# 23. DEFINITE INTEGRATION

## 1. INTRODUCTION

Let f(x) be a continuous function defined on a closed interval [a, b] and $\int f(x)\,dx = F(x) + c$ then $\int_a^b f(x)\,dx = [F(x)]_a^b$ or $\int_a^b f(x)\,dx = F(b) - F(a)$ is called the definite integral of f(x) within limits a and b. The interval [a, b] is called the range of integration. Every definite integral has a unique solution.

**Note:** $\int_a^b f(x)\,dx = F(b) - F(a)$ also represents the net area of the curve f(x) with x-axis. $\int_0^{\pi/2} \sin^2 x\,dx$

**Sol:** $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$

**Illustration 1:** If $\int_0^1 (3x^2 + 2x + k)\,dx = 0,$ find the value of k. **(JEE MAIN)**

**Sol:** Here the answer of the definite integral $\int_0^1 \left[3x^2 + 2x + k\right]dx$ is already given i.e. 0 hence by using simple integral formulas we can solve it and by comparing it to 0, we will obtain the value of k.

Here, we have, $\int_0^1 (3x^2 + 2x + k)\,dx = 0$

$\left[3\frac{x^3}{3} + 2\frac{x^2}{2} + kx\right]_0^1 = 0$ ; $\left[x^3 + x^2 + kx\right]_0^1 = 0$

(1 + 1 + k) – (0 + 0 + 0) = 0 ; 2 + k = 0 ⇒ k = –2

**Illustration 2:** Evaluate: $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx.$ **(JEE MAIN)**

**Sol:** As we know $\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ . Hence by using this method we can solve the given definite integral.

We have, $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} x^3\,dx + 2\int_0^{\frac{\pi}{4}} dx$

$= 2\left[\tan x\right]_0^{\pi/4} + \left[\frac{x^4}{4}\right]_0^{\pi/4} + 2[x]_0^{\pi/4} = 2\left(\tan\frac{\pi}{4} - \tan 0\right) + \left[\frac{(\pi/4)^4}{4} - 0\right] + 2\left[\frac{\pi}{4} - 0\right]$

$$= 2(1-0)+\left(\frac{\pi^4}{4^5}-0\right)+\frac{\pi}{2}=2+\frac{\pi^4}{1024}+\frac{\pi}{2}$$

## 2. PROPERTIES OF DEFINITE INTEGRALS

**Property 1**

$$\int_a^b f(x)\,dx=\int_a^b f(t)\,dt=\int_a^b f(u)\,du$$

Here x is a dummy variable; it can be replaced by any other variable t, u,........

$$\int_0^{\pi/2}\sin(x)\,dx=\int_0^{\pi/2}\sin t\,dt=\int_0^{\pi/2}\sin u\,du=$$

This is similar to the summation property $\sum_{T=1}^{10} r^2=\sum_{T=1}^{10} t^2=\sum_{U=1}^{10} u^2=\ldots\ldots..$

**Property 2**

$$\int_a^b f(x)\,dx=-\int_b^a f(x)dx$$

i.e. the interchange of limits of a definite integral changes only its sign.

**Property 3**

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx\ \ (a<c<b)$$

Generally, this property is used when the integrand has two or more rules in the integration interval

$\Rightarrow\quad \int_a^b f(x)\,dx=\int_a^{c_1} f(x)\,dx+\int_{c_1}^{c_2} f(x)\,dx+\ldots\ldots+\int_{c_n}^{b} f(x)\,dx$ where $a<c_1<c_2<\ldots\ldots\ c_n<b$.

**Illustration 3:** Evaluate: $\int_1^4 f(x)dx$, where $f(x)=\begin{cases}2x+8, & 1\le x\le 2\\ 6x, & 2\le x\le 4\end{cases}$ **(JEE MAIN)**

**Sol:** Here as we know, $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$ where $(a<c<b)$. Hence by using this property and solving by using the integral formula we can solve it.

We have, $I=\int_1^4 f(x)\,dx$

$$=\int_1^2 f(x)\,dx+\int_2^4 f(x)dx=\int_1^2(2x+8)dx+\int_2^4 6x\,dx$$

$$=\left[x^2+8x\right]_1^2+\left[3x^2\right]_2^4=\left[(2)^2+8(2)-(1)^2-8(1)\right]+\left[3(4)^2-3(2)^2\right]$$

= 11 + 36 = 47.

