10.	$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$ (Valid for x > a > 0)
11.	$\int e^{ax}\sin bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c \; = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$
12.	$\int e^{ax}\cos bx \quad dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c \; = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx-\tan^{-1}\frac{b}{a}\right\}+c$

Solved Examples

JEE Main/Boards

Example 1: Evaluate : $\int \frac{x+\sin x}{1+\cos x}dx$

Sol: Here by using the formula

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$

we can solve the given problem.

$$\int \frac{x+\sin x}{1+\cos x}dx \; = \int \frac{x+2\sin x/2\cos x/2}{2\cos^2 x/2}dx$$

$$= \int \frac{x}{2}\sec^2 x/2 + \tan\frac{x}{2}dx \; = x\tan x/2 + c$$

Example 2: Evaluate : $\int \sqrt{x^2+a^2}dx$

Sol: By applying integration by parts and taking $\sqrt{x^2+a^2}$ as the first function we can solve the given problem.

$$\int \sqrt{x^2+a^2}dx \; =$$

$$= \sqrt{x^2+a^2}x - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx \; \sqrt{x^2+a^2}x - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$= \frac{x\sqrt{x^2+a^2}+a^2\int \frac{dx}{\sqrt{x^2+a^2}}}{2}$$

Put x = atan θ

$$\int \frac{dx}{\sqrt{x^2+a^2}} \; = \int \sec\theta d\theta \; = \log\left(\sec\theta+\tan\theta\right)$$

$$= \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)$$

$$\therefore \; I = \frac{1}{2}\left(x\sqrt{x^2+a^2} + \log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)\right)+c$$

Example 3: Evaluate : $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx$

Sol: Here first write $\cos((\pi/2)-x)$ at the place of sin x then by using the formula $1-\cos x = 2\sin^2\frac{x}{2}$

And $1+\cos x = 2\cos^2\frac{x}{2}$ we can solve it.

$$I = \int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}dx \; = \int \tan^{-1}\sqrt{\frac{1-\cos((\pi/2)-x)}{1+\cos((\pi/2)-x)}}dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2((\pi/4)-(x/2))}{2\cos^2((\pi/4)-(x/2))}}dx$$

$$= \int \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \; = \int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \; = \frac{\pi}{4}x-\frac{x^2}{4}+C$$

Example 4: Evaluate : $\int \log(2+x^2)dx$

Sol: Here integrating by parts by taking $\log(2+x^2)$ as the first function we can solve the given problem.

$I = \int \log(2+x^2)dx = \int \log(2+x^2).1dx$

Taking $\log(2+x^2)$ as first function and integrating by parts, we get

$I = \left[\log(2+x^2)\right]x - \int x.\frac{2x}{2+x^2}dx$

$= x\log(2+x^2) - 2\int \frac{(x^2+2)-2}{2+x^2}dx$

$= x\log(2+x^2) - 2\int \left[1 - \frac{2}{x^2+2}\right]dx$

$= x\log(x^2+2) - 2\left[x - \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right] + c$

Example 5: Evaluate : $\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$

Sol: Simply put $e^{2x}+e^{-2x} = t \Rightarrow (e^{2x}-e^{-2x})dx = \frac{dt}{2}$ and then solving we will get the result.

$I = \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$

Put $e^{2x}+e^{-2x} = t \Rightarrow (e^{2x}-e^{-2x})dx = \frac{dt}{2}$

$\therefore I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + C = \frac{1}{2}\log\left|e^{2x}+e^{-2x}\right| + c$

Example 6: Evaluate : $\int \frac{x^3-1}{x^3+x}dx$

Sol: By splitting the given integration as

$\int \frac{x^3}{x(x^2+1)}dx - \int \frac{1}{x(x^2+1)}dx$

We can solve the given problem.

$\int \frac{x^3-1}{x^3+x}dx = \int \frac{x^3}{x(x^2+1)}dx - \int \frac{1}{x(x^2+1)}dx$

$= \int \frac{x^2}{x^2+1}dx - \int \frac{1}{x} - \frac{x}{(x^2+1)}dx$

$= \int \left(1 - \frac{1}{x^2+1}\right)dx - \int \frac{1}{x}dx + \int \frac{x}{x^2+1}dx$

$= x - \tan^{-1}x - \log x + \log\sqrt{x^2+1} + c$

Example 7: Evaluate: $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol: By putting $x=\sin\theta \Rightarrow dx = \cos\theta\, d\theta$, we will reduce the given integration as

$\int \frac{\sec^2\theta}{1+2\tan^2\theta}d\theta$ and then put $t = \tan\theta \Rightarrow dt = \sec^2\theta\, d\theta$

and solve it.

Put $x=\sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\Rightarrow I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{1}{1+\sin^2\theta}d\theta$

$= \int \frac{\sec^2\theta}{1+2\tan^2\theta}d\theta$

Again put $t = \tan\theta \Rightarrow dt = \sec^2\theta\, d\theta$

$I = \int \frac{1}{1+2t^2}dt = \frac{1}{\sqrt{2}}\int \frac{1}{t^2+\left(1/\sqrt{2}\right)^2}dt$

$= \frac{1}{2}\left(\frac{1}{1/\sqrt{2}}\right)\tan^{-1}\left(\frac{t}{1/\sqrt{2}}\right) + c$

$= \frac{1}{2}\tan^{-1}(\sqrt{2}\tan\theta) + c = \frac{1}{2}\tan^{-1}\left(\frac{x\sqrt{2}}{\sqrt{1-x^2}}\right) + c$

Example 8: Evaluate : $\int \frac{xdx}{(x-1)(x^2+4)}$

Sol: By partial fractions, we can reduce the given fraction as a sum of two fractions which will be easier to integrate.

$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$x = 1 \Rightarrow A = 1/5$

$x = 2i \Rightarrow B = -1/5,\ C = 4/5$

$\therefore I = \int \left(\frac{1}{5(x-1)} + \frac{4-2x}{5(x^2+4)}\right)dx$

$= \frac{1}{5}\log(x-1) - \frac{1}{10}\left[\frac{2x}{x^2+4} - \frac{8}{x^2+4}\right]$

$= \frac{1}{5}\log(x-1) - \frac{\log(x^2+4)}{10} + \frac{2}{5}\tan^{-1}\frac{x}{2}$

Example 9: Evaluate: $\int \frac{\sin x}{\sin 4x}dx$

Sol: By using the formula $\sin 2x = 2\sin x.\cos x$, we can reduce the given fraction and then by putting sin x = t we can solve it.

$$\int \frac{\sin x}{\sin 4x}dx = \int \frac{\sin x\,dx}{2\cos 2x \sin 2x} = \int \frac{dx}{4\cos x \cos 2x}$$

$$= \int \frac{\cos x\,dx}{4(1-\sin^2 x)(1-2\sin^2 x)}$$

Put sin x = t

$$\Rightarrow I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}$$

$$= \frac{1}{4}\int \left(\frac{1}{(t^2-1)} - \frac{2}{(2t^2-1)}\right)dt$$

$$= \frac{1}{4}\left(\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right|\right) + c$$

$$= \frac{1}{8}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x + 1}{\sqrt{2}\sin x - 1}\right| + c$$

Example 10: Evaluate : $\int \frac{dx}{(x)(x^4-1)}$

Sol: Here we can write the given integration as

$\int \frac{x^{-5}}{(1-x^{-4})}dx$ and then by putting $1 - x^{-4} = t$

$\Rightarrow 4x^{-5} = dt$ we can solve it.

$$\int \frac{dx}{(x)(x^4-1)} = \int \frac{x^{-5}}{(1-x^{-4})}dx$$

Put $1 - x^{-4} = t \Rightarrow 4x^{-5} = dt$

$$\Rightarrow = \int \frac{dt}{4t} = \frac{1}{4}\log|t| + c = \frac{1}{4}\log|1-x^{-4}| + c$$

JEE Advanced/Boards

Example 1: Evaluate : $\int x\sin^{-1} x\,dx$

Sol: Integrating by parts taking $\sin^{-1} x$ as the first term we can solve the given integration.

$$\int x\sin^{-1} x\,dx$$

Let $u = \sin^{-1} x$, $v = x$

$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{(1-x^2)-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4}\sin^{-1} x + c$$

$$= \frac{2x^2-1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$$

Example 2: Evaluate : $\int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}}$

Sol: We can split the given fraction as

$\int e^x\left\{\frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right\}dx$ and this integration is in the form of $e^x(f(x)+f'(x))$

$$I = \int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}} = \int e^x \frac{(1-x^2)+1}{(1-x)\sqrt{1-x^2}}dx$$

$$= \int e^x\left\{\frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right\}dx$$

But $\frac{d}{dx}\left(\frac{1+x}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}} + (1+x)\frac{x}{(1-x^2)^{3/2}}$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{(1+x)x}{(1-x)(1+x)\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{x}{(1-x)\sqrt{1-x^2}} = \frac{1-x+x}{(1-x)\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x)\sqrt{1-x^2}}$$

Hence, the integrand is of type $e^x(f(x)+f'(x))$

$$\therefore I = e^x\frac{1+x}{\sqrt{1-x^2}} + C$$

Example 3: Evaluate : $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx$

Sol: Here by taking $\cos^3 x$ and $\sin^2 x$ common from the numerator and denominator respectively and then by putting sin x =t we can solve the given problem.

$$I = \int \frac{\cos^3 x(1+\cos^2 x)}{\sin^2 x(1+\sin^2 x)}dx$$

Since power of cos x is odd, put sin x =t;

Then, cos x dx = dt

$$I = \int \frac{(1-t^2)(1+1-t^2)}{t^2(1+t^2)}dt = \int \frac{(1-t^2)(2-t^2)}{t^2(1+t^2)}dt$$

$$= \int \left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)dt = t-\frac{2}{t}-6\tan^{-1}t+c$$

$$= \sin x - 2\operatorname{cosec} x - 6\tan^{-1}(\sin x) + c$$

Example 4: Evaluate : $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$

Sol: By partial fractions we can reduce the given fraction as a sum of two fractions and then by integrating them we will get the result.

$$I = \int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$$

Let $4e^x+6e^{-x} = A\left(9e^x-4e^{-x}\right)+B\left(\frac{d}{dx}\left(9e^x-4e^{-x}\right)\right)$

By comparing the coefficients of e^x and e^{-x}, we get

$A = \frac{-19}{36}$, $B = \frac{35}{36}$

$$\therefore I = \int \frac{A(9e^x-4e^{-x})+B(9e^x+4e^{-x})}{9e^x-4e^{-x}}dx$$

$$= A\int dx + B\int \frac{9e^x+4e^{-x}}{9e^x-4e^{-x}}dx = Ax + B\log\left|9e^x-4e^{-x}\right|+c$$

$$= -\frac{19}{36}x+\frac{35}{36}\log\left|9e^x-4e^{-x}\right|+c$$

Example 5: Evaluate : $\int \frac{1+x^2}{1-x^2}\frac{dx}{\sqrt{1+3x^2+x^4}}$ for x > 0

Sol: Dividing by x^2 in the numerator and denominator and then putting x – 1/x = t we can solve the given problem.

$$I = \int \frac{1+x^2}{1-x^2}\frac{dx}{\sqrt{1+3x^2+x^4}} = \int \frac{\left((1/x^2)+1\right)dx}{\left((1/x)-x\right)\sqrt{\left(x-(1/x)\right)^2+5}}$$

Put x – 1/x = t ; $\left(1+\frac{1}{x^2}\right)dx = dt$

$$I = -\int \frac{dt}{t\sqrt{t^2+5}}$$

Put $t^2+5=z^2 \Rightarrow$ 2t dt = 2z dz

$$\Rightarrow I = -\int \frac{dt}{z^2-5} = -\frac{1}{2\sqrt{5}}\log\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right|+c$$

$$= -\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{\left(x^2+(1/x^2)+3\right)}-\sqrt{5}}{\sqrt{\left(x^2+(1/x^2)+3\right)}+\sqrt{5}}\right|+c$$

Example 6: Evaluate :

$\int \operatorname{cosec}^2 x.\log\left(\cos x+\sqrt{\cos 2x}\right)dx$ for sin x > 0

Sol: By substituting $\cos^2 x-\sin^2 x$ in place of $.\cos 2x$. we can reduce the given integration as the sum of two integrations and then by integrating them separately we will obtain the result.

$$\int \operatorname{cosec}^2 x.\log\left(\cos x+\sqrt{\cos 2x}\right)dx$$

$$= \int \operatorname{cosec}^2 x\log\left(\cos x+\sqrt{\cos^2 x-\sin^2 x}\right)dx$$

$$= \int \operatorname{cosec}^2 x\log\left(\sin x(\cot x+\sqrt{\cot^2 x-1})\right)dx$$

$$= \int \operatorname{cosec}^2 x.\log\sin x\,dx + \int \operatorname{cosec}^2 x.\log[\cot x+\sqrt{\cot^2 x-1}]dx$$

$$= I_1 + I_2$$

$$I_1 = \int \operatorname{cosec}^2 x.\log\sin x\,dx$$

$$= (-\cot x).\log\sin x - \int(-\cot x).\cot x\,dx$$

$$= -\cot x.\log\sin x + \int(\operatorname{cosec}^2 x-1)dx$$

$$= -\cot x.\log\sin x - \cot x - x + C_1$$

$$I_2 = \int \operatorname{cosec}^2 x.\log[\cot x+\sqrt{\cot^2 x-1}]dx$$

Put cot x=t; $-\operatorname{cosec}^2 x dx = dt$

$$I_2 = -\int \log[t+\sqrt{t^2-1}]dt$$

(Integrate by parts)

$$= -t.\log\left(t+\sqrt{t^2-1}\right)+\int t.\frac{1+(t/\sqrt{t^2-1})}{t+\sqrt{t^2-1}}dt$$

$$= -t.\log\left(t+\sqrt{t^2-1}\right)+\int \frac{t}{\sqrt{t^2-1}}dt$$

$$= -t.\log\left(t+\sqrt{t^2-1}\right)+\sqrt{t^2-1}+C_2$$

$$= -\cot x\log\left(\cot x+\sqrt{\cot^2 x-1}\right)+\sqrt{\cot^2 x-1}+C_2$$

Example 7: Evaluate : $\int \frac{\sin x}{\sin^3 x+\cos^3 x}dx$

Sol: By taking $\cos^3 x$ common from the denominator and then by putting tan x = t we can solve it.

Integrand contains odd powers in sin x and cos x. So, put tan x = t.

$$\Rightarrow I = \int \frac{1}{\cos^3 x}\frac{\sin x}{1+\tan^3 x}dx$$

$$= \int \frac{\tan x.\sec^2 x}{1+\tan^3 x}dx \text{ (put tan x = t)}$$

$$= \int \frac{t}{1+t^3}dt = -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{3}\int\frac{t+1}{t^2-t+1}dt$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\int\frac{(2t-1)+3}{t^2-t+1}dt$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\frac{dt}{(t-(1/2))^2+(3/4)}$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-(1/2)}{\sqrt{3}/2}\right)+C$$

$$= \frac{1}{6}\log\left|\frac{1-\tan x+\tan^2 x}{(1+\tan x)^2}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$$

Example 8: If $I_{m,n} = \int \cos^m x.\cos nx\, dx$, show that $(m+n)I_{m,n} = \cos^m x \sin nx + mI_{m-1,n-1}$

Sol: By using integration by parts and by taking $\cos^m x$ as the first term we can prove the given equation.

Integrating by parts,

$$I_{m,n} = \cos^m x\frac{\sin nx}{n} + \frac{m}{n}\int\cos^{m-1}x\sin x\sin nx\,dx \quad \text{... (i)}$$

But cos(n-1)x=cos(nx-x)

= cosnx cos nx + sin nx sin x

⇒ sinx sin nx = cos(n-1)x – cosnx cosx ... (ii)

From (i) and (ii):

$$I_{m,n} = \frac{1}{n}\cos^m x\sin nx + \frac{m}{n}\int\cos^{m-1}x[\cos(n-1)x-\cos nx\cos x]dx$$

$$= \frac{1}{n}\cos^m x\sin nx + \frac{m}{n}I_{m-1,n-1} - \frac{m}{n}I_{m,n}$$

$$\Rightarrow (m+n)I_{m,n} = \cos^m x\sin nx + mI_{m-1,n-1}$$

Example 9: Evaluate : $I = \int\left(x+\sqrt{1+x^2}\right)^n dx$

Sol: Simply putting $x+\sqrt{1+x^2}$ = t and integrating we can solve the given problem.

Let $x+\sqrt{1+x^2}$ = t, then

$$\left(1+\frac{x}{\sqrt{1+x^2}}\right)dx = dt \Rightarrow \frac{t}{\sqrt{1+x^2}}dx = dt$$

As $\sqrt{1+x^2}+x = t$

$$\frac{1}{t} = \frac{1}{\sqrt{1+x^2}+x} = \frac{\sqrt{1+x^2}-x}{1}$$

$$\Rightarrow 2\sqrt{1+x^2} = t+\frac{1}{t} = \frac{t^2+1}{t}$$

Thus $I = \int t^n\left(\frac{t^2+1}{2t}\right)\frac{dt}{t}$

$$= \frac{1}{2}\int t^{n-2}(t^2+1)dt = \frac{1}{2}\int(t^n+t^{n-2})dt = \frac{1}{2}\left(\frac{t^{n+1}}{n+1}+\frac{t^{n-1}}{n-1}\right)+C$$

Where $t = x+\sqrt{1+x^2}$

Example 10: Evaluate:

$$I = \int\frac{2\sin^3(x/2)dx}{(\cos(x/2))\sqrt{\cos^3 x+3\cos^2 x+\cos x}} \text{ for } \cos x > 0$$

Sol: Here we can reduce the given fraction by using the formula $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and then by putting cos x = t we can solve it.

$$I = \int\frac{(2\sin(x/2)\cos(x/2))(2\sin^2(x/2))dx}{(2\cos^2(x/2))\sqrt{\cos^3 x+3\cos^2 x+\cos x}}$$

$$= \int\frac{(1-\cos x)\sin x\;\;dx}{(1+\cos x)\sqrt{\cos^3 x+3\cos^2 x+\cos x}}$$

[Put cos x = t]

$$\Rightarrow I = \int\frac{(t-1)dt}{(1+t)\sqrt{t^3+3t^2+t}}$$

$$= \int \frac{\left(t^2-1\right)dt}{(1+t)^2 t\sqrt{t+3+(1/t)}}$$

$$= \int \frac{t^2\left(1-(1/t^2)\right)dt}{t(t^2+2t+1)\sqrt{t+(1/t)+3}}$$

$$= \int \frac{\left(1-(1/t^2)\right)dt}{(t+(1/t)+2)\sqrt{\left(t+(1/t)\right)+3}}$$

Put $t+\frac{1}{t}+3=z^2 : z>0$;

Then $\left(1-\frac{1}{t^2}\right)dt = 2zdz$

$$\Rightarrow I = \int \frac{2zdz}{(z^2-1)z} = 2\int \frac{dz}{z^2-1} = \log\left|\frac{z-1}{z+1}\right| + c$$

$$\Rightarrow I = \log\left|\frac{\sqrt{\cos x+\sec x+3}-1}{\sqrt{\cos x+\sec x+3}+1}\right| + c$$

Example 11: Evaluate:

$$I = \int \frac{(\sin x - \cos x)\quad dx}{(\sin x+\cos x)\sqrt{\sin x\cos x + \sin^2 x\cos^2 x}}$$

Sol: Here by putting sin x + cos x = t we can integrate the given fraction using the appropriate formula.

