Hz (b) 25 cm
42. 8·0 cm
43. (a) 50 cm
(b) (0·5 cm) $\sin[(0\cdot06\pi \text{ cm}^{-1})x] \times \cos[(1320\pi \text{ s}^{-1})t]$
44. 26·7 cm, 23·8 cm, 22·4 cm and 20·0 cm
45. 70
46. (a) 30 Hz (b) 3rd, 5th and 7th
(c) 2nd, 4th and 6th (d) 48 m/s
47. 2 : 3
48. 5 cm from the left end
49. 180 Hz
50. (a) $2L$, π/L (b) $y = A \sin(\pi x/L) \sin(2\pi \nu t)$
51. (a) 2 m, 100 Hz
(b) (0·5 cm) $\sin[(\pi \text{m}^{-1})x] \cos[(200\pi \text{ s}^{-1})t]$
52. (a) 300 Hz (b) 0, 10 cm, 20 cm, 30 cm
(c) 30 cm (d) 20 cm, 60 m/s
53. 10 cm
54. $1\cdot98 \times 10^{11}$ N/m^2
55. $5\cdot8 \times 10^{3}$ kg/m^3
56. (a) 10 Hz, 30 Hz, 50 Hz (b) 8·00 m, 2·67 m, 1·60 m
57. 240 Hz

□

SOLUTIONS TO CONCEPTS
CHAPTER 15

1. v = 40 cm/sec
As velocity of a wave is constant location of maximum after 5 sec
= 40 × 5 = 200 cm along negative x-axis.



2. Given y = $Ae^{-[(x/a)+(t/T)]^2}$
a) [A] = $[M^0L^1T^0]$, [T] = $[M^0L^0T^1]$
[a] = $[M^0L^1T^0]$
b) Wave speed, v = λ/T = a/T [Wave length λ = a]
c) If y = f(t – x/v) → wave is traveling in positive direction
and if y = f(t + x/v) → wave is traveling in negative direction
So, y = $Ae^{-[(x/a)+(t/T)]^2}$ = $Ae^{-(1/T)\left[\frac{x}{a/T}+t\right]^2}$
= $Ae^{-(1/T)\left[\frac{x}{v}+t\right]^2}$
i.e. y = f{t + (x / v)}
d) Wave speed, v = a/T
∴ Max. of pulse at t = T is (a/T) × T = a (negative x-axis)
Max. of pulse at t = 2T = (a/T) × 2T = 2a (along negative x-axis)
So, the wave travels in negative x-direction.

3. At t = 1 sec, s_1 = vt = 10 × 1 = 10 cm
t = 2 sec, s_2 = vt = 10 × 2 = 20 cm
t = 3 sec, s_3 = vt = 10 × 3 = 30 cm

4. The pulse is given by, y = $[(a^3) / \{(x - vt)^2 + a^2\}]$
a = 5 mm = 0.5 cm, v = 20 cm/s
At t = 0s, y = $a^3 / (x^2 + a^2)$
The graph between y and x can be plotted by taking different values of x.
(left as exercise for the student)
similarly, at t = 1 s, y = $a^3 / \{(x - v)^2 + a^2\}$
and at t = 2 s, y = $a^3 / \{(x - 2v)^2 + a^2\}$

5. At x = 0, f(t) = a sin (t/T)
Wave speed = v
⇒ λ = wavelength = vT (T = Time period)
So, general equation of wave
Y = A sin [(t/T) – (x/vT)] [because y = f((t/T) – (x/λ))

6. At t = 0, g(x) = A sin (x/a)
a) $[M^0L^1T^0]$ = [L]
a = $[M^0L^1T^0]$ = [L]
b) Wave speed = v
∴ Time period, T = a/v (a = wave length = λ)
∴ General equation of wave
y = A sin {(x/a) – t/(a/v)}
= A sin {(x – vt) / a}

7. At t = t_0, g(x, t_0) = A sin (x/a) …(1)
For a wave traveling in the positive x-direction, the general equation is given by
y = $f\left(\frac{x}{a}-\frac{t}{T}\right)$
Putting t = $-t_0$ and comparing with equation (1), we get
⇒ g(x, 0) = A sin {(x/a) + (t_0/T)}
⇒ g(x, t) = A sin {(x/a) + (t_0/T) – (t/T)}

As T = a/v (a = wave length, v = speed of the wave)

$$\Rightarrow y = A\sin\left(\frac{x}{a} + \frac{t_0}{(a/v)} - \frac{t}{(a/v)}\right)$$

$$= A\sin\left(\frac{x + v(t_0 - t)}{a}\right)$$

$$\Rightarrow y = A\sin\left[\frac{x - v(t - t_0)}{a}\right]$$

8. The equation of the wave is given by

$y = (0.1 \text{ mm}) \sin [(31.4 \text{ m}^{-1})x + (314 \text{ s}^{-1})t]$ $\quad$ $y = r \sin \{(2\pi x / \lambda)\} + \omega t)$

a) Negative x-direction

b) $k = 31.4 \text{ m}^{-1}$

$\Rightarrow 2\lambda/\lambda = 31.4 \Rightarrow \lambda = 2\pi/31.4 = 0.2$ mt = 20 cm

Again, $\omega = 314 \text{ s}^{-1}$

$\Rightarrow 2\pi f = 314 \Rightarrow f = 314 / 2\pi = 314 / (2 \times (3/14)\} = 50 \text{ sec}^{-1}$

$\therefore$ wave speed, $v = \lambda f = 20 \times 50 = 1000$ cm/s

c) Max. displacement = 0.10 mm

Max. velocity = $a\omega = 0.1 \times 10^{-1} \times 314 = 3.14$ cm/sec.

