6. CENTRE OF MASS AND THE LAW OF CONSERVATION OF MOMENTUM

1. INTRODUCTION

In this chapter we will study the motion of system of particles or bodies. The individual particles or bodies comprising the system in the general case move with different velocities and accelerations and exert forces on each other and are influenced by external or surrounding bodies as well. We will learn the techniques to simplify the analysis of complicated motion of such a system. We will also learn about the dynamics of extended bodies whose shape and/or mass changes during their motion. We define the linear momentum of a system of particles and introduce the concept of center of mass of a system. The dynamics of center of mass and the law of conservation of linear momentum are important tools in the study of system of particles.

2 CENTER OF MASS

When we study the dynamics of the motion of a system of particles as a whole, then we need not bother about the dynamics of individual particles of the system, but only focus on the dynamics of a unique point corresponding to that system. The motion of this unique point is identical to the motion of a single particle whose mass is equal to the sum of the masses of all the individual particles of the system and the resultant of all the forces exerted on all the particles of the system, by the surrounding bodies, or due to action of a field of force, is exerted directly to that particle. This point is called the center of mass (COM) of the system of particles. The COM behaves as if the entire mass of the system is concentrated there. The concept of COM is very useful in analyzing complicated motion of system of objects, in particular, when two or more objects collide or an object explodes into fragments.

2.1 Center of Mass of a System of Particles

For a system of n particles, having masses $m_1, m_2, m_3 \ldots\ldots m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots\ldots \vec{r}_n$ respectively with respect to the origin in a certain reference frame, the position vector of center of mass, $\vec{r}_{cm}$ with respect to the origin is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots\ldots + m_n}$$

If the total mass of the system is M, then $M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_n\vec{r}_n$

Co-ordinates of center of mass are

$$x_{cm}=\frac{x_1m_1+x_2m_2+.........x_mm_n}{m_1+m_2+........m_n};\qquad y_{cm}=\frac{y_1m_1+y_2m_2+.........y_mm_n}{m_1+m_2+........m_n};\qquad z_{cm}=\frac{z_1m_1+z_2m_2+.........z_mm_n}{m_1+m_2+........m_n}$$

For a system comprising of two particles of masses m_1, m_2, positioned at co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_1), respectively, we have

$$X_{com}=\frac{m_1x_1+m_2x_2}{m_1+m_2};\qquad Y_{com}=\frac{m_1y_1+m_2y_2}{m_1+m_2};\qquad Z_{com}=\frac{m_1z_1+m_2z_2}{m_1+m_2}$$

PLANCESS CONCEPTS

For a two-particle system, COM lies closer to the particle having more mass, which is rather obvious. If COM's co-ordinates are made zero, we would clearly observe that distances of individual particles are inversely proportional to their masses.

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 1: Two particles of masses 1 kg and 2 kg are located at x = 0 and x = 3 m respectively. Find the position of their center of mass. **(JEE MAIN)**

Sol: For the system of particle of masses m_1 and m_2, if the distance of particle from the center of mass are r_1 and r_2 respectively then it is seen that $m_1r_1 = m_2r_2$.

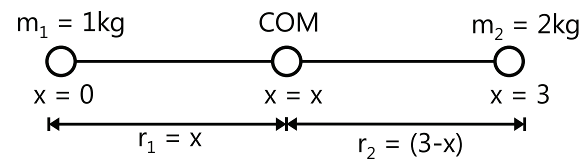


Figure 6.1

Since, both the particles lie on x-axis, the COM will also lie on the x-axis. Let the COM be located at x = x, then r_1 = distance of COM from the particle of mass 1 kg = x

and r_2 = distance of COM from the particle of mass 2 kg = (3 – x)

Using $\frac{r_1}{r_2}=\frac{m_2}{m_1}$ or $\frac{x}{3-x}=\frac{2}{1}$ or x = 2 m

Thus, the COM of the two particles is located at x = 2 m.

Illustration 2: Four particles A, B, C and D having masses m, 2m, 3m and 4m respectively are placed in order at the corners of a square of side a. Locate the center of mass. **(JEE ADVANCED)**

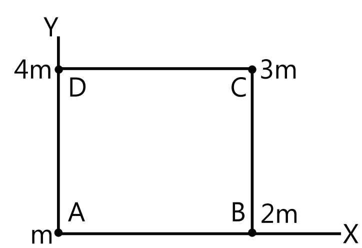


Figure 6.2

Sol: The co-ordinate of center of mass of n particle system are given as

$$X=\frac{\sum_i m_ix_i}{\sum_i m_i},\quad Y=\frac{\sum_i m_iy_i}{\sum_i m_i}$$

Take the x and y axes as shown in Fig. 6.2. The coordinates of the four particles are as follows:

Particle	Mass	x-coordinate	y-coordinate
A	m	0	0 (taking A as origin)
B	2m	a	0
C	3m	a	a
D	4m	0	a

Hence, the coordinates of the center of mass of the four-particle system are

$$X = \frac{m\,.\,0 + 2ma + 3ma + 4m\,.\,0}{m + 2m + 3m + 4m} = \frac{a}{2};\; Y = \frac{m\,.\,0 + 2m\,.\,0 + 3ma + 4ma}{m + 2m + 3m + 4m} = \frac{7a}{10}$$

The center of mass is at $\left(\frac{a}{2}, \frac{7a}{10}\right)$.

2.2. Center of Mass of a Continuous Body

For continuous mass distributions, the co-ordinates of center of mass are determined by following formulae,

$$x_{cm} = \frac{\int x\,dm}{\int dm}; \qquad y_{cm} = \frac{\int y\,dm}{\int dm}; \qquad z_{cm} = \frac{\int z\,dm}{\int dm}$$

where x, y and z are the co-ordinates of an infinitesimal elementary mass dm taken on the continuous mass distribution. The integration should be performed under proper limits, such that the elementary mass covers the entire body.

PLANCESS CONCEPTS

Many people have misconception that the center of mass of a continuous body must lie inside the body. Center of mass of a continuous body may lie outside that body also. e.g. Ring.

Vaibhav Krishnan (JEE 2009, AIR 22)

(a) Center of Mass of a Uniform Straight Rod

Center of mass of a rod of length L is at (L/2, 0, 0).

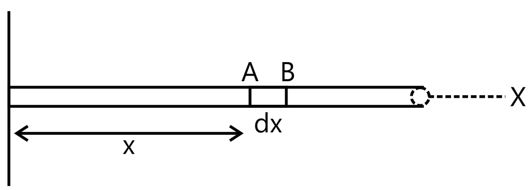


Figure 6.3

(b) Center of Mass of a Uniform Semicircular Wire

Center of mass of a Uniform Semicircular Wire of radius R is (0, 2R/π).

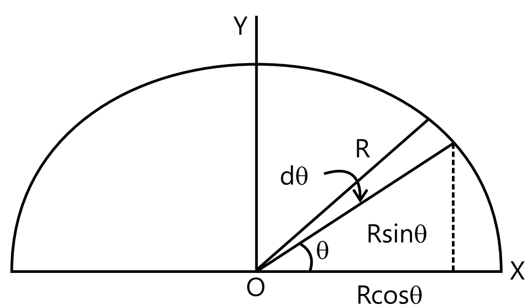


Figure 6.4

(c) Center of Mass of a Uniform Semicircular Plate

Center of mass of a uniform semicircular plate of radius R is (0, 4R/3π)

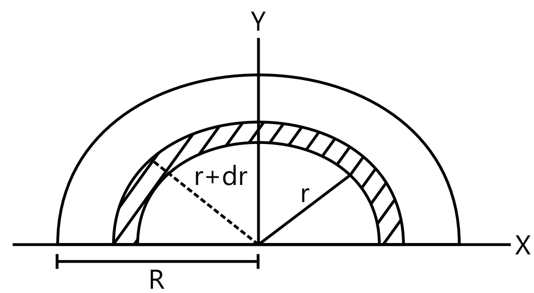


Figure 6.5

(d) Center of mass of a uniform hollow cone

Center of mass of a uniform hollow cone of height H lies on the axis at a distance of H/3 from the center of the bottom.

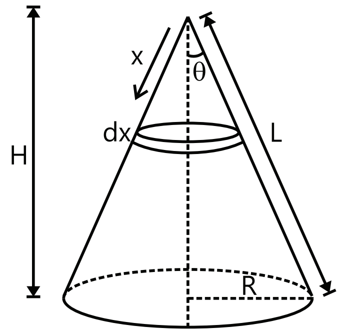


Figure 6.6

PLANCESS CONCEPTS

Student must solve the above integrations to get a better view of how we take infinitesimal segment of a body and the corresponding limits to integrate over whole body.

Please solve the integrations for hollow cone and solid cone to note the difference.