Let sin x + cos x = t

$\Rightarrow$ (cos x – sin x) dx = dt

Also, $t^2 = (\sin x+\cos x)^2 = 1+2\sin x\cos x$

$\therefore\ \sin x\cos x = \frac{t^2-1}{2}$

$$\Rightarrow I = -\int \frac{dt}{t\sqrt{\left((t^2-1)/2\right)\left(1+\left((t^2-1)/2\right)\right)}}$$

$$= -\int \frac{dt}{t\sqrt{((t^2-1)(t^2+1))/4}} = -2\int \frac{t^3dt}{t^4\sqrt{t^4-1}}$$

Put $t^4 - 1 = z^2 : z > 0$

$$\Rightarrow I = -2\int \frac{1}{4}\frac{2z\,dz}{(z^2+1)z} = -\int \frac{dz}{1+z^2}$$

$$= -\tan^{-1} z + c = -\tan^{-1}\sqrt{t^4-1} + c$$

$$= -\tan^{-1}\sqrt{(1+\sin 2x)^2-1} + c$$

$$= -\tan^{-1}\sqrt{\sin^2 2x + 2\sin 2x} + c$$

Example 12: Evaluate : $I = \int \frac{dx}{1+\sqrt{x^2+x+2}}$

Sol: We can reduce the given fraction as

$\int \frac{dx}{1+\sqrt{\left(x+(1/2)\right)^2+(7/4)}}$ and then by putting

$x+\frac{1}{2} = \frac{\sqrt{7}}{2}\tan\theta : -\frac{\pi}{2}<\theta<\frac{\pi}{2}$; and using appropriate integration formula we can integrate the given fraction.

$$I = \int \frac{dx}{1+\sqrt{\left(x+(1/2)\right)^2+(7/4)}}$$

Put $x+\frac{1}{2} = \frac{\sqrt{7}}{2}\tan\theta : -\frac{\pi}{2}<\theta<\frac{\pi}{2}$; then

$$dx = \frac{\sqrt{7}}{2}\sec^2\theta\, d\theta$$

$$\Rightarrow I = \int \frac{\sqrt{7}}{2}\frac{\sec^2\theta d\theta}{1+(\sqrt{7}/2)\sec\theta}$$

$$= \frac{\sqrt{7}}{2}\int \frac{d\theta}{\cos\theta\left(\cos\theta+(\sqrt{7}/2)\right)}$$

$$= \int\left(\frac{1}{\cos\theta} - \frac{1}{\cos\theta+(\sqrt{7}/2)}\right)d\theta$$

$$= \log\left|\sec\theta+\tan\theta\right| - \int\frac{d\theta}{a+\cos\theta};\ a=\frac{\sqrt{7}}{2}$$

$$I = \log\left|\sec\theta+\tan\theta\right| - I_1 \quad ....(i)$$

Where, $I_1 = \int \frac{d\theta}{a+\cos\theta}$

Put $\tan\frac{\theta}{2} = t;\ \cos\theta = \frac{1-t^2}{1+t^2}$

$$I_1 = \int \frac{2dt}{1+t^2}\frac{1}{a+\left((1-t^2)/(1+t^2)\right)}$$

$$= 2\int \frac{dt}{a(1+t^2)+1-t^2}$$

$$= \frac{2}{a-1}\int \frac{dt}{\left((a+1)/(a-1)\right)+t^2}$$

$$= \frac{2}{a-1}\sqrt{\frac{a-1}{a+1}}\tan^{-1}\left(\sqrt{\frac{a-1}{a+1}}t\right) + c$$

$$= \frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{\sqrt{a-1}}{a+1}\tan\frac{\theta}{2}\right) + c \quad ....(ii)$$

From (i) and (ii), we get I.

JEE Main/Boards

Exercise 1

Q.1 $\int \frac{\sec x}{\sec x+\tan x}dx$

Q.2 $\int\left(1+\frac{1}{1+x^2}-\frac{2}{\sqrt{1-x^2}}+5\frac{1}{|x|\sqrt{x^2-1}}+a^x\right)dx$

Q.3 $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx : x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Q.4 $\int \frac{1+\tan x}{x+\log\sec x}dx$

Q.5 $\int \frac{2\cos x-3\sin x}{3\cos x+2\sin x}dx$

Q.6 $\int \frac{2x-1}{\sqrt{x^2-x-1}}dx$

Q.7 $\int \frac{dx}{\sqrt{1-3x}-\sqrt{5-3x}}$

Q.8 $\int x^2e^{x^3}\cos(e^{x^3})dx$

Q.9 $\int \frac{\sec^2(2\tan^{-1}x)}{1+x^2}dx$

Q.10 $\int \frac{dx}{(2\sin x+3\cos x)^2}$

Q.11 $\int \cos^{3/5}x\sin^3 x dx$

Q.12 $\int \frac{\log x}{x^2}dx$

Q.13 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$: a > 0

Q.14 $\int e^x\frac{2+\sin 2x}{1+\cos 2x}dx$

Q.15 $\int \frac{dx}{x\left[6(\log x)^2+7\log x+2\right]}$

Q.16 $\int \frac{x^2+1}{(x+3)(x-1)^2}dx$

Q.17 $\int \frac{1}{1-\tan x}dx$

Q.18 If $f'(x)=x-\frac{1}{x^2}$ and $f(1)=\frac{1}{2}$, find f(x).

Q.19 For any natural number evaluate m

$\int\left(x^{3m}+x^{2m}+x^m\right)\left(2x^{2m}+3x^m+6\right)^{1/m}dx, x\rangle 0$

Q.20 $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$

Q.21 $\int \frac{dx}{\sin x+\sec x}$

Q.22 $\int \frac{\sqrt{\cos 2x}}{\sin x}dx$: cos x > 0

Q.23 $\int \frac{\sqrt{x^2+1}(\log(x^2+1)-2\log x)}{x^4}dx$

Q.24 $\int \frac{\sin x}{\sin x-\cos x}dx$

Q.25 $\int x\sin^{-1}\left(\frac{1}{2}\frac{\sqrt{2a-x}}{a}\right)dx$

Q.26 $\int \sec^4 x\,\mathrm{cosec}^2 x dx$

Q.27 $\int \frac{d\theta}{(a+b\cos\theta)^2}$: a > b > 0

Q.28 Evaluate $\int \frac{dx}{x\sqrt{x^4-1}}$

Q.29 $\int \frac{dx}{\sqrt{\sin^3 x\sin(x+\alpha)}}$

Q.30 $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$

Q.31 $\int \frac{\cos 8x-\cos 7x}{1+2\cos 5x}dx$

Q.32 $\int \frac{x^3+1}{x(x-1)^3}dx$

Q.33 $\int \frac{dx}{\sqrt[3]{\sin^{11} x . \cos x}}$

Q.34 Evaluate $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

Q.35 Evaluate $\int \frac{\log x}{(1+x)^3} dx$

Q.36 Evaluate $\int \frac{f(x)}{x^3 - 1} dx$, where f(x) is polynomial of the second degree in x such that $f(0) = f(1) = 3f(2) = -3$

Exercise 2

Single Correct Choice Type

Q.1 $\int \left(\frac{x}{1+x^5}\right)^{3/2} dx$ equals-

(A) $\frac{2}{5}\sqrt{\frac{x^5}{1+x^5}} + c$ (B) $\frac{2}{5}\sqrt{\frac{x}{1+x^5}} + c$

(C) $\frac{2}{5}\frac{1}{\sqrt{1+x^5}} + c$ (D) None of these

Q.2 $\frac{\cos^8 x - \sin^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ equals-

(A) $-\frac{\sin 2x}{2} + c$ (B) $\frac{\sin 2x}{2} + c$

(C) $\frac{\cos 2x}{3} + c$ (D) $-\frac{\cos 2x}{1} + c$

Q.3 Identify the correct expression

(A) $x\int \log x dx = x^2 \log - x^2 + c$

(B) $x\int \log|x| dx = xe^x + c$

(C) $x\int e^x dx = xe^x + cx$

(D) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right] + c$

Q.4 Primitive of $\sqrt{1 + 2\tan x(\sec x + \tan x)}$ w.r.t. x is

(A) $\log(\sec x + \tan x) + \log\cos x + c$

(B) $\log(\sec x + \tan x) + \log\sec x + c$

(C) $2\log\left(\sec\frac{x}{2} + \tan\frac{x}{2}\right) + c$

(D) $\log\sqrt{1 + \sin x} + c$

Q.5 $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ is equal to

(A) $\cot^{-1}(\cot^2 x) + c$ (B) $-\cot^{-1}(\tan^2 x) + c$

(C) $\tan^{-1}(\tan^2 x) + c$ (D) $-\tan^{-1}(\cos 2x) + c$

Q.6 The value of integral $\int \frac{d\theta}{\cos^3\theta\sqrt{\sin 2\theta}}$ can be expressed as irrational function of $\tan\theta$ as

(A) $\frac{\sqrt{2}}{5}\left(\sqrt{\tan^2\theta + 5}\right)\tan\theta + c$

(B) $\frac{2}{5}\left(\tan^2\theta + 5\right)\sqrt{\tan\theta} + c$

(C) $\frac{\sqrt{2}}{5}\left(\tan^2\theta + 5\right)\sqrt{\tan\theta} + c$

(D) $\frac{\sqrt{2}}{5}\left(\tan^2\theta + 5\right)^{\sqrt{\tan\theta}} + c$

Q.7 $\int \frac{dx}{a + bx^2}$ a,b≠0 and a/b > 0

(A) $\frac{1}{\sqrt{ab}}\tan^{-1} x\sqrt{\frac{b}{a}} + c$ (B) $\sqrt{\frac{b}{a}}\tan^{-1} x\sqrt{\frac{b}{a}} + c$

(C) $\sqrt{\frac{a}{b}}\tan^{-1} x\sqrt{\frac{a}{b}} + c$ (D) $\sqrt{ab}\tan^{-1} x\sqrt{\frac{b}{a}} + c$

Q.8 $\int \frac{1 - x^7}{x(1 + x^7)} dx$ equals

(A) $\log x + \frac{2}{7}\log(1 + x^7) + c$ (B) $\log x - \frac{2}{4}\log(1 - x^7) + c$

(C) $\log x - \frac{2}{7}\log(1 + x^7) + c$ (D) $\log x + \frac{2}{4}\log(1 - x^7) + c$

Q.9 $\int \frac{\log|x|}{x\sqrt{1 + \log|x|}} dx$ equal

(A) $\frac{2}{3}\sqrt{1 + \log|x|}(\log|x| - 2) + c$

(B) $\frac{2}{3}\sqrt{1 + \log|x|}(\log|x| + 2) + c$

(C) $\frac{1}{3}\sqrt{1+\log|x|}(\log|x|-2)+c$

(D) $2\sqrt{1+\log|x|}(3\log|x|-2)+c$

Q.10 If $\int\frac{x^4+1}{x(x^2+1)^2}dx = A\log|x|+\frac{B}{1+x^2}+c$,

Where c is the constant of integration then:

(A) A=1; B=-1 (B) A=-1; B=1

(C) A=1; B=1 (D) A=-1; B=-1

Q.11 $\int 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}dx$ equals

(A) $\cos x-\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+c$

(B) $\cos x-\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+c$

(C) $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+c$

(D) $\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x+c$

Q.12 $\int$ sin x. cos x. cos 2x. cos 4x. cos 8x. cos16x dx equals

(A) $\frac{\sin 16x}{1024}+c$ (B) $-\frac{\cos 32x}{1024}+c$

(C) $\frac{\cos 32x}{1096}+c$ (D) $-\frac{\cos 32x}{1096}+c$

Q.13 If $\int f(x)\,dx = F(x)$, then $\int x^3 f(x^2)dx$ is equal to

(A) $\frac{1}{2}\left((F(x))^2-\int(f(x))^2\,dx\right)$ (B) $\frac{1}{2}\left(x^2F(x^2)-\int(f(x^2)\,d(x^2)\right)$

(C) $\frac{1}{2}\left(F(x)-\frac{1}{2}\int(F(x^2)\,dx\right)$ (D) None of these

Q.14 If $\int\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = Ax + B\log(9e^{2x}-4)+C$ then A, B and C are

(A) $A=\frac{3}{2}, B=\frac{36}{35}, C=\frac{3}{2}\log 3+\text{constant}$

(B) $A=\frac{3}{2}, B=\frac{35}{36}, C=\frac{-3}{2}\log 3+\text{constant}$

(C) $A=-\frac{3}{2}, B=\frac{35}{36}, C=\frac{3}{2}1\log 3+\text{constant}$

(D) None of these

Q.15 $\int\frac{\sqrt{\cot x}-\sqrt{\tan x}}{\sqrt{2}(\cos x+\sin x)}dx$ equals

(A) $\sec^{-1}(\sin x+\cos x)+c$

(B) $\sec^{-1}(\sin x-\cos x)+c$

(C) $\log[(\sin x+\cos x)+\sqrt{\sin 2x}]+c$

(D) $\log[(\sin x-\cos x)+\sqrt{\sin 2x}]+c$

Previous Years' Questions

Q.1 The value of $\int\frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}$ ***(2006)***

(A) $2\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (B) $2\sqrt{2+\frac{2}{x^2}+\frac{1}{x^4}}+c$

(C) $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (D) None of these

Q.2 If $\int\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = Ax + B\log(9e^{2x}-4)+c$, then A=.........., B=......... and C=......... ***(1989)***

Q.3 Integrate $\frac{1}{1-\cot x}$ or $\frac{\sin x}{\sin x-\cos x}$ ***(1978)***

Q.4 Integrate the curve $\frac{x}{1+x^4}$ ***(1978)***

Q.5 Integrate

$\sin x\cdot\sin 2x\cdot\sin 3x+\sec^2 x\cdot\cos^2 2x+\sin^4 x\cdot\cos^4 x.$ ***(1979)***

Q.6 Integrate $\frac{x^2}{(a+bx)^2}$ ***(1979)***

Q.7 Evaluate $\int(\sqrt{\tan x}+\sqrt{\cot x})dx.$ ***(1988)***

Q.8 Evaluate $\int\frac{(x+1)}{x(1+xe^x)^2}dx.$ ***(1996)***

Q.9 Integrate the following ***(1997)***

$\int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}$

Q.10 The value of $\sqrt{2}\int \frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is ***(2008)***

(A) $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$ (B) $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(C) $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$ (D) $x-\log\left|\sin\cos\left(x-\frac{\pi}{4}\right)\right|+c$

Q.11 If $\frac{dy}{dx}=y+3>0$ and $y(0)=2$, then y(log2) is equal to ***(2011)***

(A) 5 (B) 13 (C) -2 (D) 7

Q.12 If the integral

$\int\frac{5\tan x}{\tan x-2}dx = x+a\log\left|\sin x-2\cos x\right|+k$

then a is equal to ***(2012)***

(A) -1 (B) -2 (C) 1 (D) 2

Q.13 If $\int f(x)dx=\psi(x)$, then $\int x^5 f(x^3)dx$ is equal to ***(2013)***

(A) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^2\psi(x^3)dx\right]+c$

(B) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)dx+c$

(C) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)dx+c$

(D) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^2\psi(x^3)dx\right]+c$

Q.14 The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to ***(2014)***

(A) $(x+1)^{x+\frac{1}{x}}+c$ (B) $-xe^{x+\frac{1}{x}}+c$

(C) $(x-1)^{x+\frac{1}{x}}+c$ (D) $xe^{x+\frac{1}{x}}+c$

Q.15 The integral $\int\frac{dx}{x^2\left(x^4+1\right)^{3/4}}$ equals ***(2015)***

(A) $\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$ (B) $\left(x^4+1\right)^{1/4}+c$

(C) $-\left(x^4+1\right)^{1/4}+c$ (D) $-\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$

Q.16 The integral $\int\frac{2x^{12}+5x^9}{\left(x^5+x^3+1\right)^3}dx$ is equal to ***(2016)***

(A) $\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+c$ (B) $\frac{x^5}{2\left(x^5+x^3+1\right)^2}+c$

(C) $\frac{-x^{10}}{2\left(x^5+x^3+1\right)^2}+c$ (D) $\frac{-x^5}{2\left(x^5+x^3+1\right)^2}+c$

JEE Advanced/Boards

Exercise 1

Q.1 (i) $\int\frac{dx}{\cot(x/2)\cdot\cot(9x/3)\cdot\cot(x/6)}$;

(ii) $\int\frac{\tan(\log x)\tan(\log(x/2))\tan(\log 2)}{x}dx$

Q.2 $\int\frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

Q.3 $\int\frac{\log\left(\log\left((1+x)/(1-x)\right)\right)}{1-x^2}dx$

Q.4 $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\log x\,dx$

Q.5 $\int\sqrt{\frac{\cos(x-a)}{\sin(x+a)}}dx$

Q.6 $\int\frac{x^5+3x^4-x^3+8x^2-x+8}{x^2+1}dx$

Q.7 $\int\frac{(\sqrt{x}+1)dx}{\sqrt{x}(\sqrt[3]{x}+1)}$

Q.8 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Q.9 $\int \frac{x\ln x}{(x^2-1)^{3/2}}\,dx$

Q.10 $\int \frac{\log 6\sqrt[6]{((\sin x)6^{\cos x})}\cos x}{\sin x}dx$

Q.11 $\int \left[\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}\right]dx$

Q.12 If $f(x)=\frac{\sin x}{\sin^2 x+4\cos^2 x}$ the antiderivative of $\left(\frac{1}{\sqrt{3}}\right)\tan^{-1}\left(\left(\frac{1}{\sqrt{3}}\right)g(x)\right)+c$ is then g (x) is equal to

Q.13 $\int \frac{\cot x\,dx}{(1-\sin x)(\sec x+1)}$

Q.14 $\int \frac{3x^2+1}{(x^2-1)^3}dx$

Q.15 $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

Q.16 Evaluate $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{3/2}}dx$

Q.17 Evaluate $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Q.18

$$\int \frac{(e^{\sqrt{x}}-e^{-\sqrt{x}})\cos\left(e^{\sqrt{x}}+e^{-\sqrt{x}}+\frac{\pi}{4}\right)+\left(e^{\sqrt{x}}+e^{-\sqrt{x}}\right)\cos\left(e^{\sqrt{x}}-e^{-\sqrt{x}}+\frac{\pi}{4}\right)}{\sqrt{x}}dx$$

Q.19 $\int \frac{x^2+x}{(e^x+x+1)^2}dx$

Q.20 $\int \frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$

Q.21 $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

Q.22 $\int \frac{a^2\sin^2 x+b^2\cos^2 x}{a^4\sin^2 x+b^4\cos^2 x}dx$

Q.23 $\int \frac{\cos^2 x}{1+\tan x}dx$

Q.24 $\int \log x.\sin^{-1}x\,dx$

Q.25 Evaluate $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$

Q.26 Evaluate $\int \frac{e^{\sin z}}{\cos^2 x}(x\cos^3 x-\sin x)dx$

Q.27 Evaluate $\int \frac{dx}{\sqrt{1-2x-x^2}}$

Q.28 $\int \frac{dx}{\sec x+\operatorname{cosec} x}$

Q.29 Evaluate $\int \sqrt{2x^2+3x+4}dx$

Q.30 Evaluate $\int \frac{dx}{x(x^n+1)}$

Q.31 $\int \frac{\cos x-\sin x}{7-9\sin 2x}dx$

Q.32 $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx$

Q.33 $\int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$

Q.34 $\int \frac{dx}{\cos^3 x-\sin^3 x}$

Q.35 $\int \frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}dx$

Q.36 $\int \cos 2\theta.\ln\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}d\theta$

Q.37 Match the columns

Column I	Column II
(A) $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx$	(p) $\log\left(\frac{(x^2+1)+\sqrt{x^4+1}}{x}\right)+c$
(B) $\int \frac{x^2-1}{x\sqrt{1+x^4}}dx$	(q) $C-\frac{1}{2}\log\left(\frac{\sqrt{x^4+1}-2\sqrt{x}}{(x^2-1)}\right)$
(C) $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}dx$	(r) $C-\tan^{-1}\left(\sqrt{\sqrt{1+\frac{1}{x^4}}-1}\right)$

Column I	Column II
(D) $\int \frac{1}{(1+x^4)\sqrt{\sqrt{1+x^4}-x^2}}dx$	(s) $\frac{\sqrt{x^4+x^2+1}}{x}+c$

