Class 12

2017-18

MATHEMATICS

FOR JEE MAIN & ADVANCED

SECOND EDITION

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(Now Revised)

Formula Sheet

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Topic Covered

Indefinite Integration

Masterjee Concepts

Tips & Tricks, Facts, Notes, Misconceptions,
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Questions recommended for revision

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22. INDEFINITE INTEGRATION

1. INTRODUCTION

Integration is a reverse process of differentiation. The integral or primitive of a function f(x) with respect to x is a differential function $\phi(x)$ such that the derivative of $\phi(x)$ with respect to x is the given function f(x). It is expressed symbolically as $\int f(x)dx = \phi(x)$

Thus. $\int f(x)dx = \phi(x) \Leftrightarrow \frac{d}{dx}\left[\phi(x)\right] = f(x)$.

The process of finding the integral of a function is called Integration and the given function is called Integrand. Now, it is obvious that the operation of integration is the inverse operation of differentiation. Hence the integral of a function is also named as the anti-derivative of that function.

Further we observe that

$$\left.\begin{array}{lcl} \frac{d}{dx}\left(x^2\right) & = & 2x \\ \frac{d}{dx}\left(x^2+2\right) & = & 2x \\ \frac{d}{dx}\left(x^2+k\right) & = & 2x \end{array}\right\} \Rightarrow \int 2x dx = x^2 + \text{constant}$$

So we always add a constant to the integral of function, which is called the constant of Integration. It is generally denoted by c. Due to the presence of this arbitrary constant such an integral is called an Indefinite Integral.

2. ELEMENTERY INTEGRATION

The following integrals are directly obtained from the derivatives of standard functions.

(a) $\int 0.dx = c$

(b) $\int 1.dx = x + c$

(c) $\int k.dx = kx + c(k \in R)$

(d) $\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1)$

(e) $\int \frac{1}{x} dx = \log_e x + c$

(f) $\int e^x dx = e^x + c$

(g) $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

(h) $\int \sin x\, dx = -\cos x + c$

(i) $\int \cos x\, dx = \sin x + c$

3. BASIC THEOREMS OF INTEGRATION

If f(x), g(x) are two functions of a variable x and k is a constant, then

(a) $\int k\, f(x)dx = k\int f(x)dx$

(b) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(c) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(d) $\int \left(\frac{d}{dx}f(x)\right)dx = f(x) + c$

MASTERJEE CONCEPTS

The results of integration are very different from differentiation. There is no standard formula for integration.

Always make sure to write the constant of integration. NEVER assume it as zero from your side.

Vaibhav Gupta (JEE 2009, AIR54)

Illustration 1: Evaluate: $\int \frac{1-\sin x}{\cos^2 x} dx$ **(JEE MAIN)**

Sol: As we know, $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ therefore we can split

$\int \frac{1-\sin x}{\cos^2 x} dx$ as $\int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$ and then by solving we can get result.

$$\int \frac{1-\sin x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx = \int \sec^2 x\, dx - \int \tan x \sec x\, dx = \tan x - \sec x + c$$

Illustration 2: Evaluate: $\int \sqrt{1+\sin 2x}\, dx$ **(JEE MAIN)**

Sol: Here $\sin^2 x + \cos^2 x = 1$ and $\sin 2x = 2\sin x \cos x$, therefore by using these formulae and solving we will get the result.

$$\int \sqrt{1+\sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx = \int \sqrt{(\sin x + \cos x)^2}\, dx$$

$$= \int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x + c$$

Illustration 3: Evaluate: $\int \sin^4 x \quad dx$ **(JEE MAIN)**

Sol: Here as we know, $\sin^2 x = \frac{1-\cos 2x}{2}$, Now by putting this in the above integration and solving we will get the term $\frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx$, After that by using the formula

$\cos^2 2x = \frac{1+\cos 4x}{2}$ we can solve the problem given above.

$$\int \sin^4 x \quad dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx = \frac{1}{8}\int (3-4\cos 2x+\cos 4x)\,dx = \frac{1}{8}\left[3x-2\sin 2x+\frac{\sin 4x}{4}\right]+C$$

Illustration 4: If $f'(x) = 4x^3 - \frac{3}{x^4}$ such that f(2)=0,then, find f(x) **(JEE ADVANCED)**

Sol: Here $f'(x) = 4x^3 - \frac{3}{x^4}$ therefore $f(x) = \int \left(4x^3 - \frac{3}{x^4}\right)dx$ hence by splitting this integration and solving we will get the result.

We have, $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4} \quad \Rightarrow f(x) = \int \left(4x^3 - \frac{3}{x^4}\right)dx = \int 4x^3\,dx - \int \frac{3}{x^4}dx = 4\int x^3 dx - 3\int x^{-4}dx$

$$= 4\frac{x^{3+1}}{3+1} - 3\frac{x^{-4+1}}{-4+1} + C = x^4 + \frac{1}{x^3} + C \qquad \text{...(i)}$$

Given $f(2) = 2^4 + \frac{1}{2^3} + C = 0 \Rightarrow 0 = 16 + \frac{1}{8} + C \Rightarrow C = -\frac{129}{8}$

Putting the value of C in (i), we get $f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$

4. METHODS OF INTEGRATION

When the integrand can't be reduced into some standard form then integration is performed using following methods

4.1 Integration by Substitution

4.1.1 Integrand is a Function of Another Function

If the integral is of the form $\int f\left[\phi(x)\right]\phi'(x)dx$, then we put $\phi(x) = t$ so that $\phi'(x)$ dx=dt. Now integral is reduced $\int f(t)\,dt$.

MASTERJEE CONCEPTS

In this method the function is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

In case of objective questions in which direct indefinite integration is asked, function being very complicated to integrate, then try differentiating the options.

If $I=\int \frac{dx}{\sin(x-a)\cos(x-b)}$, then I is Equal to

(a) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$

(b) $\frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$

(c) $\frac{1}{\sin(a+b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$

(d) $\frac{1}{\cos(a+b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$

Vaibhav KrishnanJEE 2009, AIR 22

Illustration 5: Evaluate: $\int x\tan x^2 \sec x^2 dx$ **(JEE MAIN)**

Sol: This problem is based on integration using substitution method. In this we can put x^2 =t and therefore 2x dx=dt and then solving we will get the result.

Let x^2 =t

$\Rightarrow$ 2x dx=dt $\Rightarrow$ x dx = $\frac{1}{2}$dt $\quad \therefore \int x\tan x^2 \sec x^2 dx = \frac{1}{2}\int \tan t \sec t\, dt = \frac{1}{2}\sec t + c = \frac{1}{2}\sec x^2 + c$

4.1.2 Integrand is the Product of Function and its Derivative

If the integral is of the form I= $\int f'(x)\ f(x)\, dx$ we put f(x) = t and convert it into a standard integral.

Illustration 6: Evaluate: $\int \tan x \sec^2 x\, dx$ **(JEE MAIN)**

Sol: Here $\sec^2 x$ is a derivatives of $\tan x$ hence we can put tan x = t and $\sec^2 x.dx = dt$ thereafter we can solve the given problem.

Let tan x =t $\Rightarrow$ $\sec^2 x.dx = dt$

$\therefore\ I = \int \tan x \sec^2 x\, dx = \int t\, dt = \frac{t^2}{2} + c = \frac{\tan^2 x}{2} + c$

4.1.3 Integrand is a Function of the Form f(ax+b)

Here we put ax+b =t and convert it into a standard integral. Now if,

$\int f(x)dx = \phi(x),$ then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)$

Illustration 7: Evaluate: $\int \cos 3x \cos 5x\, dx$ **(JEE MAIN)**

Sol: By multiplying and dividing by 2 in the given integration and using the formula

$2\cos A.\cos B = \cos(A+B) + \cos(A-B)$ we can solve it.

