**PROBLEM-SOLVING TACTICS**

**Integration by Parts**

(a) Integration by parts is useful for dealing with integrals of the products of the following functions:

\[ u \ll \tan^{-1}x, \sin^{-1}x, \cos^{-1}x, (\log x)^k, \sin x, \cos x, e^x \ll dv \]

Priority for choosing u and dv: ILATE

(b) Integration by parts is sometimes useful for finding integrals of functions involving inverse functions such as \( n x \) and \( \sin^{-1}x \).

(c) Sometimes when dealing with integrals, the integrand involves inverse functions (like \( \sin^{-1}x \)), it is useful to substitute \( x = \) the inverse of that inverse function (like \( x = \sin u \)), then do integration by parts.

(d) Sometimes you will have to do integration by parts more than once (for example, \( \int x^2 e^x \, dx \) and \( \int x^3 \sin x \, dx \)). Sometimes you need to do it twice by parts, then manipulate the equation (for example, \( \int e^x \sin x \, dx \)).

(e) Try u – substitution first before integration by parts.

**Trigonometric Integral**

(a) Integral type: \( \int \sin^m x \cos^n x \, dx \)

**Case 1:** One of \( m \) or \( n \) are even, and the other odd.

Use u – substitution by setting \( u = \sin \) or \( \cos \) that with an even power. Use the identity \( \sin^2 x + \cos^2 x = 1 \).

**Case 2:** Both \( m \) and \( n \) are odd.

Use u – substitution by setting \( u = \sin \) or \( \cos \) that with a higher power. Use the identity \( \sin^2 x + \cos^2 x = 1 \).

**Case 3:** Both \( m \) and \( n \) are even (hard case).

Do not use u – substitution. Use the half double angle formula to reduce the integrand into case 1 or 2:

\[ \sin x \cos x = \frac{1}{2} \sin 2x; \quad \sin^2 x = \frac{1}{2} (1 - \cos 2x); \quad \cos^2 x = \frac{1}{2} (1 - \cos 2x) \]

(Note: 0 is also an even number. For example, \( \sin^3 x = \sin^3 x \cos^0 x \), so it is in case 1.)

Just remember that when both are even, you can’t use u-substitution, but you can use the half – double angle formula. When it is not that case, let \( u = \sin x \) or \( \cos x \), and one will work (at the end there is no square root term after substitution).

(b) Integral type: \( \int \tan^m x \sec^n x \, dx \)

**Case 1:** sec is odd power, tan is even power.

Hard to do, we omit (most likely won’t pop out in the exam).

**Case 2:** Else

Set \( u = \sec x \) or tan \( x \), and use \( 1 + \tan^2 x = \sec^2 x \). One will work at the end (there is no square root term after substitution).

(c) Integral type: \( \int \sin(Ax) \cos(Bx) \, dx \), \( \int \cos(Ax) \cos(Bx) \, dx \), \( \int \sin(Ax) \sin(Bx) \, dx \)

Use the product to sum formula:

\[ \cos \theta \cos \phi = \frac{1}{2} (\cos (\theta - \phi) + (\cos (\theta + \phi)) \quad \sin \theta \cos \phi = \frac{1}{2} (\cos (\theta - \phi) - (\cos (\theta + \phi)) \]

\[ \sin \theta \cos \phi = \frac{1}{2} (\sin (\theta - \phi) + (\sin (\theta + \phi)) \]

Reduce product into sum and then integrate.
Trigonometric Substitution

(a) Trigonometric substitution is useful for quadratic form with square root:
\[ \sqrt{a^2 - x^2} : \text{Let } x = a \sin \theta \]
\[ \sqrt{x^2 + a^2} : \text{Let } x = a \tan \theta \]
\[ \sqrt{x^2 - a^2} : \text{Let } x = a \sec \theta \]

(b) General procedure for doing trig sub:

Step 1: Draw the right triangle, and decide what trigonometric function to substitute for x.
Step 2: Find dx, then substitute the integrand using triangle, convert integral into trigonometric integral.
Step 3: Solve the trigonometric integral.
Step 4: Substitute back using triangle.

(i) If the quadratic form is not in the Pythagoras form (for example, \( \sqrt{2 + 2x + x^2} \), then use the perfecting the square method to transform it into Pythagoras form).

(ii) Try u – substitution before trigonometric substitution.

(iii) Integrals involving \((1 - x^2)\) and \((x^2 - 1)\) without square roots can be solved easily with partial fractions. So don’t use trigonometric substitution.

Rational Integral and Partial Fraction

(a) General step for solving rational integral:

Step 1: Do long division for the rational function if the degree of the numerator is higher than the denominator.
Step 2: Do partial fraction decomposition.
Step 3: Evaluate the integral of each simple fraction.

