PROBLEM-SOLVING TACTICS

Integration by Parts

(a) Integration by parts is useful for dealing with integrals of the products of the following functions

$u << \tan^{-1}x,\ \sin^{-1}x, \cos^{-1}x \quad (\log x)^k \quad \sin x,\ \cos x \qquad e^x >> dv$

Priority for choosing u and dv: ILATE

(b) Integration by parts is sometimes useful for finding integrals of functions involving inverse functions such as n x and $\sin^{-1}x$.

(c) Sometimes when dealing with integrals, the integrand involves inverse functions (like $\sin^{-1}x$), it is useful to substitute x = the inverse of that inverse function (like x = sin u), then do integration by parts.

(d) Sometimes you will have to do integration by parts more than once (for example, $\int x^2e^x dx$ and $\int x^3 \sin x dx$.

Sometimes you need to do it twice by parts, then manipulate the equation (for example, $\int e^x \sin x dx$).

(e) Try u – substitution first before integration by parts.

Trigonometric Integral

(a) Integral Type : $\int \sin^m x \cos^n x dx$

Case 1: One of m or n are even, and the other odd

Use u – substitution by setting u = sin or cos that with an even power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 2: Both m and n are odd

Use u – substitution by setting u = sin or cos that with a higher power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 3: Both m and n are even (hard case)

Do not use u – substitution. Use the half double angle formula to reduce the integrand into case 1 o r2:

$$\sin x \cos x = \frac{1}{2}\sin 2x\ ;\ \sin^2 x = \frac{1}{2}(1-\cos 2x)\ ;\ \cos^2 x = \frac{1}{2}(1-\cos 2x)$$

(Note: 0 is also an even number. For example, $\sin^3 x = \sin^3 x \cos^o x$, so it is in case 1)

Just remember that when both are even, you can't use u-substitution, but you can use the half – double angle formula. When it is not that case, let u = sin x or cos x, and one will work (at the end there is no square root term after substitution).

(b) Integral type : $\int \tan^m x \sec^n x dx$

Case 1: sec is odd power, tan is even power.

Hard to do, we omit (most likely won't pop out in the exam).

Case 2: Else

Set u = sec x or tan x, and use $1+\tan^2 x = \sec^2 x$. One will work at the end (there is no square root term after substitution).

(c) Integral type : $\int \sin(Ax)\cos(Bx)dx$, $\int \cos(Ax)\cos(Bx)dx$, $\int \sin(Ax)\sin(Bx)dx$

Use the product to sum formula:

$$\cos\theta\cos\phi = \frac{1}{2}(\cos\theta-\phi)+(\cos\theta+\phi))\ ;\ \sin\theta\cos\phi = \frac{1}{2}(\cos\theta-\phi)-(\cos\theta+\phi))$$

$$\sin\theta\cos\phi = \frac{1}{2}(\sin\theta-\phi)+(\sin\theta+\phi))$$

Reduce product into sum and then integrate.

Trigonometric Substitution

(a) Trigonometric substitution is useful for quadratic form with square root:

$\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$

$\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$

$\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$

(b) General procedure for doing trig sub:

Step 1: Draw the right triangle, and decide what trigonometric function to substitute for x.

Step 2: Find dx, then substitute the integrand using triangle, convert integral into trigonometric integral.

Step 3: Solve the trigonometric integral.

Step 4: Substitute back using triangle.

(i) If the quadratic form is not in the Pythagoras form (for example, $\sqrt{2 + 2x + x^2}$, then use the perfecting the square method to transform it into Pythagoras form).

(ii) Try u – substitution before trigonometric substitution.

(iii) Integrals involving $(1 - x^2)$ and $(x^2 - 1)$ without square roots can be solved easily with partial fractions. So don't use trigonometric substitution.

Rational Integral and Partial Fraction

(a) General step for solving rational integral:

Step 1: Do long division for the rational function if the degree of the numerator is higher than the denominator.

Step 2: Do partial fraction decomposition.

Step 3: Evaluate the integral of each simple fraction.