I= $\int \cos 3x \cos 5x\, dx = \frac{1}{2}\int(\cos 8x + \cos 2x)dx = \frac{1}{2}\left[\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x\right] + c$

Illustration 8: Evaluate: $I=\int \frac{x\,dx}{x^4+x^2+1}$ **(JEE ADVANCED)**

Sol: Here by putting $x^2=t \Rightarrow$ dt = 2xdx we will get the term $\frac{1}{2}\int \frac{dt}{t^2+t+1}=\frac{1}{2}\int \frac{dt}{\left(t+(1/2)\right)^2+\left(\sqrt{3}/2\right)^2}$ and then by putting $t+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$, we can solve it.

Let $x^2=t \Rightarrow$ dt = 2xdx $\therefore$ $I=\frac{1}{2}\int \frac{dt}{t^2+t+1}=\frac{1}{2}\int \frac{dt}{\left(t+(1/2)\right)^2+\left(\sqrt{3}/2\right)^2}$

$\int \frac{1}{\left[f(x)\right]^2+a^2}dx=\frac{1}{a}\tan^{-1}\left[\frac{f(x)}{a}\right]\times\frac{1}{f'(x)}+c$ $\quad \therefore I=\frac{1}{2}\left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]+c$

$=\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]+c=\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)\right]+c$

Now put $t+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta \Rightarrow dt=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$\therefore \frac{1}{2}\int \frac{(\sqrt{3}/2)\sec^2\theta d\theta}{(3/4)\left(\tan^2\theta+1\right)}=\frac{1}{\sqrt{3}}\int d\theta=\frac{1}{\sqrt{3}}\theta+c=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+c=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$

Standard integration results

(a) $\int \frac{f'(x)}{f(x)}dx=\log_e\left[f(x)\right]+c$

(b) $\int \left[f(x)\right]^n f'(x)dx=\frac{\left[f(x)\right]^{n+1}}{n+1}+c$ (provided n≠-1)

(c) $\int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$

Illustration 9: Evaluate: $\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$ **(JEE MAIN)**

Sol: Here simply substituting t=tan x$\Rightarrow$ dt = $\sec^2 x$ dx we can solve it.

Let t=tan x$\Rightarrow$ dt = $\sec^2 x$ dx

$\therefore I=\int \frac{dt}{\sqrt{t}}=2t^{\frac{1}{2}}+c=2\sqrt{\tan x}+c$

4.1.4 Integral of the Form

$\int \frac{dx}{a\sin x+b\cos x}$ then substitute a=r cos θ and b=r sin θ, $\tan\theta=\frac{b}{a}\Rightarrow \theta=\tan^{-1}\left(\frac{b}{a}\right)$,we get

$I=\int \frac{dx}{r\sin(x+\theta)}=\frac{1}{r}\int \mathrm{cosec}(x+\theta)dx=\frac{1}{r}\log\tan\left(\frac{x+\theta}{2}\right)+c=\frac{\log\tan\left((x/2)+(1/2)\tan^{-1}(b/a)\right)}{\sqrt{a^2+b^2}}+c$

MASTERJEE CONCEPTS

$\int \sin^m x \cos^n x \quad dx$, where m, n ∈ N

⇒ If m is odd put cos x=t

If n is odd put sin x = t

If both m and n are odd, put sin x=t if m ≥ n and cos x=t otherwise.

If both m and n are even, use power reducing formulae

$\sin^2 x = \frac{1-\cos 2x}{2}$ or $\cos^2 x = \frac{1+\cos 2x}{2}$

If m+n is a negative even integer, put tan x=t

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 10: Evaluate: $\int \frac{1}{\sin x + \cos x} dx$ **(JEE ADVANCED)**

Sol: As we know, if integration is in the form of $\int \frac{dx}{a \sin x + b \cos x}$ then we can put

a=r cos θ and b=r sin θ hence the integration will be $\frac{1}{r} \log \tan\left(\frac{x+\theta}{2}\right) + c$.

Here a=1 & b=1

So $\int \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{1+1}} \log \tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1} 1\right) + c = \frac{1}{\sqrt{2}} \log \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$

4.1.5 Standard Substitutions

The following standard substitutions will be useful

Integrand form	Substitutions
$\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2 - x^2}}$	x = a sin θ or x = a cos θ
$\sqrt{x^2 + a^2}$ or $\frac{1}{\sqrt{x^2 + a^2}}$	x = a tan θ or x = a cot θ or x=a sinh θ
$\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2 - a^2}}$	x = a sec θ or x = acosec θ
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	x=a $\tan^2 \theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$	x = a $\sin^2 \theta$ or x= a $\cos^2 \theta$

$\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$ or $x = a\ \text{cosec}^2\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta > \alpha)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Some Standard Integrals

(a) $\int \tan x dx = \log\sec x + c = -\log\cos x + c$

(b) $\int \cot x dx = \log\sin x + c = -\log\text{cosec}\ x + c$

(c) $\int \sec x dx = \log(\sec x + \tan x) + c = -\log(\sec x - \tan x) + c = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(d) $\int \text{cosec} x dx = -\log(\text{cosec} x + \cot x) + c = \log(\text{cosec} x - \cot x) + c = \log\tan\left(\frac{x}{2}\right) + c$

(e) $\int \sec x \tan x dx = \sec x + c$

(f) $\int \text{cosec}\ x \cot x\ dx = -\text{cosec}\ x + c$

(g) $\int \sec^2 x dx = \tan x + c$

(h) $\int \text{cosec}^2 x dx = -\cot x + c$

(i) $\int \log x dx = x\log\left(\frac{x}{e}\right) + c = x\left(\log x + 1\right) + c$

MASTERJEE CONCEPTS

If the integral is of the form $\int R\left(x^{\frac{1}{p}}, x^{\frac{1}{q}}, x^{\frac{1}{r}} ......\right)dx$, where R is a rational function then,

Let a = lcm of (p,q,r,.......) and put $x = t^a$

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 11: Prove that: $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) + C$ **(JEE ADVANCED)**

Sol: By putting $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$, we can solve the problem given above.

Let $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta \Rightarrow \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int \sec\theta d\theta = \log(\sec\theta + \tan\theta) + C$

$= \log\left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right) + C = \log(x + \sqrt{x^2 + a^2}) + C'$

Illustration 12: Evaluate: $\int \cos 2x.\cos 4x.\cos 6x dx$ **(JEE MAIN)**

Sol: By multiplying and dividing by 2 in the given integration and then by using $2\cos A.\cos B = \cos(A+B)+\cos(A-B)$ we can solve it.

Let $I = \int \cos 2x.\cos 4x.\cos 6x dx = \frac{1}{2}\int (2\cos 2x.\cos 4x)\cos 6x dx = \frac{1}{2}\int (\cos 6x + \cos 2x)\cos 6x dx$

$\left[\therefore 2\cos A\cos B = \cos(A+B)+\cos(A-B)\right]$

$= \frac{1}{2}\int\left(\cos^2 6x + \cos 6x - \cos 2x\right)dx = \frac{1}{4}\int\left(2\cos^2 6x + 2\cos 6x - \cos 2x\right)dx$

$= \frac{1}{4}\left[\int\left(1+\cos 12x\right)+\left(\cos 8x + \cos 4a\right)dx\right] = \frac{1}{4}\int dx + \frac{1}{4}\int \cos 12x dx + \frac{1}{4}\int \cos 8x dx + \frac{1}{4}\int \cos 4x dx$

$\because \int \cos\left[f(x)dx = \frac{\sin\left[f(x)\right]}{f'(x)}\right] + C$

$I = \frac{1}{4}\left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$

Illustration 13: Evaluate: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$ **(JEE ADVANCED)**

Sol: Here in this problem by using the formulae

$\cos C - \cos D = -2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}$, sin 2A = 2 sin A cos A and 2 cos C cos D = cos (C+D)+cos (C-D)

We can solve the problem above step by step.