9. Wave speed, v = 20 m/s

A = 0.20 cm

λ = 2 cm

a) Equation of wave along the x-axis

$y = A \sin (kx - wt)$

$\therefore$ $k = 2\pi/\lambda = 2\pi/2 = \pi \text{ cm}^{-1}$

$T = \lambda/v = 2/2000 = 1/1000 \text{ sec} = 10^{-3}$ sec

$\Rightarrow \omega = 2\pi/T = 2\pi \times 10^{-3} \text{ sec}^{-1}$

So, the wave equation is,

$\therefore$ $y = (0.2 \text{ cm})\sin[(\pi \text{ cm}^{-1})x - (2\pi \times 10^{3} \text{ sec}^{-1})t]$

b) At x = 2 cm, and t = 0,

$y = (0.2 \text{ cm}) \sin (\pi/2) = 0$

$\therefore$ $v = r\omega \cos \pi x = 0.2 \times 2000\ \pi \times \cos 2\pi = 400\ \pi$

$= 400 \times (3.14) = 1256$ cm/s

$= 400\ \pi$ cm/s $= 4\pi$ m/s

10. $Y = (1 \text{ mm}) \sin \pi \left[\frac{x}{2\text{cm}} - \frac{t}{0.01\text{sec}}\right]$

a) $T = 2 \times 0.01 = 0.02$ sec = 20 ms

$\lambda = 2 \times 2 = 4$ cm

b) $v = dy/dt = d/dt\ [\sin 2\pi\ (x/4 - t/0.02)] = -\cos 2\pi\ \{x/4) - (t/0.02)\} \times 1/(0.02)$

$\Rightarrow v = -50 \cos 2\pi\ \{(x/4) - (t/0.02)\}$

at x = 1 and t = 0.01 sec, v = –50 cos 2* [(1/4) – (1/2)] = 0

c) i) at x = 3 cm, t = 0.01 sec

$v = -50 \cos 2\pi\ (3/4 - ½) = 0$

ii) at x = 5 cm, t = 0.01 sec, v = 0 (putting the values)

iii) at x = 7 cm, t = 0.01 sec, v = 0

at x = 1 cm and t = 0.011 sec

$v = -50 \cos 2\pi\ \{(1/4) - (0.011/0.02)\} = -50 \cos (3\pi/5) = -9.7$ cm/sec

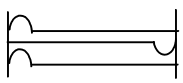
(similarly the other two can be calculated)

11. Time period, $T = 4 \times 5 \text{ ms} = 20 \times 10^{-3} = 2 \times 10^{-2}$ s

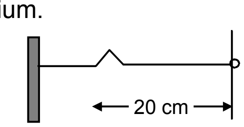
$\lambda = 2 \times 2$ cm = 4 cm

frequency, $f = 1/T = 1/(2 \times 10^{-2}) = 50 \text{ s}^{-1} = 50$ Hz

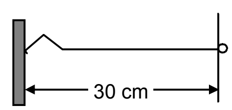
Wave speed = $\lambda f = 4 \times 50$ m/s = 2000 m/s = 2 m/s

12. Given that, v = 200 m/s
    a) Amplitude, A = 1 mm
    b) Wave length, λ = 4 cm
    c) wave number, n = $2\pi/\lambda$ = $(2 \times 3.14)/4$ = 1.57 cm^{-1} (wave number = k)
    d) frequency, f = 1/T = $(26/\lambda)/20$ = 20/4 = 5 Hz
       (where time period T = λ/v)
13. Wave speed = v = 10 m/sec
    Time period = T = 20 ms = $20 \times 10^{-3} = 2 \times 10^{-2}$ sec
    a) wave length, $\lambda = vT = 10 \times 2 \times 10^{-2}$ = 0.2 m = 20 cm
    b) wave length, λ = 20 cm
    $\therefore$ phase diff^n = $(2\pi/\lambda)$ x = $(2\pi / 20) \times 10 = \pi$ rad
       $y_1 = a \sin(\omega t - kx) \Rightarrow 1.5 = a \sin(\omega t - kx)$
    So, the displacement of the particle at a distance x = 10 cm.
    [$\phi = \frac{2\pi x}{\lambda} = \frac{2\pi \times 10}{20} = \pi$] is given by
    $y_2 = a \sin(\omega t - kx + \pi) \Rightarrow -a \sin(\omega t - kx) = -1.5$ mm
    $\therefore$ displacement = –1.5 mm
14. mass = 5 g, length l = 64 cm
    $\therefore$ mass per unit length = m = 5/64 g/cm
    $\therefore$ Tension, T = 8N = 8×10^5 dyne
    V = $\sqrt{(T/m)} = \sqrt{(8 \times 10^5 \times 64)/5} = 3200$ cm/s = 32 m/s
15. 

    a) Velocity of the wave, v = $\sqrt{(T/m)} = \sqrt{(16 \times 10^5)/0.4} = 2000$ cm/sec
    $\therefore$ Time taken to reach to the other end = 20/2000 = 0.01 sec
    Time taken to see the pulse again in the original position = 0.01×2 = 0.02 sec
    b) At t = 0.01 s, there will be a 'though' at the right end as it is reflected.
16. The crest reflects as a crest here, as the wire is traveling from denser to rarer medium.
    $\Rightarrow$ phase change = 0

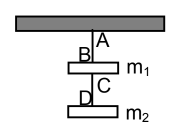
    

    a) To again original shape distance travelled by the wave S = 20 + 20 = 40 cm.
       Wave speed, v = 20 m/s $\Rightarrow$ time = s/v = 40/20 = 2 sec
    b) The wave regains its shape, after traveling a periodic distance = 2×30 = 60 cm
       $\therefore$ Time period = 60/20 = 3 sec.