(b) General step for partial fraction:

Step 1: Factorize the denominator.

Step 2: Set the partial fraction according to "rule".

Step 3: Solve the unknown of the numerator of the partial fraction.

Improper Integral

(a) General steps for evaluating improper integral:

Step 1: Change the improper integral into the appropriate limit. [Change ±∞ or singular point (where) to appropriate limit.]

Step 2: Evaluate the integral.

Step 3: Find the limit.

(b) The very first step to test improper integral involving ∞ is to check its limit. If its limit is not zero, then the integral diverges.

(c) Whenever you see improper integrals involving the quotient of a rational or irrational function, such as

$$\int_a^\infty \frac{x^3 + \sqrt{3x}}{(8x^3 + 7x)^{3/2}}\,dx$$

Use **limit comparison test**. The appropriate comparing function can be found by looking at the Integrand (quotient of rational irrational). "Discard" the lower degree terms.

(d) Sometimes, using u – substitution before using any test will be easier.

(e) Sometimes, to determine if an improper integral converges or diverges, directly evaluating the improper integral is easier.

(f) When doing a **comparison test**, beware of the comparing function that you choose. It might not give an appropriate conclusion if the comparing function is not correct.

(g) Try the **limit comparison test** before the **comparison test**.

(h) Useful comparing function, which is good to know their convergence or divergence

$$\int_a^{\infty} x^k e^{-\beta x} dx < \infty \quad \text{For } k \ge 0, \beta > 0$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p > 1; \\ = \infty & \text{if } p \le 1; \end{cases}$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p < 1; \\ = \infty & \text{if } p \ge 1; \end{cases}$$

FORMULAE SHEET

Basic theorems of Integration:

1. $\int k\, f(x)dx = k\int f(x)dx$	**2.** $\int \left[f(x) \pm g(x)\right]dx = \int f(x)dx \pm \int g(x)dx$
3. $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$	**4.** $\int \left(\frac{d}{dx} f(x)\right)dx = f(x)$

Elementary Integration:

1. $\int 0.dx = c$	**2.** $\int 1.dx = x + c$
3. $\int k.dx = kx + c(k \in R)$	**4.** $\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \ne -1)$
5. $\int \frac{1}{x} dx = \log_e x + c$	**6.** $\int e^x dx = e^x + c$
7. $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$	**8.** $\int \sin x\, dx = -\cos x + c$
9. $\int \cos x\, dx = \sin x + c$	**10.** $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
11. $\int \frac{c}{ax+b} dx = \frac{c}{a} \log \left\|ax+b\right\| + c$	**12.** $\int f'(x) e^{f(x)} dx = e^{f(x)} + c$
13. $\int \log x\, dx = x\log x - x + c$	**14.** $\int \log_a x\, dx = x\log_a x - \frac{x}{\log a} + c$

Standard substitution:

1. $\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$ or $x = a\cos\theta$
2. $\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x = a\tan\theta$ or $x = a\cot\theta$
3. $\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
4. $\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}$ and $\frac{1}{\sqrt{x(a+x)}}$	$x=a\tan^2\theta$
5. $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ $\sqrt{x(a-x)}$ and $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$ or $x= a\cos2$
6. $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x=a\sec^2\theta$ or $x=a\,\text{cosec}^2\theta$
7. $\sqrt{\frac{a-x}{a+x}}$ and $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos2\theta$
8. $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta>\alpha)$	$x = \alpha\cos^2\theta+\beta\sin^2\theta$

Some standard Integrals:

1. $\int \tan x dx = \log\sec x + c = -\log\cos x + c$	**2.** $\int \cot x dx = \log\sin x + c$
3. $\int \sec x dx = \log(\sec x + \tan x) + c$ $= -\log(\sec x - \tan x) + c = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$	**4.** $\int \text{cosec} x dx = -\log(\text{cosec} x + \cot x) + c$ $= \log(\text{cosec} x - \cot x) + c = \log\tan\left(\frac{x}{2}\right)+c$
5. $\int \sec x \tan x dx = \sec x + c$	**6.** $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$
7. $\int \sec^2 x dx = \tan x + c$	**8.** $\int \text{cosec}^2 x dx = -\cot x + c$
9. $\int \log x dx = x\log\left(\frac{x}{e}\right)+c = x\left(\log x + 1\right)+c$	**10.** $\int \sin^2 x dx = \frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+C$ $=\frac{1}{2}(x-\sin x\cos x)+C$