Exercise 2

Single Correct Choice Type

Q.1 If $\int \frac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}dx$ is equal to :

(A) $\frac{4}{\pi}x\tan^{-1}x+\frac{2}{\pi}\log(1+x^2)-x+c$

(B) $\frac{4}{\pi}x\tan^{-1}x-\frac{2}{\pi}\log(1+x^2)+x+c$

(C) $\frac{4}{\pi}x\tan^{-1}x+\frac{2}{\pi}\log(1+x^2)+x+c$

(D) $\frac{4}{\pi}x\tan^{-1}x-\frac{2}{\pi}\log(1+x^2)-x+c$

Q.2 $\int \frac{x^2-4}{x^4+24x^2+16}dx$ equals

(A) $\frac{1}{4}\tan^{-1}\left(\frac{x^2+4}{4x}\right)+c$ (B) $-\frac{1}{4}\cot^{-1}\left(\frac{x^2+4}{x}\right)+c$

(C) $-\frac{1}{4}\cot^{-1}\left(\frac{4x^2+4}{x}\right)+c$ (D) $\frac{1}{4}\cot^{-1}\left(\frac{x^2+4}{x}\right)+c$

Q.3 $\int \frac{(x-1)^2}{x^4+2x^2+1}dx$ equals

(A) $\frac{x^3}{3}+x+\frac{x}{x^2+1}+c$ (B) $\frac{x^5+x^3+x+3}{3(x^2+1)}+c$

(C) $\frac{x^5+4x^3+3x+3}{3(x^2+1)}+c$ (D) None of these

Q.4 $\int \frac{x^4-4}{x^2\sqrt{4+x^2+x^4}}dx$ equals

(A) $\frac{\sqrt{4+x^2+x^4}}{x}+c$ (B) $\sqrt{4+x^2+x^4+c}$

(C) $\frac{\sqrt{4+x^2+x^4}}{2}+c$ (D) $\frac{\sqrt{4+x^2+x^4}}{2x}+c$

Q.5 $\int \frac{\sec x+\tan x-1}{\tan x-\sec x+1}dx$ equals

(A) log [(sec x + tan x)] + log sec x + c

(B) log [sec x-tan x] – log cos x + c

(C) log (sec x + tan x) – log sec x + c

D) –log (sec x + tan x) + log cos x + c

Q.6 $\int e^x\frac{(x^2-3)}{(x+3)^2}dx$

(A) $e^x\cdot\frac{x}{x+3}+c$ (B) $e^x(2-\frac{6}{x+3})+c$

(C) $e^x(1-\frac{6}{x+3})+c$ (D) $e^x\frac{3}{x+3}+c$

Q.7 $\int \sqrt{\frac{1-\cos x}{\cos\alpha-\cos x}}dx$ where $0<\alpha<x<\pi$, equals

(A) $2\log\left(\cos\frac{\alpha}{2}-\cos\frac{x}{2}\right)+c$ (B) $2\cos^{-1}\left(\frac{\cos(x/2)}{\cos(\alpha/2)}\right)+c$

(C) $2\sqrt{2}\log\left(\cos\frac{\alpha}{2}-\cos\frac{x}{2}\right)+c$ (D) $-2\sin^{1}\left(\frac{\cos(x/2)}{\cos(\alpha/2)}\right)+c$

Q.8 Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is :

(A) $\frac{x}{x^4+x+1}+c$ (B) $-\frac{x}{x^4+x+1}+c$

(C) $\frac{x+1}{x^4+x+1}+c$ (D) $-\frac{x+1}{x^4+x+1}+c$

Q.9 If $\int e^{3x}\cos 4x\,dx = e^{3x}(A\sin 4x+B\cos 4x)+c$

Then

(A) 4A=3b (B) 2A=3B

(C) 3A=4B (D) 4B+3A=1

Q.10 The evaluation of $\int \frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}dx$ is

(A) $-\frac{x^p}{x^{p+q}+1}+c$ (B) $\frac{x^q}{x^{p+q}+1}+c$

(C) $-\frac{x^q}{x^{p+q}+1}+c$ (D) $\frac{x^p}{x^{p+q}+1}+c$

Q.11 If $\int f(x)dx = g(x)$, then $\int f^{-1}(x)dx$ is equal to –

(A) $g^{-1}(x)$ (B) $xf^{-1}(x) - g(f^{-1}(x))$

(C) $xf^{-1}(x)g^{-1}(x)$ (D) $f^{-1}(x)$

Q.12 Primitive of $\sqrt[3]{\dfrac{x}{(x^4-1)^4}}$ w.r.t x is –

(A) $\dfrac{3}{4}\left(1+\dfrac{1}{x^4-1}\right)^{\frac{1}{3}}+c$ (B) $-\dfrac{3}{4}\left(1+\dfrac{1}{x^4-1}\right)^{\frac{1}{3}}+c$

(C) $\dfrac{4}{3}\left(1+\dfrac{1}{x^4-1}\right)^{\frac{1}{3}}+c$ (D) $-\dfrac{4}{3}\left(1+\dfrac{1}{x^4-1}\right)^{\frac{1}{3}}+c$

Q.13 If $\int e^u$. Sin 2x dx can be found in terms of known functions of x then u can be :

(A) x (B) sin x (C) cos x (D) cos 2x

Previous Years' Questions

Q.1 The value of the integral $\int \dfrac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx$ is ***(1995)***

(A) $\sin x - 6\tan^{-1}(\sin x) + c$

(B) $\sin x - 2(\sin x)^{-1} + c$

(C) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$

(D) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

Q.2 Let $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and

$g(x) = \underbrace{fofo....of}_{f \text{ occurs n times}}(x)$. Then, $\int x^{n-2}g(x)dx$ equals ***(2007)***

(A) $\dfrac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+c$

(B) $\dfrac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+c$

(C) $\dfrac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+c$

(D) $\dfrac{1}{n+1}(1+nx^n)^{1-\frac{1}{n}}+c$

Q.3 Let $I = \int \dfrac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \dfrac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$

Then, for an arbitrary constant c, the value of J-I equals ***(2008)***

(A) $\dfrac{1}{2}\log\left|\dfrac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right|+c$ (B) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right|+c$

(C) $\dfrac{1}{2}\log\left|\dfrac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right|+c$ (D) $\dfrac{1}{2}\log\left|\dfrac{e^{4x}+e^{2x}+1}{e^{4x}-e2^{x}+1}\right|+c$

Q.4 $f(x)$ is the integral of

$\dfrac{2\sin x - \sin 2x}{x^3}, x \neq 0$ find $\lim\limits_{x\to 0} f'(x)$. ***(1979)***

Q.5 Evaluate the following integrals ***(1980)***

(i) $\int \sqrt{1+\sin\left(\dfrac{1}{2}x\right)}dx$ (ii) $\int \dfrac{x^2}{\sqrt{1-x}}dx$

Q.6 Evaluate $\int (e^{\log x} + \sin x)\cos x\, dx$. ***(1981)***

Q.7 Evaluate $\int \dfrac{(x-1)e^x}{(x+1)^3}dx$. ***(1983)***

Q.8 Evaluate $\int \dfrac{dx}{x^2(x^4+1)^{3/4}}$. ***(1984)***

Q.9 $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}dx$. ***(1985)***

Q.10 Evaluate $\int \dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$. ***(1986)***

Q.11 Evaluate $\int \dfrac{(\cos 2x)^{1/2}}{\sin x}dx$. ***(1987)***

Q.12 Find the indefinite integral

$\int \left(\dfrac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \dfrac{\log(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$ ***(1992)***

Q.13 Integrate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$ ***(1999)***

Q.14 Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx.$ ***(2000)***

Q.15 For any natural number m, evaluate ***(2002)***

$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6x^m)^{1/m}dx, x>0.$

Q.16 Let $I=\int \frac{e^x}{e^{4x}+e^{2x}+1}dx, J=\int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$.

Then, for an arbitrary constant C, the value of $J-I$ equals ***(2008)***

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$

(B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\frac{1}{2}\log\left(\frac{e^{x}-e^{-x}-1}{e^{x}+e^{-x}+1}\right)+c$

(D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{x}+1}\right)+c$

Q.17 The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ equals (for some arbitrary constant K) ***(2012)***

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.14 Q.19 Q.23 Q.25
Q.29 Q.32 Q.35 Q.38

Exercise 2

Q.4 Q.10 Q.13 Q.15

Previous Years' Questions

Q.5 Q.7 Q.9

JEE Advanced/Boards

Exercise 1

Q.3 Q.11 Q.18 Q.20
Q.30 Q.37 Q.38

Exercise 2

Q.1 Q.5 Q.7 Q.10
Q.12

Previous Years' Questions

Q.1 Q.2 Q.4 Q.10
Q.12

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\tan x - \sec x + A$

Q.2 $x + \tan^{-1} x - 2\sin^{-1} x + 5\sec^{-1} x + \frac{a^x}{\log a} + A$

Q.3 $\frac{x^2}{2} + A$

Q.4 $\log\left|x + \log(\sec x)\right| + A$

Q.5 $\log\left|3\cos x + 2\sin x\right| + A$

Q.6 $2\sqrt{x^2 - x - 1} + A$

Q.7 $\frac{1}{18}\left[(1-3x)^{3/2} + (5-3x)^{3/2}\right] + A$

Q.8 $\frac{1}{3}\sin(e^{x^3}) + A$

Q.9 $\frac{x}{1+x^2} + c$

Q.10 $-\frac{1}{2(2\tan x + 3)} + A$

Q.11 $-\frac{5}{8}\cos^{8/5} x + \frac{5}{18}\cos^{18/5} x + A$

Q.12 $\frac{-\log x}{x} - \frac{1}{x} + A$

Q.13 $\left(\tan^{-1}\sqrt{\frac{x}{a}}\right)(a+x) - a\sqrt{\frac{x}{a}} + c$

Q.14 $e^x \tan x + A$

Q.15 $\log\left|2\log x + 1\right| - \log\left|3\log x + 2\right| + A$

Q.16 $\frac{5}{8}\log\left|x+3\right| + \frac{3}{8}\log\left|x-1\right| - \frac{1}{2(x-1)} + A$

Q.17 $\frac{1}{2}x - \frac{1}{2}\log\left|\cos x - \sin x\right| + A$

Q.18 $f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$

Q.19 $\frac{1}{6(m+1)}(2x^{3m} + 3x^{2m} + 6x^m)^{m+1/m}$

Q.20 $\frac{1}{4}\log(x^2+1) - \frac{1}{2}\log(x+1) + \frac{3}{2}\tan^{-1} x + \frac{x}{x^2+1}$

Q.21 $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+p}{\sqrt{3}-p}\right| + \tan^{-1}(q) + c$

Q.22 $-\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + c$

Q.23 $-\frac{1}{3}\left(1 + \frac{1}{x^2}\right)^{3/2}\left[\log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3}\right] + c$

Q.24 $\frac{1}{2}x + \frac{1}{2}\log\left|\sin x - \cos x\right| + c$

Q.25 $\frac{x^2}{2}\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) + \frac{a^2}{2}\left(\sin^1\frac{x}{2a} - \frac{x}{2a}\sqrt{1 - \frac{x^2}{4a^2}}\right) + c$

Q.26 $\frac{1}{3}\tan^3 x + 2\tan x - \cot x + c$

Q.27 $\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan(\theta/2)\right] + c_1$

Q.28 $\int\frac{1}{2}d\theta = \frac{1}{2}\theta + c = \frac{1}{2}\sec^{-1}(x^2) + c$

Q.29 $\frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \cot x \sin\alpha} + c$

Q.30 $2\left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}}\right) + c$

Q.31 $\frac{1}{2}\int(\cos 3x - \cos 2x)dx = \frac{1}{3}\sin 3x - \frac{1}{2}\sin 2x + c$

Q.32 $-\log|x| + 2\log|x-1| + \frac{1}{x-1} - \frac{1}{(x-1)^2} + c$

Q.33 $2\log t-\frac{3}{2}\log(2t-1)-\frac{3}{2}\frac{1}{(2t-1)}+c$, where $t=x+\sqrt{x^2-x+1}$

Q.34 $\log(e^x+\sqrt{e^{2x}-4})-\log 2+c=\log(e^x+\sqrt{e^{2x}-4})+c'$

Q.35 $I=-\frac{1}{2}\frac{\log x}{(1+x)^2}+\frac{1}{2}\log\frac{x}{1+x}+\frac{1}{2(1+x)}+c$

Q.36 $\log\frac{x^2+x+1}{|x-1|}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c$

Exercise 2

Single Correct Choice Type

Q.1 A **Q.2** B **Q.3** C **Q.4** B **Q.5** C **Q.6** C

Q.7 A **Q.8** C **Q.9** A **Q.10** C **Q.11** B **Q.12** B

Q.13 B **Q.14** D **Q.15** A

Previous Years' Questions

Q.1 C

Q.2 $A=-\frac{3}{2}$, $B=\frac{35}{36}$ and $C\in R$

Q.3 $\frac{1}{2}\log(\sin x-\cos x)+\frac{x}{2}+c$

Q.4 $\frac{1}{2}\tan^{-1}(x^2)+c$

Q.5 $-\frac{\cos 4x}{16}-\frac{\cos 2x}{8}+\frac{\cos 6x}{24}+\sin 2x+\tan x-2x+\frac{3x}{128}-\frac{\sin 4x}{128}+\frac{\sin 8x}{1024}$

Q.6 $\frac{1}{b^3}\left(a+bx-2a\log(a+bx)-\frac{a^2}{a+bx}+c\right)$

Q.7 $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

Q.8 $\int\frac{dt}{t^2(t-1)}=\log\left|\frac{xe^x}{1+xe^x}\right|+\frac{1}{1+xe^x}+c$

Q.9 $2[\cos^{-1}\sqrt{x}-\log|1+\sqrt{1-x}|-\frac{1}{2}\log|x|]+c$

Q.10 C **Q.11** D **Q.12** D **Q.13** C **Q.14** D **Q.15** D

Q.16 A

JEE Advanced/Boards

Exercise 1

Q.1 (i) $2\log\left(\sec\frac{x}{2}\right)-3\log\left(\sec\frac{x}{3}\right)-6\log\left(\sec\frac{x}{6}\right)++c$

(ii) $\log\left|\frac{\sec(\log x)}{\sec\left(\log\frac{x}{2}\right)}\right|-x\tan\log|2|+c$

Q.2 $\frac{-2}{\alpha-\beta}\cdot\frac{\sqrt{x-\beta}}{x-\alpha}$ + c

Q.3 $\log\left(\frac{1+x}{1-x}\right)\log\left(\frac{1+x}{1-x}\right)\log\left(\frac{1+x}{1-x}\right)+c$

Q.4 $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x$ + c

Q.5 $\cos a.\text{arc}\cos\left(\frac{\cos x}{\cos a}\right)-\sin a.\log\left(\sin x+\sqrt{\sin^2 x-\sin^2 a}\right)+c$

Q.6 $\frac{x^4}{4}+x^3-x^2+5x+\frac{1}{2}\log(x^2+1)+3\tan^{-1}x+c$
+ c where t = $x^{1/6}$

Q.7 $6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\log(1+t^2)-\tan^{-1}t\right]+C$ where $t=x^{1/6}$

Q.8 $(a+x)\tan^{-1}\left[\frac{\sqrt{x}}{a}-\sqrt{ax}\right]+c$

Q.9 $\sec^{-1}x-\frac{\log x}{\sqrt{x^2-1}}+c$

Q.10 $\frac{1}{6}\left[\frac{\log^2(\sin x)}{\log 36}+\log\left[\tan\frac{x}{2}+\cos x\right]\right]+c$

Q.11 $-\frac{1}{3}\left[1+\frac{1}{x^2}\right]^{3/2}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+c$

Q.12 $\int f(x)dx=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sec x}{\sqrt{3}}+c$

Q.13 $\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+c$

Q.14 $-\frac{x}{(x^2-1)^2}$ + C

Q.15 $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$

Q.16 $1=\frac{e^{m\tan^{-1}x}}{m^2+1}\left(m\cdot\frac{1}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}\right)+c=\frac{e^{m\tan^{-1}x}(m+x)}{(m^2+1)\sqrt{x^2+1}}+c$

Q.17 $a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(\frac{a+x}{a}\right)-\sqrt{\frac{x}{a}}\right]+c$

Q.18 $\sqrt{2}\left(\cos(e^{-\sqrt{x}})\right)\left(\sin(e\sqrt{x})+\cos(e^{\sqrt{x}})\right)+c$

Q.19 $-\log\left(\frac{x+1}{e^x}+1\right)-\frac{1}{\left(\frac{x+1}{e^x}+1\right)}+c$

Q.20 $-e^{\cos x}(x+\text{cosec}\,x)+c$

Q.21 $C-\frac{x+1}{x^5+x+1}$ or $c+\frac{x^5}{x^x+x+1}$

Q.22 $\frac{1}{a^2+b^2}\left(x+\tan^{-1}\left(\frac{a^2\tan x}{b^2}\right)\right)+c$

Q.23 $\frac{1}{4}\log(\cos x+\sin x)+\frac{x}{2}+\frac{1}{8}(\sin 2x+\cos 2x)+c$

Q.24 $(x\log x-x)\text{Sin}^{-1}x+\sqrt{1-x^2}\log x-\log\left(\frac{1-\sqrt{1-x^2}}{x}\right)+c$

Q.25 $e^x f(x)+c=e^x\left(\frac{x-1}{x+1}\right)+c$

Q.26 $e^z f(z)+c=e^z\left(\sin^{-1}z-\frac{1}{\sqrt{1-z^2}}\right)+c=e^{\sin x}(x-\sec x)+c$

Q.27 $\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$

Q.28 $\frac{1}{2}\left[\sin x-\cos x-\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right]+C$ c

Q.29 $I=\frac{4x+3}{8}\sqrt{2x^2+3x+4}+\frac{23}{16\sqrt{2}}\log\left[\frac{4x+3}{4}+\frac{\sqrt{2x^2+3x+4}}{\sqrt{2}}\right]+c$

Q.30 $\frac{1}{n}\log\left|\frac{z-1}{z}\right|+c=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$

Q.31 $\frac{1}{24}\log\frac{(4+3\sin x+3\cos x)}{4-3\sin x-3\cos x)}+c$

Q.32 $\tan^{-1}\left(\frac{\sqrt{2}\sin 2x}{\sin x+\cos x}\right)+c$

Q.33 $4\log x+\frac{7}{x}+6\tan^{-1}(x)+\frac{6x}{1+x^2}+c$

Q.34 $\frac{2}{3}\tan^{-1}(\sin x+\cos x)+\frac{1}{3\sqrt{2}}\quad\log\left|\frac{\sqrt{2}+\sin x+\cos x}{\sqrt{2}-\sin x-\cos x}\right|+c$

Q.35 $\log\left|\frac{x\sin x+\cos x}{x\cos x-\sin x}\right|+c$

Q.36 $\frac{1}{2}(\sin 2\theta)\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\log(\sec 2\theta)+c=\frac{2}{\cos 2\theta}+c$

Q.37 $A\to s$; $B\to p$; $C\to q$; $D\to r$

Exercise 2

Q.1 D **Q.2** A **Q.3** D **Q.4** A **Q.5** A **Q.6** C **Q.7** B

Q.8 B **Q.9** C **Q.10** C **Q.11** B **Q.12** B **Q.13** A

Previous Years' Questions

Q.1 C **Q.2** A **Q.3** C **Q.4** 1

Q.5 (i) $4\sin\frac{x}{4}-4\cos\frac{x}{4}+c$ (ii) $-2\left\{\sqrt{1-x}-\frac{2}{2}(1-x)^{3/2}+\frac{1}{5}(1-x)^{5/2}\right\}+c$

Q.6 $x\sin x+\cos x-\frac{\cos 2x}{4}+c$

Q.7 $\frac{e^x}{(x+1)^2}+c$

Q.8 $-\frac{(x^4+1)^{1/4}}{x}+c$

Q.9 $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+c$

Q.10 $\frac{2}{\pi}[\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}]-x+c$