We have,

$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int \frac{-2\sin(x+\alpha)\sin(x-\alpha)}{-2\sin\left((x+\alpha)/2\right)\sin\left((x-\alpha)/2\right)}dx = \int \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left((x+\alpha)/2\right)\sin\left((x-\alpha)/2\right)}dx$

$= \int \frac{2\sin\left((x+\alpha)/2\right)\cos\left((x+\alpha)/2\right).2\sin\left((x-\alpha)/2\right)\cos\left((x-\alpha)/2\right)}{\sin\left((x+\alpha)/2\right)\sin\left((x-\alpha)/2\right)}dx$ (sin 2A = 2 sin A cos A)

$= 4\int \cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)dx$ $\because$ [2 cos C cos D = cos (C+D)+cos (C-D)]

$= 2\int (\cos x + \cos\alpha)dx = 2\int \cos x dx + 2\cos\alpha\int dx = 2\sin x + 2x\cos\alpha + C$

Illustration 14: Evaluate: $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x}dx$ **(JEE ADVANCED)**

Sol: Here by using the formula $a^2 - b^2 = (a+b)(a-b)$ and putting $(\sin^2 x + \cos^2 x)^2$ in place of 1 in the denominator, we can reduce the above integration and then using $\cos 2x = \cos^2 x - \sin^2 x$ we can solve it.

We have, $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x}dx = \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x.\cos^2 x}dx$

$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{(\sin^4 x + \cos^4 x)} dx = \int 1.(\sin^2 x - \cos^2 x)dx = -\int \cos 2x dx \left[\because \cos 2x = \cos^2 x - \sin^2 x\right]$

$= -\frac{\sin 2x}{2} + C$

4.2 Integration by Parts

If u and v are two functions of x, then

$$\int (u.v)dx = u\left(\int v\ dx\right) - \int \left(\frac{du}{dx}\right)\left(\int v\ dx\right)dx$$

This is also known as uv rule of integration. This method of integrating is called integration by parts.

MASTERJEE CONCEPTS

- From the first letter of the words inverse circular, logarithmic, Algebraic, Trigonometric, Exponential functions, we get a word ILATE. Therefore the preference of selecting the u function will be according to the order ILATE.
- In some problems we have to give preference to logarithmic function over inverse trigonometric functions. Hence sometimes the word LIATE is used for reference.
- For the integration of Logarithmic or Inverse trigonometric functions alone, take unity (1) as the v function.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 15: Evaluate: $\int (1+x)\log x dx$ **(JEE MAIN)**

Sol: Here we can integrate the given problem by using Integration by parts i.e.

$$\int (u.v)dx = u\left(\int v\ dx\right) - \int \left(\frac{du}{dx}\right)\left(\int v\ dx\right)dx$$

Here u = $\log x$ and v = $(1+x)$.

Let I= $\int (1+x)\log x dx$

Integrating by parts, taking log x as 1st function, (by LIATE rule) we get

$$I = \log x \int (1+x)dx - \int \left[\frac{d}{dx}(\log x).\int (1+x)dx\right]dx = \log x\left(x + \frac{x^2}{2}\right) - \int \frac{1}{x}.\left(x + \frac{x^2}{2}\right)dx = \left(x + \frac{x^2}{2}\right)\log x - \int \left(1 + \frac{x}{2}\right)dx$$

$$= \left(x + \frac{x^2}{2}\right)\log x - \left(x + \frac{x^2}{4}\right) + C$$

Illustration 16: Evaluate: $\int \sec^3 x dx$ **(JEE ADVANCED)**

Sol: Here we can solve by integrating by parts, taking sec x as the first function.

$I = \int \sec^3 x dx = \int \sec x.\sec^2 x dx$ Let $u = \sec x$ & $v = \sec^2 x$

$I = \sec x \tan x - \int (\sec x \tan x).\tan x\ dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$

$$I = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \Rightarrow I = \sec x + \tan x - I + \int \sec x\, dx$$

$$\Rightarrow 2I = \sec x.\tan x + \log(\sec x + \tan x) + C \Rightarrow I = \frac{1}{2}\left[\sec x \tan x + \log(\sec x \tan x)\right] + C$$

Illustration 17: Evaluate : $\int (\sin^{-1}x)^2 dx$ **(JEE ADVANCED)**

Sol: We can write the given integration as $\int (\sin^{-1}x)^2.1dx$ and then taking $u = \left(\sin^{-1}x\right)^2$ & $v = 1$ solving by integration by parts.

$$I = (\sin^{-1}x)^2.x - \int \left\{\frac{d}{dx}(\sin^{-1}x)^2.x\right\}dx = (\sin^{-1}x)^2.x - \left\{2\left(\sin^{-1}x\right).\frac{1}{\sqrt{1-x^2}}.x\right\}dx$$

Now, putting $\sin^{-1}x = t \Rightarrow x = \sin t$ so that $\frac{dx}{\sqrt{1-x^2}} = dt$

$$\Rightarrow I = x(\sin^{-1}x)^2 - 2\int t.\sin t dt = x(\sin^{-1}x)^2 - 2\left\{-t\cos t + \int \cos t dt\right\} \text{ (again Integrating by parts)}$$

$$= x(\sin^{-1}x)^2 - 2\left\{-t\cos t + \sin t\right\} + C = x(\sin^{-1}x)^2 + 2t\cos t - 2\sin t + C = x(\sin^{-1}x)^2 + 2\sin^{-1}x.\sqrt{1-x^2} - 2x + C$$

Illustration 18: Evaluate: $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$ **(JEE ADVANCED)**

Sol: By using the formula $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$, we can solve the above problem.

$$\text{Let } I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx = \int \frac{\sin^{-1}\sqrt{x} - \left((\pi/2) - \sin^{-1}\sqrt{x}\right)}{(\pi/2)}dx \quad \left[\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}dx - \int 1dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}dx - x \quad \text{... (i)}$$

Putting $\sin^{-1}\sqrt{x} = \theta \Rightarrow x = \sin^2\theta$ so that $dx = 2\sin\theta.\cos\theta\, d\theta = \sin 2\theta\, d\theta$.

$\therefore \int \sin^{-1}\sqrt{x}dx = \int \theta.\sin 2\theta d\theta$ Let $u = \theta$ & $V = \sin 2\theta$, then integing by parts we get

$$= \frac{-\theta}{2}\left(1 - 2\sin^2\theta\right) + \frac{1}{4}\, 2\sin\theta.\cos\theta$$

$$= -\theta.\frac{\cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta d\theta = -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta = -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= -\frac{1}{2}(\sin^{-1}\sqrt{x})(1-2x) + \frac{1}{2}\sqrt{x}.\sqrt{1-x} + C \quad \text{... (ii)}$$

From (i) and (ii), we get

$$I = \frac{4}{\pi}\left\{-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right\} - x + C = \frac{2}{\pi}\left\{\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right\} - x + C$$

4.2.1 Integration by Cancellation

Illustration 19: Evaluate : $\int\left(3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right)dx$ **(JEE MAIN)**

Sol: Let $\int\left(3x^2\tan\frac{1}{x}-x\sec^2\frac{1}{x}\right)dx$ = $\int 3x^2\tan\frac{1}{x}dx-\int x\sec^2\frac{1}{x}dx$ and then by using the integration by parts formula

i.e. $\int(u.v)dx=u\left(\int v\,dx\right)-\int\left(\frac{du}{dx}\right)\left(\int v\,dx\right)dx$ we can solve the problem above.

$$\int\left(3x^2\tan\frac{1}{x}-x\sec^2\frac{1}{x}\right)dx = \int 3x^2\tan\frac{1}{x}dx-\int x\sec^2\frac{1}{x}dx = \tan\frac{1}{x}x^3-\int\left(\sec^2\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)x^3dx-\int x\sec^2\frac{1}{x}dx = x^3\tan\frac{1}{x}+c$$