Nivvedan (JEE 2009, AIR 113)

Illustration 3: A rod of length L is placed along the x-axis between x = 0 and x = L. The linear density (mass/length) ρ of the rod varies with the distance x from the origin as ρ = a + bx. Here, a and b are constants. Find the position of center of mass of this rod. **(JEE MAIN)**

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as

$\therefore x_{COM} = \frac{\int x\,dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

Choose an infinitesimal element of the rod of length dx situated at co-ordinates (x, 0, 0) (see Fig.6.7) The linear mass density can be assumed to be constant along the infinitesimal length dx.

Thus the mass of the element dm = rdx = (a + bx) dx

As x varies from 0 to L the element covers the entire rod.

Therefore, x-coordinate of COM of the rod will be

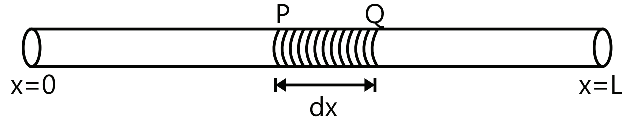


Figure 6.7

$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L (x)\,(a+bx)dx}{\int_0^L (a+bx)dx} = \frac{\left[\frac{ax^2}{2}+\frac{bx^3}{3}\right]_0^L}{\left[ax+\frac{bx^2}{2}\right]_0^L} = \frac{3aL^2+2bL^3}{6} \times \frac{2}{2aL+bL^2}$$

$$x_{COM} = \frac{3aL+2bL^2}{6a+3bL}\ \text{m}$$

Illustration 4: Determine the center of mass of a uniform solid cone of height h and semi angle α, as shown in Fig. 6.8 **(JEE MAIN)**

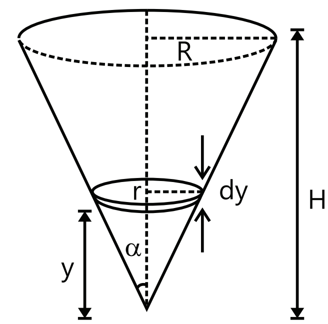


Figure 6.8

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as

$\therefore Y_{COM} = \dfrac{\int y\, dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

We place the apex of the cone at the origin and its axis along the y-axis. As the cone is a right circular cone then by symmetry it is clear that the center of mass will lie on its axis i.e. on the y-axis. We consider an elementary disk of radius r and infinitesimal thickness dy whose center is on the y-axis at distance y from the origin as shown in Fig. 6.8. The volume of such a disk is

$dV = \pi r^2\, dy = \pi\, (y \tan\alpha)^2\, dy$

The mass of this elementary disk is dm = rdV. As y varies from 0 to H, the total height of the cone, the elementary disc covers the entire cone. The total mass M of the cone is given by,

$$M = \int dm = \pi\rho \tan^2\alpha \int_0^H y^2 dy \ = \ \pi\rho\tan^2\alpha \frac{H^3}{3} \qquad ....(i)$$

The position of the center of mass is given by

$$y_{com} = \frac{1}{M}\int_0^H y\, dm = \frac{1}{M}\pi\rho\tan^2\alpha \int_0^H y^3 dy$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\, \frac{H^4}{4} \qquad ....(ii)$$

From equations (i) and (ii), we have $y_{com} = \dfrac{3H}{4}$

3. CENTER OF GRAVITY

Definition: Center of gravity is a point, near or within a body, at which its entire weight can be assumed to act when considering the motion of the body under the influence of gravity. This point coincides with the center of mass when the gravitational field is uniform.

Note: The center of mass and center of gravity for a continuous body or a system of particles will be different when there is non- uniform gravitational field.

PLANCESS CONCEPTS

You can find the center of gravity and center of mass for a very thin cylinder extending from the surface of earth to the height equal to radius of earth to get the difference. Just sum up all the individual weights of infinitesimal size disks and find the position where gravity will make the same weight of body. This will give center of gravity.

Chinmay S Purandare (JEE 2012, AIR 698)

4. CENTER OF MASS OF THE BODY WHEN A PORTION OF THE BODY IS TAKEN OUT

Suppose there is a body of total mass m and a mass m_1 is taken out from this body. The remaining body will have mass $(m - m_1)$ and its center of mass will be at coordinates,

$$x_{cm} = \frac{mx - m_1x_1}{(m - m_1)}; \quad y_{cm} = \frac{my - m_1y_1}{(m - m_1)}; \quad z_{cm} = \frac{mz - m_1z_1}{(m - m_1)}$$

where (x, y, z) are coordinates of center of mass of original (whole) body and (x_1, y_1, z_1) are coordinates of center of mass of the portion taken out.

Illustration 5: A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Fig. 6.9. Find the center of mass of the remaining position.

(JEE MAIN)

Sol: Let O be the center of circular plate and, O_1, the center of circular portion removed from the plate. The COM of the whole plate will lie at O and the COM of the circular cavity will lie at O_1. Let O be the origin. So OO_1 = 28cm – 21cm = 7cm.

The center of mass of the remaining portion will be given as

$$x_{cm} = \frac{mx - m_1x_1}{(m - m_1)} = \frac{\sigma(Ax - A_1x_1)}{\sigma(A - A_1)} = \frac{\pi((28)^2 0 - (21)^2 7)}{\pi((28)^2 - (21)^2)}$$

$$x_{cm} = -9 \text{ cm} = -0.09 \text{ m}.$$

This means that center of mass of the remaining plate is at a distance 9 cm from the center of given circular plate opposite to the removed portion i.e. in this questioon, the new Centre of Mass will shift 9 cm left.

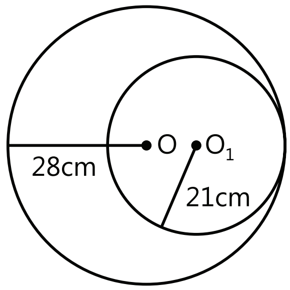


Figure 6.9

5. MOTION OF THE CENTER OF MASS

For a n-particle system of total mass M and individual particles having mass m_1, m_2, ... m_n, from the definition of center of mass we can write,

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_n\vec{r}_n$$

where $\vec{r}_{cm}$ is the position vector of the center of mass, and $\vec{r}_1$, $\vec{r}_2$, ... $\vec{r}_n$ are the position vectors of the individual particles relative to the same origin in a particular reference frame.

If the mass of each particle of the system remains constant with time, then, for our system of particles with fixed mass, differentiating the above equation with respect to time, we obtain.

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{r}_n}{dt} \quad \text{....(i)}$$

$$\text{or } M\vec{V}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + m_n\vec{v}_n$$

Where $\vec{v}_1$, $\vec{v}_2$, ... $\vec{v}_n$ are the velocities of the individual particles, and $\vec{V}_{cm}$ is the velocity of the center of mass. Again differentiating with respect to time, we obtain

$$M\frac{d\vec{V}_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{v}_n}{dt}$$

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots + m_n\vec{a}_n \quad \text{....(ii)}$$

Where $\vec{a}_1$, $\vec{a}_2$, ... $\vec{a}_n$ are the accelerations of the individual particles, and $\vec{a}_{cm}$ is the acceleration of the center of mass. Now, from Newton's second law, the force F_i acting on the i^{th} particle is given by $F_i = m_i a_i$. Then, above equation can be written as

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \vec{F}_{internal} + \vec{F}_{external} \quad ....(iii)$$

Internal forces are the forces exerted by the particles of the system on each other. However, from Newton's third law, these internal forces occur in pairs of equal and opposite forces, so their net sum is zero. $\therefore M\vec{a}_{cm} = \vec{F}_{ext}$

This equation states that the center of mass (C.O.M) of a system of particles behaves as if all the mass of the system were concentrated there and the resultant of all the external forces acting on all the particles of the system was applied to it (at C.O.M).

Concept: Whatever may be the rearrangement of the bodies in a system, due to **internal forces** (such as different parts of the system moving away or towards each other or an internal explosion taking place, breaking a body into fragments) provided net F_{ext}=0, we have two possibilities:

(a) If the system as a whole was originally at rest, i.e. the C.O.M was at rest, then the C.O.M. will continue to be at rest.

(b) If before the change, the system as a whole had been moving with a constant velocity (C.O.M was moving with a consant velocity), it will continue to move with a constant velocity.

In presence of a net external force if the C.O.M had been moving with certain acceleration at the instant of an explosion, in a particular trajectory, the C.O.M. will continue to move in the same trajectory, with the same acceleration, as if the system had never exploded at all.

Briefly saying, any internal changes of the body do not effect the motion of C.O.M.

Illustration 6: A man of mass m is standing on a platform of mass M kept on smooth ice (see Fig. 6.10). If the man starts moving on the platform with a speed v relative to the platform, with what velocity relative to the ice does the platform recoil ? **(JEE MAIN)**

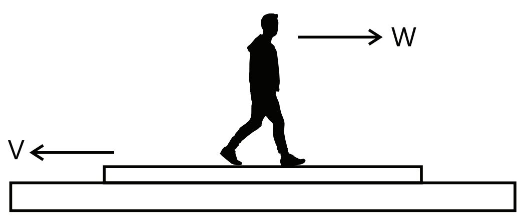


Figure 6.10

Sol: When net external force on system is zero, the C.O.M. will either remain at rest or continue the state of motion. i.e. V_{cm} = constant

Let velocity of platform be $\vec{V}$. If velocity of man relative to platform is $\vec{v}$ then the velocity of man in reference frame of ice is $\vec{w} = \vec{v} + \vec{V}$.