(b) General step for partial fraction:

Step 1: Factorize the denominator.
Step 2: Set the partial fraction according to “rule”.
Step 3: Solve the unknown of the numerator of the partial fraction.

Improper Integral

(a) General steps for evaluating improper integral:

Step 1: Change the improper integral into the appropriate limit. [Change \( \pm \infty \) or singular point (where) to appropriate limit.]
Step 2: Evaluate the integral.
Step 3: Find the limit.

(b) The very first step to test improper integral involving \( \infty \) is to check its limit. If its limit is not zero, then the integral diverges.

(c) Whenever you see improper integrals involving the quotient of a rational or irrational function, such as
\[ \int_0^\infty \frac{x^3 + \sqrt{3x}}{(8x^3 + 7x)^{3/2}} \, dx \]

Use limit comparison test. The appropriate comparing function can be found by looking at the Integrand (quotient of rational irrational). “Discard” the lower degree terms.
(d) Sometimes, using $u$– substitution before using any test will be easier.

(e) Sometimes, to determine if an improper integral converges or diverges, directly evaluating the improper integral is easier.

(f) When doing a comparison test, beware of the comparing function that you choose. It might not give an appropriate conclusion if the comparing function is not correct.

(g) Try the limit comparison test before the comparison test.

(h) Useful comparing function, which is good to know their convergence or divergence

$$\int_a^\infty x^k e^{-\beta x} \, dx < \infty \text{ for } k > 0, \beta > 0$$

$$\int_a^\infty \frac{1}{x^p} \, dx \begin{cases} < \infty & \text{if } p > 1; \\ = \infty & \text{if } p \leq 1; \end{cases}$$

$$\int_a^\infty \frac{1}{x^p} \, dx \begin{cases} < \infty & \text{if } p < 1; \\ = \infty & \text{if } p \geq 1; \end{cases}$$

FORMULAE SHEET

Basic theorems of Integration:

1. $\int k f(x) \, dx = k \int f(x) \, dx$
2. $\int \left[ f(x) \pm g(x) \right] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
3. $\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x)$
4. $\int \left( \frac{d}{dx} f(x) \right) \, dx = f(x)$

Elementary Integration:

1. $\int 0 \, dx = c$
2. $\int 1 \, dx = x + c$
3. $\int k \, dx = kx + c (k \in \mathbb{R})$
4. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c (n \neq -1)$
5. $\int \frac{1}{x} \, dx = \log_e x + c$
6. $\int e^x \, dx = e^x + c$
7. $\int a^x \, dx = \frac{a^x}{\log_a a} + c = a^x \log_a e + c$
8. $\int \sin x \, dx = -\cos x + c$
9. $\int \cos x \, dx = \sin x + c$
10. $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c$
11. $\int \frac{c}{ax + b} \, dx = \frac{c}{a} \log |ax + b| + c$
12. $\int f(x) e^{f(x)} \, dx = e^{f(x)} + c$
13. $\int \log x \, dx = x \log x - x + c$
14. $\int \log_a x \, dx = x \log_a x - \frac{x}{\log a} + c$
### Standard substitution:

| 1. $\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$ | $x = a \sin \theta$ or $x = a \cos \theta$ |
| 2. $\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$ | $x = a \tan \theta$ or $x = a \cot \theta$ |
| 3. $\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$ | $x = a \sec \theta$ or $x = a \cosec \theta$ |
| 4. $\sqrt{\frac{x}{a+x}}$, $\frac{a+x}{x}$, $\sqrt{x(a+x)}$ and $\frac{1}{\sqrt{x(a+x)}}$ | $x = a \tan^2 \theta$ |
| 5. $\frac{x}{\sqrt{a-x}}$ or $\frac{a-x}{x}$, $\sqrt{x(a-x)}$ and $\frac{1}{\sqrt{x(a-x)}}$ | $x = a \sin^2 \theta$ or $x = a \cos^2 \theta$ |
| 6. $\frac{x}{\sqrt{x-a}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$ | $x = a \sec^2 \theta$ or $x = a \cosec^2 \theta$ |
| 7. $\sqrt{\frac{a-x}{a+x}}$ and $\sqrt{\frac{a+x}{a-x}}$ | $x = a \cos 2 \theta$ |
| 8. $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ ($\beta > \alpha$) | $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$ |