**Illustration 4:** Evaluate : $\int_0^2 |1-x|dx$ **(JEE MAIN)**

**Sol:** Here $|1-x| = \begin{cases} 1-x, \text{ when } 0 \le x \le 1 \\ x-1, \text{ when } 1 \le x \le 2 \end{cases}$ therefore, similar to the problem above, we can solve it.

$$|1-x| = \begin{cases} 1-x, \text{ when } 0 \le x \le 1 \\ x-1, \text{ when } 1 \le x \le 2 \end{cases}$$

$$\therefore\ I = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 = (1/2 - 0) + (0 + 1/2) = 1$$

**Property 4**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

**Property 5**

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

**Application**: $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$

## MASTERJEE CONCEPTS

With the help of the above property, the following integrals can be obtained.

$$\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx \ ; \ \int_0^{\pi/2} f(\tan x)dx = \int_0^{\pi/2} f(\cot x)dx$$

$$\int_0^{\pi/2} f(\sin 2x)\sin x\ dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \ ; \ \int_0^1 f(\log x)dx = \int_0^1 f[\log(1-x)]dx$$

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx = \frac{\pi}{4} \ ; \ \int_0^{\pi/2} \frac{1}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}dx = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x + \sec^n x}dx = \frac{\pi}{4} \ ; \ \int_0^{\pi/4} \log(1+\tan x)dx = \frac{\pi}{8}\log 2$$

$$\int_0^{\pi/2} \log\cot x\ dx = \int_0^{\pi/2} \log\tan x\ dx = 0$$

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**Illustration 5:** Prove that $\int_0^1 \cot^{-1}(1-x+x^2)\,dx = 2\int_0^1 \tan^{-1} x.dx$ **(JEE MAIN)**

**Sol:** As we know $\cot^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{b}{a}\right)$ and $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ by using these two formulae we can solve the given problem.

$$\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \int_0^1 \tan^{-1}\left[\frac{1}{1-x+x^2}\right] = \int_0^1 \tan^{-1}\left[\frac{1+x-x}{1-x(1-x)}\right] = \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x)dx = 2\int_0^1 \tan^{-1} x.dx$$

$$\left(\because \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1} a + \tan^{-1} b\right)$$

**Illustration 6:** Find the value of $\int_0^1 \log\left(\frac{1}{x}-1\right)dx$ **(JEE MAIN)**

**Sol:** Here $\log\left(\frac{1-x}{x}\right) = \log(1-x) - \log(x)$ and $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ by using these two formulae we can solve it.

$$\int_0^1 \log\left(\frac{1-x}{x}\right)dx = \int_0^1 \log\,(1-x)dx - \int_0^1 \log\,(x)dx = \int_0^1 \log\left[1-(1-x)\right]dx - \int_0^1 \log x\,dx = \int_0^1 \log x\,dx - \int_0^1 \log x\,dx$$

$$= \int_0^1 \log\,(x)dx - \int_0^1 \log\,(x)dx = 0$$

**Illustration 7:** Evaluate: $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx$ **(JEE MAIN)**

**Sol:** As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ therefore we can write $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx$ in the form of

$\int_0^{\pi/2} \frac{a\sin(\pi/2-x) + b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx$ and then adding these two equations we can solve the given problem.

$$I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx \qquad \text{... (i)}$$

$$I = \int_0^{\pi/2} \frac{a\sin(\pi/2-x) + b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\sin x + \cos x}dx \qquad \text{... (ii)}$$

Adding (i) and (ii),

$$\therefore 2I = \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{\sin x + \cos x}dx = \int_0^{\pi/2} (a+b)dx = (a+b)\pi/2 \Rightarrow I = (a+b)\,\pi/4$$

**Illustration 8:** Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ **(JEE ADVANCED)**

**Sol:** This problem is similar to the problem above.

$$\text{Let } I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx \qquad \text{... (i)}$$

By property 4, we have

$$I = \int_0^{\pi/2} \frac{\sin^2\left((\pi/2)-x\right)}{\sin\left((\pi/2)-x\right)+\cos\left((\pi/2)-x\right)}dx = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}dx \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx \Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{(1/\sqrt{2})\sin x + (1/\sqrt{2})\cos x} dx$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos(\pi/4)\sin x + \sin(\pi/4)\cos x} dx = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sin\left(x + (\pi/4)\right)} dx$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \operatorname{cosec}\left[x + \frac{\pi}{4}\right] dx = \frac{1}{2\sqrt{2}}\left\{\log \tan\left[\frac{x}{2} + \frac{\pi}{8}\right]\right\}_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{2}}\left\{\log \tan\left[\frac{\pi}{4} + \frac{\pi}{8}\right] - \log \tan\frac{\pi}{8}\right\} = \frac{1}{2\sqrt{2}}\log\left(\frac{\tan(3\pi/8)}{\tan(\pi/8)}\right) = \frac{1}{2\sqrt{2}}\log\left(\frac{\cot(\pi/8)}{\tan(\pi/8)}\right)$$

$$= \frac{2}{2\sqrt{2}}\log\cot\frac{\pi}{8} = \frac{1}{\sqrt{2}}\log(\sqrt{2} + 1)$$

**Illustration 9:** Evaluate : $\int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx$ **(JEE ADVANCED)**

**Sol:** By putting $\tan x = \frac{\sin x}{\cos x}$ and using the property $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, we can solve the given problem.

