

5. $\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}}$	6. $\int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi^2}{6}$
7. $\int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = \Gamma(n) \left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots \right)$	8. $\int_0^\infty \frac{x dx}{e^x + 1} = \frac{\pi^2}{12}$
9. $\int_0^\infty \frac{x^{n-1}}{e^x + 1} dx = \Gamma(n) \left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \dots \right)$	10. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \sec(px)} dx = \frac{1}{2} \ln \left(\frac{b^2 + p^2}{a^2 + p^2} \right)$
11. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \csc(px)} dx = \arctan \frac{b}{p} - \arctan \frac{a}{p}$	12. $\int_0^\infty \frac{e^{-ax}(1 - \cos x)}{x^2} dx = \operatorname{arccot} a - \frac{a}{2} \ln(a^2 + 1)$

Solved Examples

JEE Main/Boards

Example 1: Evaluate:

$$(i) \int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}} \quad (ii) \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$$

Sol: (i) As we know $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$, therefore by using this formula we can solve the given problem.

(ii) Put $x = a \cos \theta : \theta \in [0, \pi]$ and solve it using the appropriate formula.

$$(i) \int_0^a \frac{dx}{\sqrt{(a^2/4) - (x - (a/2))^2}} = \left(\sin^{-1} \frac{x - (a/2)}{(a/2)} \right)_0^a = \left(\sin^{-1} \frac{2x - a}{a} \right)_0^a$$

$$= [\sin^{-1} 1 - \sin^{-1}(-1)] = 2 \sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi. \quad (ii)$$

Then $dx = -a \sin \theta d\theta$. Hence,

$$\int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx = \int_\pi^0 \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} (-a \sin \theta) d\theta$$

$$= a \int_0^\pi \sqrt{\frac{2 \sin^2(\theta/2)}{2 \cos^2(\theta/2)}} \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

$$= a \int_0^\pi 2 \sin^2 \frac{\theta}{2} d\theta = a \int_0^\pi (1 - \cos \theta) d\theta$$

$$= a(\theta - \sin \theta)_0^\pi = a(\pi) = a\pi.$$

Example 2: Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Sol: Let $\int_0^a f(x) dx = \int_0^a f(a-x) dx$.

By using this we can write $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

$$\text{as } \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]} dx \text{ and by adding}$$

we can get the result.

$$I = \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Example 3: Evaluate $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$

Sol: Here $\log\left(\frac{1-x}{x}\right) = \log(1-x) - \log(x)$ and $\int_a^a f(x) dx = \int_a^a f(a-x) dx$ by using these two formulae we can solve it.

$$I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx$$

(Put $x = \cos^2 t$: $\cos t > 0$; then $dx = -2 \cos t \sin t dt$)

$$\begin{aligned} &= - \int_{\pi/2}^0 \log(\sec^2 t - 1) \cdot 2 \cos t \sin t dt \\ &= \int_0^{\pi/2} \log(\tan^2 t) \cdot \sin 2t dt = 2 \int_0^{\pi/2} \sin 2t \cdot \log(\tan t) dt \\ &= 2 \int_0^{\pi/2} \log(\cot t) \cdot \sin 2t dt \end{aligned}$$

$$\therefore 2I = 2 \int_0^{\pi/2} \log(\tan t \cdot \cot t) \times \sin 2t dt = 0$$

Example 4: Evaluate:

(i) $I = \int_0^{\pi} |\cos x| dx$

(ii) $I = \int_{-2}^1 |2x+1| dx$

(iii) $I = \int_1^4 f(x) dx$, where $f(x) = \begin{cases} 4x+3, & 1 \leq x \leq 2 \\ 3x+5, & 2 < x \leq 4 \end{cases}$

Sol: (i) Here $|\cos(\pi - x)| = |\cos x|$ hence $|\cos x| = \cos x$ therefore using the formula $\int \cos x = \sin x$ we can solve it.

(ii) By putting $2x+1 = z$ we can solve it.

(iii) As $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ ($a < c < b$)

By using this formula we can obtain the result.

(i) $I = 2 \int_0^{\pi/2} |\cos x| dx$
 $= 2 \int_0^{\pi/2} \cos x dx = 2(\sin x)_0^{\pi/2} = 2(1) = 2$

(ii) $I = \int_{-2}^1 |2x+1| dx$ (put $2x+1 = z$)

$$= \frac{1}{2} \int_{-3}^3 |z| dz = \int_0^3 |z| dz = \frac{9}{2}$$

(iii) $I = \int_1^2 f(x) dx + \int_2^4 f(x) dx$
 $= \int_1^2 (4x+3) dx + \int_2^4 (3x+5) dx$
 $= (2x^2 + 3x)_1^2 + \left(\frac{3x^2}{2} + 5x\right)_2^4$
 $= 9 + 28 = 37$

Example 5: Evaluate $I = \int_0^{1.7} [x^2] dx$, where $[x]$ is the greatest integer function

Sol: $[x^2]$ takes constant values 0, 1, 2 in intervals (0, 1), (1, $\sqrt{2}$), ($\sqrt{2}$, $\sqrt{3}$) respectively. By substituting these values we will get the required result.

$$\begin{aligned} I &= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{1.7} [x^2] dx \\ &= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{1.7} 2 dx \\ &= 0 + (\sqrt{2} - 1) + 2(1.7 - \sqrt{2}) = 2.4 - \sqrt{2} \end{aligned}$$

Example 6: Let $f(x)$ be an odd function in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ with period T , prove that $F(x) = \int_a^x f(t) dt$ is a periodic function with period T .

Sol: As $f(x)$ is an odd function.

$$F(x+T) = \int_a^{x+T} f(t) dt = \int_a^x f(t) dt + \int_x^{x+T} f(t) dt = F(x) + I(x)$$

where $I(x) = \int_x^{x+T} f(t) dt = \int_x^{\frac{T}{2}} f(t) dt + \int_{\frac{T}{2}}^{\frac{T}{2}+T} f(t) dt = 0$ (since f is an odd function). Hence $F(x)$ is a periodic function with period T .

Example 7: Evaluate $\int_0^{\pi} \theta \sin^2 \theta \cos^2 \theta d\theta$

Sol: As we know, $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, hence by using this formula we can evaluate it.

Let $I = \int_0^{\pi} \theta \sin^2 \theta \cos^2 \theta d\theta$

$$\begin{aligned}
 &= \int_0^{\pi} (\pi - \theta) \sin^2(\pi - \theta) \cos^2(\pi - \theta) d\theta \\
 &= \int_0^{\pi} (\pi - \theta) \sin^2 \theta \cos^2 \theta d\theta \\
 &= \pi \int_0^{\pi} \sin^2 \theta \cos^2 \theta d\theta - \int_0^{\pi} \theta \sin^2 \theta \cos^2 \theta d\theta \\
 &= \pi \int_0^{\pi} \left(\frac{\sin 2\theta}{2} \right)^2 d\theta - I \\
 \Rightarrow 2I &= \frac{\pi}{4} \int_0^{\pi} \sin^2 2\theta d\theta = \frac{\pi}{4} \int_0^{\pi} \left(\frac{1 - \cos 4\theta}{2} \right) d\theta \\
 &= \frac{\pi}{8} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi} = \frac{\pi^2}{8} \\
 \therefore I &= \frac{\pi^2}{16}
 \end{aligned}$$

Example 8: Evaluate $\lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\frac{n+r}{n-r}}$

Sol: Here by using the limit as a sum method we can solve the given problem.

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\frac{n+r}{n-r}} \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\frac{1+r/n}{1-r/n}} = \int_0^1 \sqrt{\frac{1+x}{1-x}} dx \\
 &= \int_0^1 \frac{1+x}{\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{x dx}{\sqrt{1-x^2}} \\
 &= [\sin^{-1} x - \sqrt{1-x^2}]_0^1 \\
 &= [\sin^{-1} 1 - 0] - [\sin^{-1} 0 - 1] = \frac{\pi}{2} + 1
 \end{aligned}$$

Example 9: Integrate : $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\theta}{1 + \sin \theta} d\theta$

Sol: As $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ hence we can

write $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\theta}{1 + \sin \theta} d\theta$ as $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi - \theta}{1 + \sin \theta} d\theta$ and then

by putting $\theta = \frac{\pi}{2} + y$ we can solve the given problem.

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\theta}{1 + \sin \theta} d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi - \theta}{1 + \sin \theta} d\theta$$

$$2I = \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{d\theta}{1 + \sin \theta} ; \text{ Put } \theta = \frac{\pi}{2} + y$$

$$= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dy}{1 + \cos y} = 2\pi \int_0^{\frac{\pi}{4}} \frac{dy}{1 + \cos y}$$

$$I = \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sec^2 \frac{y}{2} dy = \pi \left[\tan \frac{y}{2} \right]_0^{\pi/4} = \pi \tan \frac{\pi}{8}$$

JEE Advanced/Boards

Example 1: Show that $1 < \int_0^1 e^{x^2} dx < e$.

Sol: e^{x^2} is an increasing function in $[0, 1]$. Further, $e^0 \leq e^{x^2} \leq e^1 \forall x \in [0, 1]$

$$\therefore \int_0^1 1 dx < \int_0^1 e^{x^2} dx < \int_0^1 e dx$$

$$\text{or } 1 < \int_0^1 e^{x^2} dx < e.$$

Example 2: If $F(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{4 + e^{2t}} dt$, find the critical points of $F(x)$.

Sol: By using Leibnitz rule we can write

$$F(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{4 + e^{2t}} dt,$$

$$\text{as } F'(x) = \frac{(x^2)^2 - 5x^2 + 4}{4 + e^{2x^2}} \cdot (2x) = 0.$$

By Leibnitz Rule,

$$F'(x) = \frac{(x^2)^2 - 5x^2 + 4}{4 + e^{2x^2}} \cdot (2x)$$

$$F'(x) = 0$$

$$\Rightarrow (x^4 - 5x^2 + 4) x = 0$$

$$\Rightarrow (x^2 - 4)(x^2 - 1) x = 0$$

$$\Rightarrow x = 0, \pm 1, \pm 2$$

These are the critical points of $F(x)$.

Example 3: Evaluate: $\int_0^{\pi/2} \log \sin x \, dx$

Sol: We can write $\int_0^{\pi/2} \log \sin x \, dx$

As $\int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$ and then by adding these two integration we can obtain the result.

$$\begin{aligned} I &= \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx \\ &= \int_0^{\pi/2} \log \cos x \, dx \\ \therefore 2I &= \int_0^{\pi/2} (\log \sin x + \log \cos x) dx = \int_0^{\pi/2} \log(\sin x \cos x) dx \\ &= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx \\ &= -\frac{\pi}{2} \log 2 + \int_0^{\pi/2} \log \sin 2x \, dx \quad (\text{Put } 2x = t) \\ &= -\frac{\pi}{2} \log 2 + \frac{1}{2} \int_0^{\pi} \log \sin t \, dt \\ &= -\frac{\pi}{2} \log 2 + \frac{1}{2} (2) \int_0^{\pi/2} \log \sin t \, dt. \end{aligned}$$

$$\therefore 2I = -\frac{\pi}{2} \log 2 + I \Rightarrow I = -\frac{\pi}{2} \log 2$$

Example 4: Evaluate: (i) $I = \int_1^3 (x^2 + x) dx$

(ii) $I = \int_a^b \sin x \, dx$ as limit of a sum.

Sol: By using the limit as a sum method we can solve the problems above.

(i) $f(x) = x^2 + x, a = 1, b = 3, nh = 3 - 1 = 2$

$$\begin{aligned} I &= \lim_{n \rightarrow \infty} h \sum_{r=1}^n f(a+rh) \\ &= \lim_{n \rightarrow \infty} h \sum_{r=1}^n ((a+rh)^2 + (a+rh)) \\ &= \lim_{n \rightarrow \infty} h \left(\sum_{r=1}^n r^2 h^2 + rh(2a+1) + (a^2 + a) \right) \\ &= \lim_{n \rightarrow \infty} h \left(\frac{n(2+h)(4+h)}{6} + (2a+1) \frac{n(2+h)}{2} + n(a^2 + a) \right) \end{aligned}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left(\frac{2(2+h)(4+h)}{6} + (2a+1) \frac{2(2+h)}{2} + 2(a^2 + a) \right) \\ &= \frac{8}{3} + 6 + 4 = \frac{38}{3} \end{aligned}$$

(ii) $I = \int_a^b \sin x \, dx$

$$\begin{aligned} nh &= b - a; \quad I = \lim_{h \rightarrow 0} h \left(\sum_{r=1}^n \sin(a+rh) \right) \\ &= \lim_{h \rightarrow 0} \frac{h}{\sin \frac{h}{2}} \left(\sum_{r=1}^n 2 \sin \frac{h}{2} \sin(a+rh) \right) \\ &= \lim_{h \rightarrow 0} \frac{h}{\sin \frac{h}{2}} \left(\sum_{r=1}^n \cos\left(a+hr - \frac{h}{2}\right) - \cos\left(a+hr + \frac{h}{2}\right) \right) \\ &= \lim_{h \rightarrow 0} \frac{h}{\sin \frac{h}{2}} \left(\cos\left(a + \frac{h}{2}\right) - \cos\left(a + nh + \frac{h}{2}\right) \right) \\ &= \cos a - \cos b \end{aligned}$$

Example 5: Evaluate $I = \int_{-1}^3 (|x-2| + 2[x]) dx$, where $[x]$ is the greatest integer function.

Sol: By putting $x-2 = y$ and it is negative in interval -3 to -1 and positive in interval 0 to 1.

$$I_1 = \int_{-1}^3 |x-2| dx; \text{ Put } x-2 = y$$

$$\begin{aligned} \int_{-3}^1 |y| dy &= \int_{-3}^{-1} -y \, dy + 2 \int_0^1 y \, dy \\ &= -\frac{1}{2} [y^2]_{-3}^{-1} + [y^2]_0^1 = 4 + 1 = 5 \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{-1}^3 [x] \, dx \\ &= \int_{-1}^0 -dx + \int_0^1 0 \, dx + \int_1^2 dx + \int_2^3 2 \, dx = -1 + 0 + 1 + 2 = 2 \\ \therefore I &= I_1 + 2I_2 = 9 \end{aligned}$$

Example 6: Show that $I = \int_0^{\infty} \frac{x \log x}{(1+x^2)^2} dx = 0$

Sol: By splitting the given integration into two intervals i.e. from 0 to 1 and then 1 to ∞ we can solve the given problem.

$$\int_0^{\infty} \frac{x \log x}{(1+x^2)^2} dx = \int_0^1 \frac{x \log x}{(1+x^2)^2} dx + \int_1^{\infty} \frac{x \log x}{(1+x^2)^2} dx$$

Put $x = 1/y$ in the second integral

$$\therefore \int_1^{\infty} \frac{x \log x}{(1+x^2)^2} dx = \int_1^0 \frac{y^4 \log y}{y^3(1+y^2)^2} dy = -\int_0^1 \frac{y \log y}{(1+y^2)^2} dy$$

$$\text{Thus } I = \int_0^1 \frac{x \log x}{(1+x^2)^2} dx - \int_0^1 \frac{y \log y}{(1+y^2)^2} dy = 0$$

Example 7: If $I = \int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1-x^2}\right) dx$, then find its value.

Sol: We can write $\cos^{-1}\left(\frac{2x}{1-x^2}\right)$ as $\cos^{-1}\left(\frac{-2x}{1-x^2}\right) = \left(\pi - \cos^{-1}\frac{2x}{1+x^2}\right)$ and then by solving we will get the result.

$$I = \int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1-x^2}\right) dx = \int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{-2x}{1-x^2}\right) dx$$

$$= \int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} \left(\pi - \cos^{-1}\frac{2x}{1+x^2}\right) dx$$

$$2I = \pi \int_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4} dx$$

$$I = \pi(-1) \int_0^{\frac{1}{\sqrt{3}}} \left(1 - \frac{1}{1-x^4}\right) dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-x^2} + \frac{1}{1+x^2} dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \cdot \frac{\pi}{6} + \frac{\pi}{4} \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-x} + \frac{1}{1+x} dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} + \frac{\pi}{4} \left(\log \left| \frac{1+x}{1-x} \right| \right)_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{\pi^2}{12} - \frac{\pi}{\sqrt{3}} + \frac{\pi}{4} \log \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right)$$

Example 8: Evaluate $\int_a^b (px + q) dx$ as a limit of a sum

Sol: Here as $f(x) = px + q$, therefore using the limit as sum method we can solve the given problem.

$$\begin{aligned} I &= \int_a^b (px + q) dx \\ &= \lim_{h \rightarrow 0} h[f(a) + f(a+h) + \dots + f(a+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h[(pa + q) + \{p(a+h) + q\} + \dots + \{p(a+(n-1)h) + q\}] \\ &= \lim_{h \rightarrow 0} h[p(a+a+\dots+a) + ph(1+2+\dots+(n-1)) + q(1+1+\dots+1)] \\ &= \lim_{h \rightarrow 0} h \left[pna + \frac{1}{2} pnh(n-1) + qn \right] \\ &= \lim_{h \rightarrow 0} \left[hpna + \frac{1}{2} pnh(nh-h) + qnh \right] \quad \dots(i) \end{aligned}$$

Since, $h = (b-a)/n$, or $nh = b-a$, we obtain from (i)

$$\begin{aligned} I &= \lim_{h \rightarrow 0} \left[(pa+q)(b-a) + \frac{p}{2}(b-a)(b-a-h) \right] \\ &= (pa+q)(b-a) + \frac{p}{2}(b-a)^2 \\ &= \frac{p}{2}(b-a)(2a+b-a) + q(b-a) \\ &= \frac{p}{2}(b^2-a^2) + q(b-a). \end{aligned}$$

Example 9: If $U_n = \int_0^{\pi} \frac{1 - \cos n\pi}{1 - \cos x} dx$ where n is a positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$.

Hence show that $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} d\theta = \frac{n\pi}{2}$

Sol: Here $U_n = U_{n+2} - U_{n+1}$ therefore by substituting $n+2$ and $n+1$ in place of n and solving we will get the required result.

$$\begin{aligned} \therefore U_n &= \int_0^{\pi} \frac{1 - \cos nx}{1 - \cos x} dx \\ \therefore U_{n+2} - U_{n+1} &= \int_0^{\pi} \frac{\{(1 - \cos(n+2)x)\} - \{1 - \cos(n+1)x\}}{(1 - \cos x)} dx \\ &= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{(1 - \cos x)} dx \end{aligned}$$

$$= \int_0^{\pi} \frac{2\sin(n + (3/2))x \sin(x/2)}{2\sin^2(x/2)}$$

$$\Rightarrow U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin(n + (3/2))x}{\sin(x/2)} dx$$

Similarly

$$\Rightarrow U_{n+1} - U_n = \int_0^{\pi} \frac{\sin(n + (1/2))x}{\sin(x/2)} dx \quad \dots(ii)$$

from (1) and (2), we get

$$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$$

$$= \int_0^{\pi} \frac{\sin(n + (3/2))x - \sin(n + (1/2))x}{\sin(x/2)}$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin(x/2)}{\sin(x/2)} dx = 2 \left\{ \frac{\sin(n+1)x}{(n+1)} \right\}_0^{\pi} = 0$$

$$\therefore U_{n+2} + U_n = 2U_{n+1}$$

Hence proved

$$\text{Now } U_{n+2} - U_{n+1} = U_{n+1} - U_n$$

Similarly implies

$$U_{n+2} - U_{n+1} = U_{n+1} - U_n = U_n - U_{n-1} = \dots = U_1 - U_0$$

$$\therefore U_n - U_{n-1} = U_1 - U_0 = \pi - 0$$

$$\Rightarrow U_n = \pi + U_{n-1}$$

$$= \pi + \pi + U_{n-2}$$

$$= 2\pi + U_{n-2}$$

$$U_n = n\pi + U_0 \dots(3) [\because U_0 = 0]$$

$$U_n = n\pi$$

$$\text{Hence } \therefore \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} d\theta = \int_0^{\pi/2} \frac{1 - \cos 2n\theta}{1 - \cos 2\theta} d\theta$$

$$\text{Put } 2\theta = x \therefore d\theta = \frac{dx}{2}$$

$$\text{Hence } \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} d\theta = \frac{1}{2} \int_0^{\pi} \frac{1 - \cos nx}{1 - \cos x} dx$$

$$= \frac{1}{2} U_n = \frac{1}{2} n\pi \quad \{\text{from (1)}\}$$

Example 10: Solve $\int_{-\pi/4}^{\pi/4} \frac{x + (\pi/4)}{2 - \cos 2x} dx$.

Sol: By splitting $\int_{-\pi/4}^{\pi/4} \frac{x + (\pi/4)}{2 - \cos 2x} dx$

$$= \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x} dx + \frac{\pi}{4} \int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x} dx \text{ and as we}$$

know $\left(\frac{x}{2 - \cos 2x}\right)$ is an odd function

$$\text{therefore } \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x} dx = 0.$$

$$\text{Therefore } 0 + \frac{\pi}{4} \int_0^{\pi/4} \frac{1}{2 - \cos 2x} dx$$

This is because $\left(\frac{x}{2 - \cos 2x}\right)$ is an odd function,

whereas $\left(\frac{1}{2 - \cos 2x}\right)$ is an even function

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{dx}{2 - ((1 - \tan^2 x) / (1 + \tan^2 x))}$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{(1 + \tan^2 x) dx}{2(1 + \tan^2 x) - (1 - \tan^2 x)} = \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2 x dx}{1 + 3\tan^2 x}$$

Now let $\tan x = t \therefore \sec^2 x dx = dt$

$$\Rightarrow \frac{\pi}{2} \int_0^1 \frac{dt}{1 + 3t^2} = \frac{\pi}{2\sqrt{3}} \left(\tan^{-1} \sqrt{3}t\right)_0^1 = \frac{\pi^2}{6\sqrt{3}}$$

Example 11: Show that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$$

Sol: Since $0 < x < 1$

$$\text{so } \frac{1}{\sqrt{4 - x^2}} < \frac{1}{\sqrt{4 - x^2 - x^3}} < \frac{1}{\sqrt{4 - 2x^2}}$$

Hence by using the property:

If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ we

can solve the given problem.

Integrate the above relation

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2}} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}}$$

$$\left(\sin^{-1} \frac{x}{2}\right)_0^1 < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{1}{\sqrt{2}} \left(\sin^{-1} \frac{x}{\sqrt{2}}\right)_0^1$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$$

Hence proved.

JEE Main/Boards

Exercise 1

Q.1 $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$

Q.2 $\int_0^{\pi/2} \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$

Q.3 $\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx$

Q.4 $\int_0^1 |5x - 3| dx$

Q.5 $\int_1^3 f(x) dx$, where $f(x) = \begin{cases} 2x+1, & 1 \leq x \leq 2 \\ x^2+1, & 2 \leq x \leq 3 \end{cases}$

Q.6 $\int_{-\pi/4}^{\pi/4} |\sin x| dx$

Q.7 $\int_0^{\pi} \frac{x}{(1 + \sin^2 x)} dx$

Q.8 Evaluate using limit of a sum: $\int_0^2 (x^2 + 1) dx$

Q.9 Evaluate: $\int_0^{\pi/2} |\sin x - \cos x| dx$

Q.10 If f and g are continuous function on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$ then, show that

$$\int_0^a f(x)g(x)dx = \int_0^a f(x)dx.$$

Q.11 Evaluate: $\int_0^{100\pi} \sqrt{1 - \cos 2x} dx$

Q.12 (i) Show that if $f(t)$ is an odd function then $\int_0^a f(t) dx$ is an even function w.r.t. x .

(ii) Can $\int_a^x f(t) dt$ be an odd function if $f(t) dt$ is an even function?

Q.13 If $f(x) = \int_0^{x^2} \sqrt{1+t^2} dt$, then find the value of $f'(x)$.

Q.14 Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\pi + 4x^2}{-\cos(|x| + (\pi/3))} dx$.

Q.15 If $f(x) = \int_1^x \frac{\log t}{1+t} dt$ then prove that

$$f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2.$$

Q.16 $\int_{\frac{1}{x}}^{2t} |\log x| dt$

Q.17 $\int_0^x \frac{\sin(n+(1/2)x)}{2\sin(x/2)} dx$, $n \in \mathbb{N}$.

Q.18 If $F(x) = \int_{\frac{5x}{4}}^x (3\sin t + 4\cos t) dx$. Find the least value of $F(x)$ on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

Q.19 If $I_A = \int_0^{\pi/4} \tan^n \theta d\theta$, $n \in \mathbb{N}$, then find $n(I_{n-1} + I_{n+1})$ and I_B .

Q.20 If "a" is a positive integer, solve for "a"

$$\int_0^a \left(a^2 \left(\frac{\cos 3x}{4} + \frac{3}{4} \cos x \right) + a \sin x - 20 \cos x \right) dx \leq \frac{-a^3}{3}.$$

Q.21 If $f(x) = \sin x$, then find its mean value on $(-2, 0)$.

Q.22 Evaluate $I = \int_0^{\pi} \frac{1}{x + \sqrt{a^2 - x^2}} dx$.

Q.23 Show that $I = \int_0^{a^2} [\sqrt{x}] dx = \frac{n(n-1)(4\pi+1)}{6}$, where $[x]$ is the greatest integer function.

Q.24 Show that $I = \int_0^{n\pi+\lambda} |\sin x| dx = 2n+1 - \cos \lambda$, $n \in \mathbb{N}$, $0 \leq \lambda < \pi$.

Q.25 Show that $I = \int_0^{\pi} \frac{\pi \sin 2x, \sin((\pi/2)\cos x)}{2x - \pi} dx = \frac{8}{x^2}$.

Q.26 Let f and g be function satisfying the following conditions:

- (i) $f(0) = 1$
- (ii) $f(x) = g(x), g'(x) = f(x)$
- (iii) $g(0) = 0$
- (iv) $g(x) \geq 0 \forall x \in \mathbb{R}$

Find $f(1)$.

Q.27 Show that

(i) $\int_0^{\pi} \log(1 + \cos x) dx = \pi \log(1/2)$;

(ii) $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{12}$

Q.28 Prove that

$$\int_0^{\pi} \frac{dx}{1 - 2a \cos x + x^2} - \frac{\pi}{1 - a^2} \text{ or } \frac{\pi}{a^2 - 1}; a > 0,$$

According as $a < 1$ or $a > 1$.

Q.29 (i) Evaluate $\lim_{n \rightarrow 0} \frac{\int_0^a x dx}{\alpha \sin \alpha}$

(ii) If $y = x \int_x^a \log dx$, Find $\frac{dy}{dx}$ at $x = e$.