### Some standard Integrals:

| 1. $\int \tan x \, dx = \log \sec x + c = -\log \cos x + c$ | 2. $\int \cot x \, dx = \log \sin x + c$ |
| 3. $\int \sec x \, dx = \log(\sec x + \tan x) + c$ | 4. $\int \cosec x \, dx = -\log(\cosec x + \cot x) + c$ |
| $= -\log(\sec x - \tan x) + c = \log \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) + c$ | $= \log(\cosec x - \cot x) = \log \tan \left( \frac{x}{2} \right) + c$ |
| 5. $\int \sec x \tan x \, dx = \sec x + c$ | 6. $\int \sec x \cot x \, dx = -\cosec x + c$ |
| 7. $\int \sec^2 x \, dx = \tan x + c$ | 8. $\int \cosec^2 x \, dx = -\cot x + c$ |
| 9. $\int \log x \, dx = x \log \left( \frac{x}{e} \right) + c = x(\log x + 1) + c$ | 10. $\int \sin^2 x \, dx = \frac{1}{2} \left( x - \frac{\sin 2x}{2} \right) + C$ |
| $= \frac{1}{2} (x - \sin x \cos x) + C$ | }
### Indefinite Integration

| 11. | \( \int \cos^2 x \, dx = \frac{1}{2} \left( x + \frac{\sin 2x}{2} \right) + C = \frac{1}{2} (x + \sin x \cos x) + C \) |
| 12. | \( \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log |\sec x + \tan x| + C \) |

| 13. | \( \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx \) |
| 14. | \( \int \cos^n x \, dx = -\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \) |

### Integration by Parts:

1. \( \int (u \cdot v) \, dx = u \left( \int v \, dx \right) - \int \left( \frac{du}{dx} \right) \left( \int v \, dx \right) \, dx \)

2. \( \int e^x \left[ f(x) + f'(x) \right] \, dx = e^x f(x) + C \)

### Standard Integrals:

| 1. | \( \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \) |
| 2. | \( \int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C \) |
| 3. | \( \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C \) |
| 4. | \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + C = -\cos^{-1} \left( \frac{x}{a} \right) + C \) |
| 5. | \( \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \sinh^{-1} \left( \frac{x}{a} \right) + C = \log \left( x + \sqrt{x^2 + a^2} \right) + C \) |
| 6. | \( \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \cosh^{-1} \left( \frac{x}{a} \right) + C = \log \left( x + \sqrt{x^2 - a^2} \right) + C \) |
| 7. | \( \int \frac{x^2}{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \) |
| 8. | \( \int \frac{x}{\sqrt{x^2 + a^2}} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C \) |
| 9. | \( \int \frac{x}{\sqrt{x^2 - a^2}} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C \) |
10. \[ \int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c \quad \text{(Valid for x > a > 0)} \]

11. \[ \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c \]

12. \[ \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c \]

### Solved Examples

#### JEE Main/Boards

**Example 1:** Evaluate \( \int \frac{x + \sin x}{1 + \cos x} \, dx \)

**Sol:** Here by using the formula \( \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \) and \( 1 + \cos x = 2 \cos^2 \frac{x}{2} \) we can solve the given problem.

\[
\int \frac{x + \sin x}{1 + \cos x} \, dx = \int \frac{x + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \, dx
\]

\[
= \int \frac{x}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \, dx = x \tan x / 2 + c
\]

**Example 2:** Evaluate \( \int \sqrt{x^2 + a^2} \, dx \)

**Sol:** By applying integration by parts and taking \( \sqrt{x^2 + a^2} \) as the first function we can solve the given problem.

\[
\int \sqrt{x^2 + a^2} \, dx = \frac{1}{2} \left[ x \sqrt{x^2 + a^2} + a^2 \log \left( x + \sqrt{x^2 + a^2} \right) \right] + c
\]

**Example 3:** Evaluate \( \int \tan^{-1} \left( \frac{1 - \sin x}{\sqrt{1 + \sin x}} \right) \, dx \)

**Sol:** Here first write \( \cos \left( \frac{\pi}{2} - x \right) \) at the place of \( \sin x \) then by using the formula \( 1 - \cos x = 2 \sin^2 \frac{x}{2} \) and \( 1 + \cos x = 2 \cos^2 \frac{x}{2} \) we can solve it.

\[
I = \int \tan^{-1} \left( \frac{1 - \sin x}{\sqrt{1 + \sin x}} \right) \, dx
\]

\[
= \int \tan^{-1} \left( 2 \sin^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) \right) \, dx
\]

\[
= \int \tan^{-1} \left( \frac{\pi - x}{2} \right) \, dx = \int \left( \frac{\pi - x}{2} \right) \, dx = \frac{\pi x - x^2}{4} + C
\]

**Example 4:** Evaluate \( \int \log(2 + x^2) \, dx \)

**Sol:** Here integrating by parts by taking \( \log(2 + x^2) \) as the first function we can solve the given problem.