as a sine or a cosine. Suppose such an external force F_{ext} is applied to an oscillator that moves along x axis such as a block connected to a spring. We can represent the external forces as: $F_{ext} = F_0 \cos\omega t$ Where F_0 is the maximum magnitude of the force and $\omega(= 2\pi f)$ is the angular frequency of the force. Then the equation of motion (with damping) is $ma = -kx - bV + F_0 \cos\omega t$. This equation can be written as

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0\cos\omega t \quad \text{or} \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega t \qquad \text{... (i)}$$

The solution of eq. (i) is $x = A_0\cos(\omega t + \phi)$ Where $A_0 = \dfrac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{b\omega}{m}\right)^2}}$ ... (ii)

and $\omega_0 = \sqrt{k/m}$ is the frequency of undamped (b=0) oscillator i.e., natural frequency.

(iii) Resonant oscillations: When a body is maintained in a state of oscillations by a periodic force having the same frequency as the natural frequency of the body, the oscillations are called resonant oscillations. The phenomenon of producing resonant oscillations is called resonance.

(b) The amplitude of motion (A_0) depends on the difference between the applied frequency (ω) and natural frequency (ω_0). The amplitude is the maximum when the frequency of the driving force equals the natural frequency i.e., when $\omega = \omega_0$. It is because the denominator in eq. (ii) is the minimum when $\omega = \omega_0$. This condition is called resonance. When the frequency of the driving force equals ω_0, the oscillator is said to be in resonance with the driving force.

$$A_0 = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right) + \left(\dfrac{b\omega}{m}\right)^2}} \qquad \text{At resonance, } \omega = \omega_0 \text{ and } A_0 = \frac{F_0/m}{\sqrt{(b\omega/m)^2}} = \frac{F_0}{b\omega}$$

PROBLEM-SOLVING TACTICS

To verify SHM see whether force is directly proportional to y or see if $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ in cases when the equation is directly given compare with general equation to find the time period and other required answers

FORMULAE SHEET

1. Simple Harmonic Motion (SHM):

$F = -kx^n$

n is even - Motion of particle is not oscillatory

n is odd - Motion of particle is oscillatory.

If n = 1, F = -kx or $F \propto -x$. The motion is simple harmonic.

x = 0 is called the mean position or the equilibrium position.

Condition for SHM $\dfrac{d^2x}{dt^2} \propto -x$

Acceleration, $a = \frac{F}{m} = -\frac{k}{m}x = -\omega^2 x$

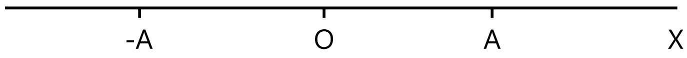


Figure 8.30

Displacement $x = A\cos\underbrace{(\omega t + \phi)}_{\text{phase angle}}$ (A is Amplitude)

Time period of SHM $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Frequency ν of SHM $\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Velocity of particle $v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$

Acceleration of particle $a = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$

2. Energy in SHM:

Kinetic energy of particle $= \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$

Potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

Total energy E= P.E + K.E = $\frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2$

E is constant throughout the SHM.

3. Simple pendulum: Time period $T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$

Here, ℓ is length of simple pendulum and $\vec{g}_{eff} = \vec{g} - \vec{a}$ where $\vec{g}$ is acceleration due to gravity and $\vec{a}$ is acceleration of the box or cabin etc. containing the simple pendulum.

4. Spring-block system: Time period $T = 2\pi\sqrt{\frac{m}{k}}$

5. Physical pendulum: Time period $T = 2\pi\sqrt{\frac{I}{mg\ell}}$

Here I is the moment of inertia about axis of rotation and ℓ is the distance of center of gravity from the point of suspension.

6. Torsional Pendulum:

$T = 2\pi\sqrt{\frac{I}{k}}$

I is the moment of Inertia about axis passing through wire, k is torsional constant of wire.

7. Springs in series and parallel

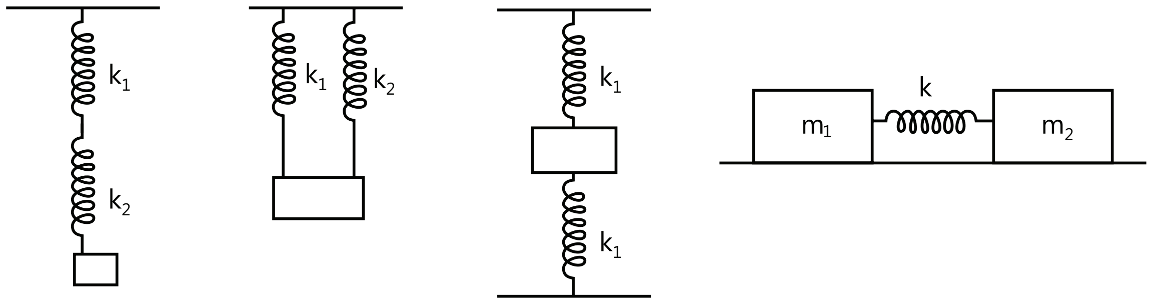


Figure 8.31

Series combination $\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$

Parallel combination $k = k_1 + k_2$

8. For two blocks of masses m_1 and m_2 connected by a spring of constant k:

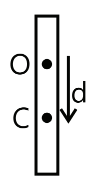
Time period $T=2\pi\sqrt{\frac{\mu}{k}}$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is reduced mass of the two-block system.

Solved Examples

JEE Main/Boards

Example 1: What is the period of pendulum formed by pivoting a meter stick so that it is free to rotate about a horizontal axis passing through 75 cm mark?



Sol: This is an example of a physical pendulum.
Find moment of inertia about point of suspension and the distance of the point of suspension from the center of gravity.

Let m be the mass and ℓ be the length of the stick. $\ell = 100\text{cm}$ The distance of the point of suspension from center of gravity is $d = 25\,\text{cm}$

Moment of inertia about a horizontal axis through O is

$$I = I_c + md^2 = \frac{m\ell^2}{12} + md^2$$

$$T = 2\pi\sqrt{\frac{I}{mgd}}; \quad T = 2\pi\sqrt{\frac{\frac{m\ell^2}{12}+md^2}{mgd}}$$

$$T = 2\pi\sqrt{\frac{\ell^2+12d^2}{12gd}} = 2\pi\sqrt{\frac{\ell^2+12(0.25)^2}{12\text{x}9.8\text{x}0.25}} = 153\text{ s.}$$

Example 2: A particle executes SHM.

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude?

(b) At what value of displacement are the kinetic and potential energies equal?

Sol: The sum of kinetic energy and potential energy is the total mechanical energy which is constant throughout the SHM.

We know that $E_{total} = \frac{1}{2}m\omega^2 A^2$

$$KE = \frac{1}{2}m\omega^2\left(A^2 - X^2\right) \text{ and } \quad U = \frac{1}{2}m\omega^2 x^2$$

(a) When $x = \frac{A}{2}$, $KE = \frac{1}{2}m\omega^2\frac{3A^2}{4} \Rightarrow \frac{KE}{E_{total}} = \frac{3}{4}$