Center of mass of the system comprising of "man and the platform" is initially at rest and as no horizontal external force acts on this system (ice is smooth), the center of mass will continue to remain at rest.

$$\vec{V}_{cm} = 0 = \frac{M\vec{V} + m(\vec{v} + \vec{V})}{M+m}$$

or $\quad (M+m)\vec{V} + m\vec{v} = 0$

or $\quad \vec{V} = -\frac{m\vec{v}}{M+m}\ ms^{-1}$

Negative sign shows that the platform will move in the opposite direction of relative velocity of man.

Illustration 7: Two block of masses m_1 and m_2 connected by a weightless spring of stiffness k rest on a smooth horizontal plane (see Fig 6.11). Block 2 is shifted by a small distance x to the left and released. Find the velocity of the center of mass of the system after block 1 breaks off the wall. **(JEE ADVANCED)**

Sol: Elastic potential energy stored in spring will get converted in kinetic energy of the blocks. If we consider the FBD of mass m_1 at the instant when it breaks off the wall, the normal reaction from the wall is zero, but normal reaction from the wall is equal to is equal to force exerted by spring on mass m_1 so at this instant, force by spring is also zero.

The initial potential energy of compression is $=\frac{1}{2}kx^2$

When the block m_1 breaks off from the wall, the normal reaction from the wall is zero, which in turn means that the tension in the spring is zero. Thus the spring has its natural length at this instant and the kinetic energy of the block m_2 is given by

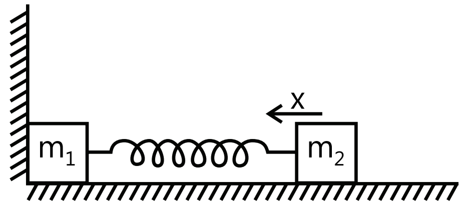


Figure 6.11

$$\frac{1}{2}m_2v_2^2=\frac{1}{2}kx^2$$

$$v_2^2=\frac{kx^2}{m_2}$$

$$v_2=x\sqrt{\frac{k}{m_2}}$$

Velocity of center of mass is

$$V_{cm}=\frac{m_1v_1+m_2v_2}{m_1+m_2}$$

At start $v_1 = 0$

$$\therefore\ V_{cm}=\frac{m_2}{m_1+m_2}v_2=\frac{m_2x}{m_1+m_2}\sqrt{\frac{k}{m_2}}$$

$\therefore$ Velocity of center of mass of system $V_{cm}=\frac{x\sqrt{km_2}}{m_1+m_2}$ ms[-1]

6. LINEAR MOMENTUM

The quantity momentum (denoted as $\vec{P}$) is a vector defined as the product of the mass of a particle and its velocity $\vec{v}$, i.e. $\vec{P}=m\vec{v}$ ....(i)

From Newton's second law of motion, if mass of a particle is constant

$$\vec{F}=m\vec{a}=m\frac{d\vec{v}}{dt}=\frac{d}{dt}(m\vec{v})=\frac{d\vec{P}}{dt}$$

Thus, for constant m, the rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force.

For a system of n particles with masses m_1, m_2 ..... etc., and velocities v_1, v_2 ..... etc. respectively, the total momentum $\vec{P}$ in a particular reference frame is,

$$\vec{P}=\vec{P}_1+\vec{P}_2+........+\vec{P}_n=m_1\vec{v}_1+m_1\vec{v}_2+.........+m_n\vec{v}_n;$$

or $\vec{P}=M\vec{V}_{cm}$

Also, $$\frac{d\vec{P}}{dt}=M\frac{d\vec{V}_{cm}}{dt}=M\vec{a}_{cm}=\vec{F}_{ext}$$

$$\therefore\quad \frac{d\vec{P}}{dt}=\vec{F}_{ext}$$

The magnitude of linear momentum may be expressed in terms of the kinetic energy as well.

$p = mv$

or $$p^2 = m^2v^2 = 2m\left(\frac{1}{2}mv^2\right) = 2mK$$

Thus, $$p = \sqrt{2Km} \text{ or } K = \frac{p^2}{2m}$$

6.1. Law of Conservation of Linear Momentum

"The law of conservation of linear momentum states that if no external forces act on a system of particles, then the vector sum of the linear momenta of the particles of the system remains constant and is not affected by their mutual interaction. In other words the total linear momentum of a closed system remains constant in an inertial reference frame."

Proof: For a system of fixed-mass particles having total mass m we have

$$\vec{F}_{ext} = m\vec{a}_{cm} = m\frac{d\vec{v}_{cm}}{dt} = \frac{d(m\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt}$$, where $\vec{P}$ is the total momentum of the system.

Thus, $$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

In case the net external force applied to the system is zero, we have

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} = 0 \quad \text{or} \quad \vec{P} = \text{constant}$$

Thus for a closed system, the total linear momentum of the system remains constant in an inertial frame of reference.

PLANCESS CONCEPTS

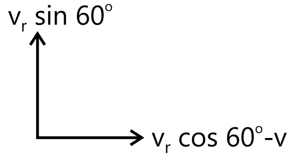
Both linear momentum and kinetic energy are dependent on the reference frame since velocity is inclusively dependent on the frame of reference.

Nitin Chandrol (JEE 2012, AIR 134)

Illustration 8: A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun. **(JEE MAIN)**

Sol: When a bullet is fired, gun recoils in backward direction. Using law of conservation of linear momentum we can find the recoil velocity of gun.

Let the recoil velocity of gun be $\vec{v}$. The relative velocity of the bullet is $\vec{v}_r$ at an angle of 60° with the horizontal. Taking gun + bullet as the system the net external force on the system in horizontal direction is zero. Let x-axis be along the horizontal and bullet be fired towards the positive direction of x-axis. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum along x-axis, we get



Components of velocity
of bullet relative to ground

Figure 6. 12

$$Mv_x + m(v_{rx} + v_x) = 0$$

$$-Mv + m(v_r \cos 60° - v) = 0$$

$$v = \frac{mv_r \cos 60°}{M+m}$$

or $$v = \frac{mv_r}{2(M+m)} \text{ ms}^{-1}$$

Illustration 9: The block of mass $m_1 = 2\text{kg}$ and $m_2 = 5\text{kg}$ are moving in the same direction along a frictionless surface with speeds 10 ms^{-1} and 3 ms^{-1}, respectively m_2 being ahead of m_1 as shown in Fig. 6.13. An ideal spring with spring constant K = 1120 N/m is attached to the back side of m_2. Find the maximum compression of the spring when the blocks move together after the collision. **(JEE ADVANCED)**

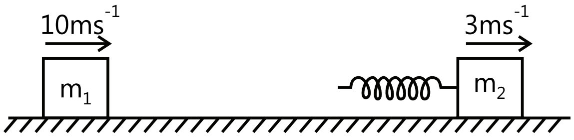


Figure 6.13

Sol: As frictional force on the blocks is zero the total momentum of blocks can be conserved during collision. At the instant of maximum compression some part of initial total K.E. of blocks is stored as elastic P.E. in the spring.

Let v be the final velocity of the system after collision when the blocks move together.

Applying the law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

Substituting the values,

$$(2 \times 10) + (5 \times 3) = (2+5)v$$

$$v = \frac{35}{7} = 5 \text{ m/s}$$

Applying the law of conservation of energy we get

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1 + m_2)v_2^2 + \frac{1}{2}Kx^2$$

$$m_1u_1^2 + m_2u_2^2 = (m_1 + m_2)v_2 + Kx^2$$

$$2 \times (10)^2 + 5 \times (3)^2 = [(2+5) \times (5)^2] + 1120x^2$$

$$x^2 = \frac{70}{1120} = \frac{1}{16} \Rightarrow x = 0.25\text{m}$$

PLANCESS CONCEPTS

In the above questions, note that the compression would be maximum when the relative velocity between the blocks is zero.

B Rajiv Reddy (JEE 2012, AIR 11)

7. VARIABLE MASS

From Newton's second law, $\vec{F}_{ext} = m\vec{a}$ is applicable to a system whose total mass m is constant. If total mass of the system is not constant, then this form of Newton's second law is not applicable. If at a certain moment of time the total mass of a system is m and a mass dm is added (or separated) to the system, then we apply $\vec{F}_{ext} = \frac{d\vec{p}}{dt}$ to the system comprising "m+dm" to get

$$\vec{F}_{ext}.dt = d\vec{p} = \vec{p}_{final} - \vec{p}_{initial} = (m+dm)(\vec{v}+d\vec{v}) - [m\vec{v} + dm(\vec{v}+\vec{u})]$$

$$\text{or } \vec{F}_{ext}.dt = md\vec{v} - dm\vec{u}; \qquad (dm.d\vec{v} \simeq 0)$$

$$\text{or } \vec{F}_{ext} = m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{u}$$

$$\text{or } m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \frac{dm}{dt}\vec{u}$$

where u is velocity of adding or separating mass dm relative to the system having instantaneous mass m and instantaneous velocity v with respect to an inertial reference frame. The term $\frac{dm}{dt}$ can be positive or negative depending upon whether mass is added to the system or mass is separating from the system.