Let $I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \quad \Rightarrow \quad I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ ... (i)

On applying $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$ we get

$$I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left((3\pi/4) - (\pi/4) - x\right)}}{\sqrt{\cos\left((3\pi/4) - (\pi/4) - x\right)} + \sqrt{\sin\left((3\pi/4) - (\pi/4) - x\right)}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left((\pi/2) - x\right)}}{\sqrt{\cos\left((\pi/2) - x\right)} + \sqrt{\sin\left((\pi/2) - x\right)}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} dx = [x]_{-\pi/4}^{3\pi/4} = \left[\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \left[\frac{3\pi}{4} + \frac{\pi}{4}\right] = \pi \Rightarrow I = \frac{\pi}{2}$$

**Illustration 10:** The value of $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ is **(JEE ADVANCED)**

**Sol:** Similar to the problems above, we can write $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ as

$\int_0^{\pi/2} \log\left(\frac{4 + 3\sin\left((\pi/2) - x\right)}{4 + 3\cos\left((\pi/2) - x\right)}\right) dx$ and then by adding these two equations we can solve the given problem.

Let $I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$

On applying property 5, we get

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin\left((\pi/2)-x\right)}{4+3\cos\left((\pi/2)-x\right)}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx = -\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx = -I \Rightarrow I = 0$$

Thus, $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx = 0$

**Illustration 11:** $I = \int_0^{\pi/2} \frac{dx}{4+5\sin x}$ **(JEE ADVANCED)**

**Sol:** Let $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and then by putting $\tan\frac{x}{2} = t$, we can solve the given problem.

$$I = \int_0^{\pi/2} \frac{dx}{4+5\ (2\tan(x/2)/1+\tan^2(x/2))} = \int_0^{\pi/2} \frac{\sec^2(\pi/2)dx}{4+4\tan^2(\pi/2)+10\tan(\pi/2)}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2} = dt$

$$\Rightarrow \int_0^1 \frac{2dt}{4+4t^2+10t} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t+(1/2)\right)\left(t+2\right)} = \frac{1}{3}\int_0^1 \frac{1}{\left(t+(1/2)\right)} - \frac{1}{\left(t+2\right)}dt = \frac{1}{3}\left[\ln\frac{t+(1/2)}{t+2}\right]_0^1 = \frac{1}{3}\log 2$$

**Illustration 12:** Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ **(JEE ADVANCED)**

**Sol:** Let $\tan x = \frac{\sin x}{\cos x}$ and then using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we can solve the given problem.

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{... (i)}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx \quad [\because \text{ here } a+b=\pi/2]$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{... (ii)}$$

$$\therefore\ 2I = \int_{\pi/6}^{\pi/3} 1\,dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \quad \Rightarrow I = \frac{\pi}{12}$$

**Property 6**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } f(-x) = -f(x) \text{ (odd function)} \end{cases}$$

**Note:** This property is to be used if the integrand is either an even or odd function of x

**Illustration 13:** $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx$ is equal to **(JEE MAIN)**

**Sol:** As $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx = 2\int_0^{\pi/2} \cos^2 x\,dx$, therefore using property 7 we can solve it.

Here $I = 2\int_0^{\pi/2} \cos^2 x\,dx\ \{\because f(-x) = f(x)\}$ ; $\int_0^{\pi/2} (1+\cos 2x)dx = \left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi/2} = \frac{\pi}{2}$

**Illustration 14:** $\int_{-1}^{1} \frac{x^3 \sin(1+x^2)}{1+x^2} dx$ is equal to **(JEE ADVANCED)**

**Sol:** Here by using the property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } f(-x) = -f(x) \text{ (odd function)} \end{cases}$

Here $f(x) = \frac{x^3 \sin(1+x^2)}{1+x^2}$ & $f(-x) = -\frac{x^3 \sin(1-x^2)}{1+x^2}$

$\because f(x) = -f(x)$

$\therefore I = 0$

**Property 7:** $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

**Note:** The above property is used to halve the limits

**Illustration 15:** Evaluate : $\int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta$ **(JEE MAIN)**

**Sol:** Let $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$. Hence by using this property we can solve the given problem.