Q.11 $-\log|\cot x+\sqrt{\cot^2 x-1}|+\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^{2x}}}\right|+c$

Q.12 $\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}-\frac{4}{3}x^{1/2}-\frac{12}{5}x^{5/12}+\frac{1}{2}x^{1/3}-4x^{1/4}-7x^{1/6}-12x^{1/12}+(2x^{1/2}-3x^{1/3}$

$-6x^{1/6}+11)\log(1+x^{1/6})+12\log(1+x^{1/2})-3[\log(1+{}^{1/6})]^2+c$

Q.13 $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+1|+\frac{3}{2}\tan^{-1}x+\frac{x}{x^2+1}+c$

Q.14 $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(4x^2+8x+13)+c$

Q.15 $\frac{1}{6(m+1)}\cdot(2x^{3m}+3x^{2m}+6x^m)^{(m+1)/m}+c$

Q.16 C **Q.17** C

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\int \frac{\sec x}{\sec x + \tan x} dx = \int \frac{\sec x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$

$= \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + C$

Sol 2: $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + 5\frac{1}{|x|\sqrt{x^2-1}} + a^x\right) dx$

$= \int dx + \int \frac{dx}{1+x^2} - \int \frac{2dx}{\sqrt{1-x^2}} + 5\int \frac{dx}{|x|\sqrt{x^2-1}} + \int a^x dx$

$= x + \tan^{-1}x - 2\sin^{-1}x + 5\sec^{-1}x + \frac{a^x}{\log a} + c$

Sol 3: $\frac{\sin 2x}{1+\cos 2x} = \tan x$

$\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx = \int \tan^{-1} \tan x dx$

$= \int x dx = \frac{x^2}{2} + c$

Sol 4: $\int \frac{1+\tan x}{x + \log \sec x} dx$

Put x + logsecx = t

$\Rightarrow \left(1 + \frac{1}{\sec x}\sec x \tan x\right) dx = dt$

$\therefore \int \frac{dt}{t} = \log t + c = \log(x + \log\sec x) + c$

Sol 5: Put 3cosx + 2sinx = t

$-3\sin x + 2\cos x = \frac{dt}{dx}$

$\therefore I = \int \frac{dt}{t} = \log t = \log(3\cos x + 2\sin x)$

Sol 6: $\int \frac{2x-1}{\sqrt{x^2-x-1}} dx$

Put $x^2 - x - 1 = t$

$\Rightarrow (2x - 1)dx = dt$

$\int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = 2\sqrt{t} + c = 2\sqrt{x^2 - x - 1} + c$

Sol 7: $\int \frac{dx}{\sqrt{1-3x} - \sqrt{5-3x}}$

$= \int \frac{\sqrt{1-3x} + \sqrt{5-3x}}{(-4)} dx$

$= -\frac{1}{4}\int \sqrt{1-3x}dx - \frac{1}{4}\int \sqrt{5-3x}dx$

$= \frac{1}{12} \times \frac{2}{3}\left(\sqrt{1-3x}\right)^{3/2} + \frac{1}{12} \times \frac{2}{3}\left(\sqrt{5-3x}\right)^{3/2}$

$= \frac{1}{18}\left[(1-3x)^{3/2} + (5-3x)^{3/2}\right] + c$

Sol 8: $\int x^2 e^{x^3} \cos(e^{x^3}) dx$

$e^{x^3} = t \Rightarrow \quad 3x^2 e^{x^3} dx = dt$

$\therefore \quad \frac{1}{3}\int \cos t dt = \frac{1}{3}\sin t + c = \frac{1}{3}\sin e^{x^3} + c$

Sol 9: $\int \frac{\sec^2(2\tan^{-1} x)}{1+x^2} dx$

$2\tan^{-1}x = t \Rightarrow \frac{2}{1+x^2} dx = dt$

$\frac{1}{2}\int \sec^2 t dt = \frac{1}{2}\tan t + c = \frac{1}{2}\tan(2\tan^{-1} x) + c$

$= \frac{1}{2}\frac{2\tan(\tan^{-1} x)}{1+\tan^2(\tan^{-1} x)} + c = \frac{x}{1+x^2} + c$

Sol 10: $\int \frac{dx}{(2\sin x + 3\cos x)^2} = \int \frac{\sec^2 x}{(2\tan x + 3)} dx$

Put 2tanx + 3 = t $\Rightarrow$ $2\sec^2 x$ dx = dt

$\therefore \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + c = -\frac{1}{2} \times \frac{1}{(2\tan x + 3)} + c$

Sol 11: $\int \cos^{3/5} x \sin^3 x dx$

$= \int \cos^{3/5} x(1 - \cos^2 x) \sin x dx$

$= \int\left(\cos^{3/5}x - \cos^{13/5}x\right)\sin x dx$

Put cosx = t; –sinxdx = dt

$= \int\left(t^{13/5} - t^{3/5}\right)dt = \frac{5}{18}t^{18/5} - \frac{5}{8}t^{8/5} + c$

$= \frac{5}{18}(\cos x)^{18/5} - \frac{5}{8}(\cos x)^{8/5} + c$

Sol 12: $\int \frac{\log x}{x^2}dx$

$\log x\int\frac{1}{x^2}dx - \int\left(\frac{d\log x}{dx}\int\frac{1}{x^2}dx\right)dx$

$= (\log x)\left(-\frac{1}{x}\right) - \int\frac{1}{x}\times\left(-\frac{1}{x}\right)dx + c$

$= -\frac{1}{x}\log x + \int\frac{1}{x^2}dx + c = -\frac{1}{x}(\log x + 1) + c$

Sol 13: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$, a > 0

$x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta d\theta$

$\int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}}\ 2a\tan\theta\sec^2\theta\ d\theta$

$= 2a\int(\sin^{-1}\sin\theta)\tan\theta\sec^2\theta\ d\theta$

$\tan\theta = t \Rightarrow \sec^2\theta d\theta = dt$

$= 2a\int t\tan^{-1}t dt$

$= 2a\left[\tan^{-1}t\int t dt - \frac{1}{2}\int\frac{1}{1+t^2}\times t^2 dt\right]$

$= 2a\left[(\tan^{-1}t)\frac{t^2}{2} - \frac{1}{2}\left[t - \tan^{-1}t\right]\right] + c$

$= at^2\tan^{-1}t + \left[\frac{1}{2}\tan^{-1}t - \frac{1}{2}t\right]2a + c$

$\therefore t = \tan\theta = \frac{\sqrt{x}}{\sqrt{a}}$

$\therefore I = a\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} + \left[\frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}} - \frac{1}{2}\sqrt{\frac{x}{a}}\right]2a + c$

$= x\tan^{-1}\sqrt{\frac{x}{a}} + \left[\frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}} - \frac{1}{2}\sqrt{\frac{x}{a}}\right]2a + c$

$= x\tan^{-1}\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + c$

$= \left(\tan^{-1}\sqrt{\frac{x}{a}}\right)(a+x) - a\sqrt{\frac{x}{a}} + c$

Sol 14: $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$

$1 + \cos 2x = 2\cos^2 x$

$2 + \sin 2x = 2 + 2\sin x\cos x$

$\Rightarrow \int e^x\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right) = \int e^x(\sec^2 x + \tan x)dx$

$\tan x = t \Rightarrow \sec^2 x dx = dt$

$\therefore$ This is of form $\int e^x(f(x) + f'(x))dx = e^x f(x)$

$\therefore I = e^x\tan x + c$

Sol 15: $\int\frac{dx}{x[6(\log x)^2 + 7\log x + 2]}$

$\log x = t; \frac{1}{x}dx = dt$

$\int\frac{dt}{(6t^2+7t+2)} = \int\frac{dt}{(6t^2+3t+4t+2)}$

$= \int\frac{dt}{(3t+2)(2t+1)} = -\int\left[\frac{3}{(3t+2)} - \frac{2}{(2t+1)}\right]dt$

$= -3\int\frac{1}{3t+2}dt + 2\int\frac{1}{2t+1}dt$

$= -3\frac{1}{3}\log(3t+2) + \frac{2}{2}\log(2t+1) + c$

$= \frac{2}{2}\left(\frac{2t+1}{3t+2}\right) + c = \frac{2}{2}\log\left(\frac{2\log x+1}{3\log x+2}\right) + c$

$= \log|2\log x + 1| - \log|3\log x + 2| + c$

Sol 16: $\int\frac{x^2+1}{(x+3)(x-1)^2}dx$

$I = \int\frac{5}{8}\times\frac{1}{(x+3)}dx + \int\frac{3}{8(x-1)}dx + \frac{1}{2}\int\frac{1}{(x-1)^2}dx$

$= \frac{5}{8}\log(x+3) + \frac{3}{8}\log(x-1) - \frac{1}{2(x-1)} + c$

Sol 17: $\int\frac{1}{(1-\tan x)}dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

or $dx = \frac{dt}{1+\tan^2 x} = \frac{dt}{1+t^2}$

$\therefore I = \int \frac{1}{(1-t)(1+t^2)} dt$

$= \frac{1}{2}\int \frac{1}{(1-t)} dt + \frac{1}{2}\int \frac{t+1}{t^2+1} dt$

$= -\frac{1}{2}\log(1-t) + \frac{1}{4}\log(1+t^2) + \frac{1}{2}\tan^{-1} t + c$

$= \frac{1}{2}\log\left(\frac{1}{1-\tan x}\right) + \frac{1}{2}\log\sqrt{1+\tan^2 x} + \frac{x}{2} + c$

$= \frac{1}{2}\log\left(\frac{\sec x}{1-\tan x}\right) + \frac{x}{2} + c$

$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + c$

Sol 18: $f'(x) = x - \frac{1}{x^2}$

$f(x) = \int f'(x)dx = \int\left(x - \frac{1}{x^2}\right)dx = \frac{x^2}{2} + \frac{1}{x} + c$

$f(1) = \frac{1}{2} + 1 + c = \frac{1}{2} \Rightarrow c = -1$

$\therefore f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$

Sol 19: $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$

Put $x^m = t$ and integrate.

Sol 20: $\int \frac{x^3 + 3x + 2}{(x^2+1)^2(x+1)} dx$

$x^3 + 3x + 2 = x^3 + x + 2x + 2$

$= x(x^2 + 1) + 2(x + 1)$

$I = \int \frac{x(x^2+1) + 2(x+1)}{(x^2+1)^2(x+1)} dx$

$= \int \frac{x}{(x^2+1)(x+1)} + 2\int \frac{1}{(1+x^2)^2} dx$

$= \int \frac{(x+1)-1}{(x^2+1)(x+1)} dx + 2\int\left(\frac{1}{1+x^2}\right)^2 dx$

$= 2\int \frac{1}{(1+x^2)^2} dx - \int \frac{dx}{(x^2+1)(x+1)} + \int \frac{1}{(1+x^2)} dx$

Put $x = \tan\theta$

$dx = \sec^2\theta d\theta$

$= 2\int \frac{\sec^2\theta}{\sec^4\theta} d\theta + \tan^{-1} x + \frac{1}{2}\int\left(\frac{(x-1)}{(1+x^2)} - \frac{1}{x+1}\right)dx$

$= \frac{2}{2}\int (\cos 2\theta + 1)d\theta + \tan^{-1}x$

$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}(1+x^2) - \tan^{-1} x - \log(x+1)\right] + c$

$= \frac{1}{4}\tan^{-1}(1+x^2) - \frac{1}{2}\tan^{-1} x - \frac{1}{2}\log(x+1) + c$

Sol 21: $I = \int \frac{1}{\sin x + \sec x} dx$

$\Rightarrow \int \frac{2\cos x}{2 + 2\sin x\cos x} dx = \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x} dx$

$\Rightarrow \int \frac{\cos x + \sin x}{2 + 2\sin x} dx + \int \frac{\cos x - \sin x}{2 + 2\sin 2x} dx$

$\Rightarrow \int \frac{1}{2 + [1 - (\sin x + \cos x)^2]} \times d(\sin x - \cos x)$

$+ \int \frac{1}{2 + [(\sin x + \cos x)^2 - 1]} \times d(\sin x + \cos x)$

$\Rightarrow \int \frac{1}{(\sqrt{3})^2 - (\sin x - \cos x)^2} \times d(\sin x - \cos x)$

$+ \int \frac{1}{(1)^2 + (\sin x + \cos x)^2} \times d(\sin x + \cos x)$

$\Rightarrow \int \frac{1}{(\sqrt{3})^2 - p^2} dp + \int \frac{1}{1^2 + q^2} dq$

$\Rightarrow \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+p}{\sqrt{3}-p}\right| + \tan^{-1}(q) + c$

Where $p = \sin x - \cos x$ & $q = \sin x + \cos x$

Sol 22:

$I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}} dx = \int \sqrt{\cot^2 x - 1} dx$

On putting

$\cot x = \sec\theta$ & $-\operatorname{cosec}^2 dx = \sec\theta\tan\theta\, d\theta$

We get, $I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta\tan\theta}{-\operatorname{cosec}^2 x} d\theta$

$= -\int \frac{\sec\theta\tan^2\theta}{1 + \sec^2\theta} d\theta$

$$= -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}d\theta = -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$$

$$= \int \frac{(1+\cos^2\theta) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)} = -\int \sec\theta d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}d\theta$$

$$\Rightarrow \ -\int \sec\theta d\theta + 2\int \frac{d(\sin\theta)}{1+\cos^2\theta}$$

$$\Rightarrow \ -\int \sec\theta d\theta + 2\int \frac{d(\sin\theta)}{2-\sin^2\theta}$$

$$\Rightarrow \ -\log|\sec\theta + \tan\theta| + 2\times\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c$$

$$\Rightarrow \ -\log|\sec\theta + \sqrt{\sec^2\theta - 1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\cos^2\theta}}{\sqrt{2}-\sqrt{1-\cos^2\theta}}\right| + c$$

$$\Rightarrow \ -\log|\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + c$$

Sol 23: $\int \frac{\sqrt{x^2+1}\left(\log(x^2+1) - \log x^2\right)}{x^4}dx$

$$= \int \frac{\left(\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)\right)}{x^3}dx$$

$1 + \frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3}dx = dt$

$$-\frac{1}{2}\int \sqrt{t}\log t dt = -\frac{1}{2}\left[(\log t)\frac{t^{3/2}}{3/2} - \int \frac{1}{t}\times\frac{t^{3/2}}{3/2}dt\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}(\log)t^{3/2} + \frac{2}{3}\int t^{1/2}dt\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}\left[\log\left(1+\frac{1}{x^2}\right)\left[1+\frac{1}{x^2}\right]^{3/2} + \frac{4}{9}\left(1+\frac{1}{x^2}\right)^{3/2} + c\right]\right]$$

$$= -\frac{1}{3}\left[1+\frac{1}{x^2}\right]^{3/2}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + c$$

Sol 24: $\int \frac{\sin x}{\sin x - \cos x}dx$

$$= \frac{1}{2}\int \frac{\sin x - \cos x + \sin x + \cos x}{(\sin x - \cos x)}dx = \frac{1}{2}\int 1dx + \frac{1}{2}\int \frac{dt}{t}$$

Put sinx – cosx = t

(cosx + sinx)dx = dt

$$\Rightarrow \frac{1}{2}x + \frac{1}{2}\log(\sin x - \cos x) + c$$

$$\because \int \frac{1}{t}dt = \log t = \log(\sin x - \cos x)$$

Sol 25: $\int x\sin^{-1}\left(\frac{1}{2}.\frac{\sqrt{2a-x}}{a}\right)dx$

$$\sin^{-1}\left(\frac{1}{2}.\frac{\sqrt{2a-x}}{a}\right)\int x dx - \int\left[\left(\frac{d\sin^{-1}\frac{1}{2}\frac{\sqrt{2a-x}}{a}}{dx}\right)\int x dx\right]dx$$

$$\Rightarrow \frac{x^2}{2}\sin^{-1}\left(\frac{1}{2}\frac{\sqrt{2a-x}}{a}\right)$$

$$-\int\left(\frac{x^2}{2}\times\frac{1}{\sqrt{1-\left(\frac{2a-x}{4a^2}\right)}}\times\frac{1}{2a}\times\frac{-1}{2\sqrt{2a-x}}\right)dx$$

$$\Rightarrow \frac{x^2}{2}\sin^{-1}\left(\frac{1}{2}\frac{\sqrt{2a-x}}{a}\right)$$

$$+\frac{2a}{8a}\int \frac{x^2}{\sqrt{4a^2-2a+x}}\times\frac{1}{\sqrt{2a-x}}dx$$

Sol 26: $\int \frac{1}{\cos^{4x}\sin^2 x}dx$

$$\int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^2 x\cos^4 x}dx$$

$$= \int\left(\frac{\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x}{\sin^2 x\cos^4 x}\right)dx$$

$$= \int\left(\frac{\sin^2 x}{\cos^4 x} + \frac{1}{\sin^2 x} + \frac{2}{\cos^2 x}\right)dx$$

$$= \int \tan^2 x\sec^2 x dx + \int \operatorname{cosec}^2 x dx + 2\int \sec^2 x dx$$

$$= \frac{\tan^3 x}{3} - \cot x + 2\tan x + c$$

Sol 27: $I = \int \frac{1}{(a+b\cos\theta)^2}d\theta$

Let $P = \frac{\sin\theta}{(a+b\cos\theta)}$

$$\frac{dP}{d\theta} = \frac{(a+b\cos\theta)\cos\theta - \sin\theta(0 - b\sin\theta)}{(a+b\cos\theta)^2}$$

$$= \frac{a\cos\theta + b(\cos^2\theta + \sin^2\theta)}{(a+b\cos\theta)^2} = \frac{a\cos\theta + b}{(a+b\cos\theta)^2}$$

Let $a + b\cos\theta = Y \Rightarrow \cos\theta = \frac{Y-a}{b}$

$$\frac{dP}{d\theta} = \frac{a\left(\frac{Y-a}{b}\right)+b}{\left[a+b\left(\frac{Y-a}{b}\right)\right]^2} = \frac{aY + b^2 - a^2}{bY^2}$$

$$\Rightarrow \quad \frac{dP}{d\theta} = \frac{a}{b}\left(\frac{1}{Y}\right) + \left(\frac{b^2-a^2}{b}\right)\frac{1}{Y^2}$$

$$\Rightarrow \quad \frac{dP}{d\theta} = \frac{a}{b}\left(\frac{1}{a+b\cos\theta}\right) + \left(\frac{b^2-a^2}{b}\right)\left(\frac{1}{a+b\cos\theta}\right)^2$$

Integrating,

$$P = \frac{a}{b}\int\frac{1}{a+b\cos\theta}d\theta + \frac{b^2-a^2}{b}\int\frac{1}{(a+b\cos\theta)}d\theta$$

$$\Rightarrow \quad \frac{-(b^2-a^2)}{b}\int\frac{1}{(a+b\cos\theta)^2}d\theta = \frac{a}{b}\int\frac{1}{a+b\cos\theta}d\theta - P$$

$$\Rightarrow \quad \int\frac{1}{(a+b\cos\theta)^2}d\theta = \frac{b}{(a^2-b^2)}\left[\frac{a}{b}\int\frac{1}{a+b\cos\theta}d\theta - P\right]$$

$$= \frac{b}{a^2-b^2}\left[\frac{a}{b}I_1 - \frac{\sin\theta}{(a+b\cos\theta)}\right] + c$$

Where $I_1 = \int\frac{1}{a+b\cos\theta}d\theta$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan(\theta/2)\right] + c_1$$

Sol 28: [Here, $\sqrt{x^4-1} = \sqrt{(x^2)^2-1}$, which is of the form, $\sqrt{x^2-a^2}$ hence substitution $x^2 = \sec\theta$ may be tried]

Now, $\int\frac{dx}{x\sqrt{x^4-1}} = \int\frac{dx}{x\sqrt{(x^2)^2-1}}$ ...(i)