Problems related to variable mass can be solved in following three steps.

(a) Make a list of all the external forces acting on the main mass and draw its FBD.

(b) Apply an additional thrust force or reaction force $\vec{F}_t$ on the main mass, due to the action of the added(separated) mass on the main mass, the magnitude of which is $\left|\vec{u}\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{u}$ in case the mass is being added or the direction of $-\vec{u}$ if mass is being separated.

(c) Apply the equation

$$m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \frac{dm}{dt}\vec{u} \quad (m = \text{instantaneous mass})$$

Illustration 10: A flat cart of mass m_0 at t=0 starts moving to the left due to a constant horizontal force F. The sand spills on the flat cart from a stationary hopper. The rate of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flat cart in the process of loading. The friction is negligibly small. **(JEE ADVANCED)**

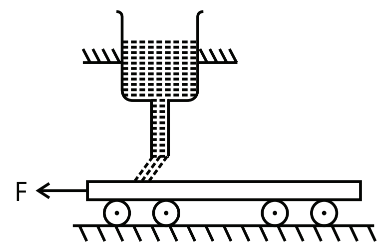


Figure 6.14

Sol: The hopper is at rest in K frame, so in the frame of the cart its initial velocity will be u=-v, where v is velocity of cart in K frame. Here we have used the equation of motion of variable mass $m\frac{dv}{dt} = F_{ext} + \frac{dm}{dt}u$

The rate of increase of mass of the flat car $\frac{dm}{dt} = \mu \text{ kgs}^{-1}$

The hopper is stationary and so its relative velocity is u = 0 - v = - v

The equation of motion is given by

$$m\frac{dv}{dt} = F + \frac{dm}{dt}u = F - \mu v \quad \left[\because \frac{dm}{dt} = \mu \text{ and } u = -v\right]$$

At the instant t, m = $m_0 + \mu t$

$$\therefore \quad \frac{dv}{F-\mu v} = \frac{dt}{m} = \frac{dt}{m_0 + \mu t} \quad \Rightarrow \quad \int_0^v \frac{dv}{F-\mu v} = \int_0^t \frac{dt}{m_0+\mu t}$$

$$\text{or,} \quad -\frac{1}{\mu}\log_e\frac{F-\mu v}{F} = \frac{1}{\mu}\log_e\frac{m_0+\mu t}{m_0}$$

or $\log_e \frac{F}{F-\mu v} = \log_e \frac{m_0+\mu t}{m_0} \Rightarrow \quad v = \frac{Ft}{m_0+\mu t}\ ms^{-1}$

The acceleration a is given by

$$a = \frac{dv}{dt} = \frac{Fm_0}{(m_0+\mu t)^2}\ ms^{-2}$$

Alternative:

$$m\frac{dv}{dt} = F + \frac{dm}{dt}u = F - \frac{dm}{dt}v \qquad \text{or } m\frac{dv}{dt} + \frac{dm}{dt}v = F$$

$$\text{or } \frac{d}{dt}(mv) = F \Rightarrow \int_0^{mv} d(mv) = \int_0^t Fdt$$

$$\text{or } mv = Ft \Rightarrow v = \frac{Ft}{(m_0+\mu t)}; \quad (\because m = m_0 + \mu t)$$

8. ROCKET PROPULSION

The propulsion of rocket is an example of a system of variable mass. In the combustion chamber of a rocket, the fuel is burnt in the presence of an oxidizing agent due to which a jet of gases emerges from the tail of the rocket. Thus the mass of the rocket is continuously decreasing. This action due to emission of gases in the backward direction produces a reaction force in the forward direction due to which the rocket moves forward.

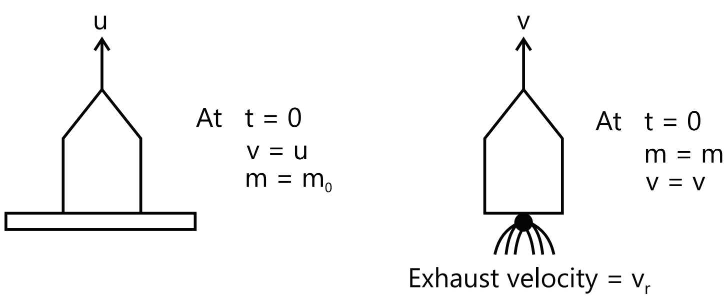


Figure 6.15: Rocket propulsion

Let m_0 be the mass of the rocket and u be its velocity at time t = 0, and m be its mass and v be its velocity at any time t. (see Fig. 6.15)

The mass of the gas ejected per unit time or the rate of change of mass of the rocket is $-\frac{dm}{dt}$ and v_r be the exhaust velocity of the gases relative to the rocket. Usually $-\frac{dm}{dt}$ and v_r are assumed constant throughout the journey of the rocket.

Now using the equation of motion for a system of variable mass derived in the previous article we get,

$$m\frac{d\vec{v}}{dt} = m\vec{g} + \vec{v}_r\frac{dm}{dt}$$

$$\text{or } \frac{d\vec{v}}{dt} = \vec{g} + \frac{\vec{v}_r}{m}\frac{dm}{dt}$$

$$\text{or } d\vec{v} = \vec{v}_r\frac{dm}{m} + \vec{g}.dt$$

This is a vector equation and we do not assume any sign of $\frac{dm}{dt}$. It is taken to be positive. After evaluating the definite integrals, when we substitute the scalar components of the vectors with proper signs we get the correct result.

Integrating on both sides, we get $\int_{\vec{u}}^{\vec{v}} d\vec{v} = \vec{v}_r \int_{m_0}^{m} \frac{dm}{m} + \vec{g}.\int_0^t dt$

or $\vec{v} - \vec{u} = \vec{v}_r \ln\frac{m}{m_0} + \vec{g}.t$

or $\vec{v} = \vec{u} + \vec{g}.t + \vec{v}_r \ln\frac{m}{m_0}$

Now taking upwards direction as positive and downwards as negative (g and v_r are downwards and u is upwards) we get,

$$v = u - g.t + (-v_r)\ln\frac{m}{m_0}$$

Thus, $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Now if $-\frac{dm}{dt} = \mu$ (constant), then m = m_0 - mt

Thus, $v = u - gt + v_r \ln\left(\frac{m_0}{m_0 - \mu t}\right)$

If the initial velocity of the rocket u = 0, and the weight of the rocket is ignored as compared to the reaction force of the escaping gases, the above equation reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$

PLANCESS CONCEPTS

The concept of variable mass can also be physically visualized by changing the reference frame to the instantaneous velocity of body. In that case mass is either being added by constant speed or being removed by a constant speed. Considering the dm mass and the body as a system, and writing the equations of conservation of momentum one can see the magic!

Anand K (JEE 2011, AIR 47)

Illustration 11: (a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2000 m/s. What should be its minimum rate of fuel consumption?

(i) To just lift it off the launching pad?

(ii) To give it an acceleration of 20 m/s^2?

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s after whole of the fuel is consumed? (Take g = 9.8 m/s^2) **(JEE ADVANCED)**

Sol: To lift the rocket upward against gravity, the thrust force in the upward direction due to exiting gases should be greater than or equal to the gravitational force. During motion the mass of rocket decreases till whole of its fuel is consumed. Final velocity of rocket is

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right).$$

(a) (i) To just lift the rocket off the launching pad

Initial weight = thrust force

or $\quad m_0 g = v_r\left(-\frac{dm}{dt}\right)$; or $-\frac{dm}{dt} = \frac{m_0 g}{v_r}$

Substituting the values, we get $-\frac{dm}{dt} = \frac{(450+50)(9.8)}{2\times10^3}$ = 2.45 kg/s

(ii) Net acceleration $a = 20\ m/s^2$

$\therefore \quad ma = F_t - mg$

or $\quad m(a+g) = F_t = v_r\left(-\frac{dm}{dt}\right)$

This gives, $\left(-\frac{dm}{dt}\right) = \frac{m(g+a)}{v_r}$

Substituting the values, we get $\left(-\frac{dm}{dt}\right) = \frac{(450+50)(9.8+20)}{2\times10^3}$ = 7.45 kg/s

(b) The rate of fuel consumption is 10 kg/s. So, the time for the consumption of entire fuel is

$$t = \frac{450}{10} = 45\,s$$

The formula for speed of the rocket at time t is, $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Here, u = 0, $v_r = 2\times10^3\ m/s$, $m_0 = 500\,kg$ and m = 50 kg

Substituting the values, we get $v = 0 - (9.8)(45) + (2\times10^3)\ln\left(\frac{500}{50}\right)$

or v = – 441 + 4605.17; or v = 4164.17 m/s; or v = 4.164 km/s

9. COLLISION

An event in which two or more bodies exert forces on each other for a relatively short time is called collision. If net external force acting on the system of bodies is zero, then according to the law of conservation of linear momentum, the total momentum of the system of bodies before and after the collision remains constant.