Let $I = \int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta \rightarrow$ Let $f(\theta) = \frac{\sin 2\theta}{a - b\cos\theta}$

$$f(2\pi - \theta) = \frac{\sin 2(2\pi - \theta)}{a - b\cos(2\pi - \theta)} = \frac{-\sin 2\theta}{a - b\cos\theta} = -f(\theta)$$

By property 7, we have

$$\therefore \int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta = 0$$

**Illustration 16:** Evaluate $\int_0^{2\pi} x\sin^4 x\cos^6 x\,dx$ **(JEE ADVANCED)**

**Sol:** Similar to the problem above.

$$I=\int_0^{2\pi} x\sin^4 x\cos^6 x\,dx=\int_0^{2\pi}(2\pi-x)\sin^4 x\cos^6 x\,dx$$

$$2I=2\pi\int_0^{2\pi}\sin^4 x\;\cos^6 x\;dx\;;\qquad I=2\pi\int_0^{\pi}\sin^4 x\;\cos^6 x\;dx\;;$$

$$I=4\pi\int_0^{\pi/2}\sin^4 x\;\cos^6 x\;dx\;;\qquad I=4\pi\int_0^{\pi/2}\cos^4 x\sin^6 x\;;$$

$$\Rightarrow I=\frac{2\pi}{16}\int_0^{\pi/2}(\sin 2x)^4 dx\;\Rightarrow 2x=t\Rightarrow dx=\frac{dt}{2}$$

$$\Rightarrow I=\frac{\pi}{16}\int_0^{\pi}\sin^4 t\,dt=\frac{\pi}{8}\int_0^{\pi/2}\sin^4 t\,dt\qquad\Rightarrow I=\frac{\pi}{8}\left[\frac{1}{2}\int_0^{\pi/2}\left(\sin^4 t+\sin^4 t\right)dt\right]=\frac{\pi}{8}.\frac{1}{2}.\frac{3\pi}{8}=\frac{3\pi^2}{128}$$

**Property 8:** If f(x) = f(x + a) (i.e. f(x) is a function with period a), then $\int_0^{na} f(x)dx=n\int_0^{a} f(x)dx$

**Illustration 17:** Evaluate: $\int_0^{4\pi}\sin^8 x\,dx$ **(JEE MAIN)**

**Sol:** Here $\sin^8(\pi-x)=\sin^8 x$, therefore by using this property, we can solve the given problem.

$$I=4\int_0^{\pi}\sin^8 x\,dx\;=8\int_0^{\pi/2}\sin^8 x\,dx=8\,\frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2}=\frac{35\pi}{32}$$

**Illustration 18:** Evaluate: $\int_0^{2\pi}\cos^5 x\,dx$ **(JEE ADVANCED)**

**Sol:** Let $I=\int_0^{2\pi}\cos^5 x\,dx$

Let $f(x)=\cos^5 x$

$f(2\pi-x)=\cos^5(2\pi-x)=\cos^5 x=f(x)$

Then $\int_0^{2\pi}\cos^5 x\;x\,dx=2\int_0^{\pi}\cos^5 x\,dx$

Now, $f(\pi-x)=\cos^5(\pi-x)=(-\cos x)^3=-\cos^5 x$

$=-f(x)\;;\;\int_0^{\pi}\cos^5 x\,dx=0$

Hence $\int_0^{2\pi}\cos^5 x\,dx=0$

**Property 9**

$$\int_a^{a+nT} f(x)dx=n\int_0^{T} f(x)dx$$ (if f(x + T) = f(x), and nÎN i.e. f(x) is a function with period T)

$$\int_{a+mT}^{b+nT} f(x)dx=(n-m)\int_0^{T} f(x)dx+\int_a^{b} f(x)dx\qquad m,n\in I$$

**Illustration 19:** $I=\int_0^{200\pi}\sqrt{1+\cos x}\;dx$ **(JEE MAIN)**

**Sol:** $I = \sqrt{2}\int_0^{200\pi}\left|\cos\frac{x}{2}\right|dx \qquad \frac{x}{2} = t$

$\Rightarrow I = 2\sqrt{2}\int_0^{100\pi}|\cos t|\,dt = 200\sqrt{2}\int_0^{\pi}|\cos t|\,dt = 400\sqrt{2}$

**Property 10:** $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

Corollary (1): $\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = h'(x)\,f(h(x))$ [a is any constant independent of x]

Corollary (2): $\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)$

**Property 11:** $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

**Property 12:** If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x)dx \ge 0$

This property is also called the domination law.