4.2.2 Integration of the Form:

If the integral is of the form $\int e^x\left[f(x)+f'(x)\right]dx$, then use the formula;

$$\int e^x\left[f(x)+f'(x)\right]dx=e^xf(x)+c$$

Illustration 20: Evaluate: $\int e^x(\log x+1/x)dx$ **(JEE MAIN)**

Sol: Solution of this problem is based on the method mentioned above, here f(x) = log x and f′(x)

= 1/x. $=\int e^x\left[\log x+\frac{1}{x}\right]dx$

I = $\int\left(e^x\log x+\frac{e^x}{x}\right)dx=e^x\log x+c$ $\begin{bmatrix}\text{Here, } f(x)=\log x\\ \&f'(x)=1/x\end{bmatrix}$

If the integral is of the form $\int\left[xf'(x)+f(x)\right]dx$ then use the formula; $\int\left[xf'(x)+f(x)\right]dx=xf(x)+c$

Illustration 21: Evaluate : $\int(x\sec^2x+\tan x)dx$ **(JEE MAIN)**

Sol: Similar to the problem above.

Here I = $\int(x\sec^2x+\tan x)dx$ = $\int\left[xf'(x)+f(x)\right]dx$ [where $f(x)=\tan x$] = $x.\tan x+c$

4.2.3 Special Integrals

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(b\sin bx+a\cos bx)+c$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(b\sin bx+a\cos bx)+c$$

Illustration 22: Evaluate : $\int e^{\sin^{-1}x}dx$ **(JEE MAIN)**

Sol: By putting $\sin^{-1}x$ = t ⇒ x=sin t ⇒ dx = cos t dt and then integrating by parts we can solve the given problem.

I = $\int e^{\sin^{-1}x}dx$

Let $\sin^{-1}x$ = t ⇒ x=sin t ⇒ dx = cos t dt

$\Rightarrow$ I = $\int e^t\cos t\,dt=\frac{e^t}{2}(\sin t+\cos t)+c=\frac{e^{\sin^{-1}x}}{2}(x+\sqrt{1-x^2})+c$

4.3 Integration of Rational Functions

4.3.1 When the Denominator can be Factorized (Using Partial Fraction)

Let the integrand be of the form $\frac{f(x)}{g(x)}$, where both f(x) and g(x) are polynomials. If degree of f(x) is greater than degree of g(x), then first divide f(x) by g(x) till the degree of the remainder becomes less than the degree of g(x). Let Q(x) be the quotient and R(x), the remainder then

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

Now in R(x)/g(x), factorize g(x) and then write partial fractions in the following manner:

(a) For every non-repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

(b) For every repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

(c) For every non-repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-d}$$

(d) For every repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)^2(x-d)} = \frac{Ax+B}{(ax^2+bx+c)^2} + \frac{Cx+D}{ax^2+bx+c} + \frac{E}{x-d}$$

MASTERJEE CONCEPTS

Consider f(x) as the function we need to factorize

1. For non- repeated linear factor in the denominator.

Let $f(x) = \frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

To obtain the value of A remove (x–a) from f(x) and find f(a).

Similarly, to obtain value of B, remove (x-b) from f(x) and find f(b).

2. For repeated linear factor in the denominator.

Let $f(x) = \frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$

To obtain value of D remove (x–b) from f(x) and find f(b).

To obtain value of c remove $(x-a)^3$ from f(x) and find f(a).

Now that we have reduced the number of unknowns from 4 to 2, we can find A and B easily by equating.

Now let's try this method for $\frac{x^4+x^3+2x^2-x+4}{x(x^2+2)(x^2+1)^3}$

Partial fraction will be of the form

$$\frac{x^4+x^3+2x^2-x+4}{x(x^2+2)(x^2+1)^3}=\frac{A}{x}+\frac{Bx+C}{(x^2+2)}+\frac{Dx+E}{(x^2+1)}+\frac{Fx+G}{(x^2+1)^2}+\frac{Hx+I}{(x^2+1)^3}$$

Now remove x and put x=0, we get A=2

Now remove $(x^2+1)^3$ and put x^2 =-1 i.e. x = i (you can also substitute x = –i).

We get Hi+I = –3i –2. Hence H = –3 and I = –2.

Now remove (x^2+2) and put x= $\sqrt{2}i$. We get B ($\sqrt{2}i$) +C=2 $\sqrt{2}i$ +3. Hence B = 2 and C = 3

Now the number of unknowns have reduced from 9 to 4 and the remaining unknowns can be solved easily.

This method very useful instead of solving for all the unknowns at the same time.

Also remember that substituting an imaginary number for x is not discussed anywhere in NCERT. So, use this method only for competitive exams.

Ravi Vooda (JEE 2009, AIR 71)

Illustration 23: Evaluate : $\int \frac{x}{x^2-x-2}dx$ **(JEE MAIN)**

Sol: Here the given integration is in the form of $\frac{1}{(x-a)(x-b)}$, hence by using partial fractions we can split it as $\frac{A}{(x-a)}+\frac{B}{(x-b)}$ and then by solving we will get the required result.

Here I = $\int \frac{x}{(x-2)(x+1)}dx = \int \frac{1}{3}\left(\frac{2}{x-2}+\frac{1}{x+1}\right)dx = \frac{1}{3}\left[2\log(x-2)+\log(x+1)\right]+c = \frac{1}{3}\log\left[(x-2)^2(x+1)\right]+c$

Illustration 24: Evaluate : $\int \frac{xdx}{3x^4-18x^2+11}$ **(JEE ADVANCED)**

Sol: Here simply by putting t= $x^2 \Rightarrow$ dt = 2x dx and then by using partial fractions we can solve the given problem.

I= $\int \frac{xdx}{3x^4-18x^2+11}dx = \int \frac{\frac{1}{2}dt}{3t^2-18t+11}$ (Put t= $x^2 \Rightarrow$ dt = 2x dx)

$$=\frac{1}{6}\int \frac{dt}{t^2-6t+(11/3)}=\frac{1}{6}\int \frac{dt}{(t-3)^2-(16/3)}=\frac{1}{6}\int \frac{dt}{(t-3)^2-\left(4/\sqrt{3}\right)^2}$$

$$=\frac{1}{6}\frac{1}{2\times(4/\sqrt{3})}\log\frac{(t-3)-(4/\sqrt{3})}{(t-3)+(4/\sqrt{3})}+C=\frac{\sqrt{3}}{48}\log\frac{\sqrt{3}t-3\sqrt{3}-4}{\sqrt{3}t-3\sqrt{3}+4}+C \quad =\frac{\sqrt{3}}{48}\log\frac{\sqrt{3}x^2-3\sqrt{3}-4}{\sqrt{3}x^2-3\sqrt{3}+4}+C$$

4.3.2 When the Denominator cannot be Factorized

In this case the integral may be in the form

(i) $\int \frac{dx}{ax^2+bx+c}$ (ii) $\int \frac{(px+q)}{ax^2+bx+c}dx$

Method:

(i) Here taking the coefficient of x^2 common from the denominator , write

$x^2+(b/a)x+c/a=(x+b/2a)^2-\frac{b^2-4ac}{4a^2}$

Now the integrand obtained can be evaluated easily by using standard formulae.

(ii) Here suppose that px + q = $A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$ = A(2ax+b)+B .....(i)

Now comparing coefficient of x and constant terms.

We get A=p/2a, B=q-(pb/2a)

$\therefore I=\frac{p}{2a}\int\frac{2ax+b}{ax^2bx+c}dx+\left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$

Now we can integrate it easily.

4.3.3 Integrand Containing Only Even Powers of x

To find integral of such functions, first we divide numerator and denominator by x^2, then express the numerator as d(x±1/x) and the denominator as a function of (x±1/x). The following examples illustrate it.