Let $x^2 = \sec\theta$, then $2x\,dx = \sec\theta\tan\theta\,d\theta$

$$dx = \frac{\sec\theta\tan\theta}{2x}d\theta = \frac{\sec\theta\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

Now, from (i)

$$\int\frac{dx}{x\sqrt{x^4-1}} = \int\left(\frac{1}{\sqrt{\sec\theta}\sqrt{\sec^2\theta-1}}\right)\frac{\sec\theta\cdot\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

$$= \int\left(\frac{1}{\sqrt{\sec\theta}\cdot\tan\theta}\right)\frac{\sec\theta\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

$$= \int\frac{1}{2}d\theta = \frac{1}{2}\theta + c = \frac{1}{2}\sec^{-1}(x^2) + c$$

Sol 29: $I = \int\frac{dx}{\sqrt{\sin^3 x\sin(x+\alpha)}}$

$$= \int\frac{1}{\sqrt{\sin^3 x[\sin x\cos\alpha + \sin\alpha\cos x]}}dx$$

$$= \int\frac{1}{\sin^4 x(\cos\alpha + \cot x\sin\alpha)}dx$$

$$= -\frac{1}{\sin\alpha}\int\frac{1}{\sqrt{\cos\alpha + \cot x\sin\alpha}}d(\cos\alpha + \cot x\sin\alpha)$$

$$= -\frac{1}{\sin\alpha}\int\frac{1}{\sqrt{t}}dt \quad ; \quad \text{where } t = \cos\alpha + \cot x\sin\alpha$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \cot x\sin\alpha} + c$$

Sol 30: $I = \int\frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx$

put $x = \sin^2\theta$ & $dx = 2\sin\theta\cos\theta\,d\theta$

$$\Rightarrow I = \int\frac{2\sin\theta\cos\theta d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta - \sin^4\theta}} = 2\int\frac{1-\sin\theta}{\cos^2\theta}d\theta$$

$$\Rightarrow = 2(\tan\theta - \sec\theta) + c = 2\left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}}\right) + c$$

Sol 31: $I = \int\frac{\cos 8x - \cos 7x}{1 + 2\cos 5x}dx$

$$= \frac{1}{2}\int\frac{2\sin\frac{5x}{2}\cos 8x - 2\sin\frac{5x}{2}\cos 7x}{\sin\frac{5x}{2} + 2\cos 5x\sin\frac{5x}{2}}dx$$

$$=\frac{1}{2}\int\frac{\left(\sin\frac{21x}{2}-\sin\frac{11x}{2}\right)-\left(\sin\frac{19x}{2}-\frac{\sin 9x}{2}\right)}{\sin\frac{15x}{2}}$$

$$=\frac{1}{2}\int\frac{2\sin\frac{15x}{2}\cos 3x-2\sin\frac{15x}{2}\cos 2x}{\sin\frac{15\,x}{2}}dx$$

$$=\frac{1}{2}\int(\cos 3x-\cos 2x)dx=\frac{1}{3}\sin 3x-\frac{1}{2}\sin 2x+c$$

Sol 32: $I=\int\frac{x^3+1}{x(x-1)^3}dx$

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

put $x=1 \Rightarrow D=2$

put $x=0 \Rightarrow A=-1$

put $x=-1$ & $x=2 \Rightarrow B=2$ & $c=1$

$$\therefore\quad I=\int\frac{-1}{x}dx+\int\frac{2}{x-1}dx+\int\frac{1}{(x-1)^2}dx+\int\frac{2}{(x-1)^3}dx$$

$$=-\log|x|+2\log|x-1|+\frac{1}{x-1}-\frac{1}{(x-1)^2}+c$$

Sol 33: $I=\int\frac{1}{x+\sqrt{x^2-x+1}}dx$

put $t=x+\sqrt{x^2-x+1}$

$$\Rightarrow\quad x=\frac{t^2-1}{2t-1}\quad\text{and}\quad dx=\frac{2t^2-2t+2}{(2t-1)^2}dt$$

$$\Rightarrow\qquad I=2\int\frac{t^2-t+1}{t(2t-1)^2}dt$$

let $\frac{t^2-t+1}{t(2t-1)^2}=\frac{A}{t}+\frac{B}{2t-1}+\frac{C}{(2t-1)^2}$

Solving by partial fraction method, we get

$A=1,\quad C=\frac{3}{2}$ and $B=-\frac{3}{2}$

$$I=2\log t-\frac{3}{2}\log(2t-1)-\frac{3}{2}\frac{1}{(2t-1)}+c$$

where $t=x+\sqrt{x^2-x+1}$

Sol 34: [Here $\sqrt{e^{2x}-4}=-\sqrt{(e^x)^2-2^2}$, which is of the form $\sqrt{x^2-a^2}$, hence substitution $e^x=2\sec\theta$ may be tried]

Now, $\int\frac{e^x}{\sqrt{e^{2x}-4}}dx=\int\frac{e^x}{\sqrt{(e^x)^2-2^2}}dx$ ...(i)

Let $e^x=2\sec\theta$, then $e^xdx=2\sec\theta\tan\theta d\theta$

Now from (i),

$$\int\frac{e^x}{\sqrt{e^{2x}-4}}dx=\int\frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta-4}}d\theta=\int\frac{2\sec\theta\tan\theta}{2\tan\theta}d\theta$$

$$=\int\sec\theta d\theta=\log|\sec\theta+\tan\theta|+c \qquad .....(ii)$$

$\because\quad e^x=2\sec\theta\qquad\therefore\quad\sec\theta=\frac{e^x}{2}$

$$\therefore\quad\tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\frac{e^{2x}}{4}-1}=\sqrt{\frac{e^{2x}-4}{2}}$$

From (ii), $\int\frac{e^x}{\sqrt{2^{2x}-4}}dx=\log\left|\frac{e^x}{2}+\frac{\sqrt{e^{2x}-4}}{2}\right|+c$

$$=\log\left(\frac{e^x+\sqrt{e^{2x}-4}}{2}\right)+c\qquad[\because\ e^x+\sqrt{e^{2x}-4}>0]$$

$$=\log(e^x+\sqrt{e^{2x}-4})-\log 2+c=\log(e^x+\sqrt{e^{2x}-4})+c'$$

Sol 35: $I=(\log x)\cdot\left[-\frac{1}{2(1+x)^2}\right]-\int\left(-\frac{1}{2(1+x)^2x}\right)dx$

[Taking u = log x]

$$=-\frac{1}{2}\cdot\frac{\log x}{(1+x)^2}+\frac{1}{2}\int\frac{dx}{x(1+x)^2}\qquad ....(i)$$

Now, $\frac{1}{x(1+x)^2}=\frac{1+x-x}{x(1+x)^2}=\frac{1}{x(1+x)}-\frac{1}{(1+x)^2}$

$$=\frac{1+x-x}{x(1+x)}-\frac{1}{(1+x)^2}=\frac{1}{x}-\frac{1}{1+x}-\frac{1}{(1+x)^2}$$

$$\therefore\qquad\int\frac{dx}{x(1+x)^2}=\int\left[\frac{1}{x}-\frac{1}{1+x}-\frac{1}{(1+x)^2}\right]dx$$

$$=\log x-\log(1+x)+\frac{1}{1+x}=\log\frac{x}{1+x}+\frac{1}{1+x}$$

[Here x > 0 a log x occurs in the integrand]

$$\therefore\text{ From (i), }I=-\frac{1}{2}\frac{\log x}{(1+x)^2}+\frac{1}{2}\log\frac{x}{1+x}+\frac{1}{2(1+x)}+c$$

Sol 36: Let $f(x) = ax^2 + bx + c$

$f(0) = c = -3$

$f(1) = a + b + c = -3$

or $a + b = 0 \because a = 1, b = -1$

$f(2) = 4a + 2b + c = -1$

or $4a + 2b = 2$

$\therefore f(x) = x^2 - x - 3$

$$\int \frac{x^2 - x - 3}{x^3 - 1}dx = \int \frac{x(x-1)}{(x-1)(x^2+x+1)}dx - \int \frac{3}{(x^3-1)}dx$$

$$= \int \frac{x}{(x^2+x+1)}dx - \int \frac{3}{(x^3-1)}dx$$

$$\frac{1}{2}\left[\int \frac{2x+1}{(x^2+x+1)}dx - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right] - \int \frac{3}{(x^3-1)}dx$$

$$= \frac{1}{2}\log(x^2+x+1) - \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$- \int\left(\frac{1}{x-1} - \frac{x+2}{(x^2+x+1)}\right)dx$$

$$= \frac{1}{2}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} - \log(x-1)$$

$$+\frac{1}{2}\int\left(\frac{2x+1}{x^2+x+1} + \frac{3}{(x^2+x+1)}\right)dx$$

$$= \log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-}\frac{2x+1}{\sqrt{3}} - \log(x-1)$$

$$+ \frac{3}{2\times\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$= \log\frac{(x^2+x+1)}{|x-1|} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c$$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $\int\left(\frac{x}{1+x^5}\right)^{3/2}dx$

$$\int \frac{x^{3/2}}{x^{15/2}\left(1+\frac{1}{x^5}\right)^{3/2}}dx = \int \frac{x^{-6}}{\left(1+\frac{1}{x^5}\right)^{3/2}}dx$$

Put $1 + \frac{1}{x^5} = t$

$\Rightarrow -5x^{-6}dx = dt$ or $x^{-6}dx = -\frac{1}{5}dt$

$$\therefore I = -\frac{1}{5}\int\frac{dt}{t^{3/2}} = -\frac{1}{5}\times(-2)\frac{1}{\sqrt{t}} + c$$

$$\therefore I = \frac{2}{5}\frac{1}{\sqrt{1+\frac{1}{x^5}}} + c = \frac{2}{5}\sqrt{\frac{x^5}{1+x^5}} + c$$

Sol 2: (B) $-\int\frac{(\sin^8 x - \cos^8 x)}{1-2\sin^2 x\cos^2 x}dx$

$$= -\int\frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{1-2\sin^2 x\cos^2 x}dx$$

$$= -\int\frac{(\sin^2 x - \cos^2 x)\left[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x)\right]}{1-2\sin^2 x\cos^2 x}dx$$

$$= -\int(\sin^2 x - \cos^2 x)dx = -\int(-\cos 2x)dx = \frac{1}{2}\sin 2x + c$$

Sol 3: (C)

(A) $x\int \log x dx = x\left[(\log x)x - \int x\times\frac{1}{x}dx\right] = x^2\log x - x^2 + cx$

(B) $x\int \ell|x|dx = x^2\log|x| - x^2 + cx$

(C) $x\int e^x dx = x\left[e^x + c\right] = xe^x + cx$

(D) $\int\frac{dx}{\sqrt{a^2+x^2}}$

$x = a\tan\theta$, $dx = a\sec^2\theta d\theta$

$$\Rightarrow I = \int\frac{\sec^2\theta d\theta}{a\sec\theta} = \int\sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + c = \log\left|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right| + c$$

Sol 4: (B) $\int\sqrt{1+2\tan x(\sec x+\tan x)}dx$

$= \int\sqrt{1+2\tan x\sec x+2\tan^2 x}\ \ dx$

$= \int\sqrt{\tan^2 x+\sec^2 x+2\tan x\sec x}dx$

$= \int(\tan x+\sec x)dx = \int\tan x dx+\int\sec x dx$

$= -\log|\cos x| + \log|\sec x + \tan x| + c$

$= \log|\sec x| + \log|\sec x + \tan x| + c$

Sol 5: (C) $I = \int\frac{2\sin x\cos x}{\sin^4 x+\cos^4 x}dx = \int\frac{2\tan x\sec^2 x}{\tan^4 x+1}dx$

Let $\tan^2 x = t$

$\therefore I = 2\tan x \sec^2 x\, dx = dt$

$\therefore I = \int\frac{1}{t^2+1}dt$

$\therefore I = \tan^{-1}(t) + c = \tan^{-1}(\tan^2 x) + c$

$\tan^{-1}(\tan^2 x) = \tan^{-1}\left(\frac{1}{\cot^2 x}\right) = \cot^{-1}(\cot^2 x)$

$\tan^{-1}(\tan^2 x) = \frac{\pi}{2} - \cot^{-1}(\tan^2 x) + c$

$= -\cot - (\tan^2 x) + c_1$

Sol 6: (C) $\int\frac{1}{\cos^3\theta\sqrt{2\sin\theta\cos\theta}}d\theta$

$= \int\frac{1}{\sqrt{2}\cos^{7/2}\theta\sin^{1/2}\theta}d\theta$

Dividing and multiplying by $\cos^4\theta$

$I = \int\frac{\sec^4\theta}{\sqrt{2}\tan^{1/2}\theta}$

let $\tan^{1/2}\theta = t$

$\therefore \frac{1}{2\sqrt{\tan\theta}}\times\sec^2\theta d\theta = dt$

$\therefore I = \int\sqrt{2}\sec^2\theta dt$

$= \int\sqrt{2}\left(1+t^4\right)dt = \sqrt{2}\times\left(t+\frac{t^5}{5}\right)+c$

$= \frac{\sqrt{2}}{5}\left(5\sqrt{\tan\theta}+\tan^2\theta\times\sqrt{\tan\theta}\right) + c$

$I = \frac{\sqrt{2}}{5}(\tan^2\theta+5)\sqrt{\tan\theta} + c$

Sol 7: (A) $\int\frac{dx}{b\left(\frac{a}{b}+x^2\right)} = \frac{1}{b\times\sqrt{\frac{a}{b}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{a}{b}}}\right)$

$= \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right)+c$

Sol 8: (C) $I = \int\frac{1+x^7-2x^7}{x(1+x^7)}dx$

$= \int\frac{1}{x}-\frac{2x^6}{1+x^7}dx = \log x - \int\frac{2x^6}{1+x^7}dx$

let $1 + x^7 = t \Rightarrow 7x^6 dx = dt$

$I = \log x - \frac{2}{7}\int\frac{1}{t}dt = \log x - \frac{2}{7}\ell n t + c$

$= \log x - \frac{2}{7}\ell n(1+x^7) + c$

Sol 9: (A) Let $\log|x| = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore I = \int\frac{t}{\sqrt{1+t}}dt = \int\sqrt{1+t}-\frac{1}{\sqrt{1+t}}dt$

$= \frac{(t+1)^{3/2}}{3}\times 2 - (t+1)^{1/2}\times 2 + c$

$= 2(t+1)^{1/2}\left(\frac{t+1}{3}-1\right)+c = \frac{2}{3}(t+1)^{1/2}(t-2)+c$

$= \frac{2}{3}(\log|x| + 1)^{1/2}(\log|x| - 2) + c$

Sol 10: (C)

$I = \int\frac{x^4+2x^2+1-2x^2}{x(x^2+1)^2}dx = \int\frac{(x^2+1)^2-2x^2}{x(x^2+1)^2}dx$

$= \int\frac{1}{x}-\frac{2x}{(x^2+1)^2}dx = \log|x| - \int\frac{2x}{(x^2+1)^2}dx$

let $x^2 + 1 = t \Rightarrow 2x dx = dt$

$\therefore I = \log|x| - \int\frac{1}{t^2}dt$

$= \log|x| + \frac{1}{t} + c = \log|x| + \frac{1}{1+x^2} + c$

A = 1, B = 1

Sol 11: (B) $I = \int 2\sin x(\cos 2x+\cos x)dx$

$= \int 2\sin x(2\cos^2 x-1+\cos x)dx$

Let cosx = t $\Rightarrow$ –sinxdx = dt

$\therefore$ I = $\int 2(1-t-2t^2)dt$ = $2t - t^2 - \frac{4t^3}{3}$ + c

= $2\cos x - \cos^2 x - \frac{4\cos^3 x}{3}$ + c

= $\cos x - \left(\frac{4\cos^3 x - 3\cos x}{3}\right) - \cos^2 x$ + c

= $\cos x - \frac{1}{3}\cos 3x - \cos^2 x - \frac{1}{2}$ + c

= $\cos x - \frac{1}{3}\cos 3x - \frac{\cos 2x}{2}$ + c

Sol 12: (B) I = $\int \frac{1}{2}\sin 2x\cos 2x\cos 4x\cos 18x\cos 16x dx$

= $\frac{1}{32}\int \sin 32x dx$ **=** $-\frac{\cos 32x}{1024}$ + c

Sol 13: (B) I = $\int x^3 f(x^2)dx$

let x^2 = t

$\therefore$ I = $\frac{1}{2}\int tf(t)dt$

= $\frac{t}{2}\int f(t)dt - \int \frac{1}{2}\left(\int f(t)dt\right)dt$

= $\frac{1}{2}\left[x^2F(x^2) - \int f(x^2)d(x^2)\right]$

Sol 14: (D) I = $\int \frac{4e^{2x}+6}{9e^{2x}-4}dx$

$\Rightarrow 4e^{2x} + 6 = a(9e^{2x} - 4) + b \times 18 \times e^{2x}$

$\Rightarrow$ 9a + 18b = 4

–4a = 6

$\therefore$ a = $-\frac{3}{2}$

18b = 4 + $\frac{27}{2}$ $\Rightarrow$ b = $\frac{35}{36}$

$\therefore$ I = $\int -\frac{3}{2}dx + \frac{35}{36}\frac{18e^{2x}}{9e^{2x}-4}dx$

= $-\frac{3x}{2} + \frac{35}{36}\log(9e^{2x}-4) + c$

A = $-\frac{3}{2}$ and B = $\frac{35}{36}$

Sol 15: (A) I = $\int \frac{\cos x - \sin x}{\sqrt{2\sin x\cos x}\left(\cos x + \sin x\right)}dx$

2sinxcosx = $(\cos x+\sin x)^2 - 1$

And d(cosx + sinx) = (–sinx + cosx)dx

$\therefore$ let cosx + sinx = t

$\therefore$ I = $\int \frac{1}{\sqrt{t^2-1}\ (t)}dt$

$\therefore$ I = $\sec^{-1}(t)$ + c = $\sec^{-1}(\cos x+\sin x)$ + c

Previous Years' Questions

Sol 1: (C) Let I = $\int \frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}$, dividing numerator and denominator by x^5

= $\int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$

Put $2-\frac{2}{x^2}+\frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx = dt$

$\therefore I = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{1}{4}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$

= $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + c$

Sol 2: Given, $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$ = Ax + B log $(9e^{2x} - 4)$ + c

LHS = $\int \frac{4e^{2x}+6}{9e^{2x}-4}dx$

Let $4e^{2x}$ + 6 = A $(9e^{2x} - 4)$ + B$(18e^{2x})$

$\Rightarrow$ 9A + 18B = 4

and –4A = 6

$\Rightarrow$ A = $-\frac{3}{2}$ and B = $\frac{35}{36}$

$\therefore \int \frac{A(9e^{2x}-4)+B(18e^{2x})}{9e^{2x}-4}dx$

= $A\int 1dx + B\int \frac{1}{t}dt$, where $t = 9e^{2x}-4$

=Ax + B log $(9e^{2x}–4)$ + c

= $-\frac{3}{2}x + \frac{35}{36}\log(9e^{2x}-4)$ + c

$\therefore A = -\frac{3}{2}, B = \frac{35}{36}$

and C = any real number

Sol 3: Let $I = \int \frac{\sin x}{\sin x - \cos x} dx$

Again, let sin x = A (cos x + sin x) + B (sin x – cos x),

then A + B = 1 and A – B = 0

$\Rightarrow A = \frac{1}{2}, B = \frac{1}{2}$

$\therefore I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{(\sin x - \cos x)} dx$

$= \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x} dx + \frac{1}{2}\int 1 dx + c$

$= \frac{1}{2}\log(\sin x - \cos x) + \frac{1}{2}x + c$

Sol 4: Let $I = \int \frac{x dx}{1 + x^4}$

$= \frac{1}{2}\int \frac{2x}{1 + (x^2)^2} dx$

Put $x^2 = u \Rightarrow 2x dx = du$

$\therefore I = \frac{1}{2}\int \frac{du}{1 + u^2} = \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}(x^2) + c$