There are a few more properties which might be helpful in solving problems

**1. Shift property:** $\int_a^b f(x)dx = \int_{a\pm c}^{b\pm c} f(x)\,dx$

**2. Reflection property:** $\int_a^b f(x)dx = -\int_{-a}^{-b} f(-x)\,dx$

**3. Expansion/Contraction property:** $\int_a^b f(x)dx = k\int_{a/k}^{b/k} f(x)\,dx \quad \forall\, k > 0$

## MASTERJEE CONCEPTS

$\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = \pi$ if $(\beta > \alpha)$

$\int_\alpha^\beta \sqrt{(x-\alpha)(x-\beta)}dx = \frac{\pi}{8}(\beta-\alpha)^2$

$\int_a^b \sqrt{\frac{x-a}{b-x}}dx = \frac{\pi}{2}(b-a)$

If f(t) is an odd function, then $\phi(x) = \int_a^x f(t)dt$ is an even function.

If f(x) is an even function, then $\phi(x) = \int_a^x f(t)dt$ is an odd function.

Every continuous function defined on [a, b] is integrable over [a, b]

Every monotonic function defined on [a, b] is integrable over [a, b]

## MASTERJEE CONCEPTS

**Change of variables:** If the function f(x) is continuous on [a, b] and the function $x = \phi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \phi(t_1)$, $b = \phi(t_2)$, then

$$\int_a^b f(x)dx = \int_{t_1}^{t_2} f(\phi(t))\phi'(t)dt.$$

**Nitish Jhawar (JEE 2009 AIR 7)**

# 3. SOME SPECIAL INTEGRALS

## 3.1 Walli's Formula

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)\,(n-3)....2}{n(n-2)....1} \quad \text{(if n is odd positive integer)}$$

$$= \frac{(n-1)\,(n-3).....1}{n(n-2).....2}\left(\frac{\pi}{2}\right) \quad \text{(if n is even positive integer)}$$

**Illustration 20:** Evaluate $\int_0^{\pi/2} \cos^7 x\, dx$ **(JEE MAIN)**

**Sol:** By using Walli's formula we can solve the given problem.

$$I = \frac{6.4.2}{7.5.3} = \frac{16}{35}$$

## 3.2 Gamma Function

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\big((m+1)/2\big)\Gamma\big((n+1)/2\big)}{2\Gamma\big((m+n+2)/2\big)}$$

where $\Gamma(n)$ is called the gamma function

OR

$$\int_0^{\pi/2} \sin^m \cos^n x\, dx = \frac{((m-1)\,(m-3).....(2 \text{ or } 1))\,(n-1)\,((n-3)...(2 \text{ or } 1))}{(m+n)\,(m+n-2).......(2 \text{ or } 1)}$$

(if m and n both are not simultaneously even positive integers)

$$\frac{((m-1)\,(m-3)....(1))((n-1)(n-3)....(1))}{(m+n)(m+n-2)....(2)}\left(\frac{\pi}{2}\right) \quad \text{(if m and n are both even positive integers)}$$

**Illustration 21:** Evaluate $I = \int_0^{\pi/2} \sin^4 x \cos^5 x\, dx.$ **(JEE MAIN)**

**Sol:** Using the gamma function formula i.e.

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left((m+1)/2\right)\Gamma\left((n+1)/2\right)}{2\Gamma\left((m+n+2)/2\right)}$$

We can solve it.

$$I = \frac{r\left((4+1)/2\right)\Gamma\left((5+1)/2\right)}{2\Gamma\left((4+5+2)/2\right)} = \frac{\Gamma\left(5/2\right)\Gamma(3)}{2\Gamma\left(11/2\right)} = \frac{\left((3/2).(1/2)\right)\left(2.1\right)}{2\left((9/2).(7/2).(5/2).(3/2).(1/2)\right)} = \frac{8}{315}$$

## 4. NEWTON LEIBNITZ FORMULA

In calculus, **Leibnitz's rule** for differentiation under the integral sign named after Gottfried Leibnitz tells us that if we have an integral $\int_{y_0}^{y_1} f(x,y)dy$ then for x in ($x_0$, $x_1$) the derivative of this integral is thus expressible as

$$\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)dy\right) = \int_{y_0}^{y_1} f_x(x,y)dy$$

provided that f and its partial derivative $f_x$ are both continuous over a region in the form [$x_0$. $x_1$] × [$y_0$, $y_1$].