MASTERJEE CONCEPTS

$\int R(\sin x,\cos x)dx$ where R is a rational function (universal substitution tan(x/2)=t)

Special cases:

(a) If R (–sin x, cos x) = –R(sinx, cos x)

Put cos x=t

(b) If R (sin x, cos x) = –R(sinx, cos x)

Put sin x=t

(c) If R (–sin x, –cos x) = R(sinx, cos x)

Put tan x=t

Akshat Kharaya (JEE 2009,AIR 235)

Illustration 25: Evaluate: $\int\frac{x^2+1}{x^4+1}dx$ **(JEE ADVANCED)**

Sol: Here dividing the numerator and denominator by x^2, we get $\int\frac{1+(1/x^2)}{x^2+(1/x^2)}dx=\int\frac{1+(1/x^2)}{\left[\left\{x-(1/x)\right\}^2+2\right]}dx$ and then by putting x-1/x=t $\Rightarrow$[1+1/x^2]dx = dt, we can solve it.

$$I = \int \frac{1+(1/x^2)}{x^2+(1/x^2)}dx = \int \frac{1+(1/x^2)}{\left[\left\{x-(1/x)\right\}^2+2\right]}dx$$

Now taking x-1/x=t ⇒[1+1/ x^2]dx = dt, we get

$$I = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+c$$

4.4 Integration of Irrational Functions

If any one term in numerator or denominator is irrational then it is made rational by a suitable substitution. Also if the integral is of the form

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} \text{ or } \int \sqrt{ax^2+bx+c}\,dx$$

Then we integrate it by expressing $ax^2+bx+c=(x+\alpha)^2+\beta^2$

Also for integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ or $\int (px+q)\sqrt{ax^2bx+c}\,dx$

First we express px+q in the form

px+q = A [$\frac{d}{dx}(ax^2bx+c)$]+B and then proceed as usual with standard form.

Illustration 26: Evaluate : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ **(JEE MAIN)**

Sol: Simply by putting e^x = t , then e^x dx = dt, we can solve the given problem.

Put e^x = t , then e^x dx = dt

$$\therefore \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{5-(t^2+4t)}} = \int \frac{dt}{\sqrt{5-(t^2+4t+4)+4}} = \int \frac{dt}{\sqrt{9-(t+2)^2}}$$

$$= \int \frac{dt}{\sqrt{(3)^2-(t+2)^2}} = \sin^{-1}\left(\frac{t+2}{3}\right)+C = \sin^{-1}\left(\frac{e^x+2}{3}\right)+C$$

Illustration 27: Evaluate : $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$ **(JEE ADVANCED)**

Sol: Here first expand $(x-a)(x-b)$ and then adding and subtracting by $\left(\frac{a+b}{2}\right)^2$, we can reduce the above integration. After that by putting $x-\left(\frac{a+b}{2}\right)=u$, we can solve the given problem.

$$\text{Let, } I = \int \frac{1}{\sqrt{(x-a)(x-b)}}dx = \int \frac{1}{\sqrt{x^2-(a+b)x+ab}}dx = \int \frac{dx}{\sqrt{x^2-(a+b)x+\left((a+b)/2\right)^2-\left((a+b)/2\right)^2+ab}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\left((a+b)/2\right)\right)^2-\left[\left((a^2+b^2+2ab)/4\right)-ab\right]}} = \int \frac{dx}{\sqrt{\left(x-\left((a+b)/2\right)\right)^2-\left[(a-b)^2/4\right]}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\left((a+b)/2\right)\right)^2-\left((a-b)/2\right)^2}} \quad \text{... (i)}$$

On putting $x-\left(\frac{a+b}{2}\right)=u$ so that dx = du in (i), we get

$$I = \int \frac{du}{\sqrt{u^2-((a-b)/2)^2}} \quad \left[\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|\right] = \log\left|u+\sqrt{u^2-\left(\frac{a-b}{2}\right)^2}\right| + c$$

Putting $u=\left(x-\left(\frac{a+b}{2}\right)\right)$, we get

$$I = \log\left|\left(x-\left(\frac{a+b}{2}\right)\right)+\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2-\left(\frac{a-b}{2}\right)^2}\right| + c = \log\left|x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+c$$

5. STANDARD INTEGRALS

(a) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(b) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

(c) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(d) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c = -\cos^{-1}\left(\frac{x}{a}\right)+c$

(e) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)+c = \log\left(x+\sqrt{x^2+a^2}\right)+c$

(f) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)+c = \log\left(x+\sqrt{x^2-a^2}\right)+c$

(g) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$ (Substitute $x=a\cos\theta$ or $x=a\sin\theta$ and proceed)

(h) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$ (Substitute $x=a\tan\theta$ or $x=a\cot\theta$ and proceed)

(i) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$ (Substitute $x=a\sec\theta$ or $x=a\,\text{cosec}\,\theta$ and proceed)

(j) $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$ (Valid for x > a > 0)

(k) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$

(l) $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c \; = \dfrac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx - \tan^{-1}\dfrac{b}{a}\right\} + c$

Integration of irrational algebraic functions:

Type 1: (a) $\int \dfrac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ (Put : $x = a\cos^2 q + b\sin^2 q$)

(b) $\int \dfrac{dx}{(x-\alpha)\sqrt{(x-\beta)}}$ (Put : $x = a\sec^2 q - b\tan^2 q$)

Type 2: $\int \dfrac{dx}{(ax+b)\sqrt{px+q}}$ (Put: $px+q=t^2$)

Type 3: $\int \dfrac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ (Put: $ax+b=\dfrac{1}{t}$)

Type 4: $\int \dfrac{dx}{(ax^2bx+c)\sqrt{px+q}}$ (Put: $px+q=t^{2.}$)

Type 5: $\int \dfrac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case I: When (ax^2bx+c) breaks up into two linear factors, e.g.

$$I = \int \frac{dx}{\left(x^2-x-2\right)\sqrt{x^2+x+1}}dx \text{ then } = \int\left(\frac{A}{x-2}+\frac{B}{x+1}\right)\frac{1}{\sqrt{x^2+x+1}}dx = A\underbrace{\int\frac{dx}{\left(x\ -2\right)\sqrt{x^2+x+1}}} + B\underbrace{\int\frac{dx}{\left(x+1\right)\sqrt{x^2+x+1}}}$$

Put $x-2=\dfrac{1}{t}$ Put $x+1=\dfrac{1}{t}$

Case II: If ax^2+bx+c is a perfect square say $(lx+m)^2$, then put $lx+m=\dfrac{1}{t}$

Case III: If b = 2, q = 0

e.g. $\int \dfrac{dx}{(ax^2+b)\sqrt{px^2+r}}$ then, put $x=\dfrac{1}{t}$ or trigonometric substitutions are also helpful.

Integral of the form $\int \dfrac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral	**Substitutions**
$\int \dfrac{1}{(ax+b)\sqrt{cx+d}}dx$	$cx+d=z^2$
$\int \dfrac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}}$	$px+q=z^2$
$\int \dfrac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px+q=\dfrac{1}{z}$
$\int \dfrac{dx}{\left(ax^2+b\right)\sqrt{cx^2+d}}$	$x=\dfrac{1}{z}$

$$\int \frac{dx}{(ax+b)^m\sqrt{ax^2+bx+c}} \qquad ax+b=1/t$$

Illustration 28: Evaluate : $\int \frac{dx}{(x+1)\sqrt{(x-2)}}$ **(JEE MAIN)**

Sol: Simply by putting $x-2=t^2$, $\therefore dx = 2t\ dt$ we can solve the given problem by using the appropriate formula.

$$\int \frac{dx}{(x+1)\sqrt{(x-2)}}$$

Put $x-2=t^2$

$\therefore dx = 2t\ dt$

$$\therefore I = \int \frac{2t\ dt}{(t^2+3)t} = 2\int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{\left(\frac{x-2}{3}\right)}\right)+c \ (\because\ t=\sqrt{(x-2)})$$

Illustration 29: Evaluate : $\int \frac{dx}{(x^2-4)\sqrt{x}}$ **(JEE MAIN)**

Sol: Here first put $x=t^2$ therefore dx = 2t dt and then using partial fractions we reduce the given integration in standard form. After that by solving we will get the result.