Q.30 Find the intervals of increase of $f(x)$ defined by $f(x)$

$$= \int_0^{\alpha} (t^2 + 2t)(t^2 - 1) dt.$$

Exercise 2

Single Correct Choice Type

Q.1 $\int_{-1}^1 f(x) dx$ is equal to where $f(x) = \begin{cases} 1 - 2x, & x < 0 \\ 1 + 2x, & x \geq 0 \end{cases}$

- (A) 4
- (B) -4
- (C) 2
- (D) -2

Q.2 $\int_{-1}^1 e^{|x|} dx$ equals

- (A) $2e$
- (B) $2e - 1$
- (C) $2e - 2$
- (D) $e - 2$

Q.3 $\int_0^1 [x] dx$ equals ; where $[\cdot]$ is G.I.F.

- (A) 0
- (B) 2
- (C) 3
- (D) 1

Q.4 $\int_0^x |\cos x| dx$ equals

- (A) 1
- (B) 2
- (C) 3
- (D) 4

Q.5 $\int_{-2}^2 |2x + 3| dx$ equals

- (A) $\frac{25}{2}$
- (B) 0
- (C) $\frac{25}{4}$
- (D) $\frac{25}{3}$

Q.6 $\int_{-2}^2 |1 - x^2| dx =$

- (A) 2
- (B) 4
- (C) 6
- (D) 8

Q.7 The point of extremum of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$ are

- (A) $x = -2$
- (B) $x = 1$
- (C) $x = 0$
- (D) All of the above

Q.8 The point of intersection

$F_1(x) = \int_2^x (2t - 5) dt$ and $F_2(x) = \int_0^x 2t dt$, are -

- (A) $(\frac{6}{5}, \frac{36}{25})$
- (B) $(\frac{2}{3}, \frac{4}{9})$
- (C) $(\frac{1}{3}, \frac{1}{9})$
- (D) $(\frac{1}{5}, \frac{1}{25})$

Q.9 If f and g are continuous function on $[0, a]$ satisfying

$f(x) = f(a - x)$ and $g(x) + g(a - x) = 2$, then $I = \int_0^a f(x)g(x) dx =$

- (A) $\int_0^a f(x) dx$
- (B) $\int_a^0 f(x) dx$
- (C) $2 \int_0^a f(x) dx$
- (D) None of these

Q.10 The value of integral $\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} dx =$

- (A) $3 + 2p$
- (B) $4 - p$
- (C) $2 + p$
- (D) None of these

Q.11 The value of the integral $\int_{-\alpha}^{\pi} \sin mx \sin nx dx$ for $m \neq n$ ($m, n \in \mathbb{I}$), is -

- (A) 0
- (B) p
- (C) $\pi/2$
- (D) $2p$

Q.12 $\int_{1/e}^e |\log x| dx =$

- (A) $1 - \frac{1}{e}$ (B) $2\left(1 - \frac{1}{e}\right)$
 (C) $e^{-1} - 1$ (D) None of these

Q.13 $\int_0^\pi \frac{dx}{1 - 2a \cos x + a^2} =$

- (A) $\frac{\pi}{2(1 - a^2)}$ (B) $\pi(1 - a^2)$
 (C) $\frac{\pi}{1 - a}$ (D) None of these

Q.14 $\int_0^1 (1 - x)^9 dx =$

- (A) x (B) $\frac{1}{10}$ (C) $\frac{11}{10}$ (D) 2

Q.15 $\int_0^\pi \frac{dx}{\left(x + \sqrt{x^2 + 1}\right)^3} =$

- (A) $\frac{3}{8}$ (B) $\frac{1}{8}$ (C) $-\frac{3}{8}$ (D) None of these

Q.16 If $[x]$ denotes the greatest integer less than or equal to x , then the value $\int_1^5 [|x - 3|] dx$ is -

- (A) 1 (B) 2 (C) 4 (D) 8

Q.17 $\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1 + \cos^2 x} e^{-\cos^2 x} dx$ is equal to -

- (A) $2e^{-1}$ (B) 1 (C) 0 (D) None of these

Q.18 The value of

$\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2 \right]^{1/2} dx$ equal

- (A) $\log(4/3)$ (B) $2 \log(4/3)$
 (C) $4 \log(4/3)$ (D) $-4 \log(4/3)$

Q.19 Let $f(x) = x - [x]$, for every real number x , where $[x]$ is integral part of x . Then $\int_{-1}^1 f(x) dx$ is

- (A) 1 (B) 2 (C) 0 (D) $1/2$

Q.20 If $[x]$ stands for the greatest integer function, the value of $\int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]} dx$ is

- (A) 0 (B) 1 (C) 3 (D) None of these

Q.21 The value of $\int_{-1}^3 (|x - 2| + [x]) dx$ is ($[x]$ stands for greatest integer less than or equal to x)

- (A) 7 (B) 5 (C) 4 (D) 3

Q.22 $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ is equal to

- (A) $\frac{\pi}{2}$ (B) $\sqrt{2} \log(\sqrt{2} + 1)$
 (C) $\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$ (D) None of these

Q.23 If $u_{10} - \int_0^{\pi/2} x^{10} \sin x dx$ then the value of $u_{10} + 90 u_8$ is

- (A) $9\left(\frac{\pi}{2}\right)^8$ (B) $\left(\frac{\pi}{2}\right)^9$ (C) $10\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^9$

Q.24 For any integer n , the integral $\int_0^\pi e^{\sin^2 x} \cos^3(2n+1)x dx$ has the value

- (A) π (B) 1 (C) 0 (D) None of these

Q.25 The value of $\int_{-\pi/2}^{\pi/2} \sin(\log(x + \sqrt{x^2 + 1})) dx$ is

- (A) 1 (B) -1 (C) 0 (D) None of these

Q.26 The value of $\alpha \in (-\pi, 0)$ satisfying

$\sin \alpha + \int_\alpha^{2\alpha} \cos 2x dx = 0$ is

- (A) $-\pi/2$ (B) $-\pi$ (C) $-\pi/3$ (D) 0

Q.27 If $f(x) = \int_{x^2}^{x^4} \sin \sqrt{t} dt$, then $f^{-1}(x)$ equals

- (A) $\sin x^2 - \sin x$ (B) $4x^3 \sin x^2 - 2x \sin x$
 (C) $x^4 \sin x^2 - x \sin x$ (D) None of these

Q.28 $\int_0^{\pi} x \sin x \cos^4 x \, dx =$

- (A) $\frac{\pi}{10}$ (B) $\frac{\pi}{5}$ (C) $-\frac{\pi}{5}$ (D) None of these

Q.29 If $f(x) = ae^{2x} + be^x + cx$, satisfies the conditions $f(0) = -1$, $f'(\log 2) = 31$, $\int_0^{\log 4} (f(x) - cx) dx = \frac{39}{2}$, then

- (A) $a = 5, b = 6, c = 3$ (B) $a = 5, b = -6, c = 3$
 (C) $a = -5, b = 6, c = 3$ (D) None of these

Q.30 $\int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x \, dx}{e^{2x} - 1}$ is equal to

- (A) 0 (B) 2 (C) e (D) None of these

Q.31 $\int_{-1}^a \log_a (x + \sqrt{1+x^2}) \, dx$ is equal to

- (A) $2 \log_a a$ (B) 0
 (C) $\log_a 2 + \log a$ (D) None of these

Q.32 The value of $\int_{-2}^x \frac{\sin^2 t}{[(t/\pi)] + (1/2)} \, dt$, where $[x]$

- = the greatest integer less than or equal to x , is
 (A) 1 (B) 0 (C) $4 - \sin 4$ (D) None of these

Q.33 If $f(x) = \int_0^x \log(1+t^2) \, dt$ then the value of $f'(1)$ is equal to

- (A) 2 (B) 0 (C) 1 (D) None of these

Q.34 $\int_0^x \frac{dx}{1+3^{\cos x}}$ is equal to

- (A) π (B) 0 (C) $\frac{\pi}{2}$ (D) None of these

Previous Years' Questions

Q.1 The value of the integral

$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} \, dx$ is (1983)

- (A) $\pi/4$ (B) $\pi/2$ (C) π (D) None of these

Q.2 For any integer n , the integral

$\int_0^x e^{\cos^2 x} \cos^3(2n+1)x \, dx$ has the value (1985)

- (A) π (B) 1 (C) 0 (D) None of these

Q.3 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and

$f(1) = 4$. Then, the value of $\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} \, dt$ is (1990)

- (A) $8f'(1)$ (B) $4f'(1)$ (C) $2f'(1)$ (D) $f'(1)$

Q.4 The value of $\int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$ is (1993)

- (A) 0 (B) 1 (C) $\pi/2$ (D) $\pi/4$

Q.5 The value of $\int_0^{2\pi} [2 \sin x] \, dx$ where $[\cdot]$ represents the greatest integral function, is (1995)

- (A) $-\frac{5\pi}{3}$ (B) $-\pi$ (C) $\frac{5\pi}{3}$ (D) -2π

Q.6 $\int_0^x f(t) \, dt + x + \int_x^1 t f(t) \, dt$, then the value of $f(1)$ is (1998)

- (A) $\frac{1}{2}$ (B) 0 (C) 1 (D) $-\frac{1}{2}$

Q.7 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to (1999)

- (A) 2 (B) -2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Q.8 If for a real number y , $[y]$ is the greatest integer less than or equal to y , then the value of the integral

$\int_{\pi/2}^{3\pi/2} [2 \sin x] \, dx$ is (1999)

- (A) $-\pi$ (B) 0 (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

Q.9 The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} \, dx$, $a > 0$, is (2001)

- (A) π (B) $a\pi$ (C) $\frac{\pi}{2}$ (D) 2π

Q.10 Let $f: (0, \infty) \rightarrow \mathbb{R}$ and $F(x) = \int_0^x f(t) \, dt$, If

$F(x^2) = x^2(1+x)$, then $f(4)$ equals (2001)

- (A) $\frac{5}{4}$ (B) 7 (C) 4 (D) 2

Q.11 Let $f(x) = \int_1^x \sqrt{2-t^2} \, dt$. Then, the real value of x if it satisfies $x^2 - f'(x) = 0$ are (2002)

- (A) ± 1 (B) $\pm \frac{1}{\sqrt{2}}$ (C) $\pm \frac{1}{2}$ (D) 0 and 1

Q.12 Let $T > 0$ be a fixed real number. Suppose, f is a continuous function such that for all $x \in \mathbb{R}$, $f(x + T) = f(x)$. If $I = \int_0^T f(x) dx$, then the value of $\int_3^{3+3T} f(2x) dx$, (2002)

- (A) $\frac{3}{2}I$ (B) I (C) $3I$ (D) $6I$

Q.13 If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in (2003)

- (A) $(2, 2)$ (B) No value of x (C) $(0, \infty)$ (D) $(-\infty, 0)$

Q.14 The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$ is (2004)

- (A) $\frac{\pi}{2} + 1$ (B) $\frac{\pi}{2} - 1$ (C) -1 (D) 1

Q.15 Match the conditions expressions in column I with statement in column II (2007)

Column I	Column II
(A) $\int_{-1}^1 \frac{dx}{1+x^2}$	(p) $\frac{1}{2} \log\left(\frac{2}{3}\right)$
(B) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(q) $2 \log\left(\frac{2}{3}\right)$
(C) $\int_2^3 \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

Q.16 The value of $\int_{-2}^2 |1-x^2| dx$ is.... (1989)

Q.17 The value of $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx$ (1993)

Q.18 The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$ is.... (1994)

Q.19 Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$.

Then which one of the following is true? (2008)

- (A) $I > \frac{2}{3}$ and $J > 2$ (B) $I < \frac{2}{3}$ and $J < 2$
 (C) $I < \frac{2}{3}$ and $J > 2$ (D) $I > \frac{2}{3}$ and $J < 2$

Q.20 $\int_0^x [\cot x] dx, [\cdot]$ denotes the greatest integer function, is equal to (2009)

- (A) $\frac{\pi}{2}$ (B) 1 (C) -1 (D) $-\frac{\pi}{2}$

Q.21 Let $p(x)$ be a function defined on \mathbb{R} such that $p(x) = p(1-x)$ for all x $p(0) = 1$ $p(1) = 41$. Then $\int_0^1 p(x) dx$ equals. (2010)

- (A) 21 (B) 41 (C) 42 (D) $\sqrt{41}$

Q.22 The value of $\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$ is (2011)

- (A) $\frac{\pi}{8} \log 2$ (B) $\frac{\pi}{2} \log 2$ (C) $\log 2$ (D) $\pi \log 2$

Q.23 If $g(x) = \int_0^x \cos 4t dt$, then $g(x+\pi)$ equals (2012)

- (A) $\frac{g(x)}{g(\pi)}$ (B) $g(x) + g(\pi)$
 (C) $g(x) - g(\pi)$ (D) $g(x) \cdot g(\pi)$

Q.24 Statement-I: The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$ is equal to $\pi/6$.

Statement-II: $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ (2013)

- (A) Statement-I is true; statement-II is true; statement-II is a correct explanation for statement-I.
 (B) Statement-I is true; statement-II is true; statement-II is not a correct explanation for statement-I.
 (C) Statement-I is true; statement-II is false.
 (D) Statement-I is false; statement-II is true.

Q.25 The integral $\int \left(1+x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx$ is equal to (2014)

- (A) $(x+1)e^{x+\frac{1}{x}} + c$ (B) $-xe^{x+\frac{1}{x}} + c$
 (C) $(x-1)e^{x+\frac{1}{x}} + c$ (D) $xe^{x+\frac{1}{x}} + c$

Q.26 The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)} dx$ is equal to (2015)

- (A) 1 (B) 4 (C) 1 (D) 6

JEE Advanced/Boards

Exercise 1

Q.1 $\int_0^1 e^{\tan^{-1} x} \sin^{-1}(\cos x) dx.$

Q.2 Prove that :

(i) $\int_0^1 \sqrt{(x-\alpha)(\beta-x)} dx - \frac{(\beta-\alpha)^2 x}{8}$

(ii) $\int_0^a \sqrt{\frac{x-\alpha}{\beta-x}} dx = (\beta-\alpha) \frac{\pi}{2}$

(iii) $\int_0^a \frac{dx}{x\sqrt{(x-\alpha)(\beta-x)}} = \frac{\pi}{\sqrt{\alpha\beta}}$ where $\alpha, \beta > 0$

(iv) $\int_0^b \frac{x dx}{\sqrt{(x-\alpha)(\beta-x)}} = (\alpha+\beta) \frac{\pi}{2}$ where $\alpha < \beta$

Q.3 (i) Let $\beta(\Pi) = \int_0^{n\pi} \sqrt{1-\sin t} dt.$

Find the value of $\beta(2) - \beta(1).$

(ii) Determine a positive integer $n \leq 5$, such that

$\int_0^1 e^x (x-1)^n dx = 16 - 6e.$

Q.4 (i) $\int_0^{\pi/2} e^x \left[\cos(\sin x) \cos^2 \frac{x}{2} + \sin(\sin x) \sin^2 \frac{x}{2} \right] dx$

(ii) $\int_0^{\pi} \left\{ (1+x)e^x + (1-x)e^{-x} \right\} \ln x dx.$

Q.5 If $P = \int_0^{\infty} \frac{x^2}{1+x^4} dx; Q = \int_0^{\infty} \frac{x dx}{1+x^4}$ and $R = \int_0^{\infty} \frac{dx}{1+x^4}$

then prove that

(i) $Q = \frac{\pi}{4},$

(ii) $P = R$

(iii) $P - \sqrt{2} Q + R = \frac{\pi}{2\sqrt{2}}$

Q.6 $\int_1^2 \frac{(x^2-1)dx}{x^2 \sqrt{2x^4-2x^2+1}} = \frac{u}{v}$ where u and v are

in their lowest form. Find the value of $\frac{(1000)u}{v}$

Q.7 Let $h(x) = (f \circ g)(x) + K$ where K is any constant. If $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^{-2}(\cos x)}$ then

compute the value of $j(0)$ where $j(x)$

$\int \frac{f(t)}{g(t)} dt$, where f and g are trigonometric functions.

Q.8 $\int_0^{\pi/2} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}} dx$

Q.9 If the value of the definite integral $I = \int_0^2 (3x^2 - 3x + 1) \cos(x^3 - 3x^3 + 4x - 2) dx$ can be expressed in the form as $p(\sin q)$ where $p, q \in \mathbb{N}$, then find $(p + q).$

Q.10 $\int_{-\sqrt{2}}^{\sqrt{3}} \frac{2x^7 + 3x^6 - 10x^5 - 7x^3 - 12x^2}{x^2 + 2} dx.$

Q.11 For $a \geq 2$, if the value of the definite integral

$\int_0^a \frac{dx}{a^2 + (x-(1x))^2}$ equals $\frac{x}{5050}$. Find the value of $a.$

Q.12 $\int_{-2}^2 \frac{x^2 - x}{\sqrt{x^2 + 4}} dx.$

Q.13 Let $u = \int_0^{\pi/4} \left(\frac{\cos x}{\sin x - \cos x} \right)^2 dx$ and

$v = \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\cos x} \right)^2 dx.$ Find the value of $\frac{v}{u}$

Q.14 $\int_0^{\pi/4} \frac{x dx}{\cos x (\cos x + \sin x)}.$

Q.15 $\int_0^1 \frac{\sin^{-1} \sqrt{x}}{x^2 - x + 1} dx$

Q.16 $\int_1^{1+\sqrt{5}} \frac{x^2 + 1}{x^4 - x^2 + 1} \ln \left(1 + x - \frac{1}{x} \right) dx$

Q.17 $\lim_{x \rightarrow 0} n^2 \int_{-1\pi}^{1\pi} (2010 \sin x + 2012 \cos x) |x| dx.$

Q.18 Find the value of the definite integral

$$\int_0^{\pi} |\sqrt{2} \sin x + 2 \cos x| dx.$$

Q.19 If $\int_0^{\pi} \sqrt{\frac{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}{dx}}$

has the value equal to $\left(\frac{\pi}{k} + \sqrt{w}\right)$. w are positive integer. Find the value of $(k^2 + w^2)$.

Q.20 $\int_0^1 \frac{1-x}{1+x} \frac{dx}{\sqrt{x+x^2+x^3}}$

Q.21 $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin\left(\frac{\pi}{4} + x\right)} dx.$

Q.22 A continuous real function f satisfies $f(2x) = 3 f(x)$ $\forall x \in \mathbb{R}$.

If $\int_0^{\pi} f(x) dx = 1$, then compute the value of definite integral $\int_1^2 f(x) dx$.

Q.23 The value of $\int_{-1}^3 \{ |x-2| + [x] \} dx$, where $[x]$ denotes the greatest integer less than or equal to x is.

Q.24 $\int_1^0 \sin^{-1} \frac{2x}{1+x^2} dx.$

Q.25 $\int_0^1 \frac{(ax+b) \sec x \tan x}{4 + \tan^2 x} dx$ ($a, b > 0$)

Q.26 $\int_0^{\pi} \frac{(2x+3) \sin x}{(1 + \cos^2 x)} dx.$

Q.27 Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x + \sqrt{\sin x}}}$

Q.28 If $\int_0^{n\pi} \frac{x |\sin x|}{1 + |\cos x|} dx$ ($n \in \mathbb{N}$) is equal to $100 \pi \log 2$, then the value of n .

Q.29 Evaluate $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx.$

Q.30 $\int_0^{\pi} \frac{\ln(1+ax)}{1+x^2} dx$, $a \in \mathbb{N}$.

Q.31 $\int_0^{\ln 3} \frac{e^x + 1}{e^{2x} + 1} dx.$

Q.32 If $\int_0^a \sqrt{x} dx = 2a \int_0^{\pi/2} \sin^3 x dx$, find the value of $\int_a^{a+1} x dx$.

Q.33 Let α, β be the distinct positive roots of the equation $\tan x = 2x$ then evaluate $\int_0^{\pi} (\sin_{\alpha} x \cdot \sin_{\beta} x) dx$, independent of α and β .

Q.34 Show that $\int_0^{p+q} |\cos x| dx = 2q + \sin p$ where $q \in \mathbb{N}$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$.

Q.35 Show that the sum of the two integrals

$$\int_{-1}^{-\pi} e^{(x+1)^2} dx + 3 \int_{1/3}^{2/3} e^{(x-2x)^2} dx$$
 is zero.

Q.36 Let $F(x) = \max(\sin px, \cos px)$. Find the value of

$$\frac{\pi}{4\sqrt{2}} \int_{-10}^{10} F(x) dx.$$

Q.37 $\int_0^{\pi/2} \tan^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] dx.$

Q.38 Comment upon the nature of roots of the quadratic

equation $x^2 + 2x = k + \int_0^1 |t+k| dt$ dependent on the value of $k \in \mathbb{R}$.

Q.39 $\int_{-1}^1 \frac{(2x^{232} + x^{998} + 4x^{1668} \sin x^{691})}{1+x^{666}}$

Q.40 $\int_0^{\pi} \frac{x^2 \sin 2x \cdot \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx$

Q.41 Evaluate $\int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4} dx$

Q.42 $\lim_{x \rightarrow \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left[k \int_k^{k+1} \sqrt{(x-k)(k+1-x)} dx \right]$

Q.43 Let $I = \int_0^{\pi/2} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} dx$ and

$I = \int_0^{\pi/2} \frac{\sin x + 3}{3 \sin x + 4 \cos x + 25} dx.$

If $25I = a\pi + b \ln \frac{c}{d}$ where a, b, c and $d \in \mathbb{N}$ and $\frac{c}{d}$ is not a perfect square of a rational then find the value of $(a + b + c + d).$

Q.44 Let $y = f(x)$ be a quadratic function with $f(2) = 1.$ Find the value of the integral

$\int_{-2-\pi}^{2+\pi} f(x) \cdot \sin\left(\frac{x-2}{2}\right) dx.$

Exercise 2

Single Correct Choice Type

Q.1 $\int_0^2 |x^2 + 2x - 3| dx$ equals

- (A) 5/3 (B) 7/3 (C) 4 (D) 0

Q.2 The correct evaluation of $\int_0^{\pi/2} \left| \sin\left(x - \frac{\pi}{4}\right) \right| dx$ is -

- (A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$ (C) $-2 + \sqrt{2}$ (D) 0

Q.3 The correct evaluation of $\int_0^{\pi} |\sin^4 x| dx$ is -

- (A) $\frac{8\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{3\pi}{8}$

Q.4 $\int_0^{1.5} [x^2] dx,$ where $[\cdot]$ denotes the greatest integer function, equals -

- (A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$
(C) $-2 + \sqrt{2}$ (D) $-2 - \sqrt{2}$

Q.5 Solve $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

- (A) $\frac{\pi^2}{2ab}$ (B) $\frac{\pi^2}{4ab}$
(C) $\frac{\pi^2}{3ab}$ (D) $\frac{\pi}{5ab}$

Q.6 $\int_0^{\pi/4} \frac{\sec x}{1 + 2 \sin^2 x}$ is equal to -

- (A) $\frac{1}{3} \left| \log(\sqrt{2} + 1) + \frac{\pi}{2\sqrt{2}} \right|$ (B) $\frac{1}{3} \left| \log(\sqrt{2} + 1) - \frac{\pi}{2\sqrt{2}} \right|$
(C) $3 \left| \log(\sqrt{2} + 1) - \frac{\pi}{2\sqrt{2}} \right|$ (D) $3 \left| \log(\sqrt{2} + 1) + \frac{\pi}{2\sqrt{2}} \right|$

Q.7 If $\int_0^1 e^{x^2} (x - \alpha) dx = 0,$ then

- (A) $1 < \alpha < 2$ (B) $\alpha < 0$
(C) $0 < \alpha < 1$ (D) None of these

Q.8 $\int_b^{\pi/2} \{x - [\sin x]\} dx$ is equal to -

- (A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{8} - 1$ (C) $\frac{\pi^2}{8} - 2$ (D) None of these

Q.9 The value of the integral $\int_b^{100} \sin\{x - [x]\} \pi dx$ is -

- (A) $\frac{100}{\pi}$ (B) $\frac{200}{\pi}$ (C) 100π (D) 200π

Q.10 The value of the integral $\int_0^{\infty} \frac{x \log x}{(1+x^2)^2} dx$ is -

- (A) 1 (B) 0 (C) 2 (D) None of these

Q.11 $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{1/n}$ is equal to -

- (A) $e/4$ (B) $4/e$ (C) $2/e$ (D) None of these

Q.12 The solution of the equation

$\int_{\log^2}^x \frac{1}{\sqrt{e^x - 1}} dx = \frac{\pi}{6}$

- (A) $x = \log 4$ (B) $x = \log 2$
(C) $x = \log \left(\frac{1}{4}\right)$ (D) None of these

Q.13 The value of

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} + \dots + n \text{ terms} \right) \text{ is -}$$

- (A) $\frac{1}{3} \log 2$ (B) 0 (C) $\frac{1}{3} \log 3$ (D) None of these

Q.14 $\lim_{n \rightarrow \infty}$

$$\left\{ \frac{n^2}{(n^2+1^2)^{3/2}} + \frac{n^2}{(n^2+2^2)^{3/2}} + \dots + \frac{n^2}{[n^2+(n-1)^2]^{3/2}} \right\} \text{ is}$$

equal -

- (A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) None of these

Q.15 $\lim_{n \rightarrow \infty} \left\{ \frac{\sqrt{n}}{\sqrt{n^2}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + \dots + \frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}} \right\}$

is equal -

- (A) $\frac{1}{10} [5 - \sqrt{5}]$ (B) $[5 - \sqrt{5}]$ (C) $\frac{1}{5} [5 - \sqrt{5}]$ (D) 0

Q.16 $\lim_{n \rightarrow \infty} \left\{ \tan\left(\frac{\pi}{2n}\right) \tan\left(\frac{2\pi}{2n}\right) \tan\left(\frac{3\pi}{2n}\right) \dots \tan\left(\frac{n\pi}{2n}\right) \right\}^{\frac{1}{6}}$

is equal -

- (A) 0 (B) 1 (C) -1 (D) 2

Previous Years' Questions

Q.1 The integral $\int_{1/2}^{1/2} \left([x] + \log\left(\frac{1+x}{1-x}\right) \right) dx$ equals **(2002)**

- (A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $\log\left(\frac{1}{2}\right)$

Q.2 If $I(m, n) = \int_0^1 t^m (1+t)^n dt$, then the expression for $I(m, n)$ in terms of $I(m+1, n-1)$ is **(2003)**

(A) $\frac{2^n}{m+1} - \frac{n}{m+1} I(m+1, n-1)$

(B) $\frac{n}{m+1} I(m+1, n-1)$

(C) $\frac{2^n}{m+1} + \frac{n}{m+1} I(m+1, n-1)$

(D) $\frac{m}{m+1} I(m+1, n-1)$

Q.3 Let f be a non-negative function defined

on the interval $[0, 1]$. If $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt$, $0 \leq x \leq 1$ and $f(0) = 0$, then **(2009)**