Sol 5: Let $I_1 = \int \sin x \sin 2x \sin 3x\, dx$

$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) dx$

$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24}$

$I_2 = \int \sec^2 x \cdot \cos^2 2x dx = \int \sec^2 x (2\cos^2 x - 1)^2 dx$

$= \int (4\cos^2 x + \sec^2 x - 4) dx = \int (2\cos 2x + \sec^2 x - 2) dx$

= sin2x + tanx – 2x

$\therefore \int \frac{A(9e^{2x} - 4) + B(18e^{2x})}{9e^{2x} - 4} dx$

$= A\int 1 dx + B\int \frac{1}{t} dt,$ where $t = 9e^{2x} - 4$

$= Ax + B \log (9e^{2x} - 4) + c$

$= -\frac{3}{2}x + \frac{35}{36}\log (9e^{2x} - 4) + c$

$\therefore A = -\frac{3}{2}, B = \frac{35}{36}$

and C = any real number

and $I_3 = \int \sin^4 x \cos^4 x\, dx$

$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x) dx = \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$

$\therefore I = I_1 + I_2 + I_3$

$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + \sin 2x + \tan x - 2x$
$+ \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$

Sol 6: Let $I = \frac{x^2}{(a + bx)^2}$

Put a + bx = t

$\Rightarrow b dx = dt$

$\therefore I = \int \frac{\left(\frac{t - a}{b}\right)^2}{t^2} \cdot \frac{dt}{b} = \frac{1}{b^3}\int \left(\frac{t^2 - 2at + a^2}{t^2}\right) dt$

$= \frac{1}{b^3}\int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) dt = \frac{1}{b^3}\left(t - 2a \log t - \frac{a^2}{t}\right) + c$

$= \frac{1}{b^3}\left(a + bx - 2a \log (a + bx) - \frac{a^2}{a + bx} + c\right)$

Sol 7: Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$\Rightarrow dx = \frac{2t}{1 + t^4} dt$

$\therefore I = \int \frac{t^2 + 1}{\sqrt{t^2}} \cdot \frac{2t}{t^4 + 1} dt = 2\int \frac{t^2 + 1}{t^4 + 1} dt$

$= 2\int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 2 + 2} dt = 2\int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + \left(\sqrt{2}\right)^2} dt$

Put $t - \frac{1}{t} = u \Rightarrow 1 + \frac{1}{t_2} dt = du$

$\therefore I = 2\int \frac{du}{u^2 + (\sqrt{2})^2}$

$\Rightarrow I = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$

$= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

Sol 8: $\int \frac{(x+1)}{x(1+xe^x)^2}dx$

This can be rewritten as $\int \frac{e^x(x+1)}{2e^x(1+xe^x)^2}dx$

let $1+xe^x = t \Rightarrow e^x(1+x)dx = dt$

Now integration becomes $\int \frac{dt}{t^2(t-1)}$

$\Rightarrow \quad \frac{1}{t^2(t-1)} = \frac{A}{t-1}+\frac{Bt+C}{t^2}$ (using partial fraction)

$\Rightarrow \quad 1 = t^2(A+B)+(C-B)t-C$

Comparing, we get C = -1, B =- 1 and A = 1

Now our integration becomes

$\int \frac{dt}{t^2(t-1)} = \int \frac{1}{t-1}dt - \int \frac{t+1}{t^2}dt = \int \frac{1}{t-1}dt - \int \frac{1}{t}dt - \int t^2 dt$

$= \log(t-1) - \log(t) - \frac{t^{-2+1}}{-2+1}+C \;= \log\frac{t-1}{t}+\frac{1}{t}+c$

Putting $t = 1+xe^x$, we get

$\int \frac{dt}{t^2(t-1)} = \log\left|\frac{xe^x}{1+xe^x}\right|+\frac{1}{1+xe^x}+c$

Sol 9: Let $I = \int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{\frac{1}{2}}\cdot\frac{dx}{x}$

Put $x = \cos^2\theta \Rightarrow dx = -2\cos\theta\ \sin\theta\ d\theta$

$\therefore\ I = \int\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{\frac{1}{2}}\cdot\frac{-2\cos\theta\cdot\sin\theta}{\cos^2\theta}d\theta$

$= \int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{-2\sin\theta}{\cos\theta}d\theta = -\int\frac{2\sin\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}\cdot\cos\theta}d\theta$

$= -2\int\frac{2\sin^2\frac{\theta}{2}}{\cos\theta}d\theta = -2\int\frac{1-\cos\theta}{\cos\theta}d\theta$

$= 2\int(1-\sec\theta)d\theta = 2[\theta-\log|\sec\theta+\tan\theta|]+c$

$\Rightarrow I = 2\left[\cos^{-1}\sqrt{x}-\log\left|\frac{1}{\sqrt{x}}+\sqrt{\frac{1}{x}-1}\right|\right]+c$

$I = 2\left[\cos^{-1}\sqrt{x}-\log\left|1+\sqrt{1-x}\right|-\frac{1}{2}\log|x|\right]+c$

Sol 10: (C) $\sqrt{2}\frac{\sin x dx}{\sin\left(x-\frac{\pi}{4}\right)} = \sqrt{2}\int\frac{\sin\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right)dx}{\sin\left(x-\frac{\pi}{4}\right)}$

$= \sqrt{2}\int\left(\cos\frac{\pi}{4}+\cot\left(x-\frac{\pi}{4}\right)\sin\frac{\pi}{4}\right)dx$

$= \int dx+\int\cot\left(x-\frac{\pi}{4}\right)dx \;= x+\ell n\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

Sol 11: (D) $\frac{dy}{dx} = y+3$

$\Rightarrow \frac{dy}{y+3} = dx$

$\log(y+3) = x+c$

$x = 0 \Rightarrow y = 2$

$\Rightarrow \log 5 = 0 + c$

$c = \log 5$

$\log(y+3) = x+\log 5$

$y + 3 = e^{x - \log 5} \Rightarrow y + 3\ e^{\log 2 + \log 5}$

$y+3 = 10 \Rightarrow y = 7$

Sol 12: (D)

$\int\frac{5\sin x}{\sin x-2\cos x}dx$

$\Rightarrow \int\left[\frac{2(\cos x+2\sin x)+(\sin x-2\cos x)}{\sin x-2\cos x}\right]dx$

$= \int\left(\frac{\cos x+2\sin x}{\sin x-2\cos}\right)dx+\int dx+k$

$= 2\log|\sin x-2\cos x|+x+k$

$\therefore$ a = 2

Sol 13: (C) $\int f(x)dx = \psi(x)$

$I = \int x^5 f(x^3)dx$

Put $x^3 = t \quad\Rightarrow\quad x^2dx = \frac{dt}{3} \;= \frac{1}{3}\int tf(t)dt$

$= \frac{1}{3}[t\psi(t)dt]$

$$=\frac{1}{3}\left[x^3\psi\left(x^3\right)-3\int x^2\psi\left(x^3\right)dx\right]+c \;=\frac{1}{3}x^3\psi\left(x^3\right)dx+c$$

Sol 14: (D) $I=\int\left\{e^{\left(x+\frac{1}{x}\right)}+x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\right\}dx$

$$=x.e^{x+\frac{1}{x}}+c$$

As $\int\left(xf'\left(x\right)+f\left(x\right)\right)dx=xf\left(x\right)+c$

Sol 15: (D) $\int\frac{dx}{x^2\left(x^4+1\right)^{3/4}}$

$$\int\frac{dx}{x^3\left(1+\frac{1}{x^4}\right)^{3/4}} \quad\Rightarrow 1+\frac{1}{x^4}=t^4$$

$$-4\frac{1}{x^5}=dx=4t^3dt$$

$$\frac{dx}{x^3}=t^3dt$$

$$\int\frac{-t^3dt}{t^3}=-t+c=-\left(1+\frac{1}{x^4}\right)^{1/4}+c$$

Sol 16: (A) $\int\frac{2x^{12}+5x^3}{\left(x^5+x^3+1\right)}dx$

$$\int\frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$$

Let $1+\frac{1}{x^2}+\frac{1}{x^5}=t$

$$\frac{dt}{dx}=\frac{-2}{x^3}-\frac{5}{x^6}$$

$$\int\frac{-dt}{t^3}=\frac{1}{2t^2}+c=\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+c=\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+c$$

JEE Advanced/Boards

Exercise 1

Sol 1: (i) $\int\tan\frac{x}{2}\tan\frac{x}{3}\tan\frac{x}{6}dx$

$$=\int\tan\frac{x}{2}\left[1-\frac{\left(\tan\frac{x}{3}+\tan\frac{x}{6}\right)}{\tan\frac{x}{2}}\right]dx$$

$$=\int\left(\tan\frac{x}{2}-\tan\frac{x}{3}-\tan\frac{x}{6}\right)dx$$

$$=\int\tan\frac{x}{2}dx-\int\tan\frac{x}{3}dx-\int\tan\frac{x}{6}dx$$

$$=2\log\sec\frac{x}{2}-3\log\sec\frac{x}{3}-6\log\sec\frac{x}{6}+c$$

(ii) $\int\frac{\tan(\log x)\tan\left(\log\frac{x}{2}\right)\tan(\log 2)}{x}dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx=dt$

$$\int\tan t\tan(t-\log 2)\tan(\log 2)dt$$

$$=\int\left[\tan t-\tan(t-\log 2)-\tan(\log 2)\right]dt$$

$$=\log\sec t-\log\sec(t-\log 2)-x\tan(\log 2)+c$$

$$=\log\frac{\sec(\log x)}{\sec\left(\log\frac{x}{2}\right)}-x\tan\log|2|+c$$

Sol 2: Put $x=\alpha\sec 2\theta-\beta\tan 2\theta$

$$\int\frac{2(\alpha-\beta)\sec^2\theta\tan\theta d\theta}{(\alpha-\beta)\tan^2\theta\sqrt{(\alpha-\beta)\tan^2\theta(\alpha-\beta)\sec^2\theta}}$$

$$=\int\frac{2\sec^2\theta\tan\theta}{\tan^2\theta\times(\alpha+\beta)\tan\theta\sec\theta}d\theta$$

$$=\frac{2}{\alpha-\beta}\int\frac{\sec\theta}{\tan^2\theta}d\theta$$

Put $\tan\theta = t$ $\sec^2\theta d\theta = dt$

Or $\sec\theta d\theta=\frac{dt}{\sqrt{1+t^2}}$

$\therefore I = \frac{2}{(\alpha-\beta)}\int\frac{dt}{t^2\sqrt{1+t^2}} = \frac{2}{(\alpha-\beta)}\int\frac{t^{-3}dt}{\sqrt{t^{-2}+1}}$

$\therefore 1 + t^{-2} = u \Rightarrow -2t^{-3}dt = du$

Or $t^{-3}dt = -\frac{1}{2}du$

$= \frac{1}{2}\times\frac{2}{(\alpha-\beta)}\int\frac{-du}{\sqrt{u}} = \frac{-1}{(\alpha-\beta)}\times 2\sqrt{u}$

$= \frac{-2}{(\alpha-\beta)}\sqrt{1+\frac{1}{t^2}}$

$= \frac{-2}{(\alpha-\beta)}\sqrt{1+\frac{1}{\tan^2\theta}} = \frac{-2}{(\alpha-\beta)}\sqrt{\frac{\sec^2\theta}{\tan^2\theta}}$

$= \frac{-2}{\alpha-\beta}\sqrt{\frac{(\alpha-\beta)\sec^2\theta}{(\alpha-\beta)\tan^2\theta}} = \frac{-2}{\alpha-\beta}\sqrt{\frac{(x-\beta)}{(x-\alpha)}}$

Sol 3: $\int\frac{\ell n\left(\ell n\left(\frac{1+x}{1-x}\right)\right)dx}{1-x^2}$

Put $\log\left(\frac{1+x}{1-x}\right) = t$

$\Rightarrow\left(\left(\frac{1-x}{1+x}\right)\times\frac{1-x+1+x}{(1-x)^2}\right)dx = dt$

Or $\left(\frac{2}{1-x^2}\right)dx = dt$

$\therefore I = \int\log(t)\,\frac{dt}{2}$

$= \frac{1}{2}\int 1.\log(t)\,\frac{dt}{2}$

Integration by parts,

$I = \frac{1}{2}\left[\log(t)\int 1dt - \left[\int\frac{d}{dt}(\log(t))\int 1dt\right]dt\right]$

$= \frac{1}{2}\left[t(\log t - 1)\right] + c$

$= \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\left\{\log\left(\log\left(\frac{1+x}{1-x}\right)\right) - 1\right\}\right] + c$

Sol 4: $d\left(\frac{x}{e}\right)^x = \left(\frac{x}{e}\right)^x[\log x]$ and $d\left(\frac{e}{x}\right)^x = \left(\frac{e}{x}\right)^x[-\log x]$

$\therefore \int\left(\frac{x}{e}\right)^x\log x dx + \int\left(\frac{e}{x}\right)^x\log x dx = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + c$

Sol 5: $\int\sqrt{\frac{\cos(x-a)}{\sin(x+a)}}dx$

$\int\sqrt{\frac{\cos x\cos a - \sin x\sin a}{\sin x\cos a + \cos x\sin a}}dx$

$= \int\sqrt{\frac{1-\tan x\tan a}{\tan x + \tan a}}dx = \int\sqrt{\cot(a+x)}\ \ dx$

Sol 6: $\int\frac{x^5+3x^4-x^3+8x^2-x+8}{(x^2+1)}dx$

$= \int\frac{(x^3+3x^2-2x+5)(x^2+1)}{(x^2+1)}dx + \int\frac{x+3}{x^2+1}dx$

$= \frac{x^4}{4} + x^3 - x^2 + 5x + \frac{1}{2}\log(x^2 + 4) + 3\tan^{-1}x + c$

Sol 7: $\int\frac{(\sqrt{x}+1)}{\sqrt{x}(\sqrt[3]{x}+1)} = \int\frac{(x)^{1/2}+1}{x^{1/2}\left(x^{1/3}+1\right)}dx$

Put $x^{1/6} = t \Rightarrow dx = 6t^5dt$

$6\int\frac{(t^3+1)t^5}{t^3(t^2+1)}dt = 6\int\frac{(t^3+1)t^2}{(t^2+1)}dt$

$= 6\int\left[\frac{(t^3-t+1)(t^2+1)}{(t^2+1)} + \frac{t-1}{(t^2+1)}\right]dt$

$= 6\left[\frac{t^4}{4} - \frac{t^2}{2} + t\right] + 3\log(1 + t^2) - 6\tan^{-1}t + c$

Where $t = x^{1/6}$

Sol 8: $\int\sin^{-1}\sqrt{\frac{x}{a+x}}dx$, $a > 0$

$x = a\tan^2\theta$

$dx = 2a\tan\theta\sec^2\theta d\theta$

$\int\sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}}\ 2a\tan\theta\sec^2\theta\, d\theta$

$2a\int(\sin^{-1}\sin\theta)\tan\theta\sec^2\theta\ \ d\theta$

$\tan\theta = t \Rightarrow \sec^2\theta d\theta = dt$

$2a\int t\tan^{-1}t\,dt$

$2a\left[\tan^{-1}t\int t\,dt - \frac{1}{2}\int\frac{1}{1+t^2}\times t^2dt\right]$

$$= 2a\left[(\tan^{-1}t)\frac{t^2}{2} - \frac{1}{2}\left[t - \tan^{-1}t\right]\right] + c$$

$$= at^2\tan^{-1}t + \left[\frac{1}{2}\tan^{-1}t - \frac{1}{2}t\right]2a + c$$

$$\therefore t = \tan\theta = \frac{\sqrt{x}}{\sqrt{a}}$$

$$\therefore I = a\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} + \left[\frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}} - \frac{1}{2}\sqrt{\frac{x}{a}}\right]2a + c$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} + \left[\frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}} - \frac{1}{2}\sqrt{\frac{x}{a}}\right]2a + c$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + c$$

$$= \left(\tan^{-1}\sqrt{\frac{x}{a}}\right)(a + x) - a\sqrt{\frac{x}{a}} + c$$

$$= \sqrt{x}\left[\sqrt{x}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{a}\right] + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

Sol 9: $\int \frac{x\log x}{(x^2-1)^{3/2}}dx$

$$\therefore \int\left(\frac{x\log x}{(x^2-1)^{3/2}} - \frac{1}{x\sqrt{x^2-1}}\right)dx + \int\frac{1}{x\sqrt{x^2-1}}dx$$

$$\int\left(-d\frac{\log x}{\sqrt{x^2-1}}\right) + \sec^{-1}x + c = -\frac{\log x}{\sqrt{x^2-1}} + \sec^{-1}x + c$$

Sol 10: $\int \frac{\log_6[(\sin x)6^{\cos x}]^{1/6}\cos x}{\sin x}dx$

$$\int\frac{\frac{1}{6}\left[\log_6(\sin x) + \log_6 6^{\cos x}\right]\cos x}{\sin x}dx$$

$$\int\left(\left(\frac{\frac{1}{6}\log(\sin x)}{(\log 6)\sin x} + \frac{1}{6}\frac{\cos x}{\sin x}\right)\cos x\right)dx$$

$$= \int\left[\frac{1}{6}\frac{1}{\log 6}\frac{\cos x}{\sin x}\log(\sin x) + \frac{1}{6}\frac{\cos^2 x}{\sin x}\right]dx$$

$$= \frac{1}{6}\frac{\log^2(\sin x)}{\log 36} + \frac{1}{6}\int(\operatorname{cosec}x - \sin x)dx$$

$$= \frac{1}{6}\left[\frac{\log^2(\sin x)}{\log 36} + \ell n\tan\frac{x}{2} + \cos x\right] + c$$

Sol 11: $\int\frac{\sqrt{x^2+1}\left(\log(x^2+1) - \log x^2\right)}{x^4}dx$

$$\int\frac{\left(\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)\right)}{x^3}dx$$

$$1 + \frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3}dx = dt$$

$$-\frac{1}{2}\int\sqrt{t}\log t\,dt = -\frac{1}{2}\left[(\log t)\frac{t^{3/2}}{3/2} - \int\frac{1}{t}\times\frac{t^{3/2}}{3/2}dt\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}(\log t)t^{3/2} + \frac{2}{3}\int t^{1/2}dt\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}\left[\log\left(1+\frac{1}{x^2}\right)\left[1+\frac{1}{x^2}\right]^{3/2} + \frac{4}{9}\left(1+\frac{1}{x^2}\right)^{3/2} + c\right]\right]$$

$$= -\frac{1}{3}\left[1+\frac{1}{x^2}\right]^{3/2}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + c$$

Sol 12: $f(x) = \frac{\sin x}{\sin^2 x + 4\cos^2 x} = \frac{\tan x\sec x}{\tan^2 x + 4}$

$$= \frac{\tan x\sec x}{\sec^2 x + 3}$$

Putting sec x = t, dx sec x tan x = dt so

$$\int f(x)dx = \int\frac{dt}{t^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\sec x}{\sqrt{3}} + c$$

Sol 13: $\int\frac{\cos^2 x}{\sin x(1-\sin x)(1+\cos x)}dx = \int\frac{(1+\sin x)}{\sin x(1+\cos x)}dx$

$$= \int\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}{2\sin\frac{x}{2}\cos\frac{x}{2}\times 2\cos^2\frac{x}{2}}dx = \frac{1}{4}\int\frac{\left(\tan\frac{x}{2}+1\right)^2}{\tan\frac{x}{2}\cos^2\frac{x}{2}}dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

$$= \frac{1}{2}\int\frac{(t+1)^2}{t}dt = \frac{1}{2}\int\left(t + 2 + \frac{1}{t}\right)dt$$

$$= \frac{t^2}{4} + t + \frac{1}{2}\log|t| + c$$

Sol 14: $\int \frac{3x^2+1}{(x^2-1)^3}dx$

$\int\left(\frac{3(x^2-1)+4}{(x^2-1)^3}\right)dx$ or $\int\frac{(3x^2+1)(x^2-1)}{(x^2-1)^4}dx$