## 5. SUMMATION OF SERIES BY INTEGRATION (LIMIT AS A SUM)

To find the sum of an infinite series with the help of definite integration, the following formula is used

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$$

The following method is used to solve the questions on summation of series.

(i) After writing (r – 1)th or rth term of the series, express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$.

Therefore the given series will take the form as $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) Now write $\int$ in place of $\lim_{n\to\infty}\Sigma$ and x in place of $\frac{r}{n}$ and dx in place of n. We get summation in the form of integral $\int_0^1 f(x)\,dx$.

Also we can write $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n}[f(a)+f(a+h)+....+f(a+\left(n-1\right)h)]$ $\left[\text{where } h = \frac{b-a}{n}\right]$

**Illustration 22:** Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{2n}\right]$ **(JEE MAIN)**

**Sol:** By using the summation of series by integration formula i.e $\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$ we can solve it.

Limit = $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \sum\left(\frac{1}{1+(r/n)}.\frac{1}{n}\right) = \int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log 2$

**Illustration 23:** $\lim_{n\to\infty} \frac{1^{100}+2^{100}+3^{100}........n^{100}}{n^{101}}$ **(JEE MAIN)**

**Sol:** By observing the given problem, we can say that it's a sum of an infinite series so by using the summation of series by integration formula we can solve it.

$T_r = \frac{r^{100}}{n^{101}} = \frac{1}{n} \times \left(\frac{r}{n}\right)^{100}$ ; $S = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^{100}$ ; $= \int_0^1 x^{100} dx = \frac{1}{101}$

**Illustration 24:** Find the value of $\lim_{n\to\infty} \left[\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + ..... + \frac{1}{4n}\right]$ **(JEE ADVANCED)**

**Sol:** Here $t_r = \frac{n}{(n+r)^2} = \frac{1}{n} \frac{1}{[1+(r/n)]^2}$, therefore similar to the problem above, we can solve it.

Therefore the given series $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{[1+(r/n)]^2} \cdot \frac{1}{n} = \int_0^1 \frac{1}{(1+x)^2} dx$

Given series $= \int_0^1 \frac{1}{(1+x)^2} dx = \left[-\frac{1}{1+x}\right]_0^1 = \frac{-1}{2} + 1 = \frac{1}{2}$

**Evaluate the following definite integrals as the limit of sums.**

**Illustration 25:** $\int_a^b \cos x \, dx$ **(JEE ADVANCED)**

**Sol:** Here $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} [f(a) + f(a+h) + .... + f(a+(n-1)h)]$ where f(x) = cos x and $h = \frac{b-a}{n}$

$$\therefore \int_a^b \cos x \, dx = \lim_{n\to\infty} \frac{b-a}{n} [\cos a + \cos(a+h) + .... + \cos(a+(n-1)h)]$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \cdot \left[\frac{\cos\left(a+((n-1)/2).h\right) . \sin\left(nh/2\right)}{\sin(h/2)}\right]$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \cdot \frac{\cos\left(a + \frac{n-1}{2} \cdot \frac{(b-a)}{n}\right) . \sin\left(\frac{n.(b-a)}{2n}\right)}{\sin\left(\frac{b-a}{2n}\right)}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{b-a}{2n} \cdot \frac{\cos\left(a + (1-(1/n))/2)\,(b-a)\right) . \sin\left((b-a)/2\right)}{\sin\left((b-a)/2n\right)}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{\cos\left(a + \left(1-(1/n)\right)\left((b-a)/2\right)\right) . \sin\left((b-a)/2\right)}{\sin\left((b-a)/2n\right) / \left((b-a)/2n\right)}$$

$$= 2\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right) = \sin b - \sin a$$

**Illustration 26:** $\int_1^2 (x^2 + x) dx$ **(JEE ADVANCED)**

**Sol:** Similar to the problem above.

$$h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} [f(a) + f(a+h) + .... + f(a+(n-1)h)]$$

$$\int_1^2 (x^2 + x)\,dx = \lim_{n\to\infty} \frac{1}{n}[f(1) + f(1+h) + .... + f(1+(n-1)h)]$$

$$= \lim_{n\to\infty} \frac{1}{n}[(1^2+1) + \{(1+h)^2 + (1+h)\} + .... + \{(1+(n-1)h)^2 + (1+(n-1)h)\}$$

$$= \lim_{n\to\infty} \frac{1}{n}[1^2.n + h(1+2+...+(n-1)) + 1\,.\,n + 2h(1+2+...+(n-1)) + h^2(1^2+2^2+...(n-1)^2)]$$

