PROBLEM-SOLVING TACTICS

A Working Rule to find the equation of a parabola when focus & directrix are given:

Step 1: Find the distance between focus and general point P(x, y) by the distance formula.

Step 2: Find the perpendicular distance from the point P(x, y) to the given directrix.

(The perpendicular distance from a point $P(x_1, y_1)$ to the line ax + by + c = 0 is $\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$)

Step 3: Equate the distances calculated in step 1 and step 2. On simplification we get the required equation of the parabola.

A Working Rule to find the equation of a parabola when the vertex and the focus are given:

Step 1: Find the slope of the axis formed by joining the focus and the vertex by the formula $\frac{y_2 - y_1}{x_2 - x_1}$

Step 2: Find the slope of the directrix by the formula $m_1.m_2 = -1$; where m_1 is the slope of the axis of the parabola and m_2 is the slope of the directrix.

Step 3: Find a point on the directrix as the vertex, which is the middle point between the focus and the point on the directrix, by means of the mid-point formula.

Step 4: Write the equation of the directrix, using the slope point formula.

Step 5: The focus and the directrix are now known so we can find the equation of the parabola by the method given above.

FORMULAE SHEET

1. **Definition:** A parabola is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed straight line.

For e.g. if the focus is (α, β) and the directrix is $ax + by + c = 0$ then the equation of the parabola is

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{(ax+by+c)^2}{a^2+b^2}$$

2. The general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $\Delta \neq 0$ and $h^2 = ab$.

3.

Equation of the parabola Properties	**$y^2 = 4ax$**	**$y^2 = -4ax$**	**$x^2 = 4ay$**	**$x^2 = -4ay$**
Vertex (Co-ordinates)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus (Co-ordinates)	(a, 0)	(–a, 0)	(0, a)	(0, –a)
Latus rectum (length)	4a	4a	4a	4a
Axis (Equation)	y =0	y =0	x = 0	x = 0
Directrix (Equation)	x = –a	x = a	y = –a	y = a
Symmetry (about)	x-axis	x-axis	y-axis	y-axis

4. The equation of the chord joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

5. If the equation of the chord joining points t_1 and t_2 on the parabola $y^2 = 4ax$ passes through the focus then $t_1t_2 = -1$.

In other words, if one end of a focal chord of the parabola $y^2 = 4ax$ is $P(at^2, 2at)$ then the co-ordinates of the other end is $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

6. The length of the focal chord passing through $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ is $a(t + 1/t)^2$.

7. The length of the chord intercepted by the parabola on the line $y = mx + c$ is $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

8. The length of the chord joining two points 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1+t_2)^2+4}$.

9. The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$

10. Parametric Form

Equation of the parabola	**Point of contact**	**Equation of the tangent**
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

11. Slope form

Equation of the parabola	Equation of the tangent	Condition of tangency	Point of contact
$y^2 = 4ax$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$y^2 = -4ax$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(\frac{-a}{m^2}, \frac{-2a}{m}\right)$
$x^2 = 4ay$	$x = mx + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$
$x^2 = -4ay$	$x = mx - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(\frac{-2a}{m}, \frac{-a}{m^2}\right)$

12. The point of intersection of tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is given by $(at_1t_2, a(t_1 + t_2))$

13. If SZ be perpendicular to the tangent at a point P of a parabola, then Z lies on the tangent at the vertex and $SZ^2 = AS \times SP$, where A is the vertex of the parabola.

14. Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $\theta = \tan^{-1} \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$

15. Equation of normal in different forms

Equation of the parabola	Equation of the normal
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$
$x^2 = -4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$

Equation of the parabola	Parametric coordinates	Equation of the normal
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Equation of the parabola	Equation of the normal	(Feet of the normal)
$y^2 = 4ax$	$y = mx - 2am - am^3$	$(am^2, -2am)$
$y^2 = -4ax$	$y = mx + 2am + am^3$	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$(-2am, am^2)$
$x^2 = -4ay$	$x = my + 2am + am^3$	$(2am, -am^2)$

16. The point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $R[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$

17. **(i)** The algebraic sum of the slopes of the normals at the co-normal point is zero.

(ii) The centroid of a triangle formed by the co-normal points on a parabola lies on its axis.

18. If the normal at the point P $(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $(at_2^2, 2at_2)$. Then $t_2 = -t_1 - \frac{2}{t_1}$.

19. If the normal drawn at the point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect at a third point on the parabola then $t_1.t_2 = 2$.

20. If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

21. The chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

22. The combined equation of the pair of tangents drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$ is $SS_1 = T^2$ where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

23. The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 \quad \text{or,} \quad T = S_1.$$

24. The polar of a point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

25. The equation of the diameter of the parabola $y^2 = 4ax$ bisecting chords of slope m is $y = 2a/m$.

26. A circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

27. If the tangents at P and Q meet at T, then

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP \times SQ$, and

(iii) Δ SPT and Δ STQ are similar.

28. Tangents and normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.

29. The semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola, i.e. $2a = \frac{2bc}{b+c}$ or, $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

30. The orthocentre of any triangle formed by tangents at any three points $P(t_1)$, $Q(t_2)$ and $R(t_3)$ on a parabola $y^2 = 4ax$ lies on the directrix and has the coordinates $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

31. If a normal drawn to a parabola passes through a point P(h, k), then $k = mh - 2am - am^3$, i.e, $am^3 + m(2a - h) + k = 0$,

$\Rightarrow m_1 + m_2 + m_3 = 0;$ $\quad m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; and $m_1m_2m_3 = -\frac{k}{a}$.

32. The equation of a circle circumscribing the triangle formed by three co-normal points and which passes through the vertex of the parabola is $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.

33. The area of a triangle formed inside the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ where y_1, y_2, y_3 are the ordinates of the vertices of the triangle.

34. If the vertex and the focus of a parabola are on the x-axis and at a distance a and b from the origin respectively then the equation of the parabola is $y^2 = 4(b - a)(x - a)$.