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Q.4 The value of

$$\int_{\sqrt{\log 3}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx \text{ is - } \quad \textbf{(2011)}$$

- (A) $\frac{1}{4} \log \frac{3}{2}$ (B) $\frac{1}{2} \log \frac{3}{2}$ (C) $\log \frac{3}{2}$ (D) $\frac{1}{6} \log \frac{3}{2}$

Q.5 Let $S_n = \sum_{k=0}^n \frac{n}{n^2 + kn + k^2}$ and

$$T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}, \text{ for } n = 1, 2, 3, \dots, \text{ then } \quad \textbf{(2008)}$$

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Q.6 If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$, $n = 0, 1, 2, \dots$, then **(2009)**

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

Q.7 The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are) **(2010)**

- (A) $\frac{22}{7} - \pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

Paragraph for Q.8

Read the following passage and answer the questions. For every function $f(x)$ which is twice differentiable, these will be good approximation of

$$\int_a^b f(x)dx = \left(\frac{b-a}{2}\right)\{f(a) + f(b)\},$$

for more accurate results for $c \in (a, b)$,

$$F(c) = \frac{c-a}{2}[f(a) - f(c)] + \frac{b-c}{2}[f(b) - f(c)]$$

When $c = \frac{a+b}{2}$

$$\int_a^b f(x)dx = \frac{b-a}{4}\{f(a) + f(b) + 2f(c)\}dx \quad (2006)$$

Q.8 Good approximation of $\int_0^{\pi/2} \sin x dx$, is **(2003)**

- (A) $\pi/4$ (B) $\pi(\sqrt{2} + 1) / 4$
 (C) $\pi(\sqrt{2} + 1) / 8$ (D) $\pi/8$

Q.9 If $f''(x) < 0$, " $x \in (a, b)$, and $(c, f(c))$ is point of maxima where $c \in (a, b)$, then $f'(c)$ is - **(2009)**

- (A) $\frac{f(b) - f(a)}{b - a}$ (B) $3\left(\frac{f(b) - f(a)}{b - a}\right)$
 (C) $2\left(\frac{f(b) - f(a)}{b - a}\right)$ (D) 0

Q.10 If $\lim_{t \rightarrow a} \frac{\int_a^1 f(x)dx - ((t-a)/2)\{f(t) + f(a)\}}{(t-a)^3} = 0$, then degree of polynomial function $f(x)$ at most is - **(2002)**

- (A) 0 (B) 1 (C) 3 (D) 2

Q.11 For any real number x , let $[x]$ denotes the largest integer less than or equal to x . Let f be a real valued function defined on the interval

$$[-10, 10] \text{ by } f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x dx$ is **(2010)**

Q.12 For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t} dt$. Find the function $f(x) + f(1/x)$ and show that $f(e) + f(1/e) = 1/2$. Here, $\ln t = \log_e t$ **(2000)**

Q.13 If f is an even function, then prove that $\int_0^{\pi/2} f(\cos 2x) \cos x dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x dx$ **(2003)**

Q.14 Evaluate $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos(|x| + (\pi/3))} dx$. **(2004)**

Q.15 Evaluate $\int_0^{\pi} e^{|\cos x|} \left(2 \sin\left(\frac{1}{2} \cos x\right) + 3 \cos\left(\frac{1}{2} \cos x\right) \right) \sin x dx$ **(2005)**

Q.16 The value of $\frac{(5050) \int_0^1 (1-x^{50})^{100} dx}{\int_0^1 (1-x^{50})^{101} dx}$ **(2006)**

Q.17 Let $g(x) = \int_0^x \frac{f'(t)}{1+t^2} dt$, then which of the following is true? **(2008)**

- (A) $g(x)$ is positive on $(-\infty, 0)$ and negative on $(\infty, 0)$
 (B) $g(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
 (C) $g(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
 (D) $g(x)$ does not change sign on $(-\infty, \infty)$

Q.18 $\int_{-1}^1 g'(x) dx =$ **(2008)**

- (A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

Q.19 The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is **(2016)**

Q.20 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function which satisfies $f(x) = \int_0^x f(t) dt$. Then the value of $f(\ln 5)$ is **(2009)**

Q.21 Let f be a non-negative function defined on the interval $[0, 1]$ $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt$ $0 \leq x \leq 1$ and $f(0) = 0$ then **(2009)**

- (A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$
 (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$
 (C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$
 (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Q.22 Match the statements/expressions in column I with the open intervals in column II.

(2009)

Column I	Column II
(A) Interval contained in the domain of definition of non-zero	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4) dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{\pi}{2}\right)$
(D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{2}\right)$
	(t) $(-\pi, \pi)$

Q.23 Match the statements in column I with those in column II.

(2010)

Column I	Column II
(A) A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ and $\frac{x-\frac{8}{2}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ = d Then d^2 is	(p) -4
(B) The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$	(q) 0
(C) Non-zero vectors \vec{a}, \vec{b} and \vec{c} satisfy $\vec{a} \cdot \vec{c} = 0$ and $(\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{c}) = 0$ and possible values of are	(r) 4
(D) Let f be the function on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \sin\left(\frac{9x}{2}\right) / \sin\left(\frac{x}{2}\right)$ $x \neq 0$ The value of $\frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx$ is	(r) 5
	(s) 6

Q.24 The value of $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t \log(1+t)}{t^4 + 4} dt$ is **(2010)**

- (A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

Q.25 The value (s) of $\int_0^1 \frac{x^{4(1-x)^4}}{1+x^2} dx$ is (are) **(2010)**

- (A) $\frac{22}{7} - \pi$ (B) $\frac{2}{105}$
 (C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

Q.26 The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$ is **(2011)**

- (A) $\frac{1}{4} \log \frac{3}{2}$ (B) $\frac{1}{2} \log \frac{3}{2}$ (C) $\log \frac{3}{2}$ (D) $\frac{1}{6} \log \frac{3}{2}$

Q.27 The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \log \frac{\pi+x}{\pi-x}\right) \cos x dx$ is **(2012)**

- (A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$

Q.28 The following integral $\int_{\pi/4}^{\pi/2} (2\operatorname{cosec}x)^{17} dx$ is equal to **(2014)**

- (A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$
 (B) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$
 (C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{16} du$
 (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

Q.29 Match the following: **(2014)**

List I	List II
(i) The number of polynomials $f(x)$ with non negative integer coefficients of degree ≤ 2 satisfying $f(0) = 0$ and $\int_0^1 f(x) dx = 1$, is	(p) 8
(ii) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(q) 2
(iii) $\int_{-2}^2 \frac{3x^2}{(1+e^x)} dx$ equals	(r) 4
(iv) $\frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}{\int_0^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}$ equals	(s) 0

- Codes:** i ii iii iv
 (A) r q s p
 (B) q r s p
 (C) r q p s
 (D) q r p s

Q.30 The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2) \right\} dx$ is _____ **(2014)**

Q.31 If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x} \right) \left(\frac{12+9x^2}{1+x^2} \right) dx$

Where \tan^{-1} takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4} \right)$ is **(2015)**

Q.32 The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at) dt} = L?$$

(A) $a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (B) $a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$
 (C) $a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (D) $a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

Q.33 The correct statement(s) is(are) **(2015)**

- (A) $f'(1) < 0$
 (B) $f'(2) < 0$
 (C) $f'(x) \neq 0$ for any $x \in (1, 3)$
 (D) $f(x) = 0$ for some $x \in (1, 3)$

Q.34 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$f(x) = \begin{cases} [x] & x \leq 2 \\ 0 & x > 2 \end{cases}$ where $[x]$ is the greatest integer less than or equal to x .

If $I = \int_{-1}^2 \frac{xf(x^2)}{2+f(x+1)} dx$, then the value of $(4I - 1)$ is **(2015)**

Q.35 The value of $\int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$ is equal to **(2016)**

- (A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$
 (C) $\pi^2 - e^{\pi/2}$ (D) $\pi^2 + e^{\pi/2}$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.3 Q.8 Q.12
 Q.17 Q.21 Q.23
 Q.26 Q.28

Exercise 2

Q.9 Q.12 Q.17
 Q.20 Q.23 Q.29
 Q.32 Q.34

Previous Years' Questions

Q.4 Q.8 Q.11

JEE Advanced/Boards

Exercise 1

Q.2 Q.7 Q.10
 Q.15 Q.22 Q.27
 Q.32 Q.34 Q.44

Exercise 2

Q.2 Q.7 Q.10
 Q.12 Q.15

Previous Years' Questions

Q.1 Q.4 Q.6
 Q.7 Q.10 Q.15

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\frac{\pi}{6}$

Q.4 $\frac{13}{10}$

Q.7 $\frac{\pi^2}{2\sqrt{2}}$

Q.11 $200\sqrt{2}$

Q.14 $\frac{4\pi}{\sqrt{3}} \tan^{-1}\left(\frac{1}{2}\right)$

Q.18 $\frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$

Q.2 $\frac{\pi}{4\sqrt{5}}$

Q.5 $\frac{34}{3}$

Q.8 $\frac{14}{3}$

Q.12 (ii) Not necessary

Q.16 $2 - \frac{2}{e} + 2e \log 2$

Q.19 $1, \frac{5}{12} - \frac{1}{2} \log 2$

Q.3 $\frac{\pi}{3\sqrt{3}}$

Q.6 $2 - \sqrt{2}$

Q.9 $2(\sqrt{2} - 1)$

Q.13 $2x\sqrt{1+x^4}$

Q.17 $\frac{\pi}{2}$

Q.20 a = 1, 2, 3 or 4

Q.21 -1	Q.22 $\frac{\pi}{4}$	Q.26 $\frac{e^2 + 1}{2e}$
Q.29 (i) $\frac{1}{2}$; (ii) $1+e$	Q.30 $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$	

Exercise 2

Single Correct Choice Type

Q.1 A	Q.2 C	Q.3 C	Q.4 B	Q.5 A	Q.6 B
Q.7 D	Q.8 A	Q.9 A	Q.10 B	Q.11 A	Q.12 B
Q.13 C	Q.14 B	Q.15 A	Q.16 B	Q.17 C	Q.18 C
Q.19 A	Q.20 C	Q.21 A	Q.22 C	Q.23 C	Q.24 C
Q.25 C	Q.26 C	Q.27 B	Q.28 B	Q.29 B	Q.30 A
Q.31 B	Q.32 C	Q.33 C	Q.34 C		

Previous Years' Questions

Q.1 A	Q.2 C	Q.3 A	Q.4 D	Q.5 A	Q.6 A
Q.7 A	Q.8 C	Q.9 C	Q.10 C	Q.11 A	Q.12 C
Q.13 D	Q.14 B	Q.15 $A \rightarrow s; B \rightarrow s; C \rightarrow p; D \rightarrow r$		Q.16 4	Q.17 $\pi(\sqrt{2}-1)$
Q.18 $\frac{1}{2}$	Q.19 B	Q.20 D	Q.21 A	Q.22 D	Q.23 B C
Q.24 D	Q.25 D	Q.26 C			

JEE Advanced/Boards

Exercise 1

Q.1 $\frac{\pi^2}{8} - \frac{\pi}{4}(1 + \log 2) + \frac{1}{2}$	Q.3 (a) 4 (b) $n = 3$	
Q.4 (i) $\frac{1}{2}[e^{\pi/2}(\cos 1 + \sin 1) - 1]$ (ii) $e^{1+e} + e^{1-e} + e^{-e} - e^e + e - e^{-1}$		Q.6 125
Q.7 $1 - \sec(1)$	Q.8 $\ln 2$	Q.9 4
Q.10 $\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$	Q.11 2525	Q.12 $4\sqrt{2} - 4\ln 1 + \sqrt{2} $
Q.13 4	Q.14 $\frac{\pi}{8}\log 2$	Q.15 $\frac{\pi^2}{6\sqrt{3}}$
Q.16 $\frac{\pi}{8}\log 2$	Q.17 2012	Q.18 $2\sqrt{6}$
Q.19 153	Q.20 $\frac{\pi}{3}$	Q.21 $\frac{\pi(a+b)}{2\sqrt{2}}$

Q.22 5

Q.23 90

Q.24 $\frac{\pi}{\sqrt{3}}$

Q.25 $\frac{(a\pi + 2b)\pi}{3\sqrt{3}}$

Q.26 $\frac{\pi(\pi+3)}{2}$

Q.27 $= \frac{1}{2} [x]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}$

Q.28 10

Q.29 $\frac{1}{2} \left[\frac{\pi}{2} - \log 2 \right]$

Q.30 $\tan^{-1}(a) \cdot \log \sqrt{1+a^2}$

Q.31 $\frac{1}{2} \left[\frac{\pi}{6} + \log 3 - \log 2 \right]$

Q.32 $\frac{9}{2}$

Q.33 0

Q.36 5

Q.37 $\frac{3\pi^2}{16}$

Q.38 Real and distinct $\forall k \in \mathbb{R}$

Q.39 $\frac{\pi+4}{666}$

Q.40 8

Q.41 6

Q.42 $\frac{\pi}{16}$

Q.43 62

Q.44 $I = 8$ as $\int_0^{\pi/2} y \sin y dy = 1$

Exercise 2**Single Correct Choice Type**

Q.1 C

Q.2 B

Q.3 D

Q.4 B

Q.5 A

Q.6 A

Q.7 C

Q.8 A

Q.9 B

Q.10 B

Q.11 B

Q.12 A

Q.13 A

Q.14 B

Q.15 A

Q.16 B

Previous Years' Questions

Q.1 A

Q.2 A

Q.3 C

Q.4 A

Q.5 A, D

Q.6 A, B, C

Q.7 A

Q.8 C

Q.9 A

Q.10 B

Q.11 4

Q.12 $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\log e)^2 = \frac{1}{2}$

Q.13 $I = \sqrt{2} \int_0^{\pi/4} f(\sin 2t) \cos t dt$

Q.14 $\frac{4\pi}{\sqrt{3}} \tan^{-1}\left(\frac{1}{2}\right)$

Q.15 $\frac{24}{5} \left(e \cos\left(\frac{1}{2}\right) + \frac{e}{2} \sin\left(\frac{1}{2}\right) - 1 \right)$

Q.16 5051

Q.17 B

Q.18 D

Q.19 A

Q.20 0

Q.21 C

Q.22 $A \rightarrow p, q, s; B \rightarrow p, t; C \rightarrow p, q, r, t; D \rightarrow s$

Q.23 A

Q.24 B

Q.25 A

Q.26 A

Q.27 B

Q.28 A

Q.29 D

Q.30 2

Q.31 9

Q.32 A, C

Q.33 A, B, C

Q.34 -2

Q.35 A

Solutions

JEE Main/Boards

Exercise 1

$$\begin{aligned} \text{Sol 1: } \int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} &= \int_{1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}} \\ &= \sin^{-1} \left(\frac{x - \frac{1}{2}}{1/2} \right) \Bigg|_{1/4}^{1/2} = \sin^{-1} 0 - \sin^{-1} \left(\frac{-1/4}{1/2} \right) \\ &= \sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \text{Sol 2: } \int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x} &= \int_0^{\pi/2} \frac{dx}{4 + \cos^2 x} \\ &= \int_0^{\pi/2} \frac{dx}{\frac{9}{2} + \frac{\cos 2\theta}{2}} = \int_0^{\pi/2} \frac{2dx}{9 + \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}} \\ &= 2 \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{10 + 8\tan^2 \theta} \\ &= \int_0^{\pi/4} \frac{2\sec^2 \theta d\theta}{10 + 8\tan^2 \theta} + \int_{\pi/4}^{\pi/2} \frac{2\operatorname{cosec}^2 \theta}{10\cot^2 \theta + 8} d\theta \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta d\theta}{5 + 4\tan^2 \theta} + \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2 \theta}{5\cot^2 \theta + 4} d\theta \\ &= \int_0^1 \frac{dt}{5 + 4t^2} + \int_1^0 \frac{dt}{5t^2 + 4} = \int_0^1 \frac{dt}{5 + 4t^2} + \int_0^1 \frac{dt}{5t^2 + 4} \\ &= \frac{1}{4} \times \frac{1}{\sqrt{5/2}} \tan^{-1} \frac{t}{\sqrt{5/2}} \Bigg|_0^1 + \frac{1}{5} \times \frac{1}{2/\sqrt{5}} \tan^{-1} \frac{t}{2/\sqrt{5}} \Bigg|_0^1 \\ &= \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} \tan^{-1} \frac{\sqrt{5}}{2} \\ &= \frac{1}{2\sqrt{5}} \left[\tan^{-1} \frac{2}{\sqrt{5}} + \cot^{-1} \frac{2}{\sqrt{5}} \right] \\ &= \frac{1}{2\sqrt{5}} \times \frac{\pi}{2} = \frac{\pi}{4\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \text{Sol 3: } \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx \\ I = \int_0^{\pi/2} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{1 + \sin x \cos x} dx \text{ or } I = \int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x} dx \\ \therefore 2I = \int_0^{\pi/2} \frac{1}{1 + \sin x \cos x} dx = \int_0^{\pi/2} \frac{\sec^2 x dx}{1 + \tan^2 x + \tan x} \\ = \lim_{x \rightarrow \infty} \int_0^x \frac{dt}{1 + t^2 + t} = \lim_{x \rightarrow \infty} \int_0^x \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t+1}{\sqrt{3}} \Bigg|_0^\infty \\ = \frac{2}{\sqrt{3}} \left[\frac{\pi}{2} - \frac{\pi}{6} \right] = \frac{2\pi}{3\sqrt{3}} \\ \therefore I = \frac{\pi}{3\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{Sol 4: } \int_0^{3/5} (3-5x)dx + \int_{3/5}^1 (5x-3)dx \\ = 3x - \frac{5}{2}x^2 \Bigg|_0^{3/5} + \frac{5x^2}{2} - 3x \Bigg|_{3/5}^1 \\ = \frac{9}{5} - \frac{5}{2} \times \frac{9}{25} + \left(\frac{5}{2} - 3 \right) - \left(\frac{5}{2} \times \frac{9}{25} - \frac{9}{5} \right) \\ = \frac{9}{5} - \frac{9}{10} - \frac{1}{2} - \frac{9}{10} + \frac{9}{5} \\ = \frac{18}{5} - \frac{9}{5} - \frac{1}{2} = \frac{9}{5} - \frac{1}{2} = \frac{13}{10} \end{aligned}$$

$$\begin{aligned} \text{Sol 5: } \int_1^2 (2x+1)dx + \int_2^3 (x^2+1)dx \\ = x^2 + x \Bigg|_1^2 + \frac{x^3}{3} + x \Bigg|_2^3 \\ = (4 + 2 - 2) + (9 + 3) - \left(\frac{8}{3} + 2 \right) \\ = 4 + 12 - 2 - \frac{8}{3} = 14 - \frac{8}{3} = \frac{34}{3} \end{aligned}$$

$$\begin{aligned}\text{Sol 6: } \int_{-\pi/4}^{\pi/4} \sin x dx &= 2 \int_0^{\pi/4} \sin x dx = 2 \left[-\cos x \right]_0^{\pi/4} \\ &= 2 \left[-\left(\frac{1}{\sqrt{2}} - 1 \right) \right] = 2 \left[1 - \frac{1}{\sqrt{2}} \right] = 2 - \sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{Sol 7: } \int_0^{\pi} \frac{x}{(1+\sin^2 x)} dx &= \int_0^{\pi} \frac{\pi-x}{(1+\sin^2 x)} dx = I \\ \therefore 2I &= \pi \int_0^{\pi} \frac{1}{1+\sin^2 x} dx = 2\pi \int_0^{\pi/2} \frac{1}{1+\sin^2 x} dx \\ \therefore I &= \pi \int_0^{\pi/2} \frac{1}{1+\sin^2 x} dx = \pi \int_0^{\pi/2} \frac{2}{3-\cos 2x} dx \\ &= 2\pi \int_0^{\pi/2} \frac{dx}{3-\frac{(1-\tan^2 x)}{2}} = 2\pi \int_0^{\pi/2} \frac{\sec^2 x dx}{2+4\tan^2 x} \\ &= \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2 x dx}{\frac{1}{2} + \tan^2 x} = \frac{\pi}{2} \left[\int_0^1 \frac{dt}{\frac{1}{2} + t^2} - \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2 x dx}{\frac{1}{2} \cot^2 x + 1} \right] \\ &= \frac{\pi}{2} \left[\left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \right]_0^1 - \int_1^0 \frac{dt}{\frac{1}{2} t^2 + 1} \right] \\ &= \frac{\pi}{2} \left[\sqrt{2} \tan^{-1} \sqrt{2} + 2 \int_0^1 \frac{dt}{t^2 + 2} \right] \\ &= \frac{\pi}{2} \left[\sqrt{2} \tan^{-1} \sqrt{2} + \frac{2}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \right]_0^1 \\ &= \frac{\pi}{2} \left[\sqrt{2} \tan^{-1} \sqrt{2} + \sqrt{2} \tan^{-1} \frac{1}{\sqrt{2}} \right] \\ &= \frac{\pi}{2} \times \sqrt{2} \times \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{Sol 8: } \int_0^2 (x^2 + 1) dx & \\ h &= \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \\ \therefore I &= \lim_{n \rightarrow \infty} \sum_{r=1}^n hf(a+rh) = \lim_{n \rightarrow \infty} \sum_{r=1}^n h((rh)^2 + 1) \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n (r^2 h^3 + h)\end{aligned}$$

$$\begin{aligned}&= \lim_{n \rightarrow \infty} \left[h^3 \times \frac{n(n+1)(2n+1)}{6} + hn \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{8}{n^3} \frac{(n)(n+1)(2n+1)}{6} + \frac{2}{n} \times n \right] \\ &= \frac{8 \times 1 \times 2}{6} + 2 = \frac{14}{3}\end{aligned}$$

$$\begin{aligned}\text{Sol 9: } \int_0^{\pi/2} |\sin x - \cos x| dx & \\ &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x) \Big|_{\pi/4}^{\pi/2} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right) - \left(1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right) \\ &= \sqrt{2} - 1 - 1 + \sqrt{2} = 2\sqrt{2} - 2\end{aligned}$$

$$\text{Sol 10: } f(x) = f(a-x)$$

$$g(x) + g(a-x) = 2$$

$$\int_0^a f(x)g(x) dx = \int_0^a f(a-x)g(a-x) dx = I$$

$$\therefore 2I = \int_0^a [f(x)g(x) + f(a-x)g(a-x)] dx = \int_0^a [f(x) \times 2] dx$$

$$I = \int_0^a f(x) dx$$

$$\text{Sol 11: } \int_0^{100\pi} \sqrt{1-\cos 2x} dx$$

$$\int_0^{100\pi} \sqrt{2\sin^2 x} dx = \sqrt{2} \int_0^{100\pi} \sqrt{\sin^2 x} dx$$

$$Q \sin^2(\pi-x) = \sin^2 x$$

$$\therefore I = 100\sqrt{2} \int_0^{\pi} \sqrt{\sin^2 x} dx = 100\sqrt{2} \int_0^{\pi} |\sin x| dx$$

$$\text{Also } |\sin(\pi-x)| = |\sin x|$$

$$\begin{aligned}\therefore I &= 200\sqrt{2} \int_0^{\pi/2} |\sin x| dx \\ &= 200\sqrt{2} (-|\cos x|)_0^{\pi/2} = 200\sqrt{2}\end{aligned}$$

Sol 12: (i) $f(-t) = -f(t)$

$$g(x) = \int_a^x f(t) dt$$

$$g(-x) = \int_a^{-x} f(t) dt = \int_a^{-a} f(t) dt + \int_{-a}^{-x} f(t) dt$$

$Qf(t) = \text{odd function}$

$$\text{So } \int_a^{-a} f(t) dt = 0$$

$$\therefore \int_a^{-x} f(t) dt = \int_{-a}^{-x} f(t) dt$$

Put $t = -p$

$$= -\int_a^x f(-p) dp \quad \because f(-p) = -f(p)$$

$$= \int_a^x f(p) dp$$

$$\therefore g(-x) = g(x)$$

(ii) $f(t) = f(-t)$

$$g(x) = \int_a^x f(t) dt ; \quad g(-x) = \int_a^{-x} f(t) dt$$

Put $t = -p$

$$= -\int_a^x f(-p) dp = -\int_a^x f(p) dp = -\int_{-a}^{-x} f(t) dt$$

$$\therefore g(-x) = \int_x^{-a} f(t) dx$$

$$\therefore g(x) + g(-x) = \int_a^x f(t) dt + \int_x^{-a} f(t) dt = \int_a^{-a} f(t) dt$$

\therefore It is not necessary that if $f(t)$ is even then $\int_a^x f(t) dt$ is odd

Sol 13: $f(x) = \int_a^{x^2} \sqrt{1+t^2} dt$

$$f'(x) = \sqrt{1+x^4} dx^2 = 2x\sqrt{1+x^4}$$

Sol 14: $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3 dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$

$$I = \int_{-\pi/3}^{\pi/3} \frac{\pi - 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$$

$$\therefore 2I = 2\pi \int_{-\pi/3}^{\pi/3} \frac{dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$$

$$\Rightarrow I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \frac{1 - \tan^2\left(\frac{x}{2} + \frac{\pi}{6}\right)}{1 + \tan^2\left(\frac{x}{2} + \frac{\pi}{6}\right)}}$$

$$= \frac{2\pi}{3} \int_0^{\pi/3} \frac{\sec^2\left(\frac{x}{2} + \frac{\pi}{6}\right) dx}{\frac{1}{3} + \tan^2\left(\frac{x}{2} + \frac{\pi}{6}\right)}$$