9.1 Classification of Collisions

Collisions are classified into following types on the basis of the degree of conservation of kinetic energy in a collision:

(a) Elastic Collision: If the total kinetic energy of the colliding particles is conserved before and after the collision, the collision is said to be an elastic collision. If two bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 respectively, collide with each other so that their final velocities after collision are v_1 and v_2 respectively, then the collision will be perfectly elastic if,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

(b) Inelastic Collision: If the total kinetic energy of the colliding particles is not conserved before and after the collision, the collision is said to be inelastic collision. The kinetic energy is partially converted into other forms of energy like sound, heat, deformation energy etc.

(c) Perfectly Inelastic Collision: The collision is said to be perfectly inelastic if the collision results in "sticking together" of the colliding particles after which they move as a single unit with the same velocity.

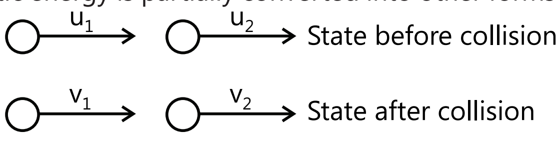


Figure 6.16: Collision in one dimension

Collisions can also be classified on the basis of the line of action of the forces of interaction.

(i) Head- on collisions: A collision is said to be head-on if the direction of the velocities of each of the colliding bodies are along the line of action of the forces of interaction acting on the bodies at the instant of collision.

(ii) Oblique collisions: A collision is said to be oblique if the direction of the velocities of the colliding bodies are not along the line of action of the forces of interaction acting on the bodies at the instant of collision. Just after collision, at least one of the colliding bodies moves in a direction different from the initial direction of motion.

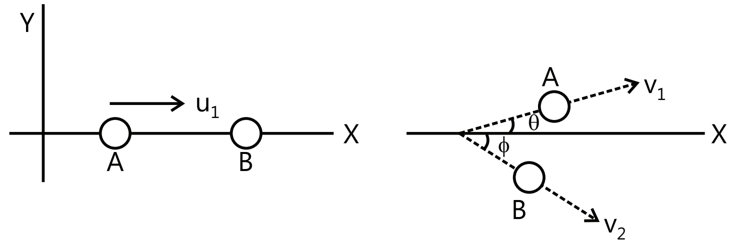


Figure 6.17: Collision in two dimensions

10. COEFFICIENT OF RESTITUTION

Coefficient of restitution is a measure of the elasticity of a collision between two particles. It is defined as the ratio of relative velocity of one of the particles with respect to the other particle after the collision to the relative velocity of the same particle before the collision and the ratio is negative.

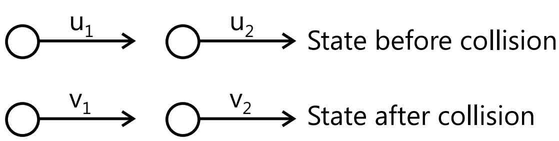


Figure 6.18: Velocities before and after collision

If the velocities of two particles before the collision are u_1 and u_2 respectively and their velocities after the collision are v_1 and v_2 respectively (see Fig. 6.18), then $\frac{v_1 - v_2}{u_1 - u_2} = -e$

The coefficient of restitution is also expressed as the ratio of velocity of separation after collision to the velocity of approach before collision.

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

PLANCESS CONCEPTS

- For perfectly inelastic collision e = 0.
- For perfectly elastic collision e = 1.
- For partially inelastic collision 0 < e < 1.

In elastic and inelastic collisioins, momentum is conserved whereas in inelastic collisions, kinetic energy is not conserved.

Yashwanth Sandupatla (JEE 2012, AIR 821)

11. ELASTIC COLLISION

Consider the collision of two small smooth spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 respectively in the same direction along the line joining their centers. Suppose m_1 is following m_2 with $u_1 > u_2$ i.e. m_1 tries to overtake m_2 but as the line of motion is same as the line joining the centers of the spheres, head-on ellastic collision takes place. Let their velocities after the elastic collision are v_1 and v_2 respectively, with $v_2 > v_1$ as shown in the Fig. 6.19

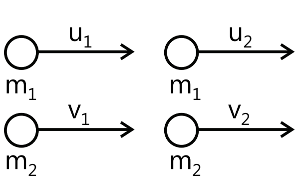


Figure 6.19: Head-on collision between two particles

Conserving momentum before and after collision we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad \text{...(i)}$$

Conserving kinetic energy before and after collision we get

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{...(ii)}$$

Dividing (ii) by (i)

$$u_1 + v_1 = v_2 + u_2$$

So $v_1 = -u_1 + u_2 + v_2$ ...(iii)

Substitute v_1 in equation (i)

$$m_1(u_1 + u_1 - u_2 - v_2) = m_2v_2 - m_2u_2$$

$$2m_1u_1 + (m_2 - m_1)u_2 = (m_1 + m_2)v_2$$

$$v_2 = \left[\frac{2m_1}{m_1 + m_2}\right]u_1 + \left[\frac{m_2 - m_1}{(m_1 + m_2)}\right]u_2 \quad \text{...(iv)}$$

Similarly, $v_1 = \left[\frac{m_1 - m_2}{m_1 + m_2}\right]u_1 + \left[\frac{2m_2}{m_1 + m_2}\right]u_2$ ...(v)

Special Cases

(i) When $m_1 = m_2$,

From equation (i)

$u_1 - v_1 = v_2 - u_2$ or $v_1 + v_2 = u_1 + u_2$ ...(vi)

Equation (iii) gives

$$v_1 - v_2 = u_2 - u_1 \quad \text{...(vii)}$$

Solving (vi) and (vii) we get

$v_1 = u_2$ and $v_2 = u_1$

$\therefore$ In one dimensional elastic collision of two bodies of equal masses, the bodies exchange there velocities after collision.

(ii) When m_2 is at rest i.e. $u_2 = 0$.

$$v_2 = \frac{2m_1u_1}{m_1 + m_2} + \left(\frac{m_2 - m_1}{m_2 + m_1}\right)u_2 \Rightarrow v_2 = \frac{2m_1u_1}{m_1 + m_2}$$

Now there are three possibilities in this case:

(a) If $m_1 = m_2 = m$; $v_2 = \frac{2mu_1}{2m} = u_1, v_1 = 0$.

The first body stops after collision. Both the momentum and the kinetic energy of the first body are completely transferred to the second body.

(b) If $m_2 >> m_1$, $v_1 \simeq -u_1$, $v_2 \simeq 0$

Thus when a light body collides with a much heavier stationary body, the velocity of light body is reversed and heavier body almost remains at rest.

(c) If $m_2 << m_1$, $v_2 \simeq u_1$ and $v_2 \simeq 2u_1$

Thus when a heavy body collides with a much lighter stationary body, the velocity of heavier body remains almost unchanged. The lighter body moves forward with approximately twice the velocity of the heavier body.

12. INELASTIC COLLISION

Consider the situation similar to previous article wherein m_1 is following m_2 with $u_1 > u_2$ i.e. m_1 tries to overtake m_2 but as the line of motion is same as the line joining the centers of the spheres, head-on collision takes place. Let their velocities after the collision are v_1 and v_2 respectively, with $v_2 > v_1$. Now suppose that the collision is inelastic, i.e. kinetic energy is not conserved.

Conserving momentum we get,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Restitution equation gives,

$$v_1 - v_2 = -e(u_1 - u_2)$$

The loss in kinetic energy ΔE in this case, is given by

$$\Delta E = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(e^2 - 1)(u_1 - u_2)^2$$

Putting e = 0 in this equation, it is clear that the loss of kinetic energy is maximum in case of pefectly inelastic collision.

Illustration 12: A block of mass m moving at a velcoity v collides head on with another block of mass 2m at rest. If the coefficient of restitution is 0.5, find the velocities of the blocks after the collision. **(JEE MAIN)**

Sol: Solve using law of conservation of momentum, before and after collision and the equation of restitution.

Suppose after the collision the block of mass m moves at a velocity u_1 and the block of mass 2m moves at a velocity u_2. By conservation of momentum,

$$mv = mu_1 + 2mu_2 \qquad \text{... (i)}$$

The velocity of sepration is $u_2 - u_1$ and the velocity of approach is v.

So, $$u_2 - u_1 = \frac{v}{2} \qquad \text{... (ii)}$$

Solving (i) and (ii) we get, $u_1 = 0\ ms^{-1}$ and $u_2 = \frac{v}{2}\ ms^{-1}$.

