Thus the quantity $\int_{t_i}^{t_f} \vec{F} \cdot dt$ is the impulse of the force $\vec{F}$ during the time interval t_i and t_f and is equal to the change in the momentum of the body in which it acts.

The magnitude of impulse $\int_{t_i}^{t_f} \vec{F} dt$ is the area under the force-time curve as shown in Fig. 6.26

Illustration 19: A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley as shown in Fig. 6.27. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v find the speed with which the system moves just after the collision. **(JEE MAIN)**

Sol: By Newton's third law, the impulse imparted to the particle in upward direction will be equal in magnitude to the total impulse imparted to the system of block and the pan.

Let N be the contact force between the particle and the pan during the collision.

Consider the impulse imparted to the particle. The force N will be in upward direction and the impulse imparted to it will be $\int N\,dt$ in the upward direction. This should be equal to the change in momentum imparted to it in the upward direction.

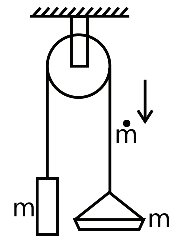


Figure 6.27

Thus, $\int N\,dt = P_f - P_i = -mV - (-mv) = mv - mV$ ....(i)

Similarly considering the impulse imparted to the pan. The forces acting on it are tension T upwards and contact force N downwards. The impulse imparted to it in the downward direction will be,

$\int (N - T)\,dt = mV - 0 = mV$ ....(ii)

Impulse imparted to the block by the tension T will be upwards,

$\int T\,dt = mV - 0 = mV$ ....(iii)

Adding (ii) and (iii) we get, $\int N dt = 2mV$ ....(iv)

Comparing (i) and (iv) we get, $mv - mV = 2mV$ or $V = \frac{v}{3}\ ms^{-1}$

PROBLEM-SOLVING TACTICS

Applying the principle of Conservation of Linear Momentum

(a) Decide which objects are included in the system.

(b) Relative to the system, identify the internal and external forces.

(c) Verify that the system is isolated.

(d) Set the final momentum of the system equal to its initial momentum. Remember that momentum is a vector.

(e) Always check whether kinetic energy is conserved or not. If it is conserved, it gives you an extra equation. Otherwise use work-energy theorem, carefully.

(f) Try to involve yourself physically in the question, imagine various events. This would help in some problems where some parameters get excluded by conditions. This will also help in checking your answer.

Impulse

(g) Ignore any finite-value forces, while dealing with impulses.

(h) Write impulse equations carefully, because integration which we are unable to calculate will always cancel out.

Collisions

(i) Remembering special cases of collisions would be nice.

FORMULAE SHEET

Position of center of mass of a system: $\vec{r}_{com} = \frac{\sum_i m_i \vec{r}_i}{M}$

$$\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j} + z_{COM}\hat{k}$$

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

For continuous bodies $x_{COM} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}$

For a two-particle system, we have

$$r_1 = \left(\frac{m_2}{m_2 + m_1}\right)d \text{ and } r_2 = \left(\frac{m_1}{m_1 + m_2}\right)d$$

where d is the separation between the particles.

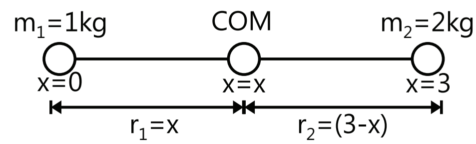


Figure 6.28

Center of Mass of a Uniform Rod $\left(\frac{L}{2},0,0\right)$

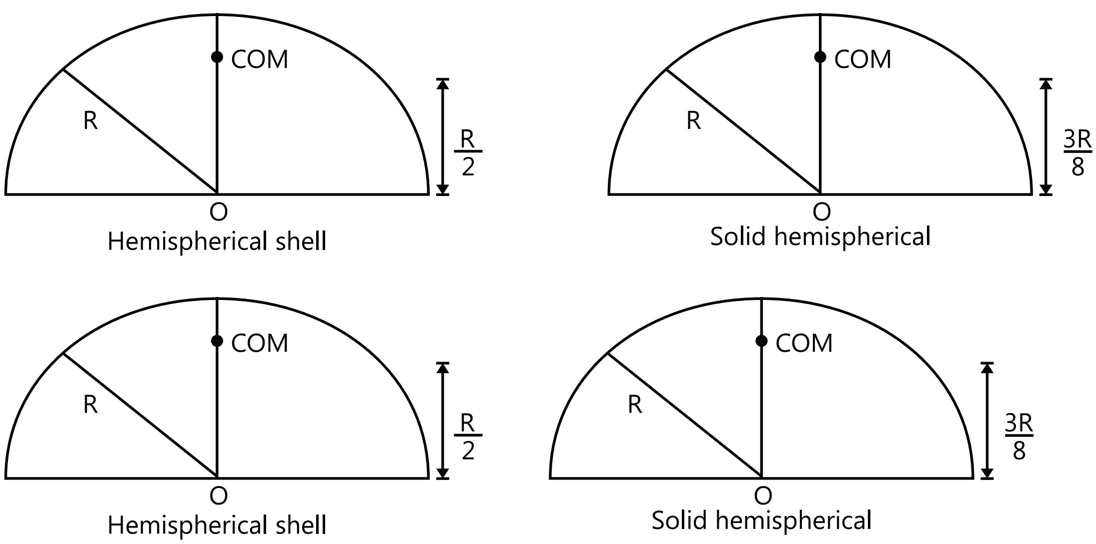


Figure 6.29

If some mass or area is removed from a rigid body, then the position of center of mass of the remaining portion is obtained from the following formula:

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$$

Where m_1 is the mass of the body after filling all cavities with same density and m_2 is the mass filled in the cavity. Cavity mass is assumed negative.