Put $\tan\left(\frac{x}{2} + \frac{\pi}{6}\right) = t$

$$\frac{1}{2} \sec^2\left(\frac{x}{2} + \frac{\pi}{6}\right) dx = dt$$

$$\therefore I = \frac{4\pi}{3} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{\frac{1}{3} + t^2} = \frac{4\pi}{3} \times \sqrt{3} \tan^{-1} \sqrt{3t} \Big|_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}} \left[\tan^{-1} 3 - \tan^{-1} 1 \right] = \frac{4\pi}{\sqrt{3}} \left[\tan^{-1} \left(\frac{3-1}{1+3 \times 1} \right) \right]$$

$$= \frac{4\pi}{3} \tan^{-1} \frac{1}{2}$$

Sol 15: $f(x) = \int_1^x \frac{\log t}{1+t} dt$

To Prove. $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2} (\log x)^2$

Put $t = \frac{1}{p} \Rightarrow dt = -\frac{1}{p^2} dp$

$$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log \frac{1}{p}}{1 + \frac{1}{p}} \left(-\frac{1}{p^2}\right) dp$$

$$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log p}{p(1+p)} dp = \int_1^x \frac{\log t}{t(t+1)} dt$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log t}{t} dt$$

$$= (\log)^2 \Big|_1^x - \int_1^x \frac{\log t}{t} dt$$

$$\therefore 2I = (\log x)^2$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2} (\log x)^2$$

$$\text{Sol 16: } - \int_{1/e}^1 \log x dx + \int_1^{2e} \log x dx$$

$$= -(x \log x - x) \Big|_{1/e}^1 + (x \log x - x) \Big|_1^{2e}$$

$$= - \left[0 - 1 - \left[\frac{1}{e} \log \frac{1}{e} - \frac{1}{e} \right] \right]$$

$$+ [(2e \log 2e - 2e) - (0 - 1)]$$

$$= 1 - \frac{1}{e} - \frac{1}{e} + 2e \log 2 + 2e - 2e + 1$$

$$= 2 - \frac{2}{e} + 2e \log 2$$

$$\text{Sol 17: } \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx \quad n \in \mathbb{N}$$

$$2 \sin\left(n + \frac{1}{2}\right)x \cos \frac{x}{2} = \sin(nx + 2) + \sin(nx)$$

$$= \frac{1}{2} \int_0^{\pi} \frac{\sin nx + \sin(nx + x)}{\sin x} dx$$

$$= \frac{1}{2} \int_0^{\pi} \frac{\sin(n+1)x + \sin nx}{\sin x} dx$$

If n is odd

$$I = \frac{1}{2} \int_0^{\pi} \frac{\sin nx - \sin(n+1)x}{\sin x} dx$$

$$\therefore 2I = \int_0^{\pi} \frac{\sin nx}{\sin x} dx = \pi \quad \Rightarrow I = \frac{\pi}{2}$$

If n is even

$$2I = \int_0^{\pi} \frac{\sin(n+1)x}{\sin x} dx = \pi; \quad I = \frac{\pi}{2}$$

$$\text{Sol 18: } F(x) = \int_{5\pi/4}^x (3 \sin t + 4 \cos t) dt$$

$$= 3(-\cos t) \Big|_{5\pi/4}^x + 4 \sin t \Big|_{5\pi/4}^x$$

$$= 3 \left[-\left(\cos x - \frac{1}{\sqrt{2}} \right) \right] + 4 \left[\sin x - \frac{1}{\sqrt{2}} \right]$$

$$= -3 \cos x + \frac{3}{\sqrt{2}} + 4 \sin x - \frac{4}{\sqrt{2}}$$

$$= \left(\frac{4 \sin x - 3 \cos x}{5} \right) 5 - \frac{1}{\sqrt{2}}$$

$$\text{From interval } \left[\frac{5\pi}{4}, \frac{4\pi}{3} \right] \sin x < \cos x$$

$$\therefore \text{We get min value of } x = \frac{4\pi}{3}$$

$$\therefore \text{Min value} = -4 \times \frac{\sqrt{3}}{2} + \frac{3}{2} - \frac{1}{\sqrt{2}} = \frac{3}{2} - 2\sqrt{3} - \frac{1}{\sqrt{2}}$$

$$\text{Sol 19: } I_n = \int_0^{\pi/4} \tan^n \theta d\theta$$

$$I_{n-1} + I_{n+1} = \int_0^{\pi/4} (\tan^{n-1} \theta + \tan^{n+1} \theta) d\theta$$

$$= \int_0^{\pi/4} (\tan^{n-1} \theta) \sec^2 \theta d\theta$$

$$\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$$

$$= \int_0^1 t^{n-1} dt = \frac{t^n}{n} \Big|_0^1 = \frac{1}{n}$$

$$\therefore n(I_{n-1} + I_{n+1}) = n \times \frac{1}{n} = 1$$

$$I_7 = \int_0^{\pi/4} \tan^7 \theta d\theta$$

$$= \int_0^{\pi/4} \tan^5 \theta \sec^2 \theta d\theta - \int_0^{\pi/4} \tan^3 \theta \sec^2 \theta d\theta$$

$$+ \int_0^{\pi/4} \tan \theta \sec^2 \theta d\theta - \int_0^{\pi/4} \tan \theta d\theta$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \log \sqrt{2} = \frac{5}{12} - \frac{1}{2} \log 2$$

Sol 20: $\int_0^{\pi/2} \left\{ a^2 \left(\frac{\cos 3x}{4} + \frac{3}{4} \cos x \right) + a \sin x - 20 \cos x \right\} dx \leq -\frac{a^2}{3}$

$$\int_0^{\pi/2} \left\{ a^2 \cos^3 x + a \sin x - 20 \cos x \right\} dx \leq -\frac{a^2}{3}$$

$$= a^2 \left[+\frac{1}{12} \sin^3 x \Big|_0^{\pi/2} + \frac{3}{4} (+\sin x) \Big|_0^{\pi/2} \right]$$

$$+ a(-\cos x) \Big|_0^{\pi/2} - 20 \sin x \Big|_0^{\pi/2} \leq -\frac{a^2}{3}$$

$$= a^2 \left[-\frac{1}{12} + \frac{3}{4} \right] + a - 20 \leq -\frac{a^2}{3}$$

$$\Rightarrow a^2 + a - 20 \leq 0$$

$$(a+5)(a-4) \leq 0 \therefore a \in [-5, 4]$$

$\therefore a$ is +ve interger

So $a = 1, 2, 3$ or 4

Sol 21: $f(x) = \sin x$

Mean value of $\sin x$ from $[-2, 0]$

$$\therefore \int_{-2}^0 \frac{\sin x}{2} dx = \frac{-1[0+2]}{2} = -1$$

Sol 22: $I = \int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} dx$

$$x = a \cos \theta$$

$$dx = -a \sin \theta d\theta$$

$$\int_{\pi/2}^a \frac{-a \sin \theta d\theta}{a \cos \theta + a \sin \theta} = \int_0^{\pi/2} \left(\frac{\sin \theta}{\cos \theta + \sin \theta} \right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\cos \theta}{\cos \theta + \sin \theta} \right) d\theta$$

$$\Rightarrow I = \frac{1}{2} \left[\int_0^{\pi/2} d\theta \right] = \frac{1}{2} \left[\frac{\pi}{2} \right] = \frac{\pi}{4}$$

Sol 23: $I = \int_0^1 0 dx + \int_1^4 1 dx + \int_4^9 2 dx \dots + \int_{(n-1)^2}^{n^2} (n-1) dx$

$$\therefore \sum_{n=0}^n \int_{(n-1)^2}^{n^2} (n-1) dx$$

$$= \sum (n-1) x \Big|_{(n-1)^2}^{n^2} = \sum (n-1) (n^2 - (n-1)^2)$$

$$= \sum (n-1)(2n-1) = \sum (2n^2 - 3n + 1)$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n$$

$$= \frac{n(n+1)}{2} \left[\frac{4n+2}{3} - 3 \right] + n = \frac{n(n+1)(4n-7)}{6} + n$$

$$= n \left(\frac{4n^2 - 3n - 7 + 6}{6} \right) = \frac{n(4n^2 - 3n - 1)}{6}$$

$$= \frac{n(n-1)(4n+1)}{6}$$

Sol 24: $\int_0^{n\pi+\lambda} |\sin x| dx = 2n + 1 - \cos \lambda$

$$n \in \mathbb{N}, 0 \leq \lambda < \pi$$

$$\text{LHS} = \int_0^{\lambda} |\sin x| dx + \int_{\lambda}^{n\pi+\lambda} |\sin x| dx$$

$$= -\cos x \Big|_0^{\lambda} + n \int_0^{\pi} |\sin x| dx$$

$$= -(\cos \lambda - 1) + 2n \int_0^{\pi/2} \sin x dx$$

$$= 2n + 1 - \cos \lambda$$

Sol 25: $I = \int_0^{\pi} \frac{x \sin 2x \cdot \sin \left(\frac{\pi}{2} \cos x \right)}{2x - \pi} dx$ (i)

$$I = \int_0^{\pi} \frac{(\pi - x)(-\sin 2x) \sin \left(\frac{\pi}{2} (-\cos x) \right)}{2(\pi - x) - \pi} dx$$
 (ii)

On adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{(2x - \pi) \sin 2x \sin \left(\frac{\pi}{2} \cos x \right)}{(2x - \pi)} dx$$

$$= \int_0^{\pi} \sin 2x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

$$\text{or } I = \int_0^{\pi/2} \sin 2x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

Let $\frac{\pi}{2} \cos x = t$

$$-\frac{\pi}{2} \sin x dx = dt \text{ or } \sin x dx = -\frac{2}{\pi} dt$$

$$\begin{aligned}
 I &= -\frac{2}{\pi} \int_{\pi/2}^0 2 \times \frac{2t}{\pi} \sin t dt \\
 &= \frac{8}{\pi^2} \int_0^{\pi/2} t \sin t dt = \frac{8}{\pi^2} \left[t(-\cos t) \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos t dt \right] \\
 &= \frac{8}{\pi^2} \left[0 + \sin t \Big|_0^{\pi/2} \right] = \frac{8}{\pi^2}
 \end{aligned}$$

Sol 26: Let $f(x) = K_1 e^x + K_2 e^{-x}$

$$g(x) + f'(x) = K_1 e^x - K_2 e^{-x}$$

$$\therefore g'(x) = K_1 e^x + K_2 e^{-x} = f(x)$$

$$\therefore f(0) = 1 \Rightarrow K_1 + K_2 = 1$$

$$\text{Also } g(0) = 0 \Rightarrow K_1 - K_2 = 0$$

$$K_1 = K_2 = \frac{1}{2}$$

$$\therefore f(x) = \frac{e^x + e^{-x}}{2}$$

$$\therefore f(1) = \frac{e + \frac{1}{e}}{2} = \frac{e^2 + 1}{2e}$$

Sol 27: (i) $\int_0^{\pi} \log(1 + \cos x) dx$

$$= \int_0^{\pi} \log(1 - \cos x) dx = I$$

$$\therefore 2I = \int_0^{\pi} \log(1 - \cos^2 x) dx$$

$$= 2 \int_0^{\pi} \log \sin x dx = 4 \int_0^{\pi/2} \log \sin x dx$$

$$= 4 \times \left(\frac{-\pi}{2} \right) \log 2$$

$$2I = -2\pi \log 2$$

$$\therefore I = -\pi \log 2 = \pi \log \frac{1}{2}$$

$$(ii) \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot\left(\frac{\pi}{2} - x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

$$\therefore 2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\tan x} + 1}{1 + \sqrt{\tan x}} \right) dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore I = \frac{\pi}{12}$$

Sol 28: $\int_0^{\pi} \frac{dx}{1 - 2a \cos x + a^2}$

$$= \int_0^{\pi} \frac{dx}{1 + a^2 - \frac{2a \left(1 - \tan^2 \frac{x}{2}\right)}{1 + \tan^2 \frac{x}{2}}}$$

$$= \int_0^{\pi} \frac{\sec^2 \frac{x}{2} dx}{(1 + a^2) \left(1 + \tan^2 \frac{x}{2}\right) - 2a \left(1 - \tan^2 \frac{x}{2}\right)}$$

$$= \int_0^{\pi} \frac{\sec^2 \frac{x}{2} dx}{(1 + a^2 - 2a) + \tan^2 \frac{x}{2} (1 + a^2 + 2a)}$$

Putting $\frac{x}{2} = t$

$$\sec^2 \frac{x}{2} dx = 2 dt$$

$$\int_0^{\infty} \frac{dt}{(1 + a^2)^2 t^2 + (1 - a)^2} = \frac{1}{1 + a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1 - a}{1 + a}\right)^2}$$

$$= \frac{1}{(1 + a^2)} \times \frac{1 + a}{|1 - a|} \tan^{-1} \frac{t}{\left|\frac{1 - a}{1 + a}\right|} \Bigg|_0^{\infty}$$

$$= \frac{1}{1 - a^2} \frac{\pi}{2} \text{ if } a < 1$$

$$= \frac{1}{a^2 - 1} \frac{\pi}{2} \text{ if } a > 1$$

Sol 29: (i) $\lim_{\alpha \rightarrow 0} \frac{\int_0^{\alpha} x dx}{\alpha \sin \alpha} = \lim_{\alpha \rightarrow 0} \frac{\alpha^2}{2\alpha \sin \alpha}$

$$= \frac{1}{2} \lim_{\alpha \rightarrow 0} \frac{1}{\left(\frac{\sin \alpha}{\alpha}\right)} = \frac{1}{2}$$

(ii) $y = \int_1^x t \log t dt$

$$\log y = \left(\int_1^x \log t dt \right) \log x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \int_1^x \log t dt + (\log x)(\log x)$$

$$\frac{dy}{dx} = x^1 \int_1^x \log t dt \left[\log^2 x + \frac{1}{x} \int_1^x \log t dt \right]$$

$$\frac{dy}{dx} \Big|_{x=e} = e^1 \int_1^e \log t dt \left[\log^2 e + \frac{1}{e} \int_1^e \log t dt \right]$$

$$= e(e \log e - e - (-1)) \left[\log^2 e + \frac{1}{e}(e \log e - e - (-1)) \right]$$

$$= e \left[\frac{1}{e} + 1 \right] = e + 1$$

Sol 30: $f(x) = \int_1^x (t^2 + 2t)(t^2 - 1) dt$

$$f'(x) = (x^2 + 2x)(x^2 - 1) > 0$$

$$x(x + 2)(x - 1)(x + 1) > 0$$

$$\therefore x \in (-\infty, -2) \cup (-1, 0) \cup (1, \infty)$$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $\int_{-1}^0 (1 - 2x) dx + \int_0^1 (1 + 2x) dx$

$$= x - x^2 \Big|_{-1}^0 + x + x^2 \Big|_0^1 = 0 - [-1 - 1] + [1 + 1] = 4$$

Sol 2: (C) $\int_{-1}^0 e^{-x} dx + \int_1^1 e^x dx$

$$= -e^{-x} \Big|_{-1}^0 + e^x \Big|_0^1 - [1 - e^{-1}] + [e^1 - 1]$$

$$= e^{-1} + e^1 - 2 = 2e - 2$$

Sol 3: (C) $\int_0^3 [x] dx = \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx$

$$= 0 + 1 + 2 = 3$$

Sol 4: (B) $\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx$

$$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{\pi} = 1 - [0 - 1] = 2$$

Sol 5: (A) $\int_{-2}^{-3/2} -(2x + 3) dx + \int_{-3/2}^2 (2x + 3) dx$

$$= -[x^2 + 3x]_{-2}^{-3/2} + x^2 + 3x \Big|_{-3/2}^2$$

$$= -\left[\frac{9}{4} - \frac{9}{2} - (4 - 6) \right] + \left[4 + 6 - \left(\frac{9}{4} - \frac{9}{2} \right) \right]$$

$$= \frac{9}{4} - 2 + 10 + \frac{9}{4} = \frac{9}{2} + 8 = \frac{25}{2}$$

Sol 6: (B) $\int_{-2}^2 |1 - x^2| dx = 2 \int_0^2 |1 - x^2| dx$

$$= 2 \left[\int_0^1 (1 - x^2) dx + \int_1^2 (x^2 - 1) dx \right]$$

$$= 2 \left[x - \frac{x^3}{3} \Big|_0^1 + \frac{x^3}{3} - x \Big|_1^2 \right]$$

$$= 2 \left[1 - \frac{1}{3} + \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= 2 \left[\frac{2}{3} + \frac{2}{3} + \frac{2}{3} \right] = 4$$

Sol 7: (D) $f'(x) = \left(\frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} \right) \times 2x = 0$

$$x = 0 \text{ or } (x^2 - 4)(x^2 - 1) = 0$$

$$\therefore x = 0, x = \pm 2, x = \pm 1$$

Sol 8: (A) $F_1(x) = \int_2^x (2t - 5) dt = t^2 - 5t \Big|_2^x$

$$= x^2 - 5x - (4 - 10) = x^2 - 5x + 6$$

$$F_2(x) = \int_0^x 2t dt = x^2$$

$$\therefore x^2 = x^2 - 5x + 6 \Rightarrow x = \frac{6}{5}, y = \frac{36}{25}$$

Sol 9: (A) $f(x) = f(a - x)$

$$g(x) = 2 - g(a - x)$$

$$I = \int_0^a f(x) g(x) dx = \int_0^a f(a - x) \cdot (2 - g(a - x)) dx$$

$$\Rightarrow 2 \int_0^a f(a - x) dx - \int_0^a f(a - x) \cdot g(a - x) dx$$

$$\text{Put } a - x = t$$

$$-dx = dt$$

$$\Rightarrow -2 \int_a^0 f(t) dt - \int_a^0 -f(t) \cdot g(t) dt$$

$$\Rightarrow -2 \int_a^0 f(t) \cdot dt + \int_a^0 f(t) \cdot g(t) \cdot dt$$

$$\Rightarrow I = 2 \int_0^a f(t) \cdot dt - \int_0^a f(t) \cdot g(t) \cdot dt$$

$$I = 2 \int_0^a f(t) \cdot dt - I \Rightarrow 2I = 2 \int_0^a f(t) \cdot dt \Rightarrow I = \int_0^a f(x) \cdot dx$$

Sol 10: (B) $\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} dx$

$$e^x + 3 = t$$

$$e^x dx = dt$$

$$\int_4^8 \frac{\sqrt{t-4}}{t} dt$$

$$t = 4 \sec^2 \theta; \quad dt = 8 \sec^2 \theta \tan \theta d\theta$$

$$\int_0^{\pi/4} \frac{2 \tan \theta \times 8 \sec^2 \theta \tan \theta}{4 \sec^2 \theta} d\theta = \int_0^{\pi/4} 4(\sec^2 \theta - 1) d\theta$$

$$= 4 \tan \theta \Big|_0^{\pi/4} - 4\theta \Big|_0^{\pi/4} = 4 - \pi$$

Sol 11: (A)

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 2 \int_0^{\pi} \sin mx \sin nx dx$$

$$= \int_0^{\pi} [\cos(m-n)x - \cos(m+n)x] dx = 0$$

Sol 12: (B) $-\int_{1/e}^1 \log x dx + \int_1^e \log x dx$

$$= -\left[x \log x - x \Big|_{1/e}^1 \right] + x \log x - x \Big|_1^e$$

$$= -\left[(-1) - \left(\frac{1}{e} \log \frac{1}{e} - \frac{1}{e} \right) \right] + [(e \log e - e) - (-1)]$$

$$= 1 + \left(-\frac{1}{e} \right) - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = 2 \left(1 - \frac{1}{e} \right)$$

Sol 13: (C) $\int_0^{\pi} \frac{dx}{1 - 2a \cos x + a^2}$

$$\Rightarrow \text{Put } \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\Rightarrow \int_0^{\pi} \frac{\sec^2(x/2) \cdot dx}{(1+a^2)(1+\tan^2(x/2)) - 2a(1-\tan^2(x/2))}$$

Put $\tan(x/2) = t$

$$\frac{1}{2} \cdot \sec^2(x/2) dx = dt \Rightarrow 2 \int_0^{\infty} \frac{dt}{(1-a^2) + (1+a^2)t^2}$$

$$\Rightarrow \frac{2}{1-a} \cdot \tan^{-1} \cdot \frac{(1+a)t}{(1-a)} \Big|_0^{\infty} \Rightarrow \frac{2}{1-a} (\tan^{-1} \infty - \tan^{-1} 0)$$

$$\Rightarrow \frac{2}{1-a} \cdot \frac{\pi}{2} \Rightarrow \frac{\pi}{1-a}$$

Sol 14: (B) $\int_0^{\pi/4} \frac{\sec x}{1 + 2 \sin^2 x} dx = \int_0^1 \frac{1}{(1-x)^2} dx$

$$= \int_0^1 x^9 dx = \frac{1}{10}$$

Sol 15: (A) $\int_0^{\infty} \frac{dx}{(x + \sqrt{x^2 + 1})^3}$

$x = \tan \theta$

$$\int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{(\tan \theta + \sec \theta)^3} = \int_0^{\pi/2} \frac{\cos \theta}{(1 + \sin \theta)^3} d\theta$$

$1 + \sin \theta = t$

$\cos \theta d\theta = dt$

$$\int_1^2 \frac{dt}{t^3} = -\frac{1}{2t^2} \Big|_1^2 = -\frac{1}{2} \left[\frac{1}{4} - 1 \right] = \frac{3}{8}$$

Sol 16: (B) $\int_1^5 [x-3] dx$

$$\int_1^2 1 dx + \int_2^3 0 dx + \int_3^4 0 dx + \int_4^5 1 dx$$

$$1 + 1 = 2$$

Sol 17: (C) $I = \int_{-\pi/2}^{\pi/2} \frac{\sin x}{1 + \cos^2 x} e^{-\cos^2 x} dx$

$$I = \int_{-\pi/2}^{\pi/2} \frac{-\sin x}{1 + \cos^2 x} e^{-\cos^2 x} dx$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

Sol 18: (C) $I = \int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1} \right)^2 + \left(\frac{x-1}{x+1} \right)^2 - 2 \right]^{1/2} dx$

$$\begin{aligned}
 I &= 2 \int_0^{1/2} \left| \frac{x+1}{x-1} - \frac{x-1}{x+1} \right|^{2/2} dx \\
 &= 2 \int_0^{1/2} \left(\frac{x+1}{x-1} - \frac{x-1}{x+1} \right) dx = 2 \int_0^{1/2} \frac{4x}{x^2-1} dx \\
 &= -4 \log |(x^2-1)|_0^{1/2} = -4 \log \frac{3}{4}
 \end{aligned}$$

$$\text{Sol 19: (A)} \int_{-1}^1 \{x - [x]\} dx$$

$$= \frac{x^2}{2} \Big|_{-1}^1 - \left[\int_{-1}^0 (-1) dx + \int_0^1 0 dx \right]$$

$$= 0 - [-(0+1) + 0] = 1$$

$$\text{Sol 20: (C)} I = \int_4^{10} \frac{[x^2]}{[(x-14)^2] + [x^2]} dx$$

$$I = \int_4^{10} \frac{[(14-x)^2]}{[x^2] + [(x-14)^2]} dx$$

$$\therefore 2I = \int_4^{10} dx = 10 - 4 = 6 \Rightarrow I = 3$$

$$\text{Sol 21: (A)} \int_{-1}^3 (|x-2| + [x]) dx$$

$$= \int_{-1}^2 (2-x) dx + \int_2^3 (x-2) dx + \int_{-1}^0 (-1) dx$$

$$+ \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx$$

$$= 2x - \frac{x^2}{2} \Big|_{-1}^2 + \frac{x^2}{2} - 2x \Big|_2^3 - 1(0+1) + 1 + 2$$

$$= [4-2] - \left(-2 - \frac{1}{2}\right) + \left(\frac{9}{2} - 6\right) - (2-4) + 2$$

$$= 2 + \frac{5}{2} - \frac{3}{2} + 4 = 6 + 1 = 7$$

$$\text{Sol 22: (C)} \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{2}{\sqrt{2}} \int_0^{\pi/4} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \log \left[\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right] \Big|_0^{\pi/4}$$

$$2I = \frac{2}{\sqrt{2}} \left[\log(\sqrt{2}+1) \right] \Rightarrow I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$\text{Sol 23: (C)} \mu_{10} = \int_0^{\pi/2} x^{10} \sin x dx$$

$$\mu_8 = \int_0^{\pi/2} x^8 \sin x dx$$

$$\mu_{10} = x^{10}(-\cos x) \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) 10x^9 dx$$

$$\mu_{10} = \left(\frac{\pi}{2}\right)^{10} (0) + \int_0^{\pi/2} \cos x 10x^9 dx$$

$$= 10 \left[\int_0^{\pi/2} \cos x \cdot x^9 dx \right]$$

$$= 10 \left[x^9 \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 9x^8 \sin x dx \right]$$

$$= 10 \times \left(\frac{\pi}{2}\right)^9 - 90\mu_8$$

$$\therefore \mu_{10} + 90\mu_8 = 10 \left(\frac{\pi}{2}\right)^9$$

$$\text{Sol 24: (C)} \int_0^{\pi} e^{\sin^2 x} \cos^3(2n+1)x dx$$

$$I = \int_0^{\pi} e^{\sin^2 x} \cos^3((2n+1)\pi - (2n+1)x) dx$$

$$= -\int_0^{\pi} \sin^2 x \cos^3((2n+1)x) dx$$

$$\therefore 2I = 0 \Rightarrow I = 0$$

$$\text{Sol 25: (C)} I = \int_{-\pi/2}^{\pi/2} \sin \log(x + \sqrt{x^2+1}) dx$$

$$= \int_{-\pi/2}^{\pi/2} \sin \log(\sqrt{x^2+1} - x) dx$$

$$= \int_{-\pi/2}^{\pi/2} \sin \log \frac{1}{\sqrt{x^2+1} + x} dx$$

$$= - \int_{-\pi/2}^{\pi/2} \sin \log(\sqrt{x^2+1} + x) dx = -I$$

$$\therefore 2I = 0 \Rightarrow I = 0$$

Sol 26: (C) $\sin \alpha + \int_{\alpha}^{2\alpha} \cos 2x dx = 0$

$$\Rightarrow \sin \alpha + \frac{1}{2} \sin 2x \Big|_{\alpha}^{2\alpha} = 0$$

$$\Rightarrow \sin \alpha + \frac{1}{2} [\sin 4\alpha - \sin 2\alpha] = 0$$

$$\Rightarrow \sin \alpha + \cos 3\alpha \sin \alpha = 0$$

$$\Rightarrow \sin \alpha = 0 \text{ or } \cos 3\alpha = -1$$

$$\Rightarrow \alpha = n\pi, \text{ or } 3\alpha = (2n+1)\pi$$

$$\therefore \alpha = -\frac{\pi}{3}$$

Sol 27: (B) $f'(x) = \sin \sqrt{x^4} dx^4 - \sin \sqrt{x^2} dx^2$
 $= 4x^3 \sin x^2 - 2x \sin x$