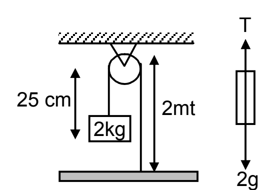
    

    c) Frequency, n = (1/3 sec^{-1})
       n = $(1/2l)\sqrt{(T/m)}$ m = mass per unit length = 0.5 g/cm
       $\Rightarrow 1/3 = 1/(2 \times 30) \sqrt{(T/0.5)}$
       $\Rightarrow T = 400 \times 0.5 = 200$ dyne = 2×10^{-3} Newton.
17. Let v_1 = velocity in the 1st string
    $\Rightarrow v_1 = \sqrt{(T/m_1)}$
    Because m_1 = mass per unit length = $(\rho_1 a_1 l_1 / l_1) = \rho_1 a_1$ where a_1 = Area of cross section
    $\Rightarrow v_1 = \sqrt{(T/\rho_1 a_1)}$ …(1)
    Let v_2 = velocity in the second string
    $\Rightarrow v_2 = \sqrt{(T/m^2)}$
    $\Rightarrow v_2 = \sqrt{(T/\rho_2 a_2)}$ …(2)
    Given that, $v_1 = 2v_2$
    $\Rightarrow \sqrt{(T/\rho_1 a_1)} = 2\sqrt{(T/\rho_2 a_2)} \Rightarrow (T/a_1\rho_1) = 4(T/a_2\rho_2)$
    $\Rightarrow \rho_1/\rho_2 = 1/4 \Rightarrow \rho_1 : \rho_2 = 1 : 4$ (because $a_1 = a_2$)

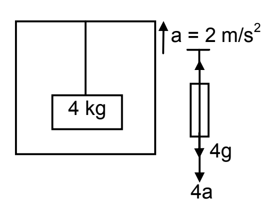
18. m = mass per unit length = 1.2×10^{-4} kg/mt
    $Y = (0.02\text{m}) \sin [(1.0\ \text{m}^{-1})x + (30\ \text{s}^{-1})t]$
    Here, $k = 1\ \text{m}^{-1} = 2\pi/\lambda$
    $\omega = 30\ \text{s}^{-1} = 2\pi f$
    $\therefore$ velocity of the wave in the stretched string
    $v = \lambda f = \omega/k = 30/I = 30$ m/s
    $\Rightarrow v = \sqrt{T/m} \Rightarrow 30\sqrt{(T/1.2)\times 10^{-4}\text{N})}$
    $\Rightarrow T = 10.8 \times 10^{-2}$ N $\Rightarrow T = 1.08 \times 10^{-1}$ Newton.
19. Amplitude, A = 1 cm, Tension T = 90 N
    Frequency, f = 200/2 = 100 Hz
    Mass per unit length, m = 0.1 kg/mt
    a) $\Rightarrow V = \sqrt{T/m} = 30$ m/s
       $\lambda = V/f = 30/100 = 0.3$ m = 30 cm
    b) The wave equation y = (1 cm) cos 2π (t/0.01 s) – (x/30 cm)
       [because at x = 0, displacement is maximum]
    c) $y = 1 \cos 2\pi(x/30 - t/0.01)$
       $\Rightarrow v = dy/dt = (1/0.01)2\pi \sin 2\pi \{(x/30) - (t/0.01)\}$
       $a = dv/dt = -\{4\pi^2 / (0.01)^2\} \cos 2\pi \{(x/30) - (t/0.01)\}$
       When, x = 50 cm, t = 10 ms = 10×10^{-3} s
       $x = (2\pi / 0.01) \sin 2\pi \{(5/3) - (0.01/0.01)\}$
       $= (p/0.01) \sin (2\pi \times 2 / 3) = (1/0.01) \sin (4\pi/3) = -200\ \pi \sin (\pi/3) = -200\ \pi x\ (\sqrt{3}/2)$
       = 544 cm/s = 5.4 m/s
       Similarly
       $a = \{4\pi^2 / (0.01)^2\} \cos 2\pi \{(5/3) - 1\}$
       $= 4\pi^2 \times 10^4 \times ½ \Rightarrow 2 \times 10^5\ \text{cm/s}^2 \Rightarrow 2\ \text{km/s}^2$
20. l = 40 cm, mass = 10 g
    $\therefore$ mass per unit length, m = 10 / 40 = 1/4 (g/cm)
    spring constant K = 160 N/m
    deflection = x = 1 cm = 0.01 m
    $\Rightarrow T = kx = 160 \times 0.01 = 1.6\ \text{N} = 16 \times 10^4$ dyne
    Again $v = \sqrt{(T/m)} = \sqrt{(16\times 10^4/(1/4)} = 8 \times 10^2$ cm/s = 800 cm/s
    $\therefore$ Time taken by the pulse to reach the spring
    t = 40/800 = 1/20 = 0/05 sec.
21. $m_1 = m_2 = 3.2$ kg



    mass per unit length of AB = 10 g/mt = 0.01 kg.mt
    mass per unit length of CD = 8 g/mt = 0.008 kg/mt
    for the string CD, $T = 3.2 \times g$
    $\Rightarrow v = \sqrt{(T/m)} = \sqrt{(3.2\times 10)/0.008} = \sqrt{(32\times 10^3)/8} = 2\times 10\sqrt{10} = 20 \times 3.14 = 63$ m/s
    for the string AB, $T = 2 \times 3.2\ g = 6.4 \times g = 64$ N
    $\Rightarrow v = \sqrt{(T/m)} = \sqrt{(64/0.01)} = \sqrt{6400} = 80$ m/s
22. Total length of string 2 + 0.25 = 2.25 mt