Velocity of COM $\vec{v}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$

Total momentum of a n-particle system $\vec{P}_{COM} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n = M\vec{v}_{COM}$

Acceleration of COM $\vec{a}_{COM} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} \dfrac{\sum_i m_i \vec{a}_i}{\sum_i m_i}$

Net force acting on the system $\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$

Net external force on center of mass is $M\vec{a}_{cm} = \vec{F}_{ext}$

If net force on the system $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = 0$ then, $\vec{P}_1 + \vec{P}_2 + \vec{P}_3 + ... + \vec{P}_n$ = constant

Equation of motion of a body with variable mass is:

$$m\left(\frac{d\vec{v}}{dt}\right) = \vec{F} + \left(\frac{dm}{dt}\right)\vec{u}$$

Where $\vec{u}$ is the velocity of the mass being added(separated) relative to the given body of instantaneous mass m and $\vec{F}$ is the external force due to surrounding bodies or due to field of force.

In case of reducing mass of a system $\dfrac{dm}{dt} = \mu\ kgs^{-1}$

For a rocket we have, $m\left(\dfrac{d\vec{v}}{dt}\right) = m\vec{g} + \left(\dfrac{dm}{dt}\right)\vec{v}_r$

Where $\vec{v}_r$ is the velocity of the ejecting gases relative to the rocket.

In scalar form we can write

$$m\left(\frac{dv}{dt}\right) = -mg + v_r\left(-\frac{dm}{dt}\right)$$

Here $-\dfrac{dm}{dt}$ = rate at which mass is ejecting and $v_r\left(-\dfrac{dm}{dt}\right)$ =Thrust force.

Final velocity of rocket $v = u - gt + v_r \ln\left(\dfrac{m_0}{m}\right)$

Impulse of a force: $\vec{J} = \int \vec{F}\,dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$

Collision

(a) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, i.e.,

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v = m_1v'_1 + m_2v'_2 \quad ...(i)$$

(b) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, i.e.

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx_m^2 = \frac{1}{2}m_1v_1^{'2} + \frac{1}{2}m_2v_2^{'2} \quad ...(ii)$$

Head on Elastic Collision

$$v'_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

$$\frac{\text{separation speed after collision}}{\text{approach speed before collision}} = e$$

$$v'_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right)v_1$$

The C-frame: Total kinetic energy of system in K-frame is related to total kinetic energy in C-frame as:

$$K_{sys} = K_{sys/c} + \frac{Mv_c^2}{2};\ M = \sum m_i$$

For a two-particle system: $\vec{P}_{1/c} = -\vec{P}_{2/c} = \frac{m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)$

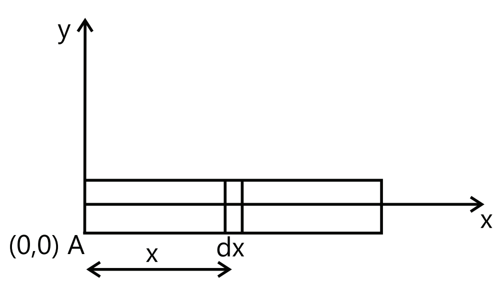
Or $P_{1/c} = P_{2/c} = \mu v_{rel} = \frac{m_1 m_2}{m_1 + m_2}\left|\vec{v}_1 - \vec{v}_2\right|$ and $K_{sys/c} = \frac{\mu v_{rel}^2}{2} = \frac{P_{1/c}^2}{2}$

Solved Examples

JEE Main/Boards

Example 1: The linear mass density of rod of a length l=2 m varies from A as (2+x) kg/m. What is the position of center of mass from end A.

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as



$\therefore x_{COM} = \frac{\int x\,dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

Take an element of the rod of infinitesimal length dx at distance x from point A. The mass of the element will be

$dm = \lambda\ dx = (2+x)dx$ As x varies from o to l the element covers the entire rod.

Center of mass of rod $X_{cm} = \frac{\int x\,dm}{\int dm}$

$$X_{cm} = \frac{\int_0^l x(2+x)dx}{\int_0^l (2+x)dx} = \frac{\left|(x^2 + \frac{x^3}{3})\right|_0^l}{\left|2x + \frac{x^2}{2}\right|_0^l}$$

$$X_{cm} = \frac{l^2 + \frac{l^3}{3}}{2l + \frac{l^2}{2}} = \frac{6l + 2l^2}{12 + 3l}$$

For l = 2 m, $X_{cm} = \frac{6\times 2 + 2\times 4}{12 + 3\times 2} = \frac{20}{18} = \frac{10}{9}$m

So center of mass is at a distance $\frac{10}{9}$ m from A.

Example 2: One fourth of the mass of square lamina is cut off (see figure). Where does the center of mass of the remaining part of the square shift.

Sol: To find the C.O.M. of a body having a cavity we first fill the cavity with the same density as body and find the C.O.M. (x,y,z) of the whole body. Then we consider the cavity as second body having negative mass and