Illustration 13: A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in Fig. 6.20. Assuming collision to be elastic, find the velocity of ball immediately after the collision. **(JEE MAIN)**

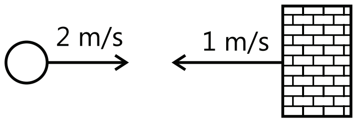


Figure 6.20

Sol: The equation of conservation of momentum will not give us any fruitful result because the mass of the wall is very large and remains at rest before and after the collision. This problem has to be solved by using equation of restitution

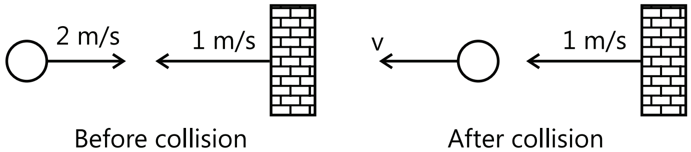


Figure 6.21

The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in Fig. 6.21. Since, collision is elastic (e = 1).

velocity of Separation=velocity of approach

or $v-1=2-(-1)$

or $v=4\ m/s$

Illustration 14: A ball of mass m is projected vertically up from smooth horizontal floor with a speed V_0. Find the total momentum delivered by the ball to the surface, assuming e as the coefficient of restitution of impact.

(JEE MAIN)

Sol: By Newton's third law the impulse delivered by the ball to the surface at each collision will be equal in magnitude to the impulse delivered to the ball by the surface i.e. change in momentum of ball at each collision. The total impulse will be the sum of DP due to all the collisions.

The momentum delivered by the ball at first, second, third impact etc. can be given as the corresponding change in its momentum (ΔP) at each impact.

$$(\Delta\vec{P})_1=\left|(mV_1)\hat{j}-m(-V_0)\hat{j}\right| \quad\Rightarrow\quad \Delta P_1=m(V_1+V_0)$$

Similarly $\Delta P_2=m(V_1+V_2)$, $\Delta P_3=m(V_2+V_3)$, ... and so on.

$\Rightarrow$ The total momentum transferred $\Delta P=\Delta P_1+\Delta P_2+\Delta P_3+.......$

Putting the values of ΔP_1, ΔP_2 etc., we obtain,

$$\Delta P=m\left[V_0+2(V_1+V_2+V_3+......)\right]$$

Putting $V_1=eV_0$, $V_2=e^2V_0$, $V_3=e^3V_0$

We obtain,

$$\Delta P=mV_0\left[1+2(e+e^2+e^3+......)\right] \quad\Rightarrow\quad \Delta P=mV_0\left(1+2\frac{e}{1-e}\right)=mV_0\left(\frac{1+e}{1-e}\right)$$

Illustration 15: A stationary body explodes into four identical fragments such that three of them fly off mutually perpendicular to each other, each with same K.E. Find the energy of explosion. **(JEE ADVANCED)**

Sol: As the body is initially at rest, the vector sum of momentum of all fragments will be zero. The energy of explosion will appear as K.E. of fragments.

Let the three fragments move along X, Y and Z axes. Therefore their velocities can be given as

$\vec{V}_1=V\hat{i}$, $\vec{V}_2=V\hat{j}$ and $\vec{V}_3=V\hat{k}$,

where V = speed of each of the three fragments. Let the velocity of the fourth fragment be $\vec{V}_4$ Since, in explosion no net external force is involved, the net momentum of the system remains conserved just before and after explosion. Initially the body is a rest,

$$\Rightarrow \quad m\vec{V}_1+m\vec{V}_2+m\vec{V}_3+m\vec{V}_4=0$$

Putting the values of $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$, we obtain, $\vec{V}_4=-V\left(\hat{i}+\hat{j}+\hat{k}\right)$

Therefore, $V_4=\sqrt{3}\,V$

The energy of explosion

$$\therefore E=KE_f-KE_i=\left(\frac{1}{2}mV_1^2+\frac{1}{2}mV_2^2+\frac{1}{2}mV_3^2+\frac{1}{2}mV_4^2\right)-(0)$$

Putting $V_1 = V_2 = V_3 = V$ and setting $\frac{1}{2}mV^2 = E_0$, we obtain, $E = 6E_0$.

13. OBLIQUE COLLISION

Let us now consider the case when the velocities of the two colliding spheres are not directed along the line of action of the forces of interaction or the line of impact (line joining the centers). As already discussed this kind of impact is said to be oblique.

Let us consider the collision of two spherical bodies. Since velocities v'_1 and v'_2 of the bodies after impact are unknown in direction and magnitude, their determination will require the use of four independent equations.

We choose the x-axis along the line of impact, i.e. along the common normal to the surfaces in contact, and the y-axis along their common tangent as shown in Fig. 6.22. Assuming the spheres to be perfectly smooth and frictionless, the impulses exerted on the spheres during the collision are along the line of impact i.e., along the x-axis. So,

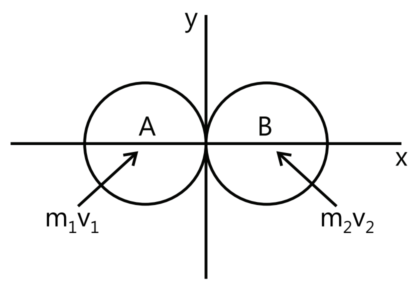


Figure 6.22: Oblique collision of two particles

(i) the component of the momentum of each sphere along the y-axis, considered separately is conserved; hence the y component of the velocity of each sphere remains unchanged. Thus we can write

$$(v_1)_y = (v'_1)_y \quad \text{....(i)}$$

$$(v_2)_y = (v'_2)_y \quad \text{....(ii)}$$

(ii) the component of total momentum of the two spheres along the x-axis is conserved. Thus we can write

$$m_1(v_1)_x + m_2(v_2)_x = m_1(v'_1)_x + m_2(v'_2)_x \quad \text{....(iii)}$$

(iii) The component along the x-axis of the relative velocity of the two spheres after impact i.e. the velocity of separation along x-axis is obtained by multiplying the x-component of their velocity of approach before impact by the coefficient of restitution. Thus we can write

$$(v'_2)_x - (v'_1)_x = e\left[(v_1)_x - (v_2)_x\right] \quad \text{....(iv)}$$

Now the four equations obtained above can be solved to find the velocities of the spheres after collision.

PLANCESS CONCEPTS

It is not advised to break the components of velocity in any other direction even though they are still valid. The only problem will be in using the coefficient of restitution.

Definition of coefficient of restitution can be applied in the normal direction in the case of oblique collision.

G.V. Abhinav (JEE 2012, AIR 329)

Illustration 16: After perfectly inelastic collision between two identical particles moving with same speed in different directions, the speed of the particles becomes half the initial speed. Find the angle between the two before collision. **(JEE MAIN)**

Sol: In case of an oblique collision, the momentum of individual particles are added vectorially in the equation of conservation of linear momentum.

Let θ be the desired angle. Linear momentum of the system will remain conserved.

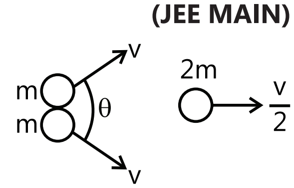


Figure 6.23

Hence $P^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\theta$

or $\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$

or $1 = 1 + 1 + 2\cos\theta$ or $\cos\theta = -\frac{1}{2}$

$\therefore$ $\theta = 120°$

Illustration 17: A ball of mass m hits the floor with a speed v making an angle of incidence θ with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.

(JEE MAIN)

Sol: In case of an oblique collision with fixed surface the component of velocity of colliding particle parallel to surface doesn't change. The impulse will act along the normal to the surface so use the equation of restitution along the normal.

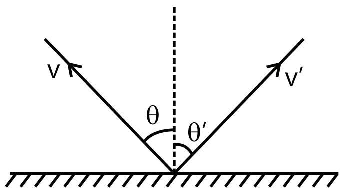


Figure 6.24

See Fig. 6.24. Let the angle of reflection is θ' and the speed after the collision is v'. The impulse on the ball is along the normal to the floor during the collision. There is no impulse parallel to the floor. Thus, the component of the velocity of the ball parallel to the surface remains unchanged before and after the collision. This gives

$v'\sin\theta' = v\sin\theta$ ...(i)

As the floor is stationary before and after the collision, the equation of conservation of momentum in the direction normal to the floor will not give any result. We have to use the formula for coefficient of restitution along the direction normal to the floor.

The velocity of separation along the normal= $v'\cos\theta'$

The velocity of approach along the normal = $v\cos\theta$

Hence, $v'\cos\theta' = ev\cos\theta$ ...(ii)

From (i) and (ii),

$v' = v\sqrt{\sin^2\theta + e^2\cos^2\theta}$ and $\tan\theta' = \frac{\tan\theta}{e}$

For elastic collision, e = 1 so that $\theta' = \theta$ and $v' = v$

PLANCESS CONCEPTS

Inelastic collision doesn't always mean that bodies will stick, which is very clear from the concept of oblique collision. Only the velocities along n-axis become same and may be different in t-direction.