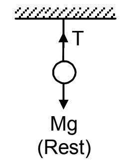


    Mass per unit length $m = \dfrac{4.5\times 10^{-3}}{2.25} = 2 \times 10^{-3}$ kg/m
    T = 2g = 20 N
    Wave speed, $v = \sqrt{(T/m)} = \sqrt{20}/(2\times 10^{-3}) = \sqrt{10^4} = 10^2$ m/s = 100 m/s
    Time taken to reach the pully, t = (s/v) = 2/100 = 0.02 sec.
23. $m = 19.2 \times 10^{-3}$ kg/m

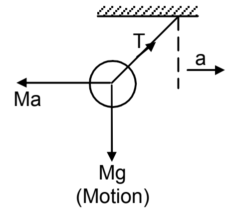


    from the freebody diagram,
    T – 4g – 4a = 0
    $\Rightarrow T = 4(a + g) = 48$ N
    wave speed, $v = \sqrt{(T/m)} = 50$ m/s

24. Let M = mass of the heavy ball
(m = mass per unit length)
Wave speed, $v_1 = \sqrt{(T/m)} = \sqrt{(Mg/m)}$ (because T = Mg)
$\Rightarrow 60 = \sqrt{(Mg/m)} \Rightarrow Mg/m = 60^2$ …(1)

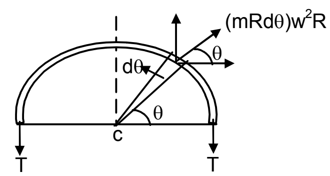


From the freebody diagram (2),
$v_2 = \sqrt{(T'/m)}$
$\Rightarrow v_2 = \dfrac{[(Ma)^2 + (Mg)^2]^{1/4}}{m^{1/2}}$ (because T' = $\sqrt{(Ma)^2 + (Mg)^2}$)
$\Rightarrow 62 = \dfrac{[(Ma)^2 + (Mg)^2]^{1/4}}{m^{1/2}}$



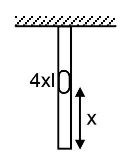
$\Rightarrow \dfrac{\sqrt{(Ma)^2 + (Mg)^2}}{m} = 62^2$ …(2)
Eq(1) + Eq(2) $\Rightarrow$ (Mg/m) $\times$ [m / $\sqrt{(Ma)^2 + (Mg)^2}$] = 3600 / 3844
$\Rightarrow g / \sqrt{(a^2 + g^2)} = 0.936 \Rightarrow g^2 / (a^2 + g^2) = 0.876$
$\Rightarrow (a^2 + 100)\ 0.876 = 100$
$\Rightarrow a^2 \times 0.876 = 100 - 87.6 = 12.4$
$\Rightarrow a^2 = 12.4 / 0.876 = 14.15 \Rightarrow a = 3.76\ m/s^2$
$\therefore$ Acce^n of the car = 3.7 m/s^2

25. m = mass per unit length of the string
R = Radius of the loop
ω = angular velocity, V = linear velocity of the string
Consider one half of the string as shown in figure.
The half loop experiences cetrifugal force at every point, away from centre, which is balanced by tension 2T.

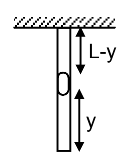


Consider an element of angular part $d\theta$ at angle θ. Consider another element symmetric to this centrifugal force experienced by the element = $(mRd\theta)\omega^2R$.
(…Length of element = $Rd\theta$, mass = $mRd\theta$)
Resolving into rectangular components net force on the two symmetric elements,
DF = $2mR^2\, d\theta\omega^2 \sin\theta$ [horizontal components cancels each other]
So, total F = $\int_0^{\pi/2} 2mR^2\omega^2 \sin\theta\, d\theta = 2mR^2\omega^2 [-\cos\theta] \Rightarrow 2mR^2\omega^2$
Again, $2T = 2mR^2\omega^2$ $\Rightarrow T = mR^2\omega^2$
Velocity of transverse vibration $V = \sqrt{T/m} = \omega R = V$
So, the speed of the disturbance will be V.

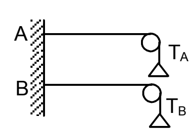
26. a) m $\rightarrow$ mass per unit of length of string
consider an element at distance 'x' from lower end.
Here wt acting down ward = (mx)g = Tension in the string of upper part
Velocity of transverse vibration = $v = \sqrt{T/m} = \sqrt{(mgx/m)} = \sqrt{(gx)}$



b) For small displacement dx, $dt = dx / \sqrt{(gx)}$
Total time $T = \int_0^L dx/\sqrt{gx} = \sqrt{(4L/g)}$



c) Suppose after time 't' from start the pulse meet the particle at distance y from lower end.
$t = \int_0^y dx/\sqrt{gx} = \sqrt{(4y/g)}$
$\therefore$ Distance travelled by the particle in this time is (L – y)



$\therefore$ S – ut + 1/2 gt^2

$\Rightarrow$ L – y (1/2)g $\times$ $\{\sqrt{(4y/g)^2}\}$ {u = 0}

$\Rightarrow$ L – y = 2y $\Rightarrow$ 3y = L

$\Rightarrow$ y = L/3. So, the particle meet at distance L/3 from lower end.