Let $I = \int \frac{dx}{(x^2-4)\sqrt{x}}$

Put $x=t^2$ $\therefore$ dx = 2t dt then $I = \int \frac{2t}{(t^4-4)t}dt = 2\int \frac{dt}{(t^2+2)(t^2-2)}$

Put $t^2 = z$ $\therefore$ $\frac{1}{(t^2+2)(t^2-2)} = \frac{1}{(z+2)(z-2)} = \frac{A}{z+2}+\frac{B}{z-2}$

$A = -\frac{1}{4}$ and $B = \frac{1}{4} \Rightarrow \frac{1}{(t^2+2)(t^2-2)} = \frac{1}{4(t^2+2)}+\frac{1}{4(t^2-2)}$

$$\therefore I = 2\int \frac{1}{(t^2+2)(t^2-2)} = -\frac{1}{2}\int \frac{dt}{t^2+2}+\frac{1}{2}\int \frac{dt}{t^2-2} = -\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c$$

$$= -\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right|+c \ (\because\ t=\sqrt{x})$$

6. SPECIAL TRIGONOMETRIC FUNCTIONS

Here we shall study the methods for evaluation of the following types of integrals.

Type 1

(i) $\int \frac{dx}{a+b\sin^2 x}$ (ii) $\int \frac{dx}{a+b\cos^2 x}$ (iii) $\int \frac{dx}{a\cos^2 x+b\sin x\cos x+c\sin^2 x}$ (iv) $\frac{dx}{\int(a\sin x+b\cos x)^2}$

Method: Divide the numerator and denominator by $\cos^2 x$ in all such types of integrals and then put tan x=t

Illustration 30: Evaluate : $\int \frac{dx}{1+3\sin^2 x}$ **(JEE MAIN)**

Sol: Here dividing the numerator and denominator by $\cos^2 x$ we can solve it.

$$I = \int \frac{\sec^2 x dx}{\sec^2 x + 3\tan^2 x} = \int \frac{\sec^2 x \quad dx}{1+4\tan^2 x} = \frac{1}{2}\tan^{-1}(2\tan x) + c$$

Type 2

(i) $\int \frac{dx}{a + b\cos x}$ (ii) $\int \frac{dx}{a + b\sin x}$ (iii) $\int \frac{dx}{a\cos x + b\sin x}$ (iv) $\int \frac{dx}{a\sin x + b\cos x + c}$

Method: In such types of integrals we use the following substitutions

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2}; \ dx = \frac{2dt}{1+t^2}$$

and integrate another method for the evaluation of the integral.

Illustration 31: Evaluate: $\int \frac{dx}{5+4\cos x} dx$ **(JEE MAIN)**

Sol: Here by putting $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$ and then by taking tan (x/2) = t we can solve the given problem

$$I = \int \frac{dx}{5+4\left[(1-\tan^2(x/2))/(1+\tan^2(x/2))\right]} = \int \frac{\sec^2(x/2)}{9+\tan^2(x/2)} dx = 2\int \frac{dt}{3^2+t^2} \qquad \text{where } \tan(x/2) = t$$

$$= 2\left(\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right) + C = 2\left[\frac{1}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right)\right] + C$$

Type 3

(i) $\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x} dx$ (ii) $\int \frac{p\sin x}{a\sin x + b\cos x} dx$ (iii) $\int \frac{q\cos x}{a\sin x + b\cos x} dx$

For their integration, we first express numerator as follows-

Numerator = A (denominator) + B (derivative of denominator)

Then integral = Ax + B log (denominator) + C

Illustration 32: Evaluate : $\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x} dx$ **(JEE ADVANCED)**

Sol: By using partial fractions, we can reduce the given integration to the standard form.

$$\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x} dx$$

$\Rightarrow 6+3\sin x+14\cos x = A(3+4\sin x+5\cos x) + B(4\cos x - 5\sin x) + c$

Solving R.H.S. & comparing both sides, we get 4A – 5B = 3 5A + 4B = 14

Also, 3A+C=6 $\quad \therefore \int \frac{A(3+4\sin x+5\cos x)+B(4\cos x-5\sin x)+c}{3+4\sin x+5\cos x}$

$\Rightarrow Ax + \log(3+4\sin x+5\cos x)+\int \underbrace{\frac{C\quad dx}{3+4\sin x+5\cos x}}_{\text{this is of type 2}}$

Illustration 33: Evaluate : $\int \frac{2\sin 2\phi-\cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi$ **(JEE ADVANCED)**

Sol: Here we can write the given integration as $\int \frac{2(\sin 2\phi-4\cos\phi)+7\cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi$ and as we know $2(\sin 2\phi-4\cos\phi)$ is the derivative of $6-\cos^2\phi-4\sin\phi$ hence by putting $6-\cos^2\phi-4\sin\phi=t$, we can solve the given problem.

$$I = \int \frac{2\sin 2\phi-\cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi \;=\; \frac{d}{d\phi}(6-\cos^2\phi-4\sin\phi)$$

$$= 2\cos\phi\sin\phi-4\cos\phi \;=\; \sin 2\phi-4\cos\phi \;=\; 2\sin 2\phi-\cos\phi = 2(\sin 2\phi-4\cos\phi)+7\cos\phi$$

$$I=\int \frac{2(\sin 2\phi-4\cos\phi)+7\cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi \;=\; \int \frac{2(\sin 2\phi-4\cos\phi)d\phi}{6-\cos^2\phi-4\sin\phi}+\int\frac{7\cos\phi d\phi}{6-\cos^2\phi-4\sin\phi} \;=2\int\frac{dt}{t}+\int\frac{7\cos\phi d\phi}{6-(1-\sin^2\phi)-4\sin\phi}$$

$$=2\log t+C_1+\int\frac{7\cos\phi d\phi}{5+\sin^2\phi-4\sin\phi} \;=\; 2\log(6-\cos^2\phi-4\sin\phi)+C_1+\int\frac{7dx}{x^2-4x+5}\quad(\sin\phi=x)$$

$$=2\log(6-\cos^2\phi-4\sin\phi)+C_1+\int\frac{7dx}{(x-2)^2+1}=2\log(6-\cos^2\phi-4\sin\phi)+C_1+7\tan^{-1}\frac{x-2}{1}+C_2$$

$$=2\log(6-\cos^2\phi-4\sin\phi)+7\tan^{-1}(\sin\phi-2)+C$$

7. SPECIAL EXPONENTIAL FUNCTIONS

(a) $\int\frac{ae^x}{b+ce^x}dx$ [put $e^x=t$]

(b) $\int\frac{1}{1+e^x}dx$ [Multiply and divide by e^{-x} and $e^{-x}=t$]

(c) $\int\frac{1}{1-e^x}dx$ [Multiply and divide by e^{-x} and $e^{-x}=t$]

(d) $\int\frac{1}{e^x-e^{-x}}dx$ [Multiply and divide by e^x]

(e) $\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$ $\left[\frac{f'(x)}{f(x)}\text{form}\right]$

(f) $\int\frac{e^x+1}{e^x-1}dx$ [Multiply and divide by $e^{-x/2}$]

(g) $\int\frac{1}{(1+e^x)(1-e^{-x})}dx$ [Multiply and divide by e^x and put $e^x=t$]

(h) $\int\frac{1}{\sqrt{1-e^x}}dx$ [Multiply and divide by $e^{-x/2}$]