$= \int\left[\frac{3x^2(x^2-1)+(x^2-1)}{(x^2-1)^4}\right]dx = \int\frac{3x^4-2x^2-1}{(x^2-1)^4}dx$

$= \int-\left[\frac{x^4+1-2x^2-4x^4+4x^2}{(x^2-1)^4}\right]dx$

$= \int-\left[\frac{(x^2-1)^2-2.2x^2(x^2-1)}{(x^2-1)^4}\right]dx$

$= \int-d\left(\frac{x}{(x^2-1)^2}\right) = -\frac{x}{(x^2-1)^2}+c$

Sol 15: $\int\frac{(ax^2-b)dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

$\int\frac{(ax^2-b)}{cx^2\sqrt{1-\frac{(ax^2+b)^2}{c^2x^2}}} \Rightarrow \int\frac{acx^2-bc}{c^2x^2\sqrt{1-\frac{(ax^2+b)^2}{c^2x^2}}}dx$

$= \int\frac{2acx^2-(acx^2+bc)}{(cx^2)\sqrt{1-\frac{(ax^2+b)^2}{c^2x^2}}}dx$

$= \int\frac{2acx^2-(ax^2+b)c}{(cx)^2}\times\frac{1}{\sqrt{1-\left(\frac{ax^2+b}{cx}\right)^2}}dx$

Put $\frac{ax^2+b}{cx}=t = \left(\frac{(2ax)cx-c(ax^2+b)}{(cx)^2}\right)dx$

$\therefore \int\frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + c = \sin^{-1}\left(\frac{ax^2+b}{cx}\right)+c$

Sol 16: Put $z = \tan^{-1}x$, then $dz = \frac{1}{1+x^2}dx$ and $x = \tan z$

Now, $I = \int\frac{e^{mz}}{\sqrt{1+\tan^2 z}}dz = \int e^{mz}\cos z\, dz$

$= \frac{e^{mz}}{m}\cos z - \int\frac{e^{mz}}{m}(-\sin z)dz$

$= \frac{e^{mz}}{m}\cos z + \frac{1}{m}\int e^{mz}\sin z\, dz$

$= \frac{e^{mz}}{m}\cos z + \frac{1}{m}\left[\frac{e^{mz}}{m}\sin z - \int\frac{e^{mz}}{m}\cos z\, dz\right]$

$= \frac{e^{mz}}{m}\cos z + \frac{e^{mz}}{m^2}\sin z - \frac{1}{m^2}1$

Or, $\left(1+\frac{1}{m^2}\right)1 = \frac{e^{mz}}{m^2}(m\cos z+\sin z)$

$\therefore 1 = \frac{e^{mz}(m\cos z+\sin z)}{m^2+1}+c$

Or,

$1 = \frac{e^{m\tan^{-1}x}}{m^2+1}\left(m\cdot\frac{1}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}\right)+c = \frac{e^{m\tan^{-1}x}(m+x)}{(m^2+1)\sqrt{x^2+1}}+c$

Sol 17: [Here $\sqrt{\frac{x}{a+x}}$ occurs, $\therefore$ put $x = a\tan^2\theta$]

Put $x = a\tan^2\theta$, then $dx = 2a\tan\theta\sec^2\theta d\theta$

Now,

$I = \int\sin^{-1}(\sin\theta)2a\tan\theta\sec^2\theta d\theta = 2a\int\theta\cdot(\tan\theta\sec^2\theta)d\theta$

$= 2a\left[\theta\frac{\sec^2\theta}{2} - \int 1\frac{\sec^2\theta}{2}d\theta\right]+c$

$[\int\tan\theta\sec^2\theta d\theta = \int z dz,$ where $z = \sec\theta]$

$= a[\theta\sec^2\theta - \tan\theta]+c$

$= a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(\frac{a+x}{a}\right) - \sqrt{\frac{x}{a}}\right]+c$

Sol 18: $\frac{d\sin\left(e^{\sqrt{x}}+e^{\sqrt{x}}+\frac{\pi}{4}\right)}{dx}$

$= \cos\left(e^{\sqrt{x}}+e^{-\sqrt{x}}+\frac{\pi}{4}\right)\left[\frac{e^{\sqrt{x}}}{2\sqrt{x}}-\frac{e^{-\sqrt{x}}}{2\sqrt{x}}\right]$

Also $\frac{d}{dx}\sin\left(e^{\sqrt{x}}-e^{-\sqrt{x}}+\frac{\pi}{4}\right)$

$= \cos\left(e^{\sqrt{x}}-e^{-\sqrt{x}}+\frac{\pi}{4}\right)\left[\frac{e^{\sqrt{x}}}{2\sqrt{x}}+\frac{2e^{-\sqrt{x}}}{2\sqrt{x}}\right]$

$$\therefore I = 2\int \begin{bmatrix} d\sin\left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4}\right) \\ + d\sin\left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4}\right) \end{bmatrix}$$

$$= 2\left[\sin\left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4}\right) + \sin\left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4}\right)\right]$$

$$= 2\sin\left(e^{\sqrt{x}} + \frac{\pi}{4}\right)\cos\left(e^{-\sqrt{x}}\right)$$

$$= \sqrt{2}\left(\sin\left(e^{\sqrt{x}}\right) + \cos\left(e^{\sqrt{x}}\right)\right)\cos\left(e^{-\sqrt{x}}\right) + c$$

Sol 19: $\int \frac{(x^2+x)}{(e^x+x+1)^2}dx$

$$= \int\left(\frac{x(e^x+x+1) - xe^x}{(e^x+x+1)^2}\right)dx$$

$$= \int\left(\frac{x}{e^x+x+1} - \frac{xe^x}{(e^x+x+1)^2}\right)dx$$

$$= \int\left[\left(\frac{1}{1+\left(\frac{x+1}{e^x}\right)}\right)\times\frac{x}{e^x} - \frac{1}{\left(1+\frac{x+1}{e^x}\right)^2}\left(\frac{x}{e^x}\right)\right]dx$$

Put $\frac{x+1}{e^x} + 1 = t \Rightarrow \frac{-x}{e^x}dx = dt$

$$\int\left(-\frac{dt}{t} + \frac{1}{t^2}\right)dt = -\log t - \frac{1}{t} + c$$

$$= -\log\left(\frac{x+1}{e^x} + 1\right) - \frac{1}{\left(\frac{x+1}{e^x} + 1\right)} + c$$

Sol 20: $\int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x}dx$

$$\int e^{\cos x}\left(\frac{\cos x}{\sin^2 x} + x\sin x\right)dx$$

$$\int e^{\cos x}(\cot x \operatorname{cosec} x + x\sin x)dx$$

$$\Rightarrow -\int e^{\cos x}(1 - \cos t x \operatorname{cosec} x - x\sin x - 1)dx$$

$$= -\int\left[e^{\cos x}(1 - \operatorname{cosec} x\cot x) + (x + \operatorname{cosec} x)e^{\cos x}(-\sin x)\right]dx$$

$$= -\int d e^{\cos x}(x + \operatorname{cosec} x) = -e^{\cos x}(x + \operatorname{cosec} x) + c$$

Sol 21: $\int \frac{5x^4 + 4x^5}{(x^5+x+1)^2}dx$

$$= \int \frac{5x^5 + 5x^4 + x + 1 - x^5 - x - 1}{(x^5+x+1)^2}dx$$

$$= \int -\frac{(x^5+x+1) + (5x^4+1)(x+1)}{(x^5+x+1)^2}dx$$

$$= \int -d\left(\frac{x+1}{x^5+x+1}\right) = -\frac{x+1}{x^5+x+1} + c \quad \text{or} \quad \frac{x^5}{x^5+x+1} + c$$

Sol 22: $\int\left(\frac{a^2\tan^2 x + b^2}{a^4\tan^2 x + b^4}\right)dx$

$$\Rightarrow \frac{1}{(a^2+b^2)}\int \frac{b^2(a^2+b^2) + a^2(b^2+a^2)\tan^2 x}{a^4\tan^2 x + b^4}dx$$

$$\Rightarrow \frac{1}{a^2+b^2}\int\left(\frac{a^2b^2(1+\tan^2 x)}{a^4\tan^2 x + b^4} + 1\right)dx$$

$$\Rightarrow \frac{1}{a^2+b^2}\int\left(\frac{a^2b^2 + b^4 + (a^2b^2 + a^4)\tan^2 x}{a^4\tan^2 x + b^4}\right)dx$$

$$\Rightarrow \frac{1}{a^2+b^2}\int\left(\left(\frac{a^2}{b^2}\right)\sec^2 x \times \frac{1}{\left(\frac{a^4}{b^4}\tan^2 x + 1\right)} + 1\right)dx$$

$$\Rightarrow \frac{1}{a^2+b^2}x + \frac{1}{a^2+b^2}\int \frac{\left(\frac{a}{b}\right)^2\sec^2 x}{1 + \left(\frac{a^2}{b^2}\tan x\right)^2}dx$$

Put $\frac{a^2}{b^2}\tan x = t \Rightarrow \frac{a^2}{b^2}\sec^2 x dx = dt$

$$\therefore \frac{1}{a^2+b^2}x + \frac{1}{a^2+b^2}\int\frac{dt}{1+t^2}$$

Or $\frac{1}{a^2+b^2}\left[x + \tan^{-1}\frac{a^2}{b^2}\tan x\right] + c$

Sol 23: $\int \frac{\cos^2 x}{1+\tan x}dx = \int \frac{\cos^3 x}{\sin x + \cos x}dx$

$$= \frac{1}{4}\int\frac{3\cos x}{\sin x + \cos x}dx + \frac{1}{4}\int\frac{\cos 3x}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int \frac{\cos x\cos 2x - \sin x\sin 2x}{(\sin x + \cos x)}dx + \frac{3}{4}\int \frac{\cos x}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int \left[(\cos 2x - \sin 2x) + \left(\frac{\sin 2x\cos x - \cos 2x\sin x}{\sin x + \cos x}\right)\right]dx$$

$$+ \frac{3}{4}\int \frac{\cos x}{\sin x + \cos x}dx$$

$$= \frac{1}{4}\int (\cos 2x - \sin 2x)dx + \frac{1}{4}\int dx$$

$$-\frac{1}{4}\int \frac{\cos x dx}{\sin x + \cos x} + \frac{3}{4}\int \frac{\cos x dx}{\sin x + \cos x}$$

$$= \frac{1}{8}(\sin 2x + \cos 2x) + \frac{x}{4} + \frac{1}{2}\int \frac{\cos x dx}{\sin x + \cos x}$$

$$= \frac{1}{8}(\sin 2x + \cos 2x) + \frac{x}{4}$$

$$+\frac{1}{4}\int \left(\frac{\cos x + \sin x + \cos x - \sin x}{\cos x + \sin x}\right)dx$$

$$= \frac{1}{8}(\sin 2x + \cos 2x) + \frac{x}{4} + \frac{1}{4}\int \left[1 + \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)\right]dx$$

$$= \frac{1}{8}(\sin 2x + \cos 2x) + \frac{x}{2} + \frac{1}{4}\log(\cos x + \sin x) + c$$

Sol 24:

$$I = (x\log x - x)\sin^{-1}x - \int \frac{x\log x}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx \quad ...(i)$$

[Integrating by parts taking $\sin^{-1}x$ as u]

Now in order to evaluate $\int \frac{x\log x}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta$, then $dx = \cos\theta d\theta$

$$\therefore \int \frac{x\log x}{\sqrt{1-x^2}}dx = \int \sin\theta \log\sin\theta\, d\theta$$

$$= -\cos\theta\log\sin\theta - \int -\cos\theta\cot\theta d\theta$$

$$= -\cos\theta\log\sin\theta + \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= -\cos\theta\log\sin\theta + \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= -\cos\theta\log\sin\theta + \int (\operatorname{cosec}\theta - \sin\theta)d\theta$$

$$= -\cos\theta\log\sin\theta + \log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta$$

$$= -\sqrt{1-x^2}\log x + \log\left(\frac{1-\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$$

Again, $\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}$

$\therefore$ from (i),

$$I = (x\log x - x)\sin^{-1}x + \sqrt{1-x^2}\log x - \log\left(\frac{1-\sqrt{1-x^2}}{x}\right) + c$$

Sol 25: $\int \frac{(x^2+1)e^x}{(x+1)^2}dx = \int e^x\frac{(x^2-1)+2}{(x+1)^2}dx$

$$= \int e^x\left[\frac{x^2-1}{(x+1)} + \frac{2}{(x+1)^2}\right]dx = \int e^x\left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right]dx$$

$$= \int e^x[f(x) + f'(x)]dx, \quad \text{where } f(x) = \frac{x-1}{x+1}$$

$$= e^x f(x) + c = e^x\left(\frac{x-1}{x+1}\right) + c$$

Sol 26: [Here $e^{f(z)}$ occurs, where f(x) = sin x

$\therefore$ Put z = f(x) = sin x]

Put z = sin x, then dz = cos x dx

$$\text{Now, } I = \int \frac{e^{\sin x}}{\cos^3 x}(x\cos^3 x - \sin x)dz$$

$$= \int e^{\sin x}(x - \tan x\sec^2 x)dz$$

$$= \int e^z\left[\sin^{-1}z - \frac{z}{\sqrt{1-z^2}}\cdot\frac{1}{1-z^2}\right]dz \qquad [\because \sin x = z]$$

$$= \int e^z\left[\sin^{-1}z + \frac{1}{\sqrt{1-z^2}} - \frac{1}{\sqrt{1-z^2}} - \frac{z}{(1-z^2)^{3/2}}\right]dz$$

$$= \int e^z\left[\left(\sin^{-1}z - \frac{1}{\sqrt{1-z^2}}\right) + \left\{\frac{1}{\sqrt{1-z^2}} - \frac{z}{(1-z^2)^{3/2}}\right\}\right]dz$$

$$= \int e^z[f(z) + f'(z)]dz, \quad \text{where } f(z) = \sin^{-1}z - \frac{1}{\sqrt{1-z^2}}$$

$$= e^z f(z) + c = e^z\left(\sin^{-1}z - \frac{1}{\sqrt{1-z^2}}\right) + c = e^{\sin x}(x - \sec x) + c$$

Sol 27: $I = \int \frac{dx}{\sqrt{1-(x^2+2x)}} = \int \frac{dx}{\sqrt{2-(x^2+2x+1)}}$

$$= \int \frac{dx}{\sqrt{2-(1+x)^2}} = \int \frac{dx}{\sqrt{(2)^2-(1+x)^2}} \quad ...(i)$$

Let z = 1 + x, then dz = dx

From (i),

$$I=\int\frac{dz}{\sqrt{(\sqrt{2})^2-z^2}}=\sin^{-1}\frac{z}{\sqrt{2}}+c=\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$$

Sol 28: $\int\frac{dx}{\sec x+\operatorname{cosec} x}=\int\left(\frac{\sin x\cos x}{\sin x+\cos x}\right)dx$

$$=\frac{1}{2}\int\left(\frac{1+2\sin x\cos x-1}{\sin x+\cos x}\right)dx$$

$$=\frac{1}{2}\int\frac{(\sin x+\cos x)^2-1}{(\sin x+\cos x)}dx$$

$$=\frac{1}{2}\int(\sin x+\cos x)dx-\frac{1}{2\sqrt{2}}\int\frac{1}{\frac{1}{\sqrt{2}}(\sin x+\cos x)}dx$$

$$=\frac{1}{2}[\sin x-\cos x]-\frac{1}{2\sqrt{2}}\int\frac{dx}{\sin\left(x+\frac{\pi}{4}\right)}$$

$$=\frac{1}{2}[\sin x-\cos x]-\frac{1}{2\sqrt{2}}\int\frac{\sec^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}{2\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}dx$$

$$=\frac{1}{2}[\sin x-\cos x]-\frac{1}{2\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$$

$$=\frac{1}{2}\left[\sin x-\cos x-\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+c\right.$$

Sol 29 $I=\int\sqrt{2x^2+3x+4}\,dx=\int\sqrt{2\left(x^2+\frac{3}{2}x+2\right)}dx$

$$=\sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+2}dx$$

$$=\sqrt{2}\int\sqrt{x^2+2\cdot x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+2}dx$$

$$=\sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}dx \quad \text{...(i)}$$

Let, $z=x+\frac{3}{4}$, then $dz=dx$. Let $\sqrt{\frac{23}{16}}=\frac{\sqrt{3}}{4}=a$

Then from (i), $I=\sqrt{2}\int\sqrt{z^2+a^2}\,dz$

Now,

$$\int\sqrt{z^2+a^2}dz=\frac{z}{2}\sqrt{z^2+a^2}+\frac{a^2}{2}\log\left(z+\sqrt{z^2+a^2}\right)+c$$

$$=\frac{\left(-x+\frac{3}{4}\right)}{2}\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}$$

$$+\frac{23}{32}\log\left[x+\frac{3}{4}+\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}\right]+c$$

$$I=\frac{4x+3}{8}\sqrt{2x^2+3x+4}$$

$$+\frac{23}{16\sqrt{2}}\log\left[\frac{4x+3}{4}+\frac{\sqrt{2x^2+3x+4}}{\sqrt{2}}\right]+c$$

Sol 30: Let $z=x^n+1$, then $dz=nx^{n-1}dx$

Now, $I=\int\frac{dx}{x\left(x^n+1\right)}=\int\frac{dx}{nx^{n-1}\cdot x\left(x^n+1\right)}$

$$=\frac{1}{n}\int\frac{dz}{x^n\left(x^n+1\right)}=\frac{1}{n}\int\frac{dz}{(z-1)z} \quad \text{...(i)}$$

Let $\frac{1}{z(z-1)}=\frac{A}{z}+\frac{B}{z-1}=\frac{A(z-1)+Bz}{z(z-1)}$ ...(ii)

$\therefore\ A(z-1)+Bz=1$ ...(iii)

Putting Z = 0 we get, –A = 1

$\therefore$ A =–1

Putting Z = 1, we get B = 1

$\therefore$ From $I=\frac{1}{n}\int\frac{dz}{z(z-1)}=\frac{1}{n}\int\left(-\frac{1}{z}+\frac{1}{z-1}\right)dz$

$$=\frac{1}{n}[-\log|z|+\log|z-1|]+c$$

$$=\frac{1}{n}\log\left|\frac{z-1}{z}\right|+c=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$$

Sol 31: $\int\frac{\cos x-\sin x}{16-9\left(1+\sin 2x\right)}dx$

$$\int\frac{(\cos x-\sin x)}{16-9(\sin x+\cos x)^2}dx$$

Let 3(sinx + cosx) = t

$\Rightarrow$ (3cosx – 3sinx)dx = dt

$$\frac{1}{3}\int\frac{dt}{16-t^2}=\frac{1}{2.4}.\frac{1}{3}\log\left|\frac{4+t}{4-t}\right|=\left(\frac{1}{24}\log\frac{4+3\cos x+3\sin x}{4-3\cos x-3\sin x}\right)$$

Sol 32: $\int\frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx$

$$\sqrt{2}\int\frac{\cos x-\sin x}{\left(\sqrt{\sin 2x}\right)(1+3\sin 2x)}dx$$

$$\sqrt{2}\int\left[\frac{(\cos 2x\cos x+\sin^2 x\sin x)+(\cos 2x\sin x-\sin 2x\cos x)}{\left(\sqrt{\sin 2x}\right)(1+3\sin 2x)}\right]dx$$

$$2\int\left[\frac{\cos 2x(s\sin x+\cos x)-\sin 2x(\cos x-\sin x)}{\sqrt{2\sin 2x}(1+3\sin 2x)}\right]dx$$

$$\Rightarrow\int\frac{1}{\frac{(\sin x+\cos x)^2+2\sin 2x}{(\sin x+\cos x)^2}}\times\int\left[\frac{(\sin x+\cos x)2.\cos 2x-2\sin^2 x(\cos x-\sin x)}{(\sin x+\cos x)^2\sqrt{2\sin 2x}}\right]dx$$

$$\Rightarrow\int\left(\frac{1}{1\;+\;\frac{2\sin 2x}{(\sin x+\cos x)^2}}\right)\left(\frac{2\sin 2x}{\sin x+\cos x}\right)dx$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\sin 2x}{\sin x+\cos x}\right)+c$$