Sol 28: (B) $\int_0^{\pi} x \sin x \cos^4 x dx = \int_0^{\pi} (\pi - x) \sin x \cos^4 x dx$

$$\therefore 2I = \pi \int_0^{\pi} \sin x \cos^4 x dx$$

Let $\cos x = t$

$$-\sin x dx = dt$$

$$2I = -\pi \int_1^{-1} t^4 dt$$

$$2I = \pi \int_{-1}^1 t^4 dt = \frac{\pi}{5} [1 + 1]$$

$$\therefore I = \frac{\pi}{5}$$

Sol 29: (B) $f(0) = a + b = -1$

$$f'(x) = 2ae^{2x} + be^x + c \quad \dots (i)$$

$$f'(\log 2) = 8a + 2b + c = 31 \quad \dots (ii)$$

$$\int_0^{\log 4} (f(x) - (x)) dx = \int_0^{\log 4} (ae^{2x} + be^x) dx$$

$$= \frac{a}{2} e^{2x} + be^x \Big|_0^{\log 4} = 8a + 4b - \left(\frac{a}{2} + b\right) = \frac{39}{2}$$

$$= 15a + 6b = 39 \quad \dots (iii)$$

$$\Rightarrow 9a = 45$$

$$a = 5; b = -6; c = 3$$

Sol 30: (A) $I = \int_{-\pi/4}^{\pi/4} \frac{e^{-x} \sec^2 x dx}{e^{-2x} - 1} = \int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x}{1 - e^{2x}} dx$

$$\therefore 2I = \int_{-\pi/4}^{\pi/4} \left(\frac{e^x \sec^2 x}{e^{2x} - 1} - \frac{e^x \sec^2 x}{e^{2x} - 1} \right) dx = 0$$

$$I = 0$$

Sol 31: (B) $I = \int_{-1}^a \log(x + \sqrt{1+x^2})$

$$= \log \left(\frac{-1}{x - \sqrt{1+x^2}} \right) = \log \left(\frac{1}{\sqrt{1+x^2} - x} \right)$$

$$= \int_{-1}^a -\log(\sqrt{1+x^2} - x)$$

$$\therefore 2I = \int_{-1}^a \log(x^2 + 1 - x^2) = \int_{-1}^a \log 1 = 0$$

Sol 32: (C) $\int_{-2}^0 \frac{\sin^2 x}{-\frac{1}{2}} dx + \int_0^2 \frac{\sin^2 x}{\frac{1}{2}} dx$

$$= -\frac{2}{2} \int_{-2}^0 \frac{1 - \cos 2x}{-\frac{1}{2}} dx + \frac{2}{2} \int_0^2 (1 - \cos 2x) dx$$

$$= -1[2] + \frac{1}{2} \sin 2x \Big|_{-2}^0 + 1[2] - \frac{1}{2} \sin 2x \Big|_0^2$$

$$= -2 + \frac{1}{2} [0 + \sin 4] + 2 - \frac{1}{2} [\sin 4] = 4 - \sin 4$$

Sol 33: (C) $f(x) = \int_0^x \log(1+t^2) dt$

$$f'(x) = \log(1+x^2)$$

$$f''(x) = \frac{1}{1+x^2} \times 2x$$

$$\therefore f''(1) = \frac{2}{2} = 1$$

Sol 34: (C) $\int_0^{\pi} \frac{dx}{1+3^{\cos x}}$

$$= \int_0^{\pi} \frac{dx}{1+3^{\cos(\pi-x)}} = \int_0^{\pi} \frac{dx}{1+3^{-\cos x}}$$

$$= \int_0^{\pi} \frac{3^{\cos x}}{1+3^{\cos x}} dx$$

$$\therefore 2I = \int_0^{\pi} dx = \pi$$

$$\therefore I = \frac{\pi}{2}$$

Previous Years' Questions

Sol 1: (A) Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ (i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} I dx$$

$$\therefore I = \frac{\pi}{4}$$

Sol 2: (C) Let $I = \int_0^{\pi} e^{\cos^2 x} \cdot \cos^3 \{(2n+1)x\} dx$

Using $\int_0^a f(x) dx$

$$= \begin{cases} 0, & f(a-x) = -f(x) \\ 2 \int_0^{a/2} f(x) dx, & f(a-x) = f(x) \end{cases}$$

Again, let $f(x) = e^{\cos^2 x} \cdot \cos^3 \{(2n+1)x\}$

$$\therefore f(\pi-x) = (e^{\cos^2 x}) \{-\cos^3 (2n+1)x\} = -f(x)$$

$$\therefore I = 0$$

Sol 3: (A) $\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt = \lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt$

(using L' Hospital's rule)

$$= \lim_{x \rightarrow 1} \frac{2f(x) \cdot f'(x)}{1} = 2f(1) \cdot f'(1)$$

$$= 8f'(1) [\because f(1) = 4]$$

Sol 4: (D) Let $I = \int_0^{\pi/2} \frac{1}{1 + \tan^3 x} dx$

$$= \int_0^{\pi/2} \frac{1}{1 + \frac{\sin^3 x}{\cos^3 x}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 \left(\frac{\pi}{2} - x\right)}{\cos^3 \left(\frac{\pi}{2} - x\right) + \sin^3 \left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx \quad \dots(ii)$$

On adding Eqs (i) and (ii), we get $2I = \int_0^{\pi/2} 1 dx$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

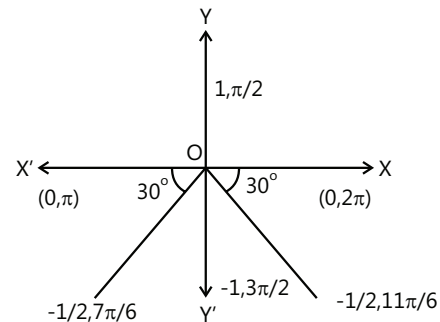
Now, $\int_0^1 f(x) dx = \frac{2A}{\pi}$

$$\Rightarrow \int_0^1 \left\{ A \sin \left(\frac{\pi x}{2} \right) + B \right\} dx = \frac{2A}{\pi}$$

$$\Rightarrow \left[-\frac{2A}{\pi} \cos \frac{\pi x}{2} + Bx \right]_0^1 = \frac{2A}{\pi}$$

$$\Rightarrow B + \frac{2A}{\pi} = \frac{2A}{\pi} \Rightarrow B = 0$$

Sol 5: (A) It is a questions of greatest integer function. We have subdivide the interval π to 2π as under keeping in view that we have to evaluate $[2 \sin x]$



We know that, $\sin \frac{\pi}{6} = \frac{1}{2}$,

$$\sin \left(\pi + \frac{\pi}{6} \right) = \sin \frac{7\pi}{6} = -\frac{1}{2}$$

$$\sin \frac{11\pi}{6} = \sin \left(2\pi - \frac{\pi}{6} \right) = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\sin \frac{9\pi}{6} = \sin \frac{3\pi}{2} = -1$$

Hence, we divide the interval π to 2π as

$$\left(\pi, \frac{7\pi}{6}\right), \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$$

$$\sin x = \left(0, -\frac{1}{2}\right), \left(-1, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right)$$

$$2\sin x = (0, -1), (-2, -1), (-1, 0)$$

$$[2\sin x] x = -1$$

$$\begin{aligned} &= \int_{\pi}^{7\pi/6} [2\sin x] dx + \int_{7\pi/6}^{11\pi/6} [2\sin x] dx \\ &+ \int_{11\pi/6}^{2\pi} [2\sin x] dx \\ &= \int_{\pi}^{7\pi/6} -1 dx + \int_{7\pi/6}^{11\pi/6} -2 dx + \int_{11\pi/6}^{2\pi} -1 dx \\ &= -\frac{\pi}{6} - 2\left(\frac{4\pi}{6}\right) - \frac{\pi}{6} = -\frac{10\pi}{6} = -\frac{5\pi}{3} \end{aligned}$$

Sol 6: (A) Given, $\int_0^x f(t) dt = x + \int_x^1 t f(t) dt$

On differentiating both sides w.r.t. x , we get

$$f(x) \cdot 1 = 1 - x f(x) \quad . \quad 1$$

$$\Rightarrow (1 + x) f(x) = 1$$

$$\Rightarrow f(x) = \frac{1}{1+x}$$

$$\Rightarrow f(1) = \frac{1}{1+1} = \frac{1}{2}$$

Sol 7: (A) Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos(\pi - x)}$$

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 - \cos x}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} \right) dx$$

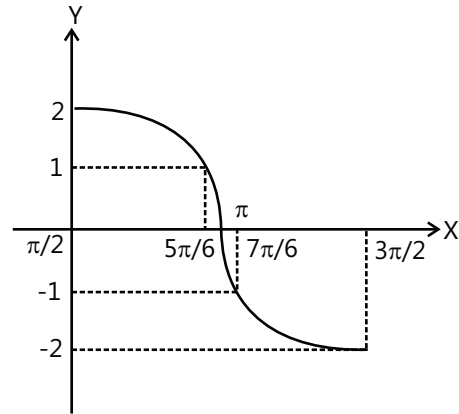
$$\Rightarrow 2I = \int_{\pi/4}^{3\pi/4} \left(\frac{2}{1 - \cos^2 x} \right) dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x dx = [-\cot x]_{\pi/4}^{3\pi/4}$$

$$= \left[-\cot \frac{3\pi}{4} + \cot \frac{\pi}{4} \right] = -(-1) + 1 = 2$$

Sol 8: (C) The graph of $y = 2\sin x$ for $\pi/2 \leq x \leq 3\pi/2$ is given in figure. From the graphs, it is clear that

$$[2\sin x] = \begin{cases} 2, & \text{if } x = \pi/2 \\ 1, & \text{if } \pi/2 < x \leq 5\pi/6 \\ 0, & \text{if } 5\pi/6 < x \leq \pi \\ -1, & \text{if } \pi < x \leq 7\pi/6 \\ -2, & \text{if } 7\pi/6 < x \leq 3\pi/2 \end{cases}$$



Therefore, $\int_{\pi/2}^{3\pi/2} [2\sin x] dx$

$$\begin{aligned} &= \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0 dx + \int_{\pi}^{7\pi/6} (-1) dx \\ &+ \int_{7\pi/6}^{3\pi/2} (-2) dx \end{aligned}$$

$$= [x]_{\pi/2}^{5\pi/6} + [-x]_{\pi}^{7\pi/6} + [-2x]_{7\pi/6}^{3\pi/2}$$

$$\dots(i) = \left(\frac{5\pi}{6} - \frac{\pi}{2} \right) + \left(-\frac{7\pi}{6} + \pi \right)$$

$$+ \left(\frac{-2 \cdot 3\pi}{2} + \frac{2 \cdot 7\pi}{6} \right)$$

\dots(ii)

$$= \pi \left(\frac{5}{6} - \frac{1}{2} \right) + \pi \left(1 - \frac{7}{6} \right) + \pi \left(\frac{7}{3} - 3 \right)$$

$$= \pi \left(\frac{5-3}{6} \right) + \pi \left(-\frac{1}{6} \right) + \pi \left(\frac{7-9}{3} \right) = -\frac{\pi}{2}$$

Sol 9: (C) Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx$ \dots(i)

$$= \int_{\pi}^{-\pi} \frac{\cos^2(-x)}{1 + a^{-x}} d(-x)$$

$$\Rightarrow I = \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{1 + a^x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned}
 2I &= \int_{-\pi}^{\pi} \left(\frac{1+a^x}{1+a^{-x}} \right) \cos^2 x \, dx \\
 &= \int_{-\pi}^{\pi} \cos^2 x \, dx = 2 \int_0^{\pi} \frac{1+\cos 2x}{2} \, dx \\
 &= \int_0^{\pi} (1+\cos 2x) \, dx \\
 &= \int_0^{\pi} 1 \, dx + \int_0^{\pi} \cos 2x \, dx \\
 &= [x]_0^{\pi} + 2 \int_0^{\pi/2} \cos 2x \, dx \\
 &= \pi + 0 \\
 \Rightarrow 2I &= \pi \Rightarrow I = \pi/2
 \end{aligned}$$

Sol 10: (C) Given, $F(x) = \int_0^{\pi} f(t) \, dt$

By Leibnitz rule,

$$F'(x) = f(x)$$

But $F(x^2) = x^2(1+x) = x^2 + x^3$ (given)

$$\Rightarrow F(x) = x + x^{3/2}$$

$$\Rightarrow F'(x) = 1 + \frac{3}{2}x^{1/2}$$

$$\Rightarrow f(x) = F'(x) = 1 + \frac{3}{2}x^{1/2} \text{ [from Eq. (i)]}$$

$$\Rightarrow f(4) = 1 + \frac{3}{4}(4)^{1/2}$$

$$\Rightarrow f(4) = 1 + \frac{3}{2} \times 2 = 4$$

Sol 11: (A) Given, $f(x) = \int_1^x \sqrt{2-t^2} \, dt$

$$\Rightarrow f'(x) = \sqrt{2-x^2}$$

$$\text{Also } x^2 - f'(x) = 0$$

$$\therefore x^2 = \sqrt{2-x^2}$$

$$\Rightarrow x^4 = 2 - x^2 \Rightarrow x^4 + x^2 - 2 = 0$$

$$\Rightarrow x = \pm 1$$

Sol 12: (C) $\int_3^{3+3T} f(2x) \, dx$ put $2x = y \Rightarrow dx = \frac{1}{2} \, dy$

$$\therefore \frac{1}{2} \int_6^{6+6T} f(y) \, dy = \frac{6I}{2} = 3I$$

Sol 13: (D) Given, $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} \, dt$

On differentiating both sides using Newton's Leibnitz formula, we get

$$\begin{aligned}
 f'(x) &= e^{-(x^2+1)^2} \left\{ \frac{d}{dx}(x^2+1) \right\} - e^{-(x^2)^2} \left\{ \frac{d}{dx}(x^2) \right\} \\
 &= e^{-(x^2+1)^2} \cdot 2x - e^{-(x^2)^2} \cdot 2x \\
 &= 2xe^{-(x^4+2x^2+1)}(1 - e^{2x^2+1})
 \end{aligned}$$

[where, $e^{2x^2+1} > 1, \forall x$ and $e^{-(x^4+2x^2+1)} > 0 \forall x$]

$$\therefore f'(x) > 0$$

which shows $2x < 0$ or $x < 0$

$$\Rightarrow x \in (-\infty, 0)$$

... (i)

Sol 14: (B) $I = \int_0^1 \sqrt{\frac{1-x}{1+x}} \, dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}} \, dx$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$$

$$I = [\sin^{-1} x]_0^1 + \int_1^0 \frac{t}{t} \, dt$$

(where, $t^2 = 1 - x^2 \Rightarrow t \, dt = -x \, dx$)

$$I = (\sin^{-1} 1 - \sin^{-1} 0) + [t]_1^0 = \frac{\pi}{2} - 1$$

Sol 15: (A) Let $I = \int_{-1}^1 \frac{dx}{1+x^2}$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

$$\therefore I = 2 \int_0^{\pi/4} d\theta = \frac{\pi}{2}$$

(B) Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Put $x = \sin \theta \Rightarrow dx = \cos \theta \, d\theta$

$$\therefore I = \int_0^{\pi/2} 1 \, d\theta = \frac{\pi}{2}$$

$$\begin{aligned}
 \text{(C)} \int_2^3 \frac{dx}{1-x^2} &= \frac{1}{2} \left[\log \left(\frac{1+x}{1-x} \right) \right]_2^3 \\
 &= \frac{1}{2} \left[\log \left(\frac{4}{-2} \right) - \log \left(\frac{3}{-1} \right) \right] = \frac{1}{2} \left[\log \left(\frac{2}{3} \right) \right]
 \end{aligned}$$

$$\text{(D)} \int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$\begin{aligned}
 \text{Sol 16: } & \int_{-2}^2 |1-x^2| dx \\
 &= \int_{-2}^{-1} (x^2-1) dx + \int_{-1}^1 (1-x^2) dx \\
 & \quad + \int_1^2 (x^2-1) dx \\
 &= \left[\frac{x^3}{3} - x \right]_{-2}^{-1} + \left[x - \frac{x^3}{3} \right]_{-1}^1 + \left[\frac{x^3}{3} - x \right]_1^2 \\
 &= \left(-\frac{1}{3} + 1 + \frac{8}{3} - 2 \right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) \\
 & \quad + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1 \right) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{Sol 17: Let } I &= \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx \quad \dots (i) \\
 \Rightarrow I &= \int_{\pi/4}^{3\pi/4} \frac{\left(\frac{\pi}{4} + \frac{3\pi}{4} - x \right)}{1 + \sin\left(\frac{\pi}{4} + \frac{3\pi}{4} - x \right)} dx \\
 \left[\because \int_a^b f(x) dx &= \int_a^b f(a+b-x) dx \right] \\
 \Rightarrow I &= \int_{\pi/4}^{3\pi/4} \frac{\pi-x}{1+\sin(\pi-x)} dx \\
 \Rightarrow I &= \int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin x} dx - \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx \\
 \Rightarrow 1 &= \pi \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\sin x} - 1 \quad [\text{from Eq. (i)}] \\
 \Rightarrow 1 &= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{dx}{(1+\sin x)} \\
 \Rightarrow 1 &= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx \\
 &= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{(1-\sin x)}{1-\sin^2 x} dx \\
 &= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx \\
 &= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} (\sec^2 x - \sec x \tan x) dx \\
 &= \frac{\pi}{2} [\tan x - \sec x]_{\pi/4}^{3\pi/4} \\
 &= \frac{\pi}{2} [-1 - 1 - (-\sqrt{2} - \sqrt{2})]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi}{2} (-2 + 2\sqrt{2}) \\
 &= \pi(\sqrt{2} - 1)
 \end{aligned}$$

$$\text{Sol 18: Let } I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \dots (i)$$

$$\Rightarrow I = \int_2^3 \frac{\sqrt{2+3-x}}{\sqrt{(2+3)-(5-x)} + \sqrt{2+3-x}} dx$$

$$\Rightarrow I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$\Rightarrow 2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5-x}}{\sqrt{5-x} + \sqrt{x}} dx$$

$$\Rightarrow 2I = \int_2^3 1 dx = 1 \Rightarrow I = \frac{1}{2}$$

Sol 19: (B)

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx = \int_0^1 \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

$$\Rightarrow I < \frac{2}{3}$$

$$J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx \Big|_0^1 = 2$$

$$\therefore J \leq 2.$$

$$\text{Sol 20: (D) Let } \int_0^x [\cot x] dx \quad \dots (i)$$

$$= \int_0^x [\cot(\pi-x)] dx, \int_0^x [-\cot x] dx \quad \dots (ii)$$

Adding (1) and (2)

$$2I = \int_0^x [\cot x] dx + \int_0^x [-\cot x] dx = \int_0^x (-1) dx$$

$$\left[\because [x] + [-x] = -1 \text{ if } x \notin \mathbb{Z} \right. \\
 \left. = 0 \text{ if } x \in \mathbb{Z} \right]$$

$$= [-x]_0^x = -\pi$$

$$\therefore = -\frac{\pi}{2}$$

Sol 21: (A) $p'(x) = p'(1-x)$

$\Rightarrow p(x) = -p(1-x) + c$

at $x = 0$

Now $p(0) = -p(1-x) + 42$

$\Rightarrow p(x) + p(1-x) = 42$

$I = \int_0^1 p(x) dx + \int_0^1 p(1-x) dx$

$2I = \int_0^1 (42) dx \Rightarrow I = 21.$

Sol 22: (D) $I = 8 \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

$= 8 \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta d\theta$ (let $x = \tan\theta$)

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta = \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) d\theta$

$= 8 \log 2 \frac{\pi}{4} - 1$

$2I = 2\pi \log 2$

$I = \pi \log 2$

Sol 23: (B, C) $g(x) = \int_0^x \cos 4t dt$

$\Rightarrow g'(x) = \cos 4x \Rightarrow g(x) = \frac{\sin 4x}{4} + k$

$\Rightarrow g(x) = \frac{\sin 4x}{4} [\because g(0) = 0]$

$= g(x) + g(\pi) = g(x) - g(\pi) (\because g(\pi) = 0)$

Sol 24: (D) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} dx}{1 + \sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} dx}{1 + \sqrt{\tan x}}$

$2I = \int_{\pi/6}^{\pi/3} dx$

$\Rightarrow I = \frac{1}{2} \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{12}$, statement-1 is false

$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ it is property

Sol 25: (D) $\int \left\{ e^{\left(x-\frac{1}{x}\right)} + \left(1-\frac{1}{x^2}\right) e^{\frac{1}{x}} \right\} dx$
 $= x \cdot e^{\frac{1}{x}} + c$

As $\int (xf'(x) + f(x)) dx = x f(x) + c$

Sol 26: (C) $I = \int_2^4 \frac{\log x^2}{2 \log x^2 + \log(36 - 12x + x^2)} dx$

$I = \frac{2}{2} \int_2^4 \frac{\log x}{\log x + \log(6-x)} dx \dots (i)$

$I = \int_2^4 \frac{\log(6-x)}{\log x + \log(6-x)} dx \left\{ \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right\} \dots (ii)$

Equation (i) & (ii) gives

$= \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)} dx = \int_2^4 dx = 2$

Hence $I = 1$

JEE Advanced/Boards

Exercise 1

Sol 1: $\int_0^1 e^{\tan^{-1} x} \sin^{-1}(\cos x) dx.$

$\int_0^1 (\tan^{-1} x) \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$

$= \int_0^1 \left(\frac{\pi}{2} \tan^{-1} x - x \tan^{-1} x\right) dx$

$$\begin{aligned}
&= \frac{\pi}{2} \left[x \tan^{-1} x \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx \right] - \int_0^1 x \tan^{-1} x dx \\
&= \frac{\pi}{2} \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] - \left[(\tan^{-1} x) \frac{x^2}{2} \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2 + 1 - 1}{1+x^2} dx \right] \\
&= \frac{\pi^2}{8} - \frac{\pi}{4} \log - \frac{1}{2} \times \frac{\pi}{4} + \frac{1}{2} \left[(x)_0^1 - \tan^{-1} \Big|_0^1 \right] \\
&= \frac{\pi^2}{8} - \frac{\pi}{4} \log - \frac{\pi}{8} + \frac{1}{2} - \frac{1}{2} \times \frac{\pi}{4} \\
&= \frac{\pi^2}{8} - \frac{\pi}{4} ((\log 2) + 1) + \frac{1}{2}
\end{aligned}$$

Sol 2: (i) Put $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

$$dx = 2(\beta - \alpha) \sin \theta \cos \theta d\theta$$

$$I = \int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)} dx$$

$$= \int_0^{\pi/2} \sqrt{(\beta-\alpha) \cos^2 \theta (\beta-\alpha) \sin^2 \theta}$$

$$\times (\beta - \alpha) \sin 2\theta d\theta$$

$$= \frac{(\beta-\alpha)^2}{2} \int_0^{\pi/2} \sin^2 2\theta d\theta$$

$$= \frac{(\beta-\alpha)^2}{4} \int_0^{\pi/2} (1 - \cos 4\theta) d\theta$$

$$= \frac{(\beta-\alpha)^2}{4} \times \frac{\pi}{2} = \frac{(\beta-\alpha)^2 \pi}{8}$$

$$(ii) I = \int_{\alpha}^{\beta} \frac{\sqrt{(x-\alpha)}}{\sqrt{(\beta-x)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\beta-\alpha) \cos^2 \theta}}{\sqrt{(\beta-\alpha) \sin^2 \theta}} \times (\beta-\alpha) \sin 2\theta d\theta$$

$$= 2(\beta-\alpha) \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta} \times \sin \theta \cos \theta d\theta$$

$$= 2(\beta-\alpha) \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= (\beta-\alpha) \frac{\pi}{2}$$

$$(iii) I = \int_0^{\pi/2} \frac{2(\beta-\alpha) \sin \theta \cos \theta d\theta}{(\alpha \cos^2 \theta + \beta \sin^2 \theta) \times (\beta-\alpha) \cos \theta \sin \theta}$$

$$= \int_0^{\pi/2} \frac{2d\theta}{\alpha \cos^2 \theta + \beta \sin^2 \theta} = \frac{1}{\beta} \int_0^{\pi/2} \frac{2 \sec^2 \theta d\theta}{\frac{\alpha}{\beta} + \tan^2 \theta}$$

Put $\tan \theta = t$

$$\frac{2}{\beta} \int_0^{\infty} \frac{dt}{\frac{\alpha}{\beta} + t^2} = \frac{2}{\beta} \times \frac{1}{\sqrt{\frac{\alpha}{\beta}}} \tan^{-1} \frac{t}{\sqrt{\frac{\alpha}{\beta}}} \Big|_0^{\infty}$$

$$= \frac{2}{\sqrt{\alpha\beta}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{\alpha\beta}}$$

$$(iv) I = \int_0^{\pi/2} \frac{(\alpha \cos^2 \theta + \beta \sin^2 \theta) \times (\beta - \alpha) \sin 2\theta d\theta}{(\beta - \alpha) \sin \theta \cos \theta}$$

$$= 2 \int_0^{\pi/2} (\alpha \cos^2 \theta + \beta \sin^2 \theta) d\theta$$

$$= 2 \int_0^{\pi/2} ((\beta - \alpha) \sin^2 \theta + \alpha) d\theta$$

$$= 2\alpha \times \frac{\pi}{2} + 2 \int_0^{\pi/2} \frac{(\beta - \alpha)}{2} (1 - \cos 2\theta) d\theta$$

$$= \alpha\pi + 2 \left(\frac{\beta - \alpha}{2} \right) \int_0^{\pi/2} (1 - \cos 2\theta) d\theta$$

$$= \alpha\pi + \frac{\beta - \alpha}{2} \times \frac{\pi}{2} \times 2 = \left(\frac{\beta}{2} + \frac{\alpha}{2} \right) \pi = (\alpha + \beta) \frac{\pi}{2}$$

$$\text{Sol 3: (i) } \int_0^{2\pi} \sqrt{1 - \sin t} dt - \int_0^{\pi} \sqrt{1 - \sin t} dt$$

$$= \frac{1}{2} \int_0^{2\pi} (\sqrt{1 - \sin t} + \sqrt{1 + \sin t}) dt - \int_0^{\pi} \sqrt{1 - \sin t} dt$$

$$= \int_0^{\pi} \sqrt{1 - \sin t} + \sqrt{1 + \sin t} dt - \int_0^{\pi} \sqrt{1 - \sin t} dt$$

$$= \int_0^{\pi} \sqrt{1 - \sin t} dt$$

$$= \int_0^{\pi} \left| \sin \frac{t}{2} + \cos \frac{t}{2} \right| dt$$

$$= -2 \cos \frac{t}{2} + 2 \sin \frac{t}{2} \Big|_0^{\pi}$$

$$= -2[0 - 1] + 2[1 - 0] = 4$$

$$\begin{aligned}
 \text{(ii) } & \int_0^1 e^x (x-1)^n \\
 &= \left[(x-1)^n e^x \Big|_0^1 - n \int_0^1 (x-1)^{n-1} e^x dx \right] \\
 &= -(-1)^n - n \left[(x-1)^{n-1} e^x \Big|_0^1 - (n-1) \int_0^1 (x-1)^{n-2} e^x dx \right] \\
 &= -(-1)^n + n(-1)^{n-1} \\
 &+ n(n-1) \left[(x-1)^{n-2} e^x \Big|_0^1 - (n-2) \int_0^1 (x-1)^{n-3} e^x dx \right] \\
 &= -(-1)^n + n(-1)^{n-1} - n(n-1)(-1)^{n-2} \\
 &\quad - n(n-1)(n-2) \int_0^1 (x-1)^{n-3} e^x dx
 \end{aligned}$$