27. $m_A = 1.2 \times 10^{-2}$ kg/m, $T_A = 4.8$ N

$\Rightarrow V_A = \sqrt{T/m}$ = 20 m/s

$m_B = 1.2 \times 10^{-2}$ kg/m, $T_B = 7.5$ N

$\Rightarrow V_B = \sqrt{T/m}$ = 25 m/s

t = 0 in string A

$t_1 = 0 + 20$ ms = 20×10^{-3} = 0.02 sec

In 0.02 sec A has travelled 20 $\times$ 0.02 = 0.4 mt

Relative speed between A and B = 25 – 20 = 5 m/s

Time taken for B for overtake A = s/v = 0.4/5 = 0.08 sec

28. r = 0.5 mm = 0.5×10^{-3} mt

f = 100 Hz, T = 100 N

v = 100 m/s

v = $\sqrt{T/m} \Rightarrow v^2 = (T/m) \Rightarrow m = (T/v^2)$ = 0.01 kg/m

$P_{ave} = 2\pi^2 mvr^2f^2$

$= 2(3.14)^2(0.01) \times 100 \times (0.5 \times 10^{-3})^2 \times (100)^2 \Rightarrow 49 \times 10^{-3}$ watt = 49 mW.

29. A = 1 mm = 10^{-3} m, m = 6 g/m = 6×10^{-3} kg/m

T = 60 N, f = 200 Hz

$\therefore V = \sqrt{T/m}$ = 100 m/s

a) $P_{average} = 2\pi^2$ mv A^2f^2 = 0.47 W

b) Length of the string is 2 m. So, t = 2/100 = 0.02 sec.

Energy = $2\pi^2 mvf^2A^2t$ = 9.46 mJ.

30. f = 440 Hz, m = 0.01 kg/m, T = 49 N, r = 0.5×10^{-3} m

a) v = $\sqrt{T/m}$ = 70 m/s

b) v = λf $\Rightarrow \lambda$ = v/f = 16 cm

c) $P_{average} = 2\pi^2 mvr^2f^2$ = 0.67 W.

31. Phase difference $\phi = \pi/2$

f and λ are same. So, ω is same.

y_1 = r sin wt, y_2 = rsin(wt + π/2)

From the principle of superposition

$y = y_1 + y_2$ $\rightarrow$ = r sin wt + r sin (wt + π/2)

= r[sin wt + sin(wt + π/2)]

= r[2sin{(wt + wt + π/2)/2} cos {(wt – wt – π/2)/2}]

$\Rightarrow$ y = 2r sin (wt + π/4) cos (–π/4)

Resultant amplitude = $\sqrt{2}$ r = 4 $\sqrt{2}$ mm (because r = 4 mm)

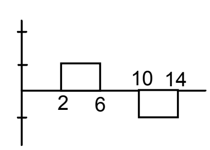
32. The distance travelled by the pulses are shown below.

t = 4 ms = 4×10^{-3} s s = vt = $50 \times 10 \times 4 \times 10^{-3}$ = 2 mm

t = 8 ms = 8×10^{-3} s s = vt = $50 \times 10 \times 8 \times 10^{-3}$ = 4 mm

t = 6 ms = 6×10^{-3} s s = 3 mm

t = 12 ms = 12×10^{-3} s s = $50 \times 10 \times 12 \times 10^{-3}$ = 6 mm



The shape of the string at different times are shown in the figure.

33. f = 100 Hz, λ = 2 cm = 2×10^{-2} m

$\therefore$ wave speed, v = fλ = 2 m/s

a) in 0.015 sec 1^{st} wave has travelled

x = 0.015 $\times$ 2 = 0.03 m = path $diff^n$

$\therefore$ corresponding phase difference, $\phi = 2\pi x/\lambda = \{2\pi / (2 \times 10^{-2})\} \times 0.03 = 3\pi$.

b) Path different x = 4 cm = 0.04 m

$\Rightarrow \phi = (2\pi/\lambda)x = \{(2\pi/2 \times 10^{-2}) \times 0.04\} = 4\pi$.

c) The waves have same frequency, same wavelength and same amplitude.

Let, $y_1 = r \sin wt$, $y_2 = r \sin (wt + \phi)$

$\Rightarrow y = y_1 + y_2 = r[\sin wt + (wt + \phi)]$

$= 2r \sin (wt + \phi/2) \cos (\phi/2)$

$\therefore$ resultant amplitude = $2r \cos \phi/2$

So, when $\phi = 3\pi$, $r = 2 \times 10^{-3}$ m

$R_{res} = 2 \times (2 \times 10^{-3}) \cos (3\pi/2) = 0$

Again, when $\phi = 4\pi$, $R_{res} = 2 \times (2 \times 10^{-3}) \cos (4\pi/2) = 4$ mm.

34. $l = 1$ m, $V = 60$ m/s

$\therefore$ fundamental frequency, $f_0 = V/2l = 30 \text{ sec}^{-1} = 30$ Hz.

35. $l = 2$m, $f_0 = 100$ Hz, $T = 160$ N

$f_0 = 1/2l\sqrt{(T/m)}$

$\Rightarrow m = 1$ g/m. So, the linear mass density is 1 g/m.