(i) $\int \frac{1}{\sqrt{1+e^x}}dx$ [Multiply and divide by $e^{-x/2}$]

(j) $\int \frac{1}{\sqrt{e^x-1}}dx$ [Multiply and divide by $e^{-x/2}$]

(k) $\int \frac{1}{\sqrt{2e^x-1}}dx$ [Multiply and divide by $\sqrt{2}e^{-x/2}$]

(l) $\int \sqrt{1-e^x}dx$ [Integrand = $(1-e^x)/\sqrt{1-e^x}$]

(m) $\int \sqrt{1+e^x}dx$ [Integrand = $(1+e^x)/\sqrt{1+e^x}$]

(n) $\int \sqrt{e^x-1}dx$ [Integrand = $(e^x-1)/\sqrt{e^x-1}$]

(o) $\int \sqrt{\frac{e^x+a}{e^x-a}}dx$ [Integrand = $(e^x+a)/\sqrt{e^{2x}-a^2}$]

Illustration 34: Evaluate : $\int \sqrt{e^x-1}dx$ **(JEE MAIN)**

Sol: Here by multiplying and dividing by $\sqrt{e^x-1}$ in the given integration and then by putting $e^x-1 = t^2$ we can evaluate the given integration.

Here I = $\int \sqrt{e^x-1}dx$ **=** $\int \frac{e^x-1}{\sqrt{e^x-1}}dx$ **=** $\int \frac{e^x}{\sqrt{e^x-1}}dx - \int \frac{1}{\sqrt{e^x-1}}dx$

Let $e^x-1 = t^2$, then $e^x = dx = 2t \quad dt$

$$\therefore I = 2\int dt - \int \frac{2}{t^2+1}dt = 2t - 2\tan^{-1}(t) + c = 2\left[\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1}\right] + c$$

Illustration 35: Evaluate : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ **(JEE MAIN)**

Sol: We have, $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Put $e^x = t$, then $e^x\ dx = dt$

$$\therefore \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx = \int \frac{dt}{\sqrt{5-4t-t^2}}dx$$

$$= \int \frac{dt}{\sqrt{5-(t^2+4t)}} = \int \frac{dt}{\sqrt{5-(t^2+4t+4)+4}}$$

$$= \int \frac{dt}{\sqrt{9-(t+2)^2}} = \int \frac{dt}{\sqrt{(3)^2-(t+2)^2}} = \sin^{-1}\left(\frac{t+2}{3}\right) + C = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C$$

PROBLEM-SOLVING TACTICS

Integration by Parts

(a) Integration by parts is useful for dealing with integrals of the products of the following functions

$u << \tan^{-1}x,\ \sin^{-1}x, \cos^{-1}x \quad (\log x)^k \quad \sin x,\ \cos x \qquad e^x >> dv$

Priority for choosing u and dv: ILATE

(b) Integration by parts is sometimes useful for finding integrals of functions involving inverse functions such as n x and $\sin^{-1}x$.

(c) Sometimes when dealing with integrals, the integrand involves inverse functions (like $\sin^{-1}x$), it is useful to substitute x = the inverse of that inverse function (like x = sin u), then do integration by parts.

(d) Sometimes you will have to do integration by parts more than once (for example, $\int x^2e^x dx$ and $\int x^3 \sin x dx$.

Sometimes you need to do it twice by parts, then manipulate the equation (for example, $\int e^x \sin x dx$).

(e) Try u – substitution first before integration by parts.

Trigonometric Integral

(a) Integral Type : $\int \sin^m x \cos^n x dx$

Case 1: One of m or n are even, and the other odd

Use u – substitution by setting u = sin or cos that with an even power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 2: Both m and n are odd

Use u – substitution by setting u = sin or cos that with a higher power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 3: Both m and n are even (hard case)

Do not use u – substitution. Use the half double angle formula to reduce the integrand into case 1 o r2:

$$\sin x \cos x = \frac{1}{2}\sin 2x\ ;\ \sin^2 x = \frac{1}{2}(1-\cos 2x)\ ;\ \cos^2 x = \frac{1}{2}(1-\cos 2x)$$

(Note: 0 is also an even number. For example, $\sin^3 x = \sin^3 x \cos^o x$, so it is in case 1)

Just remember that when both are even, you can't use u-substitution, but you can use the half – double angle formula. When it is not that case, let u = sin x or cos x, and one will work (at the end there is no square root term after substitution).

(b) Integral type : $\int \tan^m x \sec^n x dx$

Case 1: sec is odd power, tan is even power.

Hard to do, we omit (most likely won't pop out in the exam).

Case 2: Else

Set u = sec x or tan x, and use $1+\tan^2 x = \sec^2 x$. One will work at the end (there is no square root term after substitution).

(c) Integral type : $\int \sin(Ax)\cos(Bx)dx$, $\int \cos(Ax)\cos(Bx)dx$, $\int \sin(Ax)\sin(Bx)dx$

Use the product to sum formula:

$$\cos\theta\cos\phi = \frac{1}{2}(\cos\theta-\phi)+(\cos\theta+\phi))\ ;\sin\theta\cos\phi = \frac{1}{2}(\cos\theta-\phi)-(\cos\theta+\phi))$$

$$\sin\theta\cos\phi = \frac{1}{2}(\sin\theta-\phi)+(\sin\theta+\phi))$$

Reduce product into sum and then integrate.

Trigonometric Substitution

(a) Trigonometric substitution is useful for quadratic form with square root:

$\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$

$\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$

$\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$

(b) General procedure for doing trig sub:

Step 1: Draw the right triangle, and decide what trigonometric function to substitute for x.

Step 2: Find dx, then substitute the integrand using triangle, convert integral into trigonometric integral.

Step 3: Solve the trigonometric integral.

Step 4: Substitute back using triangle.

(i) If the quadratic form is not in the Pythagoras form (for example, $\sqrt{2+2x+x^2}$, then use the perfecting the square method to transform it into Pythagoras form).

(ii) Try u – substitution before trigonometric substitution.

(iii) Integrals involving $(1-x^2)$ and (x^2-1) without square roots can be solved easily with partial fractions. So don't use trigonometric substitution.

Rational Integral and Partial Fraction

(a) General step for solving rational integral:

Step 1: Do long division for the rational function if the degree of the numerator is higher than the denominator.

Step 2: Do partial fraction decomposition.

Step 3: Evaluate the integral of each simple fraction.

(b) General step for partial fraction:

Step 1: Factorize the denominator.

Step 2: Set the partial fraction according to "rule".

Step 3: Solve the unknown of the numerator of the partial fraction.

Improper Integral

(a) General steps for evaluating improper integral:

Step 1: Change the improper integral into the appropriate limit. [Change ±∞ or singular point (where) to appropriate limit.]

Step 2: Evaluate the integral.

Step 3: Find the limit.

(b) The very first step to test improper integral involving ∞ is to check its limit. If its limit is not zero, then the integral diverges.

(c) Whenever you see improper integrals involving the quotient of a rational or irrational function, such as

$$\int_a^{\infty} \frac{x^3 + \sqrt{3x}}{(8x^3 + 7x)^{3/2}} dx$$

Use **limit comparison test**. The appropriate comparing function can be found by looking at the Integrand (quotient of rational irrational). "Discard" the lower degree terms.

(d) Sometimes, using u – substitution before using any test will be easier.

(e) Sometimes, to determine if an improper integral converges or diverges, directly evaluating the improper integral is easier.

(f) When doing a **comparison test**, beware of the comparing function that you choose. It might not give an appropriate conclusion if the comparing function is not correct.

(g) Try the **limit comparison test** before the **comparison test**.