Taking $n = 3$

$$\begin{aligned}
 &= -(-1)^3 + 3(-1)^2 - 3(3-1)(-1)^1 - 3(2)(1) \int_0^1 e^x dx \\
 &= +1 + 3 + 6 - 6(e^1 - 1) \\
 &= 16 - 6e
 \end{aligned}$$

Sol 4: (i) $\int_0^{\pi/2} e^x \left\{ \begin{array}{l} \cos(\sin x) \cos^2 \frac{x}{2} \\ + \sin(\sin x) \sin^{-2} \frac{x}{2} \end{array} \right\} dx$

$$\begin{aligned}
 & \frac{1}{2} \int_0^{\pi/2} e^x \left\{ \begin{array}{l} \cos(\sin x) [\cos x + 1] \\ + \sin(\sin x) [1 - \cos x] \end{array} \right\} dx \\
 & \frac{1}{2} \int_0^{\pi/2} e^x \left\{ \begin{array}{l} [\cos(\sin x) + \sin(\sin x)] \\ + \cos x [\cos(\sin x) - \sin(\sin x)] \end{array} \right\} dx
 \end{aligned}$$

Put $\cos(\sin x) + \sin(\sin x) = t$

$$\begin{aligned}
 & (-\sin(\sin x) \cos x + \cos(\sin x) \cos x) dx = dt \\
 & \frac{1}{2} \int_0^{\pi/2} e^x \{f(x) + f'(x)\} dx \\
 &= \frac{1}{2} e^x f(x) \Big|_0^{\pi/2} = \frac{1}{2} e^x \{ \cos(\sin x) + \sin(\sin x) \} \Big|_0^{\pi/2} \\
 &= \frac{1}{2} [e^{\pi/2} (\cos 1 + \sin 1) - e^0 (\cos 0)] \\
 &= \frac{1}{2} [e^{\pi/2} (\cos 1 + \sin 1) - 1]
 \end{aligned}$$

(ii) $\int_1^e \left\{ (1+x)e^x + (1-x)e^{-x} \right\} \log x dx$

$\log x = t$

$x = e^t \Rightarrow dx = e^t dt$

$$\begin{aligned}
 & \int_0^1 \left\{ (1+e^t) e^{e^t} + (1-e^t) e^{-e^t} \right\} e^t dt \\
 &= \int_0^1 \left\{ (e^{e^t} + e^{-e^t}) t + (e^t e^{e^t} - e^t e^{-e^t}) t \right\} e^t dt
 \end{aligned}$$

Sol 5: $R = \int_0^\infty \frac{dx}{1+x^4}$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$$\begin{aligned}
 &= \int_\infty^0 \frac{-t^2}{1+t^2} dt = \int_0^\infty \frac{x^2}{1+x^2} = P \\
 \therefore 2I &= 2P = \int_0^\infty \frac{1+x^2}{1+x^4} dx \\
 &= \int_0^\infty \frac{1+x^{-2}}{\left(1-\frac{1}{x}\right)^2 + 2} dx \\
 &= \int_{-\infty}^\infty \frac{dt}{t^2 + 2} \quad (\text{Put } x - \frac{1}{x} = t) \\
 \therefore 2I &= \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \Big|_{-\infty}^\infty \\
 &= \frac{1}{\sqrt{2}} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{\sqrt{2}} \\
 \therefore I &= \frac{\pi}{2\sqrt{2}} \\
 \int_0^\infty \frac{xdx}{1+x^4} \\
 \text{Put } x^2 &= t \Rightarrow 2xdx = dt \\
 &= \frac{1}{2} \int_0^\infty \frac{dt}{1+t^2} = \frac{1}{2} \tan^{-1} t \Big|_0^\infty = \frac{\pi}{2} \\
 \therefore P + R - \sqrt{2}Q &= \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - \frac{\sqrt{2}\pi}{4} \\
 &= \frac{\pi}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}
 \end{aligned}$$

$$\text{Sol 6: } \int_1^2 \frac{(x^2 - 1)}{x^5 \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$$

$$\text{Put } 2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5} \right) dx = dt$$

$$\frac{1}{4} \int_1^{25} \frac{dt}{\sqrt{t}} = \frac{1}{4} \times 2\sqrt{t} \Big|_1^{25} = \frac{1}{2} \left[\frac{5}{4} - 1 \right] = \frac{1}{8}$$

$$\therefore \frac{1000}{8} = 125$$

$$\text{Sol 7: } h(x) = f \circ g(x) + k$$

$$\frac{dh(x)}{dx} = f' \{g(x)\} g'(x) = \frac{-\sin x}{\cos^2(\cos x)}$$

$$j(x) = \int_{g(x)}^{f(x)} \frac{f(t)}{g(t)} dt$$

$$h(x) = - \int \frac{\sin x}{\cos^2(\cos x)} dx = \int \frac{dt}{\cos^2 t} = \tan t$$

$$= \tan(\cos x) + c$$

$$\therefore f(x) = \tan x, g(x) = \cos x$$

$$J(x) = \int_{\cos x}^{\tan x} \frac{\tan t}{\cos t} dt$$

$$j(0) = \int_1^0 \frac{\sin t}{\cos^2 t} dt$$

$$\cos t = u \Rightarrow -\sin t dt = du$$

$$= \int_0^1 \frac{du}{u^2} = \frac{-1}{u} \Big|_1^{\cos 1} = - \left[\frac{1}{\cos 1} - 1 \right] = 1 - \sec 1$$

$$\text{Sol 8: } \int_0^{\pi/2} \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}} dx = \int_0^{\pi/2} \sqrt{\frac{(1 - \tan x)^2}{(1 + \tan x)^2}} dx$$

$$= \int_0^{\pi/2} \left| \tan \left(x - \frac{\pi}{4} \right) \right| dx = 2 \log \sec \left(x - \frac{\pi}{4} \right) \Big|_0^{\pi/4}$$

$$= 2 \ln \sqrt{2} = \log 2$$

$$\text{Sol 9: } 2I = \int_0^2 (3x^2 - 3x + 1) \cos(x^3 - 3x^2 + 4x - 2) dx$$

$$+ \int_0^2 (3x^2 - 9x + 7) \cos(x^3 - 3x^2 + 4x - 2) dx$$

$$2I = 2 \int_0^2 (3x^2 - 6x + 4) \cos(x^3 - 3x^2 + 4x - 2) dx$$

$$\text{Put } x^3 - 3x^2 + 4x - 2 = t$$

$$I = \int_{-2}^2 \cos t dt = \sin t \Big|_{-2}^2 = \sin 2 + \sin 2 = 2 \sin 2$$

$$\therefore p = q = 2 \Rightarrow p + q = 4$$

$$\text{Sol 10: } I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6 - 12x^2 + 1}{x^2 + 2} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6 + 6x^4 - 6x^4 - 12x^2 + 1}{x^2 + 2} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^4(x^2 + 2) - 6x^2(x^2 + 2) + 1}{x^2 + 2} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 3x^4 - 6x^2 + \frac{1}{x^2 + 2} dx$$

$$= 2 \times 3 \left[\frac{x^5}{5} - \frac{2}{3} x^3 \right]_0^{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} \Big|_0^{\sqrt{2}}$$

$$= 6 \left[\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} \right] + \frac{\pi}{2\sqrt{2}} = \frac{-16\sqrt{2}}{5} + \frac{\pi}{2\sqrt{2}}$$

$$\text{Sol 11: } \int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_0^{\infty}$$

$$\frac{1}{a} \frac{\pi}{2} = \frac{\pi}{5050} \Rightarrow a = 2525$$

$$\text{Sol 12: } \int_{-2}^2 \frac{x^2 - x}{\sqrt{x^2 + 4}} dx = \int_{-2}^2 \frac{x^2 + x}{\sqrt{x^2 + 4}} dx$$

$$\therefore I = \int_{-2}^2 \frac{x^2}{\sqrt{x^2 + 4}} dx = 2 \int_0^2 \frac{x^2}{\sqrt{x^2 + 4}} dx$$

$$= 2 \int_0^2 \left(\sqrt{x^2 + 4} - \frac{4}{\sqrt{x^2 + 4}} \right) dx$$

$$= 2 \left[\frac{x}{2} \sqrt{x^2 + 4} + 2 \log |x + \sqrt{x^2 + 4}| - 8 \log |x + \sqrt{x^2 + 4}| \right]_0^2$$

$$= 2\sqrt{8} - 4 \log |2 + 2\sqrt{2}| + 4 \log 2$$

$$= 4\sqrt{2} - 4 \log |1 + \sqrt{2}|$$

$$\text{Sol 13: } u = \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos x}{\sin\left(x + \frac{\pi}{4}\right)} \right)^2 dx$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos\left(\frac{\pi}{4} - x\right)}{\cos x} \right)^2 dx$$

$$v = 2 \int_0^{\pi/4} \left(\frac{\cos\left(\frac{\pi}{4} - x\right)}{\cos x} \right)^2 dx$$

$$\therefore \frac{v}{u} = \frac{2}{1/2} = 4$$

$$\text{Sol 14: } \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{x dx}{\cos x \cos\left(\frac{\pi}{4} - x\right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x\right) dx}{\cos\left(\frac{\pi}{4} - x\right) \cos x}$$

$$\therefore 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{\frac{\pi}{4} dx}{\cos x \cos\left(\frac{\pi}{4} - x\right)}$$

$$I = \frac{\pi}{8\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\cos x \cos\left(\frac{\pi}{4} - x\right)}$$

$$= \frac{\pi}{8} \int_0^{\pi/4} \frac{dx}{\cos^2 x + \cos x \sin x}$$

$$= \frac{\pi}{8} \int_0^{\pi/4} \frac{\sec^2 x dx}{1 + \tan x}$$

$$= \frac{\pi}{8} \int_0^1 \frac{dt}{1+t} = \frac{\pi}{8} \log(1+t) \Big|_0^1 = \frac{\pi}{8} \log 2$$

$$\text{Sol 15: } \int_0^1 \frac{\sin^{-1} \sqrt{x}}{x^2 - x + 1} dx$$

$$\text{Let } \sin^{-1} \sqrt{x} = t$$

$$\frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow dx = \left(\sqrt{1 - \sin^2 t} \right) 2 \sin t dt = \sin 2t dt$$

$$\int_0^{\pi/2} \frac{t \sin 2t dt}{\sin^4 t - \sin^2 t + 1}$$

$$= \int_0^{\pi/2} \frac{t \sin 2t dt}{1 - \sin^2 t \cos^2 t} = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - t\right) \sin 2t dt}{1 - \sin^2 t \cos^2 t}$$

$$\therefore 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin 2t dt}{1 - \sin^2 t \cos^2 t}$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin 2t dt}{1 - \frac{\sin^2 2t}{4}} = 4 \cdot \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin 2t dt}{3 + \cos^2 2t}$$

$$2I = \left[2\pi \int_1^{-1} \left(\frac{dt}{3+t^2} \right) \right] - \frac{1}{2}$$

$$\therefore I = \frac{\pi}{2} \int_{-1}^1 \frac{dt}{3+t^2} = \frac{\pi}{2} \times \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \Big|_{-1}^1$$

$$= \frac{1}{2} \frac{\pi}{\sqrt{3}} \left[\frac{\pi}{6} + \frac{\pi}{6} \right] = \frac{\pi^2}{6\sqrt{3}}$$

$$\text{Sol 16: } \int_0^{\frac{1+\sqrt{5}}{2}} \frac{x^2 + 1}{\frac{1+\sqrt{5}}{2} x^2 \left(x^2 + \frac{1}{x^2} - 1 \right)} \log \left(1 + x - \frac{1}{x} \right) dx$$

$$= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{1+x^2}{\left(\left(x - \frac{1}{x} \right)^2 + 1 \right)} \log \left(1 + \left(x - \frac{1}{x} \right) \right) dx$$

$$x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2} \right) dx = dt$$

$$= \int_0^1 \frac{\log(1+t)}{(t^2+1)} dt$$

$$t = \tan \theta$$

$$= \int_0^{\pi/4} \log(1 + \tan \theta) d\theta$$

$$= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left[\log(2) - \log(1 + \tan \theta) \right] d\theta$$

$$\therefore 2I = \int_0^{\pi/4} \log 2 d\theta = \frac{\pi}{4} \log 2 \Rightarrow I = \frac{\pi}{8} \log 2$$

$$\text{Sol 17: } \lim_{n \rightarrow \infty} n^2 \int_{-1/n}^{1/n} (2010 \sin x + 2012 \cos x) |x| dx$$

$$= \lim_{n \rightarrow \infty} n^2 \int_{-1/n}^{1/n} (2012 \cos x) |x| dx$$

$$= 2012 \lim_{n \rightarrow \infty} 2n^2 \int_0^{1/n} x \cos x dx$$

$$= 2012 \times 2 \lim_{n \rightarrow \infty} n^2 \left[x \sin x \Big|_0^{1/n} - \int_0^{1/n} \sin x dx \right]$$

$$= 2012 \times 2 \lim_{n \rightarrow \infty} n^2 \left[\frac{1}{n} \sin \frac{1}{n} + \cos \frac{1}{n} - 1 \right]$$

$$= 2012 \times 2 \lim_{n \rightarrow \infty} \left[\frac{\sin \frac{1}{n} + \cos \frac{1}{n} - 1}{\frac{1}{n}} \right]$$

$$= 2012 \times 2 \left[1 - \frac{1}{2} \right] = 2012$$

$$\text{Sol 18: } \int_0^{\pi} |\sqrt{2} \sin x + 2 \cos x| dx$$

$$\sqrt{2} \sin x + 2 \cos x > 0$$

$$\Rightarrow \tan x > -\sqrt{2}$$

$$\therefore x < \pi - \tan^{-1} \sqrt{2}$$

$$= \int_0^{\pi - \tan^{-1} \sqrt{2}} (\sqrt{2} \sin x + 2 \cos x) dx$$

$$+ \int_{\pi}^{\pi - \tan^{-1} \sqrt{2}} (\sqrt{2} \sin x + 2 \cos x) dx$$

$$= -\sqrt{2} \cos x \Big|_0^{\pi - \tan^{-1} \sqrt{2}} + 2 \sin x \Big|_0^{\pi - \tan^{-1} \sqrt{2}}$$

$$+ 2 \sin x \Big|_{\pi}^{\pi - \tan^{-1} \sqrt{2}} - \sqrt{2} \cos x \Big|_{\pi}^{\pi - \tan^{-1} \sqrt{2}}$$

$$= -\sqrt{2} [-\cos \tan^{-1} \sqrt{2} - 1] + 2 \sin \tan^{-1} \sqrt{2}$$

$$+ 2 [\sin \tan^{-1} \sqrt{2}] - \sqrt{2} [-\cos \tan^{-1} \sqrt{2} + 1]$$

$$= 2\sqrt{2} \times \frac{1}{\sqrt{3}} + 4 \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{6\sqrt{2}}{\sqrt{3}} = 2\sqrt{6}$$

$$\text{Sol 19: } \cos x + \cos 3x = 2 \cos 2x \cos x$$

$$\sin x + \sin 3x = 2 \sin 2x \cos x$$

$$\therefore I = \int_0^{\pi} \sqrt{(2 \cos x + 1)^2 [\cos^2 2x + \sin^2 2x]} dx$$

$$= \int_0^{\pi} |2 \cos x + 1| dx$$

$$= \int_0^{2\pi/3} (2 \cos x + 1) dx + \int_{2\pi/3}^{\pi} (-2 \cos x - 1) dx$$

$$= 2 \sin x \Big|_0^{2\pi/3} + \frac{2\pi}{3} - 2 \sin x \Big|_{2\pi/3}^{\pi} - \left(\pi - \frac{2\pi}{3} \right)$$

$$= 2 \left[\frac{\sqrt{3}}{2} \right] + \left(\frac{4\pi}{3} - \pi \right) - 2 \left(0 - \frac{\sqrt{3}}{2} \right)$$

$$= \sqrt{3} + \sqrt{3} + \frac{\pi}{3} = 2\sqrt{3} + \frac{\pi}{3} = \sqrt{12} + \frac{\pi}{3}$$

$$\therefore w = 12, k = 3$$

$$\Rightarrow k^2 + w^2 = 9 + 144 = 153$$

$$\text{Sol 20: } \int_0^1 \frac{(1-x)(1+x)}{x(1+x)(1+x)} \frac{dx}{\sqrt{\frac{1}{x} + 1 + x}}$$

$$= \int_0^1 \frac{1-x^2}{x^2} \left(\frac{1}{\left(\frac{1}{x} + 1\right)(1+x)} \right) \frac{dx}{\sqrt{\frac{1}{x} + x + 1}}$$

$$= \int_0^1 \frac{x^{-2} - 1}{\left(\frac{1}{x} + x + 1\right) \sqrt{\frac{1}{x} + x + 1}} dx$$

$$\text{Put } \frac{1}{x} + x + 1 = t \Rightarrow -\int_{\infty}^3 \frac{dt}{(t+1)\sqrt{t}}$$

$$\text{Put } t = \tan^2 \theta \Rightarrow dt = 2 \tan \theta \sec^2 \theta d\theta$$

$$= -\int_{\pi/2}^{\pi/3} \frac{2 \tan \theta \sec^2 \theta}{\sec^2 \theta \tan} d\theta = 2 \int_{\pi/3}^{\pi/2} d\theta$$

$$= 2 \times \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{\pi}{3}$$

$$\text{Sol 21: } \int_0^{\pi/2} \left(\frac{a \sin x + b \cos x}{\sin x + \cos x} \right) \sqrt{2} dx$$

$$= \int_0^{\pi/2} \frac{(a \cos x + b \sin x) \sqrt{2}}{\sin x + \cos x} dx$$

$$\therefore 2l = \sqrt{2}(a+b)\frac{\pi}{2} \Rightarrow l = \frac{(a+b)\pi}{2\sqrt{2}}$$

Sol 22: $\int_0^1 f(x)dx = 1$

$$\Rightarrow \int_0^1 \frac{f(2x)}{3} dx = 1 \Rightarrow \int_0^1 f(2x)dx = 3$$

$$\therefore 2l \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{(\sin x + \cos x)} \sqrt{2} dx$$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$$\int_1^2 f(t) \frac{dt}{2} = 3 \Rightarrow \int_1^2 f(t)dt = 6$$

$$\therefore \int_1^2 f(t)dt = 6 - 1 = 5$$

Sol 23: $\int_{-1}^3 \{ |x-2| + [x] \} dx = \int_{-1}^0 \{ |x-2| + [x] \} dx +$

$$\int_0^{-1} \{ |x-2| + [x] \} dx = \int_1^2 \{ |x-2| + [x] \} dx + \int_2^3 \{ |x-2| + [x] \} dx$$

$$\int_{-1}^0 (2-x-1)dx + \int_0^1 (2-x+0)dx +$$

$$\int_1^2 (2-x+1)dx + \int_2^3 (x-2+2)dx +$$

$$= x - \frac{x^2}{2} \Big|_{-1}^0 + 2x - \frac{x^2}{2} \Big|_0^1 + 3x - \frac{x^2}{2} \Big|_1^2 + \frac{x^2}{2} \Big|_2^3$$

$$= -\left(-1 - \frac{1}{2}\right) + \left(2 - \frac{1}{2}\right) + (6-2) - \left(3 - \frac{1}{2}\right) + \frac{9}{2} - 2$$

$$= -\left(-1 - \frac{1}{2}\right) + \left(2 - \frac{1}{2}\right) + (6-2) - \left(3 - \frac{1}{2}\right) + \frac{9}{2} - 2$$

$$= 7$$

Sol 24: $x = \tan\theta$

$$dx = \sec^2\theta d\theta$$

$$\int_0^{\pi/3} \left(\sin^{-1} \frac{2 \tan \theta}{1 + \tan \theta} \right) \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} 2\theta \sec^2 \theta d\theta + \int_{\pi/4}^{\pi/3} (\pi - 2\theta) \sec^2 \theta d\theta$$

$$= -2 \left[\theta \tan \theta \Big|_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3} \tan \theta d\theta \right] + \pi \tan \theta \Big|_{\pi/4}^{\pi/3} + 2 \left[\theta \tan \theta \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan \theta d\theta \right]$$

$$= -2 \left[\left(\frac{\pi}{3} \times \sqrt{3} - \frac{\pi}{4} \right) \right] + \left[\log 2 - \frac{1}{2} \log 2 \right]^2$$

$$+ \pi(\sqrt{3} - 1) + 2 \left[\frac{\pi}{4} \right] - \left[\frac{1}{2} \log 2 \right] 2$$

$$= -\frac{2\pi}{\sqrt{3}} + \frac{\pi}{2} + \log 2 + \sqrt{3} \pi - \pi + \frac{\pi}{2} - \log 2$$

$$= \frac{\pi}{\sqrt{3}}$$

Sol 25: $\int_0^{\pi} \frac{(ax+b)\sec x \tan x}{4 + \tan^2 x} dx$

$$\frac{2l}{a\pi + 2b} = \left[\int_0^{\pi} \frac{\sec x \tan x}{4 + \tan^2 x} dx \right]$$

$$= \int_1^{-1} \frac{dt}{3+t^2} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \Big|_1^{-1}$$

$$= \frac{1}{\sqrt{3}} \left[\pi - \frac{\pi}{6} - \frac{\pi}{6} \right] = \frac{1}{\sqrt{3}} \times \frac{2\pi}{3} \therefore l = \frac{(a\pi + 2b)\pi}{3\sqrt{3}}$$

Sol 26: $\int_0^{\pi} \frac{(2x-3)\sin x}{(1+\cos^2 x)} dx$

$$2l = \int_0^{\pi} \frac{(2\pi+6)\sin x}{1+\cos^2 x} dx$$

$$\frac{l}{\pi+3} = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = \int_1^{-1} \frac{-dt}{1+t^2} = \int_1^{-1} \frac{dt}{1+t^2}$$

$$= \tan^{-1} t \Big|_{-1}^1 = \frac{\pi}{4} + \frac{\pi}{4} \Rightarrow l = (\pi+3) \frac{\pi}{2}$$

Sol 27: Let $f(x) = \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}$ (i)

$$\text{Then, } f\left(\frac{\pi}{2} - x\right) = \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}$$

$$= \frac{\sqrt{\sin x}}{\sqrt{\sin x + \sqrt{\cos x}}}$$

Now, $f(x) + f\left(\frac{\pi}{2} - x\right) = \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x + \sqrt{\sin x}}} = 1$

$$\therefore I = \frac{1}{2} \int_0^{\pi/2} \left[f(x) + f\left(\frac{\pi}{2} - x\right) \right] dx = \frac{1}{2} \int_0^{\pi/2} x dx$$

$$= \frac{1}{2} [x]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}$$

Sol 28: $2I = \int_0^{n\pi} \frac{n\pi |\sin x|}{1 + (\cos x)} dx$

$$2I = n^2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos x} dx = 2n^2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos x} dx$$

$$= 2n^2\pi \int_1^0 \frac{-dt}{1+t} = 2n^2\pi \log(t+1) \Big|_0^1 = 2n^2\pi \log 2$$

$$\therefore I = n^2\pi \log 2 = 100\pi \log 2$$

$$\therefore n = 10$$

Sol 29: $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx$

$$= \int_0^{\pi/2} \frac{\cos x}{(1 + \cos x) + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}} dx$$

$$= \int_0^{\pi/2} \frac{1 - \tan^2 \frac{x}{2}}{2 + 2\tan \frac{x}{2}} dx$$

[Dividing numerator and denominator By $\cos^2 \frac{x}{2}$]

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\left(1 - \tan \frac{x}{2}\right)\left(1 + \tan \frac{x}{2}\right)}{1 + \tan \frac{x}{2}} dx$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(1 - \tan \frac{x}{2}\right) dx$$

...(ii) $= \frac{1}{2} \left[x + 2 \log \cos \frac{x}{2} \right]_0^{\pi/2}$

$$= \frac{1}{2} \left[\left(\frac{\pi}{2} + 2 \log \cos \frac{\pi}{4} \right) - (0 + 2 \log 1) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} + 2 \log \frac{1}{\sqrt{2}} \right] = \frac{1}{2} \left[\frac{\pi}{2} + \log \frac{1}{2} \right] = \frac{1}{2} \left[\frac{\pi}{2} - \log 2 \right]$$

Sol 30: $\int_0^a \frac{\log(1+ax)}{1+x^2} dx$

$$ax = \tan \theta \Rightarrow abx = \sec^2 \theta d\theta$$

$$\int_0^{\tan^{-1} a^2} \frac{a \log(1 + \tan \theta)}{(a^2 + \tan^2 \theta)} \times \sec^2 \theta = d\theta$$

$$a \left[\log(1 + \tan \theta) \int_0^{\tan^{-1} a^2} \frac{\sec^2 \theta}{a^2 + \tan^2 \theta} d\theta \Big|_0^{\tan^{-1} a^2} - \int_0^{\tan^{-1} a^2} \frac{\sec^2 \theta}{(1 + \tan \theta)} \int_0^{\tan^{-1} \theta} \frac{\sec^2 \theta}{a^2 + \tan^2 \theta} d\theta \right]$$

$$a \left[\log(1 + \tan \theta) \times \frac{1}{a} \tan^{-1} \frac{\tan \theta}{a} \Big|_0^{\tan^{-1} a^2} - \int_0^{\tan^{-1} a^2} \left(\frac{\sec^2 \theta}{1 + \tan \theta} \times \frac{1}{a} \tan^{-1} \frac{\tan \theta}{a} \right) d\theta \right]$$

$$2I = \log(1 + a^2) \tan^{-1} a$$

$$I = \tan^{-1} a \log \sqrt{1 + a^2}$$

Sol 31: $\int_0^{\ln 3} \frac{e^x + 1}{e^{2x} + 1} dx = \int_0^{\ln 3} \frac{e^x dx}{e^{2x} + 1} + \int_0^{\ln 3} \frac{1}{e^{2x} + 1} dx$

$$e^{2x} = t$$

$$2e^{2x} dx = dt$$

$$dx = \frac{1}{2t} dt$$

$$\tan^{-1} e^x \Big|_0^{\log 3} + \frac{1}{2} \int_1^3 \frac{1}{(t+1)t} dt$$

$$= \tan^{-1} \sqrt{3} - \frac{\pi}{4} + \frac{1}{2} \int_1^3 \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

$$= \frac{1}{2} \left[\frac{\pi}{6} + \log 3 - \log 2 \right]$$

Sol 32: Given, $\int_a^a \sqrt{x} \, dx = 2a \int_0^{\pi/2} \sin^3 x \, dx$

$$\Rightarrow \left[\frac{x^{3/2}}{3/2} \right]_0^a = 2a \int_0^{\pi/2} \frac{3 \sin x - \sin 3x}{4} \, dx$$

$$[\because \sin 3x = 3 \sin x - 4 \sin^3 x]$$

$$\Rightarrow \frac{2}{3} [a^{3/2} - 0] = \frac{a}{2} \left[3(-\cos x) - \left(-\frac{\cos 3x}{3} \right) \right]_0^{\pi/2}$$

$$\Rightarrow \frac{2}{3} a^{3/2} = \frac{a}{2} \left[-3 \left(\cos \frac{\pi}{2} - \cos 0 \right) + \frac{1}{3} \left(\cos \frac{3\pi}{2} - \cos 0 \right) \right]$$

$$\Rightarrow \frac{2}{3} a^{3/2} = \frac{a}{2} \left(-3(0-1) + \frac{1}{3}(0-1) \right)$$

$$\Rightarrow \frac{2}{3} a^{3/2} = \frac{4a}{3} \Rightarrow a\sqrt{a} - 2a = 0$$

$$\Rightarrow a(\sqrt{a} - 2) = 0 \Rightarrow a = 0 \text{ or } \sqrt{a} = 2 \Rightarrow a = 0 \text{ or } a = 4$$