36. $m = (4/80)$ g/ cm = 0.005 kg/m

$T = 50$ N, $l = 80$ cm = 0.8 m

$v = \sqrt{(T/m)} = 100$ m/s

fundamental frequency $f_0 = 1/2l\sqrt{(T/m)} = 62.5$ Hz

First harmonic = 62.5 Hz

f_4 = frequency of fourth harmonic = $4f_0 = F_3 = 250$ Hz

$V = f_4 \lambda_4 \Rightarrow \lambda_4 = (v/f_4) = 40$ cm.

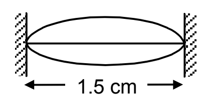
37. $l = 90$ cm = 0.9 m

$m = (6/90)$ g/cm = (6/900) kg/mt

$f = 261.63$ Hz

$f = 1/2l\sqrt{(T/m)} \Rightarrow T = 1478.52$ N = 1480 N.

38. First harmonic be f_0, second harmonic be f_1



$\therefore f_1 = 2f_0$

$\Rightarrow f_0 = f_1/2$

$f_1 = 256$ Hz

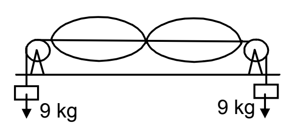
$\therefore$ 1st harmonic or fundamental frequency

$f_0 = f_1/2 = 256 / 2 = 128$ Hz

$\lambda/2 = 1.5$ m $\Rightarrow \lambda = 3$m (when fundamental wave is produced)

$\Rightarrow$ Wave speed = $V = f_0Ql = 384$ m/s.

39. $l = 1.5$ m, mass – 12 g



$\Rightarrow m = 12/1.5$ g/m $= 8 \times 10^{-3}$ kg/m

$T = 9 \times g = 90$ N

$\lambda = 1.5$ m, $f_1 = 2/2l\sqrt{T/m}$

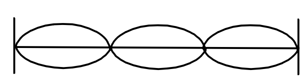
[for, second harmonic two loops are produced]

$f_1 = 2f_0 \Rightarrow 70$ Hz.

40. A string of mass 40 g is attached to the tuning fork

$m = (40 \times 10^{-3})$ kg/m

The fork vibrates with $f = 128$ Hz



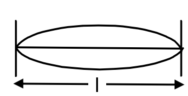
$\lambda = 0.5$ m

$v = f\lambda = 128 \times 0.5 = 64$ m/s

$v = \sqrt{T/m} \Rightarrow T = v^2m = 163.84$ N $\Rightarrow$ 164 N.

41. This wire makes a resonant frequency of 240 Hz and 320 Hz.

The fundamental frequency of the wire must be divisible by both 240 Hz and 320 Hz.

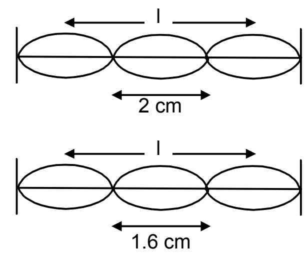


a) So, the maximum value of fundamental frequency is 80 Hz.

b) Wave speed, $v = 40$ m/s

$\Rightarrow 80 = (1/2l) \times 40 \Rightarrow 0.25$ m.

42. Let there be 'n' loops in the 1st case



$\Rightarrow$ length of the wire, $l = (n\lambda_1)/2$ $[\lambda_1 = 2 \times 2 = 4 \text{ cm}]$

So there are (n + 1) loops with the 2nd case

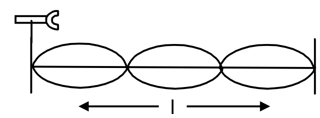
$\Rightarrow$ length of the wire, $l = \{(n+1)\lambda_2/2$ $[\lambda = 2 \times 1.6 = 3.2 \text{ cm}]$

$\Rightarrow n\lambda_1/2 = \dfrac{(n+1)\lambda_2}{2}$

$\Rightarrow n \times 4 = (n + 1)(3.2) \Rightarrow n = 4$

$\therefore$ length of the string, $l = (n\lambda_1)/2 = 8$ cm.

43. Frequency of the tuning fork, f = 660 Hz



Wave speed, v = 220 m/s $\Rightarrow \lambda = v/f = 1/3$ m

No.of loops = 3

a) So, $f = (3/2l)v \Rightarrow l = 50$ cm

b) The equation of resultant stationary wave is given by

$y = 2A \cos(2\pi x/Ql) \sin(2\pi vt/\lambda)$

$\Rightarrow y = (0.5 \text{ cm}) \cos(0.06\,\pi \text{ cm}^{-1}) \sin(1320\,\pi s^{-1} t)$

44. $l_1 = 30$ cm = 0.3 m

$f_1 = 196$ Hz, $f_2 = 220$ Hz

We know $f \propto (1/l)$ (as V is constant for a medium)

$\Rightarrow \dfrac{f_1}{f_2} = \dfrac{l_2}{l_1} \Rightarrow l_2 = 26.7$ cm

Again $f_3 = 247$ Hz

$\Rightarrow \dfrac{f_3}{f_1} = \dfrac{l_1}{l_3} \Rightarrow \dfrac{0.3}{l_3}$

$\Rightarrow l_3 = 0.224$ m = 22.4 cm and $l_3 = 20$ cm

45. Fundamental frequency $f_1 = 200$ Hz

Let l_4 Hz be nth harmonic

$\Rightarrow F_2/F_1 = 14000/200$

$\Rightarrow NF_1/F_1 = 70 \Rightarrow N = 70$

$\therefore$ The highest harmonic audible is 70th harmonic.