(h) Useful comparing function, which is good to know their convergence or divergence

$$\int_a^{\infty} x^k e^{-\beta x} dx < \infty \text{ For } k \ge 0, \beta > 0$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p > 1; \\ = \infty & \text{if } p \le 1; \end{cases}$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p < 1; \\ = \infty & \text{if } p \ge 1; \end{cases}$$

FORMULAE SHEET

Basic theorems of Integration:

1. $\int k\, f(x)dx = k\int f(x)dx$	**2.** $\int \left[f(x) \pm g(x)\right]dx = \int f(x)dx \pm \int g(x)dx$
3. $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$	**4.** $\int \left(\frac{d}{dx} f(x)\right)dx = f(x)$

Elementary Integration:

1. $\int 0.dx = c$	**2.** $\int 1.dx = x + c$
3. $\int k.dx = kx + c(k \in R)$	**4.** $\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \ne -1)$
5. $\int \frac{1}{x} dx = \log_e x + c$	**6.** $\int e^x dx = e^x + c$
7. $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$	**8.** $\int \sin x\, dx = -\cos x + c$
9. $\int \cos x\, dx = \sin x + c$	**10.** $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
11. $\int \frac{c}{ax+b} dx = \frac{c}{a} \log \lvert ax+b \rvert + c$	**12.** $\int f'(x) e^{f(x)} dx = e^{f(x)} + c$
13. $\int \log x\, dx = x\log x - x + c$	**14.** $\int \log_a x\, dx = x\log_a x - \frac{x}{\log a} + c$

Standard substitution:

1. $\sqrt{a^2-x^2}$ or $\dfrac{1}{\sqrt{a^2-x^2}}$	x = a sin θ or x = a cos θ
2. $\sqrt{x^2+a^2}$ or $\dfrac{1}{\sqrt{x^2+a^2}}$	x = a tan θ or x = a cot θ
3. $\sqrt{x^2-a^2}$ or $\dfrac{1}{\sqrt{x^2-a^2}}$	x = a sec θ or x = acosec θ
4. $\sqrt{\dfrac{x}{a+x}}, \sqrt{\dfrac{a+x}{x}}, \sqrt{x(a+x)}$ and $\dfrac{1}{\sqrt{x(a+x)}}$	$x=a\ \tan^2\theta$
5. $\sqrt{\dfrac{x}{a-x}}$ or $\sqrt{\dfrac{a-x}{x}}$ $\sqrt{x(a-x)}$ and $\dfrac{1}{\sqrt{x(a-x)}}$	$x = a\ \sin^2\theta$ or x= a cos2
6. $\sqrt{\dfrac{x}{x-a}}$ or $\sqrt{\dfrac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\dfrac{1}{\sqrt{x(x-a)}}$	$x=a\ \sec^2\theta$ or $x=a\ \text{cosec}^2\theta$
7. $\sqrt{\dfrac{a-x}{a+x}}$ and $\sqrt{\dfrac{a+x}{a-x}}$	x = a cos2 θ
8. $\sqrt{\dfrac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta>\alpha)$	$x = \alpha\cos^2\theta+\beta\sin^2\theta$

Some standard Integrals:

1. $\int \tan x dx = \log\sec x + c = -\log\cos x + c$	**2.** $\int \cot x dx = \log\sin x + c$
3. $\int \sec x dx = \log(\sec x+\tan x)+c$ $=-\log(\sec x-\tan x)+c=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)+c$	**4.** $\int \text{cosec} x dx = -\log(\text{cosec} x+\cot x)+c$ $=\log(\text{cosec}x-\cot x)+c=\log\ \tan\left(\dfrac{x}{2}\right)+c$
5. $\int \sec x\tan x dx = \sec x + c$	**6.** $\int \text{cosec}\ x\cot x\ dx = -\text{cosec}\ x + c$
7. $\int \sec^2 x dx = \tan x + c$	**8.** $\int \text{cosec}^2 x dx = -\cot x + c$
9. $\int \log x dx = x\log\left(\dfrac{x}{e}\right)+c = x(\log x+1)+c$	**10.** $\int \sin^2 x dx = \dfrac{1}{2}\left(x-\dfrac{\sin 2x}{2}\right)+C$ $=\dfrac{1}{2}(x-\sin x\cos x)+C$

11. $\int \cos^2 x dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C =$ $\frac{1}{2}(x + \sin x \cos x) + C$	**12.** $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x$ $+ \frac{1}{2}\log\lvert\sec x + \tan x\rvert + c$
13. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} +$ $\frac{n-1}{n}\int \sin^{n-2} x dx$	**14.** $\int \cos^n x dx = -\frac{\cos^{n-1} x \sin x}{n} +$ $\frac{n-1}{n}\int \cos^{n-2} x dx$

Integration by Parts:

1. $\int (u.v)dx = u\left(\int v\, dx\right) - \int \left(\frac{du}{dx}\right)\left(\int v\, dx\right)dx$	**2.** $\int e^x \left[f(x) + f'(x)\right]dx = e^x f(x) + c$

Standard Integrals:

1. $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$
3. $\int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$
4. $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c = -\cos^{-1}\left(\frac{x}{a}\right) + c$
5. $\int \frac{1}{\sqrt{x^2 + a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right) + c = \log\left(x + \sqrt{x^2 + a^2}\right) + c$
6. $\int \frac{1}{\sqrt{x^2 - a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + c = \log\left(x + \sqrt{x^2 - a^2}\right) + c$
7. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$
8. $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left\lvert x + \sqrt{x^2 + a^2}\right\rvert + c$
9. $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + c$

10.	$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$ (Valid for x > a > 0)
11.	$\int e^{ax}\sin bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c \ = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$
12.	$\int e^{ax}\cos bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c \ = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx-\tan^{-1}\frac{b}{a}\right\}+c$

Solved Examples

JEE Main/Boards

Example 1: Evaluate : $\int \frac{x+\sin x}{1+\cos x}dx$

Sol: Here by using the formula

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$

we can solve the given problem.

$$\int \frac{x+\sin x}{1+\cos x}dx \ = \int \frac{x+2\sin x/2\cos x/2}{2\cos^2 x/2}dx$$

$$= \int \frac{x}{2}\sec^2 x/2 + \tan\frac{x}{2}dx \ = x\tan x/2 + c$$

Example 2: Evaluate : $\int \sqrt{x^2+a^2}dx$

Sol: By applying integration by parts and taking $\sqrt{x^2+a^2}$ as the first function we can solve the given problem.

$$\int \sqrt{x^2+a^2}dx \ =$$

$$= \sqrt{x^2+a^2}x - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx \ \sqrt{x^2+a^2}x - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$= \frac{x\sqrt{x^2+a^2}+a^2\int \frac{dx}{\sqrt{x^2+a^2}}}{2}$$

Put x = atan θ

$$\int \frac{dx}{\sqrt{x^2+a^2}} \ = \int \sec\theta d\theta \ = \log\left(\sec\theta+\tan\theta\right)$$

$$= \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)$$

$$\therefore \ I = \frac{1}{2}\left(x\sqrt{x^2+a^2} + \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)\right)+c$$

Example 3: Evaluate : $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx$

Sol: Here first write $\cos\left((\pi/2)-x\right)$ at the place of sin x then by using the formula $1-\cos x = 2\sin^2\frac{x}{2}$

And $1+\cos x = 2\cos^2\frac{x}{2}$ we can solve it.

$$I = \int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx \ = \int \tan^{-1}\sqrt{\frac{1-\cos\left((\pi/2)-x\right)}{1+\cos\left((\pi/2)-x\right)}}dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left((\pi/4)-(x/2)\right)}{2\cos^2\left((\pi/4)-(x/2)\right)}}dx$$

$$= \int \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \ = \int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \ = \frac{\pi}{4}x-\frac{x^2}{4}+C$$

Example 4: Evaluate : $\int \log(2+x^2)dx$

Sol: Here integrating by parts by taking $\log(2+x^2)$ as the first function we can solve the given problem.