When $a = 0$:

$$\int_a^{a+1} x \, dx = \int_0^1 x \, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

When $a = 4$:

$$\int_0^{a+1} x \, dx = \int_4^5 x \, dx = \left[\frac{x^2}{2} \right]_4^5 = \frac{1}{2}(25-16) = \frac{9}{2}$$

Sol 33: $\tan x = 2x$

$$\frac{1}{2} \int_0^1 [\cos(\alpha - \beta)x - \cos(\alpha + \beta)x] \, dx$$

$$\frac{1}{2} \left[\frac{\sin(\alpha - \beta)x}{(\alpha - \beta)} \Big|_0^1 - \frac{\sin(\alpha + \beta)x}{(\alpha + \beta)} \Big|_0^1 \right]$$

$$\frac{1}{2} \left[\frac{\sin(\alpha - \beta)}{\alpha - \beta} - \frac{\sin(\alpha + \beta)}{(\alpha + \beta)} \right]$$

$$\sin \alpha = 2\alpha \cos \alpha$$

$$\sin \beta = 2\beta \cos \beta$$

$$= \frac{1}{2} \left[\left(\frac{\sin \alpha \frac{\sin \beta}{2\beta} - \frac{\sin \alpha}{2\alpha} \sin \beta \right) \Big/ (\alpha - \beta) - \left(\frac{\sin \alpha \sin \beta}{2\beta} + \frac{\cos \alpha \cos \beta}{2\alpha} \right) \Big/ (\alpha + \beta) \right]$$

$$= \frac{1}{2} \sin \alpha \sin \beta \left[\frac{1}{2\alpha\beta} - \frac{1}{2\alpha\beta} \right] = 0$$

Sol 34: $\int_0^{p+q\pi} |\cos x| \, dx$

$$\int_0^p \cos x + \int_p^{p+q\pi} |\cos x| \, dx$$

$$= \sin x \Big|_0^p + \int_0^{q\pi} (\cos x) \, dx$$

$$= \sin p + q \times 2 \int_0^{q\pi} (\cos x) \, dx$$

$$= 2q + \sin p$$

Sol 35: $\int_{-4}^{-5} e^{(x+5)^2} \, dx + 3 \int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} \, dx$

Let $x + 5 = t$ and $3\left(x - \frac{2}{3}\right) = t$

$$= \int_1^0 e^{t^2} \, dt + (-1) \int_1^0 e^{t^2} \, dt = 0$$

Put $t = -2$

$$\int_0^1 e^{t^2} \, dt + (-1) \int_1^0 e^{z^2} \, dz = 0$$

Sol 36: $\sin \pi x > \cos \pi x$

$$2n\pi + \frac{\pi}{4} < \pi x < 2n\pi + \frac{\pi}{4}$$

$$2n + \frac{1}{4} < x < 2n + \frac{1}{4}$$

$$\therefore \frac{\pi}{4\sqrt{2}} \int_{-10}^{10} F(x) \, dx$$

$$= \frac{\pi}{4\sqrt{2}} \times 2 \times 10 \int_0^1 f(x) \, dx$$

$$= \frac{5\pi}{\sqrt{2}} \left[\int_0^{1/4} \cos \pi x \, dx + \int_{1/4}^1 \sin \pi x \, dx \right]$$

$$= \frac{5\pi}{\sqrt{2}} \left[\frac{1}{\pi} \left[\sin \frac{\pi}{4} - 0 \right] - \frac{1}{\pi} \left[\cos \pi - \cos \frac{\pi}{4} \right] \right]$$

$$= \frac{5\pi}{\sqrt{2}} \left[\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \right] = 5$$

$$\text{Sol 37: } \int_0^{\pi/2} \tan^{-1} \left[\frac{1 + \sin x - (1 - \sin x)}{1 + \sin x + (1 - \sin x) - 2\sqrt{1 - \sin^2 x}} \right] dx$$

$$= \int_0^{\pi/2} \tan^{-1} \frac{2\sin x}{2 - 2\cos x} dx = \int_0^{\pi/2} \tan^{-1} \frac{2\sin x}{2 - 2\cos x} dx$$

$$= \int_0^{\pi/2} \tan^{-1} \tan \left(\frac{\pi}{2} - \frac{x}{2} \right) dx = \int_0^{\pi/2} \left(\frac{\pi}{2} - \frac{x}{2} \right) dx$$

$$= \frac{\pi^2}{4} - \frac{1}{2} \times \frac{\pi^2}{4} \times \frac{1}{2} = \frac{\pi^2}{4} - \frac{\pi^2}{16} = \frac{3\pi^2}{16}$$

$$\text{Sol 38: } x^2 + 2x = k + \int_0^1 |t+k| dt$$

$$t = k = 0 \Rightarrow dt = du$$

$$kt \int_k^{k+1} u du = \frac{1}{2} [(k+1)^2 - k^2]$$

$$= \frac{2k+1}{2}$$

$$x^2 + 2x = \frac{4k+1}{2} \Rightarrow x^2 + 2x - \left(\frac{4k+1}{2} \right) = 0$$

$$\Rightarrow x = -2 \pm \frac{\sqrt{4 + 2(4k+1)}}{2a}$$

$$\Rightarrow x = \text{real and distinct}$$

$$\text{Sol 39: } I = \int_{-1}^1 \frac{2x^{332} + x^{998} + 4x^{1668} \sin x^{691}}{1 + x^{666}} dx$$

$$I = \int_{-1}^1 \frac{2x^{332} + x^{998}}{1 + x^{666}} dx = 2 \int_0^1 \frac{2x^{332} + x^{998}}{1 + x^{666}} dx$$

$$= 2 \left[\int_0^1 \left(\frac{x^{332}}{1 + x^{666}} + x^{332} \right) dx \right]$$

$$= 2 \frac{1}{333} + 2 \int_0^1 \frac{x^{332}}{1 + (x^{333})^2} dx$$

$$= \frac{2}{333} + 2 \left[\int_0^1 \left(\frac{dt}{1+t^2} \right) \right] \frac{1}{333}$$

$$= \frac{2}{333} + \frac{2}{333} \tan^{-1} t \Big|_0^1 = \frac{2}{333} \left[1 + \frac{\pi}{4} \right] = \frac{\pi + 4}{666}$$

$$\text{Sol 40: } 2I = \pi \int_0^{\pi} \frac{[x^2 - (x - \pi)^2] \sin 2x \sin \left(\frac{\pi}{2} \cos x \right) dx}{2x - \pi}$$

$$= \pi^2 \int_0^{\pi} \sin 2x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

$$\text{Let } \frac{\pi}{2} \cos x = t \Rightarrow -\frac{\pi}{2} \sin x dx = dt$$

$$= -\pi^2 \times \frac{2}{\pi} \int_{\pi/2}^{-\pi/2} 2 \cdot \frac{2}{\pi} t \sin t dt$$

$$2I = 8 \int_{-\pi/2}^{\pi/2} t \sin t dt$$

$$I = 4 \left[-t \cos t \Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos t dt \right] = 4[+2] = 8$$

$$\text{Sol 41: } \int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4} dx = \int_{1/3}^1 \frac{[x^3 \left(\frac{1}{x^2} - 1 \right)]^{1/3}}{x^4} dx$$

$$dx = \int_{1/3}^1 \frac{\left(\frac{1}{x^2} - 1 \right)^{1/3}}{x^3} dx$$

$$\text{Put } \frac{1}{x^2} - 1 = t, \text{ then } -\frac{2}{x^3} dx = dt \text{ or } \frac{1}{x^3} dx = -\frac{1}{2} dt$$

$$\text{When } x = 1, t = \frac{1}{1^2} - 1 = 0 \text{ and when } x = \frac{1}{3}, t = 9 - 1 = 8$$

$$\text{Now, } \int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4} dx = -\frac{1}{2} \int_8^0 t^{1/3} dt = -\frac{1}{2} \left[\frac{t^{4/3}}{\frac{4}{3}} \right]_8^0$$

$$= -\frac{3}{8} [0 - 8^{4/3}] = -\frac{3}{8} [-2^4] = -\frac{3}{8} (-16) = 6$$

$$\text{Sol 42: } \lim_{n \rightarrow \infty} \frac{1}{n^2} \left[\sum_{k=0}^{n-1} k \int_k^{k+1} \sqrt{(x-k)(k+1-x)} dx \right]$$

$$x - k = t$$

$$\int_0^1 \sqrt{t(1-t)} dt = \int_0^1 \sqrt{\left(\frac{1}{2} \right)^2 - (t - \frac{1}{2})^2} dx$$

$$= \frac{1}{2} \left(t - \frac{1}{2} \right) \sqrt{\frac{1}{4} - \left(t - \frac{1}{2} \right)^2} + \frac{1}{8} \sin^{-1} \left(\frac{t - \frac{1}{2}}{\frac{1}{2}} \right) \Big|_0^1$$

$$= \frac{\pi}{2} \times \frac{1}{8} + \frac{\pi}{2} \times \frac{1}{8} = \frac{\pi}{8}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} k \times \frac{\pi}{8} = \int_0^1 \frac{\pi}{8} x dx = \frac{\pi}{8} \times \frac{1}{2} = \frac{\pi}{16}$$

$$\text{Sol 43: } I = \int_0^{\pi/2} \frac{\sin x + 3}{5 \sin(x + \alpha) + 25} dx \quad \cos \alpha = \frac{3}{5}$$

$$4I + 3J = \int_0^{\pi/2} \frac{4 \cos x + 3 \sin x + 25}{4 \cos x + 3 \sin x + 25} dx = \frac{\pi}{2}$$

$$3I - 4J = \int_0^{\pi/2} \frac{3 \cos x - 4 \sin x}{4 \cos x + 3 \sin x + 25} dx$$

$$= \log(4 \cos x + 3 \sin x + 25) \Big|_0^{\pi/2}$$

$$= \log(28) - \log(29) = \log \frac{28}{29}$$

$$16I + 9I = 2\pi + 3 \log \frac{28}{29} = 2\pi + 3 \log \frac{28}{29}$$

$$a + b + c + d = 2 + 3 + 28 + 29 = 62$$

$$\text{Sol 44: } f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

$$f'(2) = 4a + b = 1$$

$$f'(2) = 4a + b = 1$$

$$\int_{2-\pi}^{2+\pi} f(x) \sin\left(\frac{x-2}{2}\right) dx = - \int_{2-\pi}^{2+\pi} f(4-x) \sin\left(\frac{x-2}{2}\right) dx$$

$$2I = - \int_{2-\pi}^{2+\pi} [f(x) - f(4-x)] \sin\left(\frac{x-2}{2}\right) dx$$

$$= - \int_{2-\pi}^{2+\pi} \left\{ \begin{array}{l} ax^2 + bx + c \\ -[a(4-x)^2 + b(4-x) + c] \end{array} \right\} \sin\left(\frac{x-2}{2}\right) dx$$

$$= \int_{2-\pi}^{2+\pi} \left[\begin{array}{l} a(x-4+x)(x+4-x) \\ +b(x-4+x) \end{array} \right] \sin\left(\frac{x-2}{2}\right) dx$$

$$= \int_{2-\pi}^{2+\pi} (a(2x-4)4 + 2bx - 4b) \sin\left(\frac{x-2}{2}\right) dx$$

$$= \int_{2-\pi}^{2+\pi} (8ax + 2bx - 4) \sin\left(\frac{x-2}{2}\right) dx$$

$$= 4 \int_{2-\pi}^{2+\pi} \frac{(x+2)}{2} \sin\left(\frac{x-2}{2}\right) dx \quad \frac{x-2}{2} = t$$

$$= 4 \int_{-\pi/2}^{\pi/2} t \sin t dt = 4 \left[t(-\cos t) \Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos t dt \right]$$

$$= 4 \left[\sin t \Big|_{-\pi/2}^{\pi/2} \right] = 8$$

Exercise 2

Single Correct Choice Type

$$\text{Sol 1: (C)} \int_0^2 |(x+3)(x-1)| dx$$

$$= \int_0^1 (x+3)(1-x) dx + \int_1^2 (x+3)(x-1) dx$$

$$= - \int_0^1 (x^2 + 2x - 3) dx + \int_1^2 (x^2 + 2x - 3) dx$$

$$= - \left[\frac{x^3}{3} + x^2 - 3x \Big|_0^1 \right] + \left[\frac{x^3}{3} + x^2 - 3x \Big|_1^2 \right]$$

$$= - \left[\frac{1}{3} + 1 - 3 \right] + \left[\frac{8}{3} + 4 - 6 - \left(\frac{1}{3} + 1 - 3 \right) \right]$$

$$= \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4$$

$$\text{Sol 2: (B)} \int_0^{\pi/2} \left| \frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x \right| dx$$

$$= \frac{1}{\sqrt{2}} \left[\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \right]$$

$$= \frac{1}{\sqrt{2}} \left[\sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x) \Big|_{\pi/4}^{\pi/2} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right] - \left[1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right] \right]$$

$$= \frac{1}{\sqrt{2}} \left[\sqrt{2} - 1 - 1 + \sqrt{2} \right] = \frac{2\sqrt{2} - 2}{\sqrt{2}} = 2 - \sqrt{2}$$

$$\text{Sol 3: (D)} 2 \int_0^{\pi/2} (\sin^4 x) dx = 2 \times \left(\frac{(4-1)(4-3)}{4 \times (4-2)} \right) \times \frac{\pi}{2} = \frac{3\pi}{8}$$

$$\text{Sol 4: (B)} \int_0^1 0 dx = \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{1.5} 2 dx$$

$$= \sqrt{2} - 1 + 2(1.5 - \sqrt{2}) = 2 - \sqrt{2}$$

$$\text{Sol 5: (A)} \quad I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\text{Then } I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\text{Or } I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$= 2\pi \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x) \right]$$

$$\therefore 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

[Dividing num. and denom. By $\cos^2 x$]

$$\text{Or } I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

Let $\tan x = z$. Then, $\sec^2 x dx = dz$

$$\text{Also } x = 0 \Rightarrow z = \tan 0 = 0 \text{ and } x \rightarrow \frac{\pi}{2} \Rightarrow z \rightarrow \tan \frac{\pi}{2}$$

or $z \rightarrow \infty$

$$\therefore I = \pi \int_0^{\infty} \frac{dz}{a^2 + b^2 z^2} = \frac{\pi}{b^2} \int_0^{\infty} \frac{dz}{(a/b)^2 + z^2}$$

$$= \frac{\pi}{b^2} \times \frac{1}{(a/b)} \left[\tan^{-1} \left(\frac{z}{a/b} \right) \right]_0^{\infty}$$

$$\Rightarrow I = \frac{\pi}{ab} \left[\tan^{-1} \left(\frac{bz}{a} \right) \right]_0^{\infty} = \frac{\pi}{ab} (\tan^{-1} \infty - \tan^{-1} 0)$$

$$= \frac{\pi}{ab} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2ab}$$

...(i)

$$\text{Sol 6: (A)} \quad \int_0^{\pi/4} \frac{\sec x}{1 + 2 \sin^2 x} dx$$

$$= \int_0^{\pi/4} \frac{dx}{\cos x + 2 \sin^2 x \cos x}$$

$$= \int_0^{\pi/4} \frac{\cos x dx}{\cos^2 x + 2 \sin^2 x \cos^2 x}$$

...(ii)

$$= \int_0^{\pi/4} \frac{\cos x dx}{(1 - \sin^2 x)(1 + 2 \sin^2 x)}$$

$$= \int_0^{1/\sqrt{2}} \frac{dt}{(1 - t^2)(1 + 2t^2)}$$

$$= \frac{1}{3} \int_0^{1/\sqrt{2}} \left(\frac{1}{1 - t^2} + \frac{2}{1 + 2t^2} \right) dt$$

$$= \frac{1}{3} \int_0^{1/\sqrt{2}} \frac{1}{1 - t^2} dt + \frac{1}{3} \int_0^{1/\sqrt{2}} \frac{1}{\frac{1}{2} + t^2} dt$$

$$= \frac{1}{6} \ln \frac{1+t}{1-t} \Big|_0^{1/\sqrt{2}} + \frac{1}{3} \times \frac{1}{1/\sqrt{2}} \tan^{-1} \sqrt{2} t \Big|_0^{1/\sqrt{2}}$$

$$= \frac{1}{6} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}}{3} \times \frac{\pi}{4} = \frac{1}{3} \left[\ln(\sqrt{2}+1) + \frac{\pi}{2\sqrt{2}} \right]$$

$$\text{Sol 7: (C)} \quad \int_0^1 e^{x^2} (x - \alpha) dx = 0$$

For this integral to be zero

If $\alpha < 0$ then $x - \alpha$ when $x \in (0, 1) > 0$

\therefore It is not possible that integral reduce to zero

If $2 > \alpha > 1$ then $x - \alpha$

when $x \in (0, 1) < 0$ function gives negative value and so cannot reduced zero.

\therefore If $0 < \alpha < 1$, f^1 can take both positive and negative value and it is possible that integral reduced to zero

$$\text{Sol 8: (A)} \quad \int_0^{\pi/2} \{x - [\sin x]\} dx$$

[] \rightarrow greatest integer function

$$[\sin x] = 0 \text{ } \sin x \in [0, 1) \text{ i.e. } x \in \left[0, \frac{\pi}{2} \right)$$

$$= \int_0^{\pi/2} (x - 0) dx = \frac{x^2}{2} \Big|_0^{\pi/2} = \frac{\pi^2}{8}$$

$$\text{Sol 9: (B)} \int_0^{100} \sin(x[x]) \pi dx$$

Since $x - [x]$ has a period of 1

$$\begin{aligned} \therefore I &= 100 \int_0^1 \sin \pi x dx = \frac{100}{\pi} \left(-\cos \pi x \Big|_0^1 \right) \\ &= \frac{100}{\pi} (-(-1) - (-1)) = \frac{200}{\pi} \end{aligned}$$

$$\text{Sol 10: (B)} \int_0^{\infty} \frac{x \log x}{(1+x^2)^2} dx$$

$$x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$I = \int_0^{\pi/2} \frac{\tan \theta \log(\tan \theta) \sec^2 \theta d\theta}{\sec^4 \theta}$$

$$= \int_0^{\pi/2} \tan \theta \log(\tan \theta) \cos^2 \theta d\theta$$

$$= \int_0^{\pi/2} \sin \theta \cos \theta \log(\tan \theta) d\theta$$

$$= \int_0^{\pi/2} \cos \theta \sin \theta \log \cot \theta d\theta$$

$$\therefore 2I = \int_0^{\pi/2} \sin \theta \cos \theta [\log \tan \theta + \log \cot \theta] d\theta$$

$$= \int_0^{\pi/2} \sin \theta \cos \theta [\log \tan - \log \tan \theta] d\theta = 0$$

$$\text{Sol 11: (B)} \log I = \lim_{n \rightarrow \infty} \sum_{r=1}^n \log \left(1 + \frac{r}{n} \right) \times \frac{1}{n}$$

$$\log I = \int_0^1 \log(1+x) dx$$

$$\log I = x \log(1+x) \Big|_0^1 - \int_0^1 \frac{x}{1+x} dx$$

$$= \log 2 - \int_0^1 1 - \frac{1}{1+x} dx = \log 2 - [1] + \log(1+x) \Big|_0^1$$

$$= 2 \log 2 - \log e = \log \frac{4}{e} \Rightarrow I = \frac{4}{e}$$

$$\text{Sol 12: (A)} \int_{\log_2}^x \frac{1}{\sqrt{e^x - 1}} dx$$

$$\text{Put } e^x - 1 = t^2 \Rightarrow e^x dx = 2t dt \text{ or } dx = \frac{2t}{1+t^2} dt$$

$$\int_1^{\sqrt{e^x-1}} \frac{2}{1+t^2} dt = 2 \tan^{-1} t \Big|_1^{\sqrt{e^x-1}} = \frac{\pi}{6}$$

$$2 \left[\tan^{-1} t - \frac{\pi}{4} \right] = \frac{\pi}{6}$$

$$\therefore \tan^{-1} t = \frac{\pi}{3} \Rightarrow \therefore t = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\therefore e^x - 1 = 3 \Rightarrow x = \log 4$$

$$\text{Sol 13: (A)} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^2}{r^3 + n^3} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3 + 1} \times \frac{1}{n}$$

$$= \int_0^1 \frac{x^2}{x^3 + 1} dx = \frac{1}{3} \ell n(x^3 + 1) \Big|_0^1 = \frac{\ell n 2}{3}$$

$$\text{Sol 14: (B)} \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{n^2}{(n^2 + r^2)^{3/2}} \right] - \frac{1}{2n}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\left(1 + \left(\frac{r}{n}\right)^2\right)^{3/2}} \times \frac{1}{n} = \int_0^1 \frac{1}{(1+x^2)^{3/2}} dx$$

$$\text{Put } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\frac{\pi}{4} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int_0^{\pi/4} \cos \theta d\theta = \sin \theta \Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}$$

$$\text{Sol 15: (A)} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\sqrt{n}}{\sqrt{[n+4(r-1)]^3}} - \frac{\sqrt{n}}{\sqrt{(n+4n)^3}}$$

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{\left(1 + \frac{4r}{n}\right)^3}} \times \frac{1}{n} - \frac{1}{(5)^{3/2} \times n^{1/2}}$$

$$= \int_0^1 \frac{1}{\sqrt{(1+4x)^3}} dx = \int_0^1 (1+4x)^{-3/2} dx$$

$$= \frac{(1+4x)^{-1/2}}{-1/2} \times \frac{1}{4} \Big|_0^1 = -\frac{1}{2} \left(\frac{1}{1+4x} \right)^{1/2} \Big|_0^1$$

$$= -\frac{1}{2} \left[\frac{1}{\sqrt{5}} - 1 \right] = \frac{1}{10} (5 - \sqrt{5})$$

$$\text{Sol 16: (B)} \log l = \lim_{n \rightarrow \infty} \left(\sum_{r=1}^n \log \tan \left(\frac{\pi r}{2n} \right) \right) \frac{1}{n}$$

$$= \int_0^1 \log \tan \left(\frac{\pi x}{2} \right) dx$$

$$\frac{\pi}{2} x = t \Rightarrow dx = \frac{2}{\pi} dt$$

$$\Rightarrow \int_0^{\pi/2} \log \tan t dt = 0$$

$$\therefore l = e^0 = 1$$

Previous Years' Questions

$$\text{Sol 1: (A)} \int_{-1/2}^{1/2} \left[x + \log \left(\frac{1+x}{1-x} \right) \right] dx$$

$$= \int_{-1/2}^{1/2} x dx + \int_{-1/2}^{1/2} \log \left(\frac{1+x}{1-x} \right) dx$$

$$= \int_{-1/2}^{1/2} x dx + 0$$

$$\left[\because \log \left(\frac{1+x}{1-x} \right) \text{ is an odd function} \right]$$

$$= \int_{-1/2}^0 x dx + \int_0^{1/2} x dx$$

$$= \int_{-1/2}^0 (-1) dx + \int_0^{1/2} (0) dx$$

$$= [x]_{-1/2}^0$$

$$= - \left(0 + \frac{1}{2} \right) = -\frac{1}{2}$$

Sol 2: (A) Here, $I(m, n) = \int_0^1 t^m (1+t)^n dt$ reduce into $I(m+1, n-1)$ [we apply integration by parts taking $(1+t)^n$ as first and t^m as second function]

$$\therefore I(m, n) = \left[(1+t)^n \cdot \frac{t^{m+1}}{m+1} \right]_0^1$$

$$- \int_0^1 n(1+t)^{(n-1)} \cdot \frac{t^{m+1}}{m+1} dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} \int_0^1 (1+t)^{(n-1)} \cdot t^{m+1} dt$$

$$\therefore I(m, n) = \frac{2^n}{m+1} - \frac{n}{m+1} \cdot I(m+1, n-1)$$

$$\text{Sol 3: (C)} \text{ Given } \int_0^x \sqrt{1 - (f'(t))^2} dt$$

$$= \int_0^x f(t) dt, 0 \leq x \leq 1$$

Differentiating both sides w.r.t. x by using Leibnitz rule, we get

$$\sqrt{1 - (f'(x))^2} = f(x)$$

$$\Rightarrow f'(x) = \pm \sqrt{1 - (f(x))^2}$$

$$\Rightarrow \int \frac{f'(x)}{\sqrt{1 - (f(x))^2}} dx = \pm \int dx$$

$$\Rightarrow \sin^{-1}(f(x)) = \pm x + c$$

$$\text{Put } x = 0$$

$$\Rightarrow \sin^{-1}(f(0)) = c$$

$$\Rightarrow c = \sin^{-1}(0) = 0 \quad (\because f(0) = 0)$$

$$\therefore f(x) = \pm \sin x$$

$$\text{but } f(x) \geq 0, \forall x \in [0, 1]$$

$$\therefore f(x) = \sin x$$

As we know that,

$$\sin x < x \quad \forall x > 0$$

$$\therefore \sin \left(\frac{1}{2} \right) < \frac{1}{2} \text{ and } \sin \left(\frac{1}{3} \right) < \frac{1}{3}$$

$$\Rightarrow f \left(\frac{1}{2} \right) < \frac{1}{2} \text{ and } f \left(\frac{1}{3} \right) < \frac{1}{3}$$

Sol 4: (A) $x^2 = t \Rightarrow 2x dx = dt$

$$I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin t}{\sin t + \sin(\log 6 - t)} dt$$

\therefore

$$I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$$

$$2I = \frac{1}{2} \int_{\log 2}^{\log 3} 1 dt \Rightarrow I = \frac{1}{4} \log \frac{3}{2}$$

Sol 5: (A, D) Gives, $S_n = \sum_{k=0}^n \frac{n}{n^2 + kn + k^2}$

$$= \sum_{k=0}^n \frac{1}{n} \left(\frac{1}{1 + \frac{k}{n} + \frac{k^2}{n^2}} \right) < \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n}$$

$$\left(\frac{1}{1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2} \right)$$

$$= \int_0^1 \frac{1}{1+x+x^2} dx = \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) \right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \cdot \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{3\sqrt{3}}$$

i.e. $S_n < \frac{\pi}{3\sqrt{3}}$

Similarly, $T_n > \frac{\pi}{3\sqrt{3}}$

Sol 6: (A, B, C) Given $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$... (i)

Using $\int_a^b f(x) dx = \int_a^b f(b+a-x) dx$

we get $I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1 + \pi^x) \sin x} dx$... (ii)

Adding Eqs. (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx = 2 \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

(∵ $f(x) = \frac{\sin nx}{\sin x}$ is an even function)

$$\Rightarrow I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

Now, $I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx$

$$= \int_0^{\pi} \frac{2 \cos(n+1)x \cdot \sin x}{\sin x} dx$$

$$= 2 \int_0^{\pi} \cos(n+1)x dx$$

$$= 2 \left[\frac{\sin(n+1)x}{(n+1)} \right]_0^{\pi} = 0$$

$$\therefore I_{n+2} = I_n \quad \dots \text{(iii)}$$

Since, $I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx \Rightarrow I_1 = \pi$ and $I_2 = 0$

∴ From Eq. (iii) $I_1 = I_3 = I_5 = \dots = \pi$

and $I_2 = I_4 = I_6 = \dots = 0$

$$\Rightarrow \sum_{m=1}^{10} I_{2m+1} = 10\pi \text{ and } \sum_{m=1}^{10} I_{2m} = 0$$

∴ Correct options are A, B, C.