11. $\int \cos^2 x dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C = \frac{1}{2}(x + \sin x \cos x) + C$	**12.** $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\lvert\sec x + \tan x\rvert + c$
13. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$	**14.** $\int \cos^n x dx = -\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx$

Integration by Parts:

1. $\int (u.v)dx = u\left(\int v\, dx\right) - \int\left(\frac{du}{dx}\right)\left(\int v\, dx\right)dx$	**2.** $\int e^x \left[f(x) + f'(x)\right]dx = e^x f(x) + c$

Standard Integrals:

1. $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$
3. $\int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$
4. $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c = -\cos^{-1}\left(\frac{x}{a}\right) + c$
5. $\int \frac{1}{\sqrt{x^2 + a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right) + c = \log\left(x + \sqrt{x^2 + a^2}\right) + c$
6. $\int \frac{1}{\sqrt{x^2 - a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + c = \log\left(x + \sqrt{x^2 - a^2}\right) + c$
7. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$
8. $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left\lvert x + \sqrt{x^2 + a^2}\right\rvert + c$
9. $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + c$

10.	$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$ (Valid for x > a > 0)
11.	$\int e^{ax}\sin bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c \; = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$
12.	$\int e^{ax}\cos bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c \; = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx-\tan^{-1}\frac{b}{a}\right\}+c$

Solved Examples

JEE Main/Boards

Example 1: Evaluate : $\int \frac{x+\sin x}{1+\cos x}dx$

Sol: Here by using the formula

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$

we can solve the given problem.

$$\int \frac{x+\sin x}{1+\cos x}dx \;=\; \int \frac{x+2\sin x/2\cos x/2}{2\cos^2 x/2}dx$$

$$= \int \frac{x}{2}\sec^2 x/2+\tan\frac{x}{2}dx \;=\; x\tan x/2+c$$

Example 2: Evaluate : $\int \sqrt{x^2+a^2}dx$

Sol: By applying integration by parts and taking $\sqrt{x^2+a^2}$ as the first function we can solve the given problem.

$$\int \sqrt{x^2+a^2}dx \;=$$

$$= \sqrt{x^2+a^2}x - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx \;\; \sqrt{x^2+a^2}x - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$= \frac{x\sqrt{x^2+a^2}+a^2\int \frac{dx}{\sqrt{x^2+a^2}}}{2}$$

Put x = atan θ

$$\int \frac{dx}{\sqrt{x^2+a^2}} \;=\; \int \sec\theta\, d\theta \;=\; \log\left(\sec\theta+\tan\theta\right)$$

$$= \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)$$

$$\therefore\;\; I = \frac{1}{2}\left(x\sqrt{x^2+a^2} + \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)\right)+c$$

Example 3: Evaluate : $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx$

Sol: Here first write $\cos\left((\pi/2)-x\right)$ at the place of sin x then by using the formula $1-\cos x = 2\sin^2\frac{x}{2}$

And $1+\cos x = 2\cos^2\frac{x}{2}$ we can solve it.

$$I = \int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx \;=\; \int \tan^{-1}\sqrt{\frac{1-\cos\left((\pi/2)-x\right)}{1+\cos\left((\pi/2)-x\right)}}dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left((\pi/4)-(x/2)\right)}{2\cos^2\left((\pi/4)-(x/2)\right)}}dx$$

$$= \int \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \;=\; \int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \;=\; \frac{\pi}{4}x-\frac{x^2}{4}+C$$

Example 4: Evaluate : $\int \log(2+x^2)dx$

Sol: Here integrating by parts by taking $\log(2+x^2)$ as the first function we can solve the given problem.