$\int_a^b [x]dx = (b-a)\int_0^1 x\,dx$, where [ ] denotes the fractional part of x.

e.g., $\int_0^5 [x]\,dx = 5\int_0^1 x\,dx = \frac{5}{2}$

Integral of an inverse function is given by $\int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) - \int_a^b f(x)\,dx$

Derivation of the given formula is given in the solved examples

8. GEOMETRICAL APPLICATION

The area of the figure bounded by the graphs of two continuous functions $y = f_1(x)$ and $y = f_2(x)$, $f_1(x) \le f_2(x)$, and two straight lines x= a and x = b is determined by the formula $S = \int_a^b (f_2(x) - f_1(x))dx$. It is sometimes convenient to use formulae analogous to x.with respect to y, i.e., regarding x as a function of y. In particular, the area bounded by the curve x =f(y), the y-axis and the two abscissae y = c and y = d is given by $\int_c^d f\ (y)dy$. The area of the figure bounded by the graphs of two continuous functions $x = f_1(y)$ and $f_2(y)$ (with $f_1(y) \le f_2(y)$), and the two straight lines y = c, y = d is given by $\int_c^d (f_2(y) - f_1(y))dy$

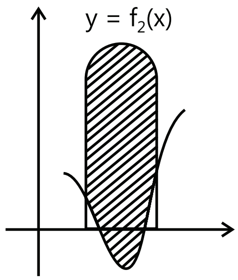

Figure 23.1

From the view of geometry we get an important inequality as if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

FORMULAE SHEET

Important results

1. $\int_a^b \{f(x) \pm g(x) \pm h(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx + \int_a^b h(x)\,dx$	2. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (a < c < b)	4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
5. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\ dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } (-x) = -f(x) \text{ (odd function)} \end{cases}$	6. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
7. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$	8. $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$
9. $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)\,dx$ (if f(x + T) = f(x), and $n \in N$ i.e. f(x) is a function with period T)	10. If f(x) = f(x + a) then $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

11. $\left\| \int_a^b f(x)\,dx \right\| \le \int_a^b \| f(x) \|\,dx$	12. $\int_a^b f(x)dx = k\int_{a/k}^{b/k} f(x)dx \quad \forall\, k > 0$
13. $\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)dy\right) = \int_{y_0}^{y_1} f_x(x,y)dy$ (Leibnitz formula)	

Definite integral of rational functions

1. $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$	2. $\int_0^\infty \frac{x^{p-1}dx}{1+x} = \frac{\pi}{\sin(p\pi)},\ 0<p<1$
3. $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$	4. $\int_0^\infty \frac{\sin(px)}{x}dx = \begin{cases} \pi/2 & p>0 \\ 0 & p=0 \\ -\pi/2 & p<0 \end{cases}$
5. $\int_0^\infty \frac{\sin^2 px}{x^2} = \frac{\pi p}{2}$	6. $\int_0^{2x} \frac{dx}{a+b\sin x} = \frac{2\pi}{\sqrt{a^2-b^2}}$
7. $\int_0^\infty \sin ax^2 dx = \int_0^\infty \cos(ax^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$	8. $\int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$
9. $\int_0^\infty \frac{\tan x}{x}dx = \frac{\pi}{2}$	

Advanced formulas

1. $\int_0^\pi \sin(mx).\sin(nx)dx = \begin{cases} 0 & \text{m,n integers and } m \ne n \\ \pi/2 & \text{m,n integers and } m = n \end{cases}$
2. $\int_0^\pi \cos(mx).\cos(nx)dx = \begin{cases} 0 & \text{m,n integers and } m \ne n \\ \pi/2 & \text{m,n integers and } m = n \end{cases}$
3. $\int_0^\pi \sin(mx).\cos(nx)dx = \begin{cases} 0 & \text{m,n integers and } m+n \text{ odd} \\ 2m/(m^2-n^2) & \text{m,n integers and } m+n \text{ even} \end{cases}$
4. $\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2} \cos^{2m} x\,dx = \frac{1.3.5....2m-1}{2.4.6....2m}\frac{\pi}{2}$

Definite integrals of exponential functions

1. $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$	2. $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$
3. $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$	4. $\int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$

5. $\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}}$	6. $\int_0^\infty \frac{x\,dx}{e^x - 1} = \frac{\pi^2}{6}$
7. $\int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = \Gamma(n)\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + ...\right)$	8. $\int_0^\infty \frac{x\,dx}{e^x + 1} = \frac{\pi^2}{12}$
9. $\int_0^\infty \frac{x^{n-1}}{e^x + 1} dx = \Gamma(n)\left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - ....\right)$	10. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \sec(px)} dx = \frac{1}{2}\ln\left(\frac{b^2 + p^2}{a^2 + p^2}\right)$
11. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \csc(px)} dx = \arctan\frac{b}{p} - \arctan\frac{a}{p}$	12. $\int_0^\infty \frac{e^{-ax}(1 - \cos x)}{x^2} dx = \text{arc}\cot a - \frac{a}{2}\ln(a^2 + 1)$

Solved Examples

JEE Main/Boards

Example 1: Evaluate:

(i) $\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}}$ (ii) $\int_{-a}^a \sqrt{\frac{a - x}{a + x}} dx$

Sol: (i) As we know $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$, therefore by using this formula we can solve the given problem.

(ii) Put x = a cos θ : θ ∈ [0, p] and solve it using the appropriate formula.

(i) $\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}}$

$= \left(\sin^{-1}\frac{x - (a/2)}{(a/2)}\right)_0^a; = \left(\sin^{-1}\frac{2x - a}{a}\right)_0^a$

$= [\sin^{-1} 1 - \sin^{-1}(-1)] = 2\sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi$. (ii)

Then dx = –a sin θ dθ. Hence,

$\int_{-a}^a \sqrt{\frac{a - x}{a + x}} dx = \int_\pi^0 \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} (-a\sin\theta) d\theta$

$= a\int_0^\pi \sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$

$= a\int_0^\pi 2\sin^2\frac{\theta}{2} d\theta = a\int_0^\pi (1 - \cos\theta) d\theta$

$= a(\theta - \sin\theta)_0^\pi = a(\pi) = a\pi$.

Example 2: Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Sol: Let $\int_0^a f(x) dx = \int_0^a f(a - x) dx$.

By using this we can write $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

as $\int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]} dx$ and by adding we can get the result.

$I = \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]} dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$

$\therefore\ 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$

$\therefore\quad I = \frac{\pi}{4}$