Sol 7: (A) Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$

$$= \int_0^1 \frac{(x^4 - 1)(1-x)^4 + (1-x)^4}{(1+x^2)} dx$$

$$= \int_0^1 (x^2 - 1)(1-x)^4 dx + \int_0^1 \frac{(1+x^2 - 2x)^2}{(1+x^2)} dx$$

$$= \int_0^1 \left\{ (x^2 - 1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4x^2}{(1+x^2)} \right\} dx$$

$$= \int_0^1 \left((x^2 - 1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1-x^2} \right) dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= \int_0^1 \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4 \tan^{-1} x \right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4 \left(\frac{\pi}{4} - 0 \right) = \frac{22}{7} - \pi$$

$$= (\operatorname{cosec} x \cdot \cot x + \sec^2 x - \cos x)$$

$$\cdot (\cos^3 x - \cos x) \cdot \cos x$$

$$= - \left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \cdot \sin^2 x}{\sin^2 x \cdot \cos^2 x} \right]$$

$$\cdot \cos^2 x \cdot \sin^2 x$$

$$= - \sin^2 x - \cos^3 x (1 - \sin^2 x)$$

$$= - \sin^2 x - \cos^5 x$$

$$\therefore \int_0^{\pi/2} f(x) dx = - \int_0^{\pi/2} (\sin^2 x + \cos^5 x) dx$$

$$\left[\begin{aligned} \therefore \int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx &= \frac{\sqrt{\frac{m+1}{2}} \sqrt{\frac{n+2}{2}}}{2\sqrt{\frac{m+n+2}{2}}} \\ \therefore \int_0^{\pi/2} f(x) \, dx &= - \left\{ \frac{\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}}}{2\sqrt{2}} + \frac{\sqrt{\frac{6}{2}} \cdot \sqrt{\frac{1}{2}}}{2\sqrt{\frac{7}{2}}} \right\} \end{aligned} \right]$$

Sol 8: (C) $\int_0^{\pi/2} \sin x \, dx$

$$= \frac{\pi - 0}{4} \left(\sin 0 + \sin\left(\frac{\pi}{2}\right) + 2 \sin\left(\frac{0 + \frac{\pi}{2}}{2}\right) \right)$$

$$= \frac{\pi}{8} (1 + \sqrt{2})$$

Sol 9: (A) $F'(c) = (b - a) f'(c) + f(a) - f(b)$

$$F''(c) = f''(c) (b - a) < 0$$

$$\Rightarrow F'(c) = 0$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sol 10: (B) Given, $\lim_{t \rightarrow a} \frac{\int_a^t f(x) \, dx - \frac{(t-a)}{2} \{f(t) + f(a)\}}{(t-a)^3} = 0$

Using L' Hospital's rule

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\int_a^{a+h} f(x) \, dx - \frac{h}{2} \{f(a+h) + f(a)\}}{h^3} = 0$$

$$f(a+h) - \frac{1}{2} \{f(a+h) + f(a)\}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-\frac{h}{2} \{f'(a+h)\}}{3h^2} = 0$$

Again, using L'Hospital's rule

$$f'(a+h) - \frac{1}{2} f'(a+h) - \frac{1}{2} f'(a+h)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-\frac{h}{2} f''(a+h)}{6h} = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-\frac{h}{2} f''(a+h)}{6h} = 0$$

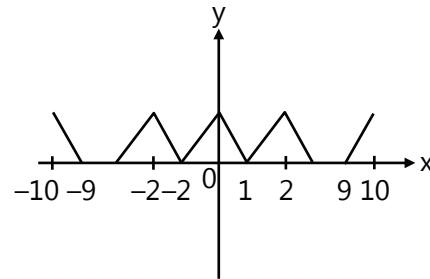
$$\Rightarrow f''(a) = 0, \forall a \Rightarrow R$$

$\Rightarrow f(x)$ must have maximum degree 1

Sol 11: Given, $f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$

$f(x)$ and $\cos \pi x$ both are periodic with period 2 and both are even.

$$\therefore \int_{-10}^{10} f(x) \cos \pi x \, dx = 2 \int_0^{10} f(x) \cos \pi x \, dx$$



$$= 10 \int_0^3 f(x) \cos \pi x \, dx$$

Now, $\int_0^1 f(x) \cos \pi x \, dx$

$$= \int_0^1 (1-x) \cos \pi x \, dx = - \int_0^1 u \cos \pi u \, du \text{ and}$$

$$\int_1^2 f(x) \cos \pi x \, dx = \int_1^2 (x-1) \cos \pi x \, dx = - \int_0^1 u \cos \pi u \, du$$

$$\therefore \int_{-10}^{10} f(x) \cos \pi x \, dx = -20 \int_0^1 u \cos \pi u \, du = \frac{40}{\pi^2}$$

$$\Rightarrow \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx = 4$$

Sol 12: $f(x) = \int_1^x \frac{\ln t}{1+t} \, dt$ for $x > 0$ (given)

Now, $f(1/x) = \int_1^{1/x} \frac{\ln t}{1+t} \, dt$

Put $t = 1/u$

$$\Rightarrow dt = (-1/u^2) du$$

$$\therefore f(1/x) = \int_1^x \frac{\ln(1/u)}{1+1/u} \cdot \frac{(-1)}{u^2} du$$

$$= \int_1^x \frac{\ln u}{u(u+1)} du = \int_1^x \frac{\ln t}{t(1+t)} dt$$

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log t}{(1+t)} dt + \int_1^x \frac{\log t}{(1+t)} dt$

$$= \int_1^x \frac{(1+t)\log t}{t(1+t)} dt + \int_1^x \frac{x \log t}{t} dt$$

$$= \frac{1}{2} \left[(\log t)^2 \right]_1^x = \frac{1}{2} (\log x)^2$$

Put $x = e$

$$\therefore f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2} (\log e)^2 = \frac{1}{2}$$

Hence proved.

Sol 13: Let $I = \int_0^{\pi/2} f(\cos 2x) \cos x dx$... (i)

$$I = \int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2} - x\right)\right) \cos\left(\frac{\pi}{2} - x\right) dx$$

$$\left[\text{using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$I = \int_0^{\pi/2} f(\cos 2x) \sin x dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\cos 2x) (\sin x + \cos x) dx$$

$$= \sqrt{2} \int_0^{\pi/2} f(\cos 2x) [\cos(x - \pi/4)] dx$$

Put $-x + \frac{\pi}{4} = t \Rightarrow -dx = dt$

$$\therefore 2I = -\sqrt{2} \int_{\pi/4}^{-\pi/4} f\left(\cos\left(\frac{\pi}{2} - 2t\right)\right) \cos t dt$$

$$\therefore 2I = \sqrt{2} \int_{-\pi/4}^{\pi/4} f(\sin 2t) \cot t dt$$

$$\therefore I = \sqrt{2} \int_0^{\pi/4} f(\sin 2t) \cot t dt$$

Sol 14:

Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} + 4 \int_{-\pi/3}^{\pi/3} \frac{x^3 dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$

Using $\int_{-a}^a f(x) dx = \begin{cases} 0, & f(-x) = -f(x) \\ 2 \int_0^a f(x) dx, & f(-x) = f(x) \end{cases}$

$$\therefore I = 2 \int_0^{\pi/3} \frac{\pi dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} + 0$$

$$\left[\frac{x^3 dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} \text{ is odd} \right]$$

$$I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos(x + \pi/3)}$$

Put $x + \frac{\pi}{3} = t \Rightarrow dx = dt$

$$\therefore I = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dt}{2 - \cos t} = 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 \frac{t}{2} dt}{1 + 3 \tan^2 \frac{t}{2}}$$

Put $\tan \frac{t}{2} = u \Rightarrow \sec^2 \frac{t}{2} dt = 2du$

$$\Rightarrow I = 2\pi \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2du}{1 + 3u^2} = \frac{4\pi}{3} [\sqrt{3} \tan^{-1} \sqrt{3}u]_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}} (\tan^{-1} 3 - \tan^{-1} 1) = \frac{4\pi}{\sqrt{3}} \tan^{-1} \left(\frac{1}{2}\right)$$

$$\therefore \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx = \frac{4\pi}{\sqrt{3}} \tan^{-1} \left(\frac{1}{2}\right)$$

Sol 15: Let

$$I = \int_0^{\pi} e^{|\cos x|} \left(2 \sin\left(\frac{1}{2} \cos x\right) + 3 \cos\left(\frac{1}{2} \cos x\right) \right) \sin x dx$$

$$\Rightarrow I = \int_0^{\pi} e^{|\cos x|} \sin x \cdot 2 \sin\left(\frac{1}{2} \cos x\right) dx$$

$$+ \int_0^{\pi} e^{|\cos x|} \cdot 3 \cos\left(\frac{1}{2} \cos x\right) \cdot \sin x dx$$
 ... (i)

$$\Rightarrow I = I_1 + I_2$$

$$\left(\text{using } \int_0^{2a} f(x) dx = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2 \int_0^a f(x) dx, & f(2a-x) = +f(x) \end{cases} \right)$$

where $I_1 = 0$ [$\because f(\pi - x) = -f(x)$] ... (ii)

and

$$I_2 = 6 \int_0^{\pi/2} e^{\cos x} \cdot \sin x \cdot \cos\left(\frac{1}{2} \cos x\right) dx$$

Now, $I_2 = 6 \int_0^1 e^t \cdot \cos\left(\frac{t}{2}\right) dt$

(Put $\cos x = t \Rightarrow -\sin x dx = dt$)

$$= 6 \left[e^t \cos\left(\frac{t}{2}\right) + \frac{1}{2} \int e^t \sin\frac{t}{2} dt \right]_0^1$$

$$= 6 \left[e^t \cos\left(\frac{t}{2}\right) + \frac{1}{2} \left(e^t \sin\frac{t}{2} - \int \frac{e^t}{2} \cos\frac{t}{2} dt \right) \right]_0^1$$

$$= 6 \left[e^t \cos\frac{t}{2} + \frac{1}{2} e^t \sin\frac{t}{2} \right]_0^1 - \frac{I_2}{4}$$

$$= \frac{24}{5} \left(e \cos\left(\frac{1}{2}\right) + \frac{e}{2} \sin\left(\frac{1}{2}\right) - 1 \right) \quad \dots \text{(iii)}$$

From Eqs. (i), we get

$$I = \frac{24}{5} \left(e \cos\left(\frac{1}{2}\right) + \frac{e}{2} \sin\left(\frac{1}{2}\right) - 1 \right)$$

Sol 16: Let $I_2 = \int_0^1 (1 - x^{50})^{101} dx$,

using integration by parts

$$= \left[(1 - x^{50})^{101} \cdot x \right]_0^1$$

$$+ \int_0^1 (1 - x^{50})^{100} \cdot 50 \cdot x^{49} \cdot x dx$$

$$= 0 - \int_0^1 (50)(101)(1 - x^{50})^{100} (-x^{50}) dx$$

$$= -50(101) \int_0^1 (1 - x^{50})^{101} dx + (50)(101) \int_0^1 (1 - x^{50})^{100} dx$$

$$= 5050I_2 + 5050I_1$$

$$\therefore I_2 + 5050I_2 = 5050I_1$$

$$\therefore \frac{(5050)I_1}{I_2} = 5051$$

Sol 17: (B) $g(x) = \frac{f'(e^x)e^x}{1 + e^{2x}}$

Hence positive for $(0, \infty)$ and negative for $(-\infty, 0)$

Consider the line

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}, L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

Sol 18: (D) Hence $\int_{-1}^1 g'(x) = g(1) - g(-1) = 2g(1)$

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the

equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

Further, it is given that the origin and the centre of C are on the same side of the line PQ.

Sol 19: (A) Let $f(x) = \int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$

$$f(x) = \frac{x^2 + x^4 - 2}{x^4 + 1} \Rightarrow \frac{-2x^4 + x - 2}{x^4 + 1} < \forall x \in \mathbb{R}$$

$f(0) > 0, f(1) < 0$

\therefore One solution in $(0, 1)$

Sol 20: $f(x) = \int_0^x f(t) dt \Rightarrow f(0) = 0$

Also $f(x) = f(x), x > 0 \Rightarrow f(x) = ke^x, x > 0$

$\therefore f(0) = 0$ and $f(x)$ is continuous

$\Rightarrow f(x) = 0 \forall x > 0$

$\therefore f(\ln 5) = 0$

Sol 21: (C) $f' = \pm \sqrt{1 - f^2}$

$\Rightarrow f(x) = \sin x$ or $f(x) = -\sin x$ (not possible)

$\Rightarrow f(x) = \sin x$

Also $x > \sin x \forall > 0$.

Sol 22: $A \rightarrow p, q, s; B \rightarrow p, t; C \rightarrow p, q, r, t; D \rightarrow s$

(A) $(x-3)^2 \frac{dy}{dx} + y = 0$

$$\int \frac{dx}{(x-3)^2} = \int \frac{dy}{y} \Rightarrow \frac{1}{x-3} = \ln|y| + c$$

So domain is $\mathbb{R} - \{3\}$.

(B) Put $x = t + 3$

$$\int (t+2)(t+1)t(t-1)(t-2) dt = \int t(t^2-1)(t^2-4) dt = 0$$

(C) $f(x) = \frac{5}{4} - \left(\sin x - \frac{1}{2}\right)^2$

Maximum value occurs when $\sin x = \frac{1}{2}$

(D) $f(x) > 0$ if $\cos > \sin x$

Sol 23: (A) Let the line be $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ intersects the lines

$$\Rightarrow S.D = 0 \Rightarrow a + 3b + 5c = 0 \text{ and}$$

$$3a + b - 5c = 0 \Rightarrow a : b : c :: 5r : -5r : 2r$$

on solving with given lines we get points of intersection

$$P \equiv (5, -5, 2) \text{ and } Q \equiv \left(\frac{10}{3}, -\frac{10}{3}, \frac{8}{3}\right) \Rightarrow PQ^2 = d^2 = 6$$

(B) (p, r)

$$\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$$

$$\Rightarrow \tan^{-1} \frac{(x+3) - (x-3)}{1 + (x^2-9)} = \tan^{-1} \frac{3}{4} \Rightarrow \frac{6}{x^2-8} = \frac{3}{4}$$

$$\therefore x^2 - 8 = 8$$

$$\text{Or } x = \pm 4$$

(C) (q, s)

$$\text{As } \bar{a} = \mu \bar{b} + 4 \bar{c} \Rightarrow \mu (|\bar{b}|) = -4 \bar{b} \cdot \bar{c} \text{ and } |\bar{b}|^2 = 4 \bar{a} \cdot \bar{c}$$

$$\text{and } |\bar{b}|^2 + \bar{b} \cdot \bar{c} - \bar{d} \cdot \bar{c} = 0$$

$$\text{Again, as } 2|\bar{b} + \bar{c}| = |\bar{b} - \bar{a}|$$

Solving and eliminating $\bar{b} \cdot \bar{c}$ and eliminating $|\bar{a}|^2$

$$\text{We get } (2\mu^2 - 10\mu) |\bar{b}|^2 = 0 \Rightarrow \mu = 0 \text{ and } 5.$$

$$(D) I = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{\sin 9(x/2)}{\sin(x/2)} dx = \frac{2}{\pi} \times 2 \int_0^{\pi} \frac{\sin 9(x/2)}{\sin(x/2)} dx$$

$$x/2 = \theta \Rightarrow dx = 2d\theta$$

$$x = \pi\theta = \pi/2$$

$$= \frac{8}{\pi} \int_0^{\pi/2} \frac{\sin 9\theta - \sin 7\theta}{\sin \theta} + \frac{(\sin 7\theta - \sin 5\theta)}{\sin \theta} + \frac{(\sin 5\theta - \sin 3\theta)}{\sin \theta} + \frac{(\sin 3\theta - \sin \theta)}{\sin \theta} + \frac{\sin \theta}{\sin \theta} d\theta = \frac{16}{\pi}$$

$$\int_0^{\pi/2} (\cos 8\theta + \cos 6\theta + \cos 4\theta + \cos 2\theta + 1) + \frac{8}{\pi} \int_0^{\pi/2} d\theta$$

$$= \frac{16}{\pi} \left[\frac{\sin 8\theta}{8} + \frac{\sin 6\theta}{6} + \frac{\sin 4\theta}{4} + \frac{\sin 2\theta}{2} \right]$$

$$+ \frac{8}{\pi} [\theta]^{\pi/2} = 0 \frac{8}{\pi} \times \left[\frac{\pi}{2} - 0 \right] = 4$$

$$\text{Sol 24: (B)} \lim_{x \rightarrow 0} \frac{x \ln(1+x)}{(x^4+4) \times 3x^2} = \lim_{x \rightarrow 0} \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

Sol 25: (A)

$$\int_0^1 \frac{x^4(1-x)^1}{1+x^2} = \int_0^1 \frac{x^4[(1+x^2)-2x]^2}{1+x^2} =$$

$$\int_0^1 \frac{x^4(1-x)^1}{1+x^2} = \int_0^1 \frac{x^4[(1+x^2)-2x]^2}{1+x^2}$$

$$= \int_0^1 x^4 \left[(1+x^2) - 4x + \frac{4x^2}{1+x^2} \right] dx$$

$$= \int \left[x^6 + x^4 - 4x^5 + \frac{4x^6}{1+x^2} \right] dx$$

Now on polynomial division of x^6 by $1+x^2$, we obtain

$$\int \left[x^6 + x^4 - 4x^5 + 4 \left[(x^4 - x^2 + 1) - \frac{1}{1+x^2} \right] \right]$$

$$dx = \int \left[(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - \frac{4}{1+x^2} \right] dx$$

$$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x \right]_0^1 - 4 \left[\tan^{-1} x \right]$$

$$= \left(\frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4 \right) - 4 \left(\frac{\pi}{4} \right) = \left(\frac{1}{7} + 3 \right) - \pi = \frac{22}{7} - \pi$$

Sol 26: (A) $x^2 = t \Rightarrow 2x dx = dt$

$$I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt$$

$$\text{and } I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin t(\ln 6 - t) + \sin t} dt$$

$$x^2 = t \Rightarrow 2x dx = dt$$

$$2I = \frac{1}{2} \int_{\ln 2}^{\ln 3} 1 dt \Rightarrow I = \frac{1}{4} \ln \frac{3}{2}$$

Sol 27: (B)

$$\int_{-\pi/2}^{\pi/2} \left(x^2 + \ln \left(\frac{\pi+x}{\pi-x} \right) \right) \cos x dx$$

$$= 2 \int_0^{\pi/2} x^2 \cos x dx + 0 \left(\because \ln \left(\frac{\pi+x}{\pi-x} \right) \text{ is an odd function} \right)$$

$$= 2 \left[(x^2 \sin x) \int_0^{\pi/2} 2x \sin x dx \right] = 2 \left(\frac{\pi^2}{4} - 0 \right) - 4 \int_0^{\pi/2} x \sin x dx$$

$$= \frac{\pi^2}{2} - 4 \left[(-x \cos x) \int_0^{\pi/2} \cos x dx \right] = \frac{\pi^2}{2} - 4$$

$$\text{Sol 28: (A)} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec}x)^{17} dx$$

Let

$$e^n + e^{-n} = 2\operatorname{cosec}x, x = \frac{\pi}{4} \Rightarrow u = \log(1 + \sqrt{2}), x = \frac{\pi}{2} \Rightarrow u = 0$$

$$\Rightarrow \operatorname{cosec}x + \cot x = e^n \text{ and}$$

$$x - \cot x = e^{-n} \Rightarrow \cot x = \frac{e^u - e^{-u}}{2}$$

$$(e^u - e^{-u}) dx = -2\operatorname{cosec}x \cot x dx$$

$$\Rightarrow -\int (e^u + e^{-u})^{17} \frac{(e^u - e^{-u})}{2\operatorname{cosec}x \cot x} du$$

$$= -2 \int_{\log(1+\sqrt{2})}^0 (e^u + e^{-u}) du = \int_{\log(1+\sqrt{2})}^{\log(1+\sqrt{2})} 2(e^u + e^{-u}) du$$

$$\text{Sol 29: (D)} \text{ (p)} f(x) = ax^2 + bx, \int_0^1 f(x) dx = 1$$

$$\Rightarrow 2a + 3b = 6$$

$$\Rightarrow (a, b) \equiv (0, 2) \text{ and } (3, 0)$$

$$\text{(q)} f(x) = \sqrt{2} \cos\left(x^2 - \frac{\pi}{4}\right)$$

$$x^2 - \frac{\pi}{4} = 2n\pi \Rightarrow x^2 = 2n\pi + \frac{\pi}{4}$$

$$\Rightarrow x = \pm \sqrt{\frac{\pi}{4}}, \pm \sqrt{\frac{9\pi}{4}} \text{ as } x \in [-\sqrt{3}, \sqrt{13}]$$

$$\text{(r)} \int_0^2 \left(\frac{3x^2}{1+e^x} + \frac{3x^2}{1+e^{-x}} \right) dx = \int_0^2 3x^2 dx = 8$$

$$\text{(s)} \int_{-1/2}^{1/2} \cos 2x \ln\left(\frac{1+x}{1-x}\right) dx = 0 \text{ as it is an odd function}$$

$$\text{Sol 30: (2)} \int_0^1 4x^3 \frac{d^2}{dx^2} (1-x^2) dx$$

$$= \left[4x^3 \frac{d}{dx} (1-x^2)^5 \right]_0^1 - \int_0^1 12x^2 \frac{d}{dx} (1-x^2) dx$$

$$= \left[4x^3 \times 5(1-x^2)^4 (-2x) \right]_0^1 - 12 \left[x^2 (1-x^2)^5 \right]_0^1 - \int_0^1 2x(1-x)^5 dx$$

$$= 0 - 0 - 12[0 - 0] + 12 \int_0^1 2x(1-x^2) dx$$

$$= 12 \times \left[-\frac{(1-x^2)^6}{6} \right]_0^1 = 12 \left[0 + \frac{1}{6} \right] = 2$$

$$\text{Sol 31: } \alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x} \right) \left(\frac{12+9x^2}{1+x^2} \right) dx$$

$$\text{Put } 9x+3\tan^{-1}x = t \Rightarrow \left(9 + \frac{3}{1+x^2} \right) dx = dt$$

$$\Rightarrow \alpha = \int_0^4 e^t dt = e^{9+\frac{3\pi}{4}} - 1 \Rightarrow \left(\log_e |1+\alpha| - \frac{3\pi}{4} \right) = 9$$

$$\text{Sol 32: (A, C)} \text{ Let } \int_0^\pi e^t \left((\sin^6 at + \cos^4 at) \right) dt = A$$

$$I = \int_\pi^{2\pi} e^t (\sin^6 at + \cos^4 at) dt$$

$$\text{Put } t = \pi + x$$

$$dt = dx$$

$$\text{For } a = 2 \text{ as well as } a = 4$$

$$I = e^x \int_0^\pi e^x (\sin^6 ax + \cos^4 at) dt = e^{2\pi} A \Rightarrow I = e^\pi A$$

$$\text{Similarly } \int_0^\pi e^\pi e^\pi (\sin^6 at + \cos^4 at) dt = e^{2\pi} A$$

$$L = \frac{A + e^\pi A + e^{2\pi} A + e^{3\pi} A}{A} = \frac{e^{4\pi} - 1}{e^\pi - 1} \therefore \text{For both } a = 2, 4$$

$$\text{Sol 33: (A, B, C)} \text{ (A)} f'(x) = F(x) + xF'(x)$$

$$f(1) = F(1) + F'(1)$$

$$f'(1) = F(1) < 0$$

$$\text{(B)} f(2) = 2F(2)$$

$$F(x) \text{ is decreasing and } F(1) = 0$$

$$\text{Hence } F(2) < 0$$

$$\Rightarrow f'(2) < 0$$

$$\text{(C)} f(x) = F(x) + xF'(x)$$

$$F(x) < 0 \forall x \in (1, 0)$$

$$F'(x) < 0 \forall x \in (1,3)$$

$$\text{Hence } f(x) < 0 \forall x \in (1,3)$$

$$\text{Sol 34: } I = \int_{-1}^0 \frac{x \cdot 0}{2+0} dx + \int_0^1 \frac{x \cdot 0}{2+1} dx + \int_1^{\sqrt{2}} \frac{x \cdot 1}{2+0} dx + 0 = \frac{1}{4}$$

$$\Rightarrow 4I - 1 = 0$$

$$\text{Sol 35: (A) } I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx \quad \dots \text{ (i)}$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + \frac{1}{e^x}} dx \quad \dots \text{ (ii)}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x \cdot e^x}{1+e^x} dx$$

(i) and (ii)

$$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx$$

$$I = \int_0^{\pi/2} x^2 \cos x dx \quad (\text{even fn})$$

$$= x^2 \cdot \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2 \left[(-x \cos x) \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) dx \right]$$

$$= \frac{\pi^2}{4} - 2 \left[0 + \sin x \Big|_0^{\pi/2} \right] = \frac{\pi^2}{4} - 2[1] = \frac{\pi^2}{4} - 2$$