Anurag Saraf JEE 2011, AIR 226

14. CENTER OF MASS FRAME

We can rigidly fix a frame of reference to the center of mass of a system. This frame is called the C-frame of reference and in general is a non-inertial reference frame. Relative to this frame, the center of mass is at rest $(\vec{V}_{com,C} = 0)$ and according to equation $\vec{P} = M\vec{V}_{com}$ the total momentum of a system of particle in the C-frame of reference is always zero.

$\vec{P} = \Sigma\vec{P}_i = 0$ in the C-frame of reference.

Note : When the net external force acting on the system is zero, the C-frame becomes an inertial frame.

14.1 A System of Two Particles

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the given reference frame K be $\vec{v}_1$ and $\vec{v}_2$ respectively. Let us find the expressions defining their momentum and total kinetic energy in the C-frame. The velocity of C-frame relative to K-frame is $\vec{v}_c$.

The momentum of the first particle in the C-frame is $\vec{P}_{1/c} = m_1\vec{v}_{1/c} = m_1(\vec{v}_1 - \vec{v}_c)$ where $\vec{v}_c$ is the velocity of the center of mass of the system in the K frame. Substituting the expression for $\vec{v}_c$

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

we get $\vec{P}_{1/c} = \mu(\vec{v}_1 - \vec{v}_2)$

where μ is the reduced mass of the system, given by $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Similarly, the momentum of the second particle in the C-frame is $\vec{P}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1)$

Thus, the momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$P_{1/c} = \mu v_{rel}$$

where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the modulus of velocity of one particle relative to another.

Finally, let us consider total kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = K_{1/c} + K_{2/c} = \frac{1}{2}m_1 v_{1/c}^2 + \frac{1}{2}m_2 v_{2/c}^2 = \frac{P_{1/c}^{\ 2}}{2m_1} + \frac{P_{1/c}^{\ 2}}{2m_2}$$

Now, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ or $\frac{1}{m_2} + \frac{1}{m_2} = \frac{1}{\mu}$

Then $K_{sys/c} = \frac{\vec{P}_{1/c}^{\ 2}}{2\mu} = \frac{\mu v_{rel}^2}{2}$

The total kinetic energy of the partices of the system in the K-frame is related to the total kinetic energy in C-frame. The velocity of the ith particle of the system in K-frame can be expressed as:

$$\vec{v}_i = \vec{v}_{i/c} + \vec{v}_c$$

So we can write $K_{sys} = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i(\vec{v}_{i/c} + \vec{v}_c)^2 = \frac{1}{2}\sum m_i v_{i/c}^2 + \vec{v}_c \sum m_i \vec{v}_{i/c} + \frac{v_c^2}{2}\sum m_i$

In the C-frame, the summation $\sum m_i \vec{v}_{i/c} = M\vec{V}_{com,C} = 0$.

So we get $K_{sys} = \frac{1}{2}\sum m_i v_{i/c}^2 + \frac{v_c^2}{2}\sum m_i = K_{sys/c} + \frac{v_c^2}{2}\sum m_i$

For a two-particle system, we get

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{M v_c^2}{2} \quad (\text{where } M = m_1 + m_2)$$

Illustration 18: Two blocks of mass m_1 and m_2 connected by an ideal spring of spring constant k are kept on a smooth horizontal surface. Find maximum extension of the spring when the block m_2 is given an initial velocity of v_0 towards right as shown in Fig. 6.25. **(JEE ADVANCED)**

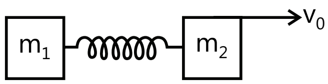


Figure 6.25

Sol: In absence of frictional forces on block, the total mechanical energy of the system comprising the blocks and spring will be conserved. At the time of maximum expansion of spring, the mechanical energy in C frame will be totally stored as elastic P.E. of the spring

This problem can be best solved in the C-frame or the reference frame rigidly fixed to the center of mass of the system of two blocks.

Initially at t=0 when the block m_2 is given velocity v_0, the total kinetic energy of the blocks in C-frame is related to the total kinetic energy in the given frame K by the relation,

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{(m_1+m_2)v_c^2}{2} = K_{sys/c}(0) + \frac{(m_1+m_2)v_c^2}{2} \qquad \left(\mu = \frac{m_1 m_2}{m_1+m_2};\ v_{rel} = v_0;\ v_c = \frac{m_2 v_0}{m_1+m_2}\right)$$

where the first term on the right hand side of this relation is the total kinetic energy in C-frame at t=0, $K_{sys/c}(0)$, and the second term is the kinetic energy associated with the motion of the system of blocks as a whole in the K-frame. As there are no dissipative external forces acting on the system, the total mechanical energy will remain constant, both in the C-frame and the K-frame. In the C-frame the blocks will oscillate under the action of spring force and the kinetic energy in the C-frame will get converted into the elastic potential energy of the spring and vice-versa, the total mechanical energy remaining constant at each instant, equal to the total kinetic energy in C-frame at t=0, $K_{sys/c}(0)$.

Initially at t=0 when the block m_2 is given velocity v_0, the mechanical energy in C frame will be totally kinetic ($K_{sys/c}(0)$), and at the instant of maximum extension of the spring, the mechanical energy in C-frame will be totally converted into elastic potential energy of the spring. So we have,

$$K_{sys/c}(0) = \frac{1}{2}kx_{max}^2 \Rightarrow \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v_0^2 = \frac{1}{2}kx_{max}^2$$

Thus, maximum extension is $x_{max} = v_0\sqrt{\frac{m_1 m_2}{k(m_1+m_2)}}$

15. IMPULSE AND MOMENTUM

When two bodies collide during a very short time period, large impulsive forces are exerted between the bodies along the line of impact. Common examples are a hammer striking a nail or a bat striking a ball. The line of impact is a line through the common normal to the surfaces of the colliding bodies at the point of contact.

When two bodies collide, the momentum of each body is changed due to the force on it exerted by the other. On an ordinary scale, the time duration of the collision is very small and yet the change in momentum is sizeable. This means that the magnitude of the force of interaction must be very large on an ordinary scale. Such large forces acting for a very short duration are called impulsive forces. The force may not be uniform during the interaction.

We know that the force is related to momentum as $\vec{F} = \frac{d\vec{P}}{dt} \quad \Rightarrow \quad \vec{F}dt = d\vec{P}$

We can find the change in momentum of the body during a collision (from $\vec{P}_i$ to $\vec{P}_f$) by integrating over the time of collision and assuming that the force during collision has a constant direction, $\vec{P}_f - \vec{P}_i = \int_{P_i}^{P_f} d\vec{P} = \int_{t_i}^{t_f} \vec{F}dt$;

Here the subscripts i (= initial) and f (= final) refer to the times before and after the collision. The integral of a force over the time interval during which the force acts is called impulse.

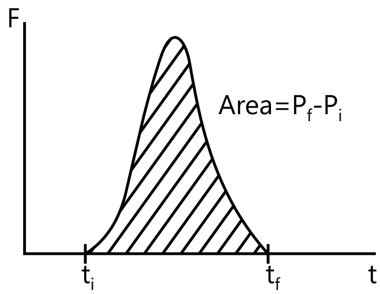


Figure 6.26: Impulse imparted to the particle

Thus the quantity $\int_{t_i}^{t_f} \vec{F}\,.\,dt$ is the impulse of the force $\vec{F}$ during the time interval t_i and t_f and is equal to the change in the momentum of the body in which it acts.

The magnitude of impulse $\int_{t_i}^{t_f} \vec{F}dt$ is the area under the force-time curve as shown in Fig. 6.26

Illustration 19: A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley as shown in Fig. 6.27. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v find the speed with which the system moves just after the collision. **(JEE MAIN)**

Sol: By Newton's third law, the impulse imparted to the particle in upward direction will be equal in magnitude to the total impulse imparted to the system of block and the pan.

Let N be the contact force between the particle and the pan during the collision.

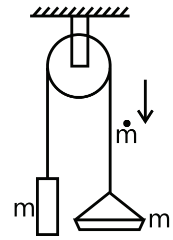


Figure 6.27

Consider the impulse imparted to the particle. The force N will be in upward direction and the impulse imparted to it will be $\int N\,dt$ in the upward direction. This should be equal to the change in momentum imparted to it in the upward direction.

Thus, $\int N\,dt = P_f - P_i = -mV - (-mv) = mv - mV$ ....(i)

Similarly considering the impulse imparted to the pan. The forces acting on it are tension T upwards and contact force N downwards. The impulse imparted to it in the downward direction will be,

$\int (N - T)dt = mV - 0 = mV$ ....(ii)

Impulse imparted to the block by the tension T will be upwards,

$\int T\,dt = mV - 0 = mV$ ....(iii)

Adding (ii) and (iii) we get, $\int N dt = 2mV$ ....(iv)

Comparing (i) and (iv) we get, $mv - mV = 2mV$ or $V = \frac{v}{3}\ ms^{-1}$

PROBLEM-SOLVING TACTICS

Applying the principle of Conservation of Linear Momentum

(a) Decide which objects are included in the system.