46. The resonant frequencies of a string are

$f_1 = 90$ Hz, $f_2 = 150$ Hz, $f_3 = 120$ Hz

a) The highest possible fundamental frequency of the string is f = 30 Hz

[because f_1, f_2 and f_3 are integral multiple of 30 Hz]

b) The frequencies are $f_1 = 3f$, $f_2 = 5f$, $f_3 = 7f$

So, f_1, f_2 and f_3 are 3rd harmonic, 5th harmonic and 7th harmonic respectively.

c) The frequencies in the string are f, 2f, 3f, 4f, 5f, ……….

So, 3f = 2nd overtone and 3rd harmonic

5f = 4th overtone and 5th harmonic

7f = 6th overtone and 7th harmonic

d) length of the string is l = 80 cm

$\Rightarrow f_1 = (3/2l)v$ (v = velocity of the wave)

$\Rightarrow 90 = \{3/(2\times80)\} \times K$

$\Rightarrow K = (90 \times 2 \times 80) / 3 = 4800$ cm/s = 48 m/s.

47. Frequency $f = \dfrac{1}{lD}\sqrt{\dfrac{T}{\pi\rho}} \Rightarrow f_1 = \dfrac{1}{l_1 D_1}\sqrt{\dfrac{T_1}{\pi\rho_1}} \Rightarrow f_2 = \dfrac{1}{l_2 n_2}\sqrt{\dfrac{T_2}{\pi\rho_2}}$

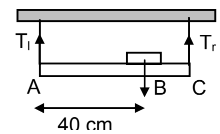
Given that, $T_1/T_2 = 2$, $r_1 / r_2 = 3 = D_1/D_2$

$\dfrac{\rho_1}{\rho_2} = \dfrac{1}{2}$

So, $\dfrac{f_1}{f_2} = \dfrac{l_2 D_2}{l_1 D_1}\sqrt{\dfrac{T_1}{T_2}}\sqrt{\dfrac{\pi\rho_2}{\pi\rho_1}}$ ($l_1 = l_2$ = length of string)

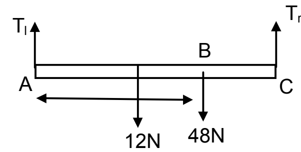
$\Rightarrow f_1 : f_2 = 2 : 3$

48. Length of the rod = L = 40 cm = 0.4 m
Mass of the rod m = 1.2 kg
Let the 4.8 kg mass be placed at a distance 'x' from the left end.
Given that, $f_l = 2f_r$



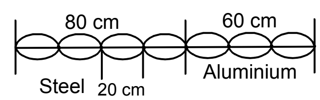
$\therefore \dfrac{1}{2l}\sqrt{\dfrac{T_l}{m}} = \dfrac{2}{2l}\sqrt{\dfrac{T_r}{m}}$

$\Rightarrow \sqrt{\dfrac{T_l}{T_r}} = 2 \Rightarrow \dfrac{T_l}{T_r} = 4$ …(1)



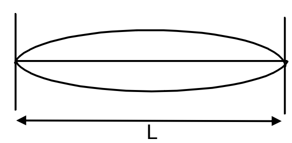
From the freebody diagram,
$T_l + T_r = 60$ N
$\Rightarrow 4T_r + T_r = 60$ N
$\therefore T_r = 12$ N and $T_l = 48$ N
Now taking moment about point A,
$T_r \times (0.4) = 48x + 12\,(0.2) \Rightarrow x = 5$ cm
So, the mass should be placed at a distance 5 cm from the left end.

49. $\rho_s = 7.8$ g/cm^3 , $\rho_A = 2.6$ g/cm^3
$m_s = \rho_s A_s = 7.8 \times 10^{-2}$ g/cm (m = mass per unit length)
$m_A = \rho_A A_A = 2.6 \times 10^{-2} \times 3$ g/cm $= 7.8 \times 10^{-3}$ kg/m



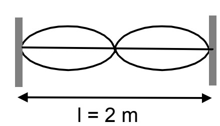
A node is always placed in the joint. Since aluminium and steel rod has same mass per unit length, velocity of wave in both of them is same.
$\Rightarrow v = \sqrt{T/m} \Rightarrow 500/7$ m/x
For minimum frequency there would be maximum wavelength for maximum wavelength minimum no of loops are to be produced.
$\therefore$ maximum distance of a loop = 20 cm
$\Rightarrow$ wavelength = $\lambda = 2 \times 20 = 40$ cm = 0.4 m
$\therefore f = v/\lambda = 180$ Hz.

50. Fundamental frequency
$V = 1/2l\ \sqrt{T/m} \Rightarrow \sqrt{T/m} = v2l$ [$\sqrt{T/m}$ = velocity of wave]



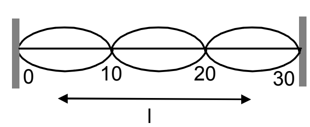
a) wavelength, λ = velocity / frequency = $v2l / v = 2l$
and wave number = $K = 2\pi/\lambda = 2\pi/2l = \pi/l$
b) Therefore, equation of the stationary wave is
$y = A \cos (2\pi x/\lambda) \sin (2\pi Vt / L)$
$= A \cos (2\pi x / 2l) \sin (2\pi Vt / 2L)$
$v = V/2L$ [because $v = (v/2l)$]

51. V = 200 m/s, 2A = 0.5 m
a) The string is vibrating in its 1st overtone
$\Rightarrow \lambda = 1 = 2$m
$\Rightarrow f = v/\lambda = 100$ Hz



b) The stationary wave equation is given by
$y = 2A\ \cos\dfrac{2\pi x}{\lambda} \sin\dfrac{2\pi Vt}{\lambda}$
$= (0.5\text{ cm}) \cos [(\pi m^{-1})x] \sin [(200\ \pi s^{-1})t]$