Here $h = \frac{1}{n}$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{1}{n}\frac{(n-1)(n)}{2} + n + \frac{2}{n}\,.\,\frac{n(n-1)}{2} + \frac{1}{n^2}\frac{(n-1)n(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{\left(1-(1/n)\right)(1)}{2} + 1 + \frac{2\left(1-(1/n)\right)}{2} + \frac{\left(1-(1/n)\right)(1)\left(2-(1/n)\right)}{6}\right]$$

$$= 1 + \frac{1}{2} + 1 + 1 + \frac{1}{3} = \frac{23}{6}$$

## 6. INTEGRAL WITH INFINITE LIMITS

If a function f(x) is continuous for $a \le x < \infty$, then by definition,

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad ... (i)$$

If there exists a finite limit on the right-hand side of (i), then the improper integral is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (i) for f(x) > 0, is the area of the figure bounded by the graph of the function y = f(x), the straight line x = a, and the x-axis. Similarly, we can define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \text{ and } \int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

## 7. IMPORTANT RESULTS

If $f(x) \ge 0$ and a < b, then $\int_a^b f(x)dx \ge 0$, e.g. $\int_0^{\pi/2} \sin x\, dx = 1$

If $f(x) \ge 0$ and a < b, then $\int_b^a f(x)dx \le 0$, e.g. $\int_{\pi/2}^0 \cos x\, dx = -1$

If $f(x) \le 0$ and a < b, then $\int_b^a f(x)dx \ge 0$, e.g. $\int_{\pi/2}^0 \sin x\, dx = 1$

$\int_0^x [x]dx = \int_0^1 (0)dx + \int_1^2 (1)dx + \int_2^3 2dx + ... + \int_{[x]}^x [x]dx$, where [ ] denotes the greatest integer of x.

$$\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$$

$$\int_0^{\pi/2} \log(\tan x)dx = \int_0^{\pi/2} (\cot x)dx = 0$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \int_0^a f(x)dx + \int_0^a f(a+x)dx$$

$\int_a^b [x]dx = (b-a)\int_0^1 x\,dx$, where [ ] denotes the fractional part of x.

e.g., $\int_0^5 [x]\,dx = 5\int_0^1 x\,dx = \frac{5}{2}$

Integral of an inverse function is given by $\int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) - \int_a^b f(x)\,dx$

Derivation of the given formula is given in the solved examples

## 8. GEOMETRICAL APPLICATION

The area of the figure bounded by the graphs of two continuous functions $y = f_1(x)$ and $y = f_2(x)$, $f_1(x) \le f_2(x)$, and two straight lines x= a and x = b is determined by the formula $S = \int_a^b (f_2(x) - f_1(x))dx$. It is sometimes convenient to use formulae analogous to x.with respect to y, i.e., regarding x as a function of y. In particular, the area bounded by the curve x =f(y), the y-axis and the two abscissae y = c and y = d is given by $\int_c^d f\ (y)dy$. The area of the figure bounded by the graphs of two continuous functions $x = f_1(y)$ and $f_2(y)$ (with $f_1(y) \le f_2(y)$), and the two straight lines y = c, y = d is given by $\int_c^d (f_2(y) - f_1(y))dy$

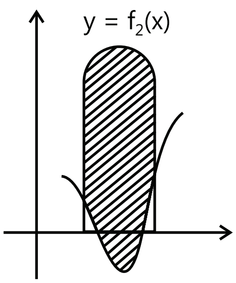

**Figure 23.1**

From the view of geometry we get an important inequality as if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

# FORMULAE SHEET

**Important results**

| | |
|---|---|
| 1. $\int_a^b \{f(x) \pm g(x) \pm h(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx + \int_a^b h(x)dx$ | 2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ |
| 3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (a < c < b) | 4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ |
| 5. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } (-x) = -f(x) \text{ (odd function)} \end{cases}$ | 6. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ |
| 7. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$ | 8. $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$ |
| 9. $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$ (if f(x + T) = f(x), and n∈N i.e. f(x) is a function with period T) | 10. If f(x) = f(x + a) then $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ |

| | |
|---|---|
| 11. $\left\| \int_a^b f(x)\,dx \right\| \le \int_a^b \| f(x) \| \,dx$ | 12. $\int_a^b f(x)dx = k\int_{a/k}^{b/k} f(x)dx \quad \forall\, k > 0$ |
| 13. $\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)dy\right) = \int_{y_0}^{y_1} f_x(x,y)dy$ (Leibnitz formula) | |