(b) Relative to the system, identify the internal and external forces.

(c) Verify that the system is isolated.

(d) Set the final momentum of the system equal to its initial momentum. Remember that momentum is a vector.

(e) Always check whether kinetic energy is conserved or not. If it is conserved, it gives you an extra equation. Otherwise use work-energy theorem, carefully.

(f) Try to involve yourself physically in the question, imagine various events. This would help in some problems where some parameters get excluded by conditions. This will also help in checking your answer.

Impulse

(g) Ignore any finite-value forces, while dealing with impulses.

(h) Write impulse equations carefully, because integration which we are unable to calculate will always cancel out.

Collisions

(i) Remembering special cases of collisions would be nice.

FORMULAE SHEET

Position of center of mass of a system: $\vec{r}_{com} = \dfrac{\sum_i m_i \vec{r}_i}{M}$

$$\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j} + z_{COM}\hat{k}$$

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

For continuous bodies $x_{COM} = \dfrac{\int x\,dm}{\int dm} = \dfrac{\int x\,dm}{M}$

For a two-particle system, we have

$$r_1 = \left(\frac{m_2}{m_2 + m_1}\right)d \text{ and } r_2 = \left(\frac{m_1}{m_1 + m_2}\right)d$$

where d is the separation between the particles.

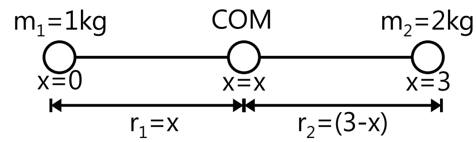


Figure 6.28

Center of Mass of a Uniform Rod $\left(\dfrac{L}{2}, 0, 0\right)$

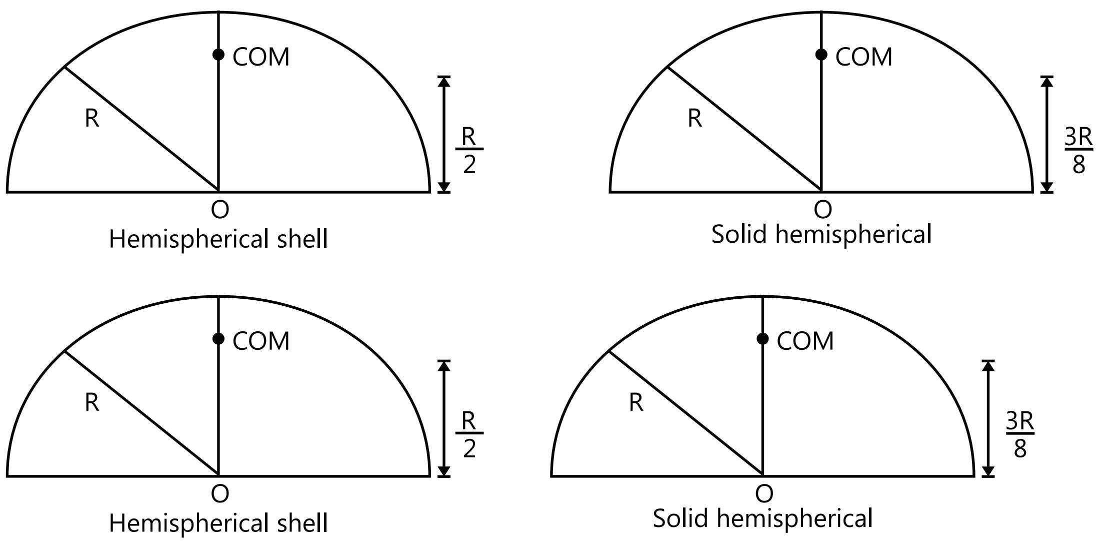


Figure 6.29

If some mass or area is removed from a rigid body, then the position of center of mass of the remaining portion is obtained from the following formula:

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$$

Where m_1 is the mass of the body after filling all cavities with same density and m_2 is the mass filled in the cavity. Cavity mass is assumed negative.

Velocity of COM $\vec{v}_{COM} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{m_1 + m_2 + ... + m_n} = \dfrac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$

Total momentum of a n-particle system $\vec{P}_{COM} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n = M\vec{v}_{COM}$

Acceleration of COM $\vec{a}_{COM} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} \dfrac{\sum_i m_i \vec{a}_i}{\sum_i m_i}$

Net force acting on the system $\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$

Net external force on center of mass is $M\vec{a}_{cm} = \vec{F}_{ext}$

If net force on the system $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = 0$ then, $\vec{P}_1 + \vec{P}_2 + \vec{P}_3 + ... + \vec{P}_n$ = constant

Equation of motion of a body with variable mass is:

$$m\left(\frac{d\vec{v}}{dt}\right) = \vec{F} + \left(\frac{dm}{dt}\right)\vec{u}$$

Where $\vec{u}$ is the velocity of the mass being added(separated) relative to the given body of instantaneous mass m and $\vec{F}$ is the external force due to surrounding bodies or due to field of force.

In case of reducing mass of a system $\dfrac{dm}{dt} = \mu\ kgs^{-1}$

For a rocket we have, $m\left(\dfrac{d\vec{v}}{dt}\right) = m\vec{g} + \left(\dfrac{dm}{dt}\right)\vec{v}_r$

Where $\vec{v}_r$ is the velocity of the ejecting gases relative to the rocket.

In scalar form we can write

$$m\left(\frac{dv}{dt}\right) = -mg + v_r\left(-\frac{dm}{dt}\right)$$

Here $-\dfrac{dm}{dt}$ = rate at which mass is ejecting and $v_r\left(-\dfrac{dm}{dt}\right)$ =Thrust force.

Final velocity of rocket $v = u - gt + v_r \ln\left(\dfrac{m_0}{m}\right)$

Impulse of a force: $\vec{J} = \int \vec{F}\, dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$

Collision

(a) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, i.e.,

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v = m_1v'_1 + m_2v'_2 \quad ...(i)$$

(b) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, i.e.

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx_m^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \quad ...(ii)$$

Head on Elastic Collision

$$v'_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

$$\frac{\text{separation speed after collision}}{\text{approach speed before collision}} = e$$

$$v'_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right)v_1$$

The C-frame: Total kinetic energy of system in K-frame is related to total kinetic energy in C-frame as:

$$K_{sys} = K_{sys/c} + \frac{Mv_c^2}{2};\ M = \sum m_i$$

For a two-particle system: $\vec{P}_{1/c} = -\vec{P}_{2/c} = \frac{m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)$

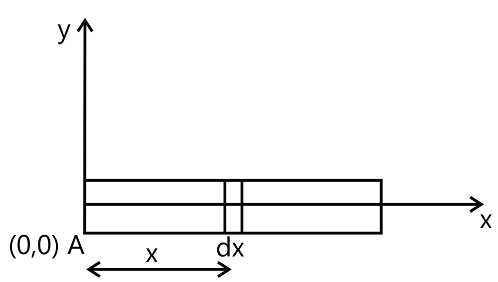
Or $P_{1/c} = P_{2/c} = \mu v_{rel} = \frac{m_1 m_2}{m_1 + m_2}\left|\vec{v}_1 - \vec{v}_2\right|$ and $K_{sys/c} = \frac{\mu v_{rel}^2}{2} = \frac{P_{1/c}^2}{2}$

Solved Examples

JEE Main/Boards

Example 1: The linear mass density of rod of a length l=2 m varies from A as (2+x) kg/m. What is the position of center of mass from end A.

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as



$\therefore x_{COM} = \frac{\int x\, dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

Take an element of the rod of infinitesimal length dx at distance x from point A. The mass of the element will be

$dm = \lambda\ dx = (2 + x)dx$ As x varies from o to l the element covers the entire rod.

Center of mass of rod $X_{cm} = \frac{\int x\,dm}{\int dm}$

$$X_{cm} = \frac{\int_0^l x(2+x)dx}{\int_0^l (2+x)dx} = \frac{\left|(x^2 + \frac{x^3}{3})\right|_0^l}{\left|2x + \frac{x^2}{2}\right|_0^l}$$

$$X_{cm} = \frac{l^2 + \frac{l^3}{3}}{2l + \frac{l^2}{2}} = \frac{6l + 2l^2}{12 + 3l}$$

For l = 2 m, $X_{cm} = \frac{6 \times 2 + 2 \times 4}{12 + 3 \times 2} = \frac{20}{18} = \frac{10}{9}$m

So center of mass is at a distance $\frac{10}{9}$ m from A.

Example 2: One fourth of the mass of square lamina is cut off (see figure). Where does the center of mass of the remaining part of the square shift.

Sol: To find the C.O.M. of a body having a cavity we first fill the cavity with the same density as body and find the C.O.M. (x,y,z) of the whole body. Then we consider the cavity as second body having negative mass and