52. The stationary wave equation is given by
$y = (0.4\text{ cm}) \sin [(0.314\text{ cm} - 1)x] \cos [(6.00\ \pi s^{-1})t]$
a) $\omega = 600\ \pi \Rightarrow 2\pi f = 600\ \pi \Rightarrow f = 300$ Hz
wavelength, $\lambda = 2\pi/0.314 = (2 \times 3.14) / 0.314 = 20$ cm



b) Therefore nodes are located at, 0, 10 cm, 20 cm, 30 cm
c) Length of the string = $3\lambda/2 = 3 \times 20/2 = 30$ cm
d) $y = 0.4 \sin (0.314\ x) \cos (600\ \pi t) \Rightarrow 0.4 \sin \{(\pi/10)x\} \cos (600\ \pi t)$
since, λ and v are the wavelength and velocity of the waves that interfere to give this vibration λ = 20 cm

$v = \omega/k = 6000$ cm/sec = 60 m/s

53. The equation of the standing wave is given by

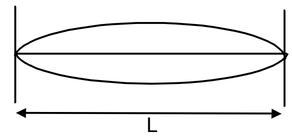
$y = (0.4 \text{ cm}) \sin [(0.314 \text{ cm}^{-1})x] \cos \; [(6.00 \; \pi s^{-1})t]$

$\Rightarrow k = 0.314 = \pi/10$

$\Rightarrow 2\pi/\lambda = \pi/10 \Rightarrow \lambda = 20$ cm

for smallest length of the string, as wavelength remains constant, the string should vibrate in fundamental frequency

$\Rightarrow l = \lambda/2 = 20 \text{ cm} / 2 = 10$ cm



54. L = 40 cm = 0.4 m, mass = 3.2 kg = 3.2×10^{-3} kg

$\therefore$ mass per unit length, $m = (3.2)/(0.4) = 8 \times 10^{-3}$ kg/m

change in length, $\Delta L = 40.05 - 40 = 0.05 \times 10^{-2}$ m

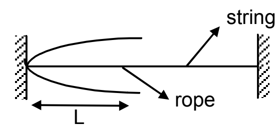
strain = $\Delta L/L = 0.125 \times 10^{-2}$ m

f = 220 Hz

$$f = \frac{1}{2l'}\sqrt{\frac{T}{m}} = \frac{1}{2 \times (0.4005)}\sqrt{\frac{T}{8 \times 10^{-3}}} \Rightarrow T = 248.19 \text{ N}$$

Strain = 248.19/1 mm^2 = 248.19×10^6

Y = stress / strain = 1.985×10^{11} N/m^2



55. Let, $\rho \rightarrow$ density of the block

Weight ρ Vg where V = volume of block

The same turning fork resonates with the string in the two cases

$$f_{10} = \frac{10}{2l}\sqrt{\frac{T - \rho_w Vg}{m}} = \frac{11}{2l}\sqrt{\frac{(\rho - \rho_w)Vg}{m}}$$

As the f of tuning fork is same,

$$f_{10} = f_{11} \Rightarrow \frac{10}{2l}\sqrt{\frac{\rho Vg}{m}} = \frac{11}{2l}\sqrt{\frac{(\rho - \rho_w)Vg}{m}}$$

$$\Rightarrow \frac{10}{11} = \sqrt{\frac{\rho - \rho_w}{m}} \Rightarrow \frac{\rho - 1}{\rho} = \frac{100}{121} \qquad \text{(because, } \rho_w = 1 \text{ gm/cc)}$$

$\Rightarrow 100\rho = 121\,\rho - 121 \Rightarrow 5.8 \times 10^3$ kg/m^3

56. l = length of rope = 2 m

M = mass = 80 gm = 0.8 kg

mass per unit length = m = 0.08/2 = 0.04 kg/m

Tension T = 256 N

Velocity, V = $\sqrt{T/m}$ = 80 m/s

For fundamental frequency,

$l = \lambda/4 \Rightarrow \lambda = 4l = 8$ m

$\Rightarrow f = 80/8 = 10$ Hz

a) Therefore, the frequency of 1st two overtones are

1st overtone = 3f = 30 Hz

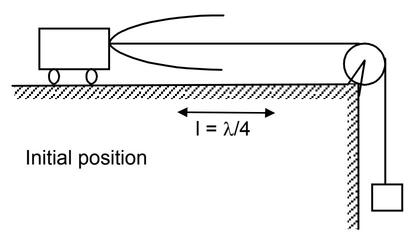
2nd overtone = 5f = 50 Hz

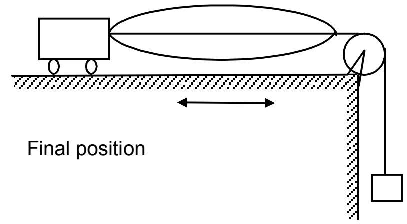
b) $\lambda_1 = 4l = 8$ m

$\lambda_1 = V/ f_1 = 2.67$ m

$\lambda_2 = V/f_2 = 1.6$ mt

so, the wavelengths are 8 m, 2.67 m and 1.6 m respectively.





57. Initially because the end A is free, an antinode will be formed.

So, $l = Ql_1 / 4$

Again, if the movable support is pushed to right by 10 m, so that the joint is placed on the pulley, a node will be formed there.

So, $l = \lambda_2 / 2$

Since, the tension remains same in both the cases, velocity remains same.

As the wavelength is reduced by half, the frequency will become twice as that of 120 Hz i.e. 240 Hz.

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