**Definite integral of rational functions**

| | |
|---|---|
| 1. $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ | 2. $\int_0^\infty \frac{x^{p-1}dx}{1+x} = \frac{\pi}{\sin(p\pi)},\ 0<p<1$ |
| 3. $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$ | 4. $\int_0^\infty \frac{\sin(px)}{x}dx = \begin{cases} \pi/2 & p>0 \\ 0 & p=0 \\ -\pi/2 & p<0 \end{cases}$ |
| 5. $\int_0^\infty \frac{\sin^2 px}{x^2} = \frac{\pi p}{2}$ | 6. $\int_0^{2x} \frac{dx}{a+b\sin x} = \frac{2\pi}{\sqrt{a^2-b^2}}$ |
| 7. $\int_0^\infty \sin ax^2dx = \int_0^\infty \cos(ax^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$ | 8. $\int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$ |
| 9. $\int_0^\infty \frac{\tan x}{x}dx = \frac{\pi}{2}$ | |

**Advanced formulas**

| |
|---|
| 1. $\int_0^\pi \sin(mx).\sin(nx)dx = \begin{cases} 0 & \text{m,n integers and } m \ne n \\ \pi/2 & \text{m,n integers and } m = n \end{cases}$ |
| 2. $\int_0^\pi \cos(mx).\cos(nx)dx = \begin{cases} 0 & \text{m,n integers and } m \ne n \\ \pi/2 & \text{m,n integers and } m = n \end{cases}$ |
| 3. $\int_0^\pi \sin(mx).\cos(nx)dx = \begin{cases} 0 & \text{m,n integers and } m+n \text{ odd} \\ 2m/(m^2-n^2) & \text{m,n integers and } m+n \text{ even} \end{cases}$ |
| 4. $\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2} \cos^{2m} x\,dx = \frac{1.3.5....2m-1}{2.4.6....2m}\frac{\pi}{2}$ |

**Definite integrals of exponential functions**

| | |
|---|---|
| 1. $\int_0^\infty e^{-ax} \cos bx\,dx = \frac{a}{a^2+b^2}$ | 2. $\int_0^\infty e^{-ax} \sin bx\,dx = \frac{b}{a^2+b^2}$ |
| 3. $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ | 4. $\int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$ |

| | |
|---|---|
| 5. $\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}}$ | 6. $\int_0^\infty \frac{x\,dx}{e^x - 1} = \frac{\pi^2}{6}$ |
| 7. $\int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = \Gamma(n)\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + .....\right)$ | 8. $\int_0^\infty \frac{x\,dx}{e^x + 1} = \frac{\pi^2}{12}$ |
| 9. $\int_0^\infty \frac{x^{n-1}}{e^x + 1} dx = \Gamma(n)\left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - .......\right)$ | 10. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \sec(px)} dx = \frac{1}{2}\ln\left(\frac{b^2 + p^2}{a^2 + p^2}\right)$ |
| 11. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \csc(px)} dx = \arctan\frac{b}{p} - \arctan\frac{a}{p}$ | 12. $\int_0^\infty \frac{e^{-ax}(1 - \cos x)}{x^2} dx = \text{arc}\cot a - \frac{a}{2}\ln(a^2 + 1)$ |

# Solved Examples

## JEE Main/Boards

**Example 1:** Evaluate:

(i) $\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}}$ (ii) $\int_{-a}^a \sqrt{\frac{a-x}{a+x}}dx$

**Sol:** (i) As we know $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$, therefore by using this formula we can solve the given problem.

(ii) Put x = a cos θ : θ ∈ [0, p] and solve it using the appropriate formula.

(i) $\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}}$

$= \left(\sin^{-1}\frac{x - (a/2)}{(a/2)}\right)_0^a ; = \left(\sin^{-1}\frac{2x - a}{a}\right)_0^a$

= [sin$^{-1}$ 1–sin$^{-1}$(–1)] = 2 sin$^{-1}$(1) = 2 × $\frac{\pi}{2} = \pi$. (ii)

Then dx = –a sin θ dθ. Hence,

$\int_{-a}^a \sqrt{\frac{a-x}{a+x}}dx = \int_\pi^0 \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-a\sin\theta)d\theta$

$= a\int_0^\pi \sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}} . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$

$= a\int_0^\pi 2\sin^2\frac{\theta}{2}d\theta = a\int_0^\pi (1 - \cos\theta)d\theta$

$= a(\theta - \sin\theta)_0^\pi = a(\pi) = a\pi.$

**Example 2:** Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx$

**Sol:** Let $\int_0^a f(x)dx = \int_0^a f(a - x)dx$.

By using this we can write $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx$

as $\int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}dx$ and by adding we can get the result.

$I = \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}dx$

$\therefore\ 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$

$\therefore\quad I = \frac{\pi}{4}$