Sol 33: $\int\frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$

$$\therefore\ \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}+\frac{Ex^2+Fx+G}{(x^2+1)^2}$$

$$\int\frac{(4x^5-7x^2+8x^3-2x^2+4x-7)}{x^2(x^2+1)^2}dx$$

$$=\int\left(\frac{4}{x}+\left(-\frac{7}{x^2}\right)+\frac{6}{x^2+1}+\frac{6(1-x^2)}{(1+x^2)^2}\right)dx$$

$$=4\log x+\frac{7}{x}+6\tan^{-1}x+6\int\left(\frac{1-x^2}{(1+x^2)^2}\right)dx$$

Put x = tanθ

$\Rightarrow$ dx = $\sec^2\theta d\theta$

$$=4\log x+\frac{7}{x}+6\tan^{-1}x+6\int\frac{1-\tan^2\theta}{(1+\tan^2\theta)}d\theta$$

$$=4\log x+\frac{7}{x}+6\tan^{-1}x+6\int\cos 2\theta d\theta$$

$$=4\log x+\frac{7}{x}+6\tan^{-1}x+6\frac{1}{2}\sin 2\theta$$

$$\because\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2x}{1+x^2}$$

$$\therefore I=4\log x+\frac{7}{x}+6\tan^{-1}x+\frac{6x}{1+x^2}+c$$

Sol 34: $\int\frac{dx}{\cos^3 x-\sin^3 x}$

$$\int\frac{dx}{(\cos x-\sin x)(1+\cos x\sin x)}$$

$$=\int\frac{(\cos x-\sin x)^2+2\sin x\cos x}{(\cos x-\sin x)(1+\cos x\sin x)}dx$$

$$\int\frac{(\cos x-\sin x)}{(1+\cos x\sin x)}dx+2\int\frac{\sin x\cos x}{(\cos x-\sin x)(1+\cos x\sin x)}dx$$

$$=\frac{2}{3}\int\frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}dx+\frac{2}{3}\int\frac{(\cos x-\sin x)}{2-(\sin x+\cos x)^2}dx$$

$$=\frac{2}{3}\tan^{-1}(\sin x+\cos x)+\frac{1}{3\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin x+\cos x}{\sqrt{2}-\sin x-\cos x}\right|+c$$

Sol 35: $\int\frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}dx$

$$\int\frac{(x\cos x-\sin x)^2+(x\sin x+\cos x)^2}{(x\cos x-\sin x)(x\sin x+\cos x)}dx$$

$$\int\frac{x\cos x-\sin x}{x\sin x+\cos x}dx+\int\frac{x\sin x+\cos x}{x\cos x-\sin x}dx$$

$$+\int\frac{-1}{(x\cos x-\sin x)(x\sin x+\cos x)}dx$$

$$=\int\left(\frac{x\cos x+\sin x-\sin x}{x\sin x+\cos x}\right)dx+\int\frac{x\sin x+\sin x-\cos x}{x\cos x-\sin x}dx$$

$$+\int\frac{\cos^2 x+\sin^2 x-1}{(x\sin x+\cos x)(x\cos x-\sin x)}dx$$

$$=\log\left|\frac{x\sin x+\cos x}{x\cos x-\sin x}\right|+c$$

Sol 36: $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

$\Rightarrow \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\int \cos 2\theta d\theta$

$-\int\left(\frac{d\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)}{d\theta}\int \cos 2\theta d\theta\right)d\theta$

$\Rightarrow \frac{1}{2}\sin 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\left(\frac{2}{\cos 2\theta}\times\frac{\sin 2\theta}{2}\right)d\theta$

$\Rightarrow \frac{1}{2}\sin 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int \tan 2\theta d\theta$

$\Rightarrow \frac{1}{2}\sin 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\log(\sec 2\theta)d\theta$

$=\left(\frac{1-\tan\theta}{1+\tan\theta}\right)\left(\frac{(1-\tan\theta\times\sec^2\theta}{(1-\tan\theta)^2}-(1+\tan\theta)\sec^2\theta\right)$

$=\frac{(1-\tan\theta)}{(1-\tan\theta)}\times\frac{2\sec^2\theta}{(1-\tan\theta)^2}=\frac{2(1+\tan^2\theta)}{(1-\tan^2\theta)}=\frac{2}{\cos 2\theta}$

Sol 37: A → s; B → p; C → q; D → r

(A) $\int \frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}+1}}dx$

$\int \frac{x-x^{-3}}{\sqrt{x^2+\frac{1}{x^2}+1}}dx$

Put $x^2+\frac{1}{x^2}+1=t$

$\left(2x-\frac{2}{x^3}\right)dx=\frac{dt}{2}$ or $\left(x-\frac{1}{x^3}\right)dx=\frac{dt}{2}$

$\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\sqrt{t}+c = \sqrt{x^2+\frac{1}{x^2}+1}+c = \sqrt{\frac{x^4+x^2+1}{x}}+c$

(B) $\int \frac{x^2-1}{x\sqrt{1+x^4}}dx$

$\int \frac{x^2-1}{x^2\sqrt{x^2+\frac{1}{x^2}}}dx \Rightarrow \int\frac{(1-x^{-2})}{\sqrt{\left(x+\frac{1}{x}\right)^2-2}}dx$

Put $x+\frac{1}{x}=t$

$\Rightarrow (1-x^{-2})dx=dt$

$\int\frac{dt}{\sqrt{t^2-2}}$

Put $t=\sqrt{2}\sec\theta$

$\int\frac{\sqrt{2}\sec\theta\tan\theta d\theta}{\sqrt{2}\tan\theta}=\log|\sec\theta+\tan\theta|$

$=\log\left|\frac{x^2+1}{\sqrt{2}x}+\frac{\sqrt{x^4+1}}{\sqrt{2}x}\right|+c$

Or $\log\left|\frac{x^2+1}{x}+\frac{\sqrt{x^4+1}}{x}\right|+c$

(C) $\int\frac{(1+x^2)}{x^2\left(\frac{1}{x}-x\right)\sqrt{x^2+\frac{1}{x^2}}}dx$

$\int\frac{(1+x^{-2})}{\left(\frac{1}{x}-x\right)\sqrt{\left(x-\frac{1}{x}\right)^2+2}}dx$

$x-\frac{1}{x}=t \Rightarrow (1+x^{-2})dx=dt$

$=-\int\frac{dt}{t\sqrt{t^2+2}}$

$t=\sqrt{2}\tan\theta$

$dt=\sqrt{2}\sec^2\theta\, d\theta$

$=-\int\frac{\sqrt{2}\sec^2\theta d\theta}{\sqrt{2}\tan\theta\sqrt{2}\sec\theta}-\frac{1}{\sqrt{2}}\int \operatorname{cosec} d\theta$

$=\frac{1}{\sqrt{2}}\log|\operatorname{cosec}\theta-\cot\theta|+c$

$\Rightarrow -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{x^4+1}-\sqrt{x}}{(x^2-1)}\right|+c$

(D) $\int\frac{\sqrt{\sqrt{1+x^4}+x^2}}{1+x^4}dx$

$=\int\left(\frac{1}{\sqrt{1+\frac{1}{x^4}}}\right)\frac{1}{2\sqrt{\left(\sqrt{1+\frac{1}{x^4}}-1\right)}}\times\frac{1}{2\sqrt{1+\frac{1}{x^4}}}\times\left(\frac{-4}{x^5}\right)$

$-\tan^{-1}\left(\sqrt{\sqrt{1+\frac{1}{x^4}}-1}\right)+c$

$\because \dfrac{d\tan^{-1}\left(\sqrt{\sqrt{1+\frac{1}{x^4}}-1}\right)}{dx}$

$= \dfrac{1}{1+\left(\sqrt{1+\frac{1}{x^4}}-1\right)}\times\left(\sqrt{\sqrt{1+\frac{1}{x^4}}-1}\right)$

$= \left(\dfrac{1}{\sqrt{1+\frac{x}{4}}}\right)\left(\dfrac{1}{2\sqrt{1\quad+\frac{1}{x^4}-1}}\right)\dfrac{1}{2\sqrt{1+\frac{1}{x^4}}}\times\dfrac{-4}{x^5}$

Exercise 2

Single Correct Choice Type

Sol 1: (D) $\int\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}dx$

$\because\ \tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}$

$I=\int\left(\dfrac{2\tan^{-1}\frac{x}{2}-\frac{\pi}{2}}{\frac{\pi}{2}}\right)dx = \dfrac{4}{\pi}\int\tan^{-1}x\,dx-\int dx$

$= \dfrac{4}{\pi}\int\tan^{-1}x\int dx-\int\left[\left(\left(\dfrac{d\tan^{-1}x}{dx}\right)\int dx\right)dx\right]-x+c$

$= \dfrac{4}{\pi}x\tan^{-1}x-\dfrac{4}{x}\int\left(\dfrac{x}{1+x^2}\right)dx-x+c$

$= \dfrac{4}{\pi}x\tan^{-1}x-\dfrac{2}{\pi}\log(1+x^2)-x+c$

Sol 2: (A) $\int\dfrac{x^2-4}{x^4+24x^2+16}dx = \int\dfrac{x^2-4}{x^2\left(x^2+24+\frac{16}{x^2}\right)}dx$

$= \int\dfrac{\left(1-\frac{4}{x^2}\right)}{\left(x^2+\frac{16}{x^2}+8\right)+16}dx = \int\dfrac{\left(1-\frac{4}{x^2}\right)}{\left(x+\frac{4}{x}\right)^2+(4)^2}dx$

Put $x+\dfrac{4}{x}=t$

$\Rightarrow\left(1-\dfrac{4}{x^2}\right)dx = dt$

$\therefore\int\dfrac{dt}{t^2(4)^2} = \dfrac{1}{4}\tan^{-1}\dfrac{t}{4}+c$

$\therefore I=\dfrac{1}{4}\tan^{-1}\left(\dfrac{x+\frac{4}{x}}{4}\right)+c$

Or $I=\dfrac{1}{4}\tan^{-1}\left(\dfrac{x^2+4}{4x}\right)+c$

Sol 3: (D) $\int\dfrac{(x-1)^2}{x^4+2x^2+1}dx = \int\dfrac{x^2-1-2x+2}{(x^2+1)^2}dx$

$= \int\dfrac{(x^2-1)}{x^2\left(x+\frac{1}{x}\right)^2}dx + \int\dfrac{-2x}{(x^2+1)^2}dx+\int\dfrac{2dx}{(1+x^2)^2}$

Put $x^2+1=t$ and Put $x=\tan\theta$

$dx=\sec^2\theta d\theta$

$= \int\dfrac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2}dx+\int-\dfrac{dt}{t^2}+2\int\dfrac{\sec^2\theta d\theta}{\sec^4\theta}$

Put $x+\dfrac{1}{x}=u$

$= \int\dfrac{du}{u^2}+\dfrac{1}{x^2+1}+2\int\cos^2\theta d\theta$

$= -\dfrac{1}{\left(x+\frac{1}{x}\right)}+\dfrac{1}{(x^2+1)}+\dfrac{2}{2}\int(\cos2\theta+1)d\theta$

$= \dfrac{(1-x)}{x^2+1}+\dfrac{1}{2}\sin2\theta+\theta+c$

$= \dfrac{(1-x)}{1+x^2}+\dfrac{1}{2}\dfrac{2\times x}{1+x^2}+\tan^{-1}x+c$

$= \dfrac{(1-x)}{1+x^2}+\dfrac{x}{1+x^2}+\tan^{-1}x+c = \dfrac{1}{1+x^2}+\tan^{-1}x+c$

Sol 4: (A) $\int\dfrac{x^4-4}{x^2\sqrt{x^4+x^2+4}}dx$

$\int\dfrac{x^4-4}{x^2\times x\sqrt{x^2+1+\frac{4}{x^2}}}dx$

$$\int \frac{\left[x - \frac{4}{x^3}\right]}{\sqrt{x^2 + 1 + \frac{4}{x^2}}} dx$$

Put $x^2 + \frac{4}{x^2} + 1 = t \Rightarrow \left(2x - \frac{8}{x^3}\right)dx = dt$

$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}.2\sqrt{t} + c = \sqrt{x^2 + 1 + \frac{4}{x^2}} + c$

$= \frac{\left(\sqrt{x^4 + x^2 + 4}\right)}{x} + c$

Sol 5: (A) $\int \left(\frac{\sec x + \tan x - 1}{\tan x - \sec x + 1}\right) dx$

$= \int \frac{\sec x + \tan x - (\sec^2 x - \tan^2 x)}{(\tan x - \sec x + 1)} dx$

$= \int \frac{(\sec x + \tan x)[1 - \sec x + \tan x]}{[\tan x - \sec x + 1]} dx$

$= \int (\sec x + \tan x) dx$

= log|secx + tanx| + logsecx + c

$s = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots$

$s_x = x + 2x^2 + 3x^3 + \ldots\ldots\ldots\ldots$

$s(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$

$s(1 - x) = \frac{1}{1 - x} \quad \therefore s = \frac{1}{(1 - x)^2}$

$\therefore \int \frac{1}{(1 - x)^2} dx = \frac{(1 - x)^{-1}}{-1 \times -1} + c = (1 - x)^{-1} + c$

Sol 6: (C) $\int e^x \frac{(x^2 - 3)}{(x + 3)^2} dx = \int e^x \frac{(x^2 - 9 + 6)}{(x + 3)^2} dx$

$= \int e^x \left(\frac{x - 3}{x + 3} + \frac{6}{(x + 3)^2}\right) dx$

$= \int e^x \left(f(x) + f'(x)\right) dx = e^x f(x) + c$

$= e^x \frac{(x - 3)}{(x + 3)} + c$

Sol 7: (B) $I = \int \sqrt{\frac{1 - \cos x}{\cos \alpha - \cos x}} dx$

$I = \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{\alpha}{2} - 2\cos^2\frac{x}{2}}}$

$= \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2}\cos\frac{\alpha}{2}\sqrt{1 - \left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)^2}} dx$

$= \int \frac{\frac{1}{2} \times 2x\left(\frac{-\sin\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)}{\sqrt{1 - \left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)^2}} dx$

Let $\frac{\cos\frac{x}{3}}{\cos\frac{\alpha}{2}} = t \quad \therefore I = \int -\frac{2}{\sqrt{1 - t^2}} dt$

$= 2\cos^{-1}(t) + c = 2\cos^{-1}\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right) + c$

Sol 8: (B) $I = \int \frac{3x^4 - 1}{(x^4 + x + 1)^2} dx$

$= \int \frac{4x^4 + x - (x^4 + x + 1)}{(x^4 + x + 1)^2} dx$

$= \int \frac{x(4x^2 - 1)}{(x^4 + x + 1)^2} - \frac{1}{(x^4 + x + 1)} dx$

Let $\int \frac{1}{x^4 + x + 1} = I_1$

$\therefore I = x\int \frac{(4x^3 + 1)}{(x^4 + x + 1)^2} - \int\int \frac{4x^3 + 1}{(x^4 + x + 1)^2} dx - I_1$

$= x \times \frac{(x^4 + x + 1)^{-1}}{-1} - \int \frac{(x^4 + x + 1)^{-1}}{-1} dx - I_1$

$= \frac{-x}{(x^4 + x + 1)} + c + I_1 - I_1$

Sol 9: (C) $\int e^{3x} \cos 4x dx$

Let 3x =t

$\therefore I = \frac{1}{3}\int e^t \cos\frac{4t}{3}dt$

$= \frac{1}{3}\int e^t \cos\frac{4t}{3}dt = \frac{1}{3}\int e^t\left(\cos\frac{4t}{3} - \frac{4}{3}\sin\frac{4t}{3}\right)dt$

$+ \frac{4}{9}\int e^t\left(\sin\frac{4\pi}{3}\right)dt$

$= \frac{1}{3}\int\left[e^t\cos\frac{4t}{3}\right] + \frac{4}{9}\int e^t\left(\sin\frac{4t}{3} + \frac{4}{3}\cos\frac{4t}{3}\right)dt$

$-\int\frac{16}{27}e^t\cos\frac{4t}{3}dt$

$\frac{25}{9}I = \frac{1}{3}e^t\cos\frac{4t}{3} + \frac{4}{9}e^t\sin\frac{4t}{3} + c$

$I = \frac{e^t}{25}\left(3\frac{4t}{3} + 4\sin\frac{4t}{3}\right) + c$

$\therefore 3A = 4B$

Sol 10: (C)

$I = \int\frac{p + x^{p+2q-1} + qx^{p+2q-1} - q(x^{q-1} + x^{p+2q-1})}{(x^{p+q}+1)^2}dx$

$= \int\frac{(p+q)x^{p+q-1}x^q - qx^{q-1}(x^{p+q}) + 1}{(x^{p+q}+1)^2}dx$

It is of the form $\frac{uv' - vu'}{u^2}$

$\therefore$ Where $u = x^{p+q} + 1$ and $v = -x^q$

$\therefore I = \frac{v}{u} + c = \frac{-x^q}{x^{p+q}+1} + c$

Sol 11: (B) Let $f^{-1}(x) = t$

$\therefore f(f^{-1}(x)) = x$

$\therefore f(t) = x$

$\therefore \int f^{-1}(x)dx$

$\int t\,dx = \int tf'(t)dt$

$= tf(t) - \int f(t)dt = tf(t) - g(t)$

$= f^{-1}(x)\ (x - g)(f^{-1}(x))$

Sol 12: (B) $\left(\frac{x}{(x^4-1)^4}\right)^{1/3} dx$

$I = \int\left(\frac{x}{(x^4-1)^4}\right)^{1/3}dx = \int\left(\frac{x}{x^{16}\left(1-\frac{1}{x^4}\right)^4}\right)^{1/3}dx$

$= \int\frac{1}{\left(1-\frac{1}{x^4}\right)^{4/3}}\frac{1}{x^5}dx$

Let $1 - \frac{1}{x^4} = t \quad \therefore \frac{4}{x^5}dx = dt$

$\therefore I = \int\frac{1}{t^{4/3}}\times\frac{1}{4}dt = \frac{1}{4}\times\frac{t^{-1/3}}{-\frac{1}{3}} + c = -\frac{3}{4}t^{-1/3} + c$

$= -\frac{3}{4}\left(1-\frac{1}{x^4}\right)^{-1/3} + c = -\frac{3}{4}\left(\frac{x^4}{x^4-1}\right)^{1/3} + c$

Sol 13: (A) $I = \int e^u \sin 2x dx$

When u = x

$I = \int e^x \sin 2x dx = \int e^x(\sin 2x + 2\cos 2x - 2\cos 2x)dx$

$= \int e^x(\sin 2x + 2\cos 2x) - 2\int e^x\cos 2x$

$= \int e^x(\sin 2x + 2\cos 2x)$

$-2\int e^x(\cos 2x - 2\sin 2x + 2\sin 2x)dx$

$\therefore 5I = \int e^x(\sin 2x + 2\cos 2x)dx$

$-2\int e^x(\cos 2x - 2\sin 2x)dx$

$= e^x\sin 2x - 2e^x\cos 2x + c$

When u = sinx

$I = \int 2e^{\sin x}\cos x\sin x dx$

Put sinx = t $\therefore$ cosxdx = dt

$\therefore I = 2\int te^t dt$ which is solvable

Previous Years' Questions

Sol 1: (C) Let $I = \int\frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx$

$= \int\frac{(\cos^2 x + \cos^4)\cdot\cos x\,dx}{(\sin^2 x + \sin^4 x)}$

Put sinx = t $\Rightarrow$ cosx dx = dt

$$\therefore\ I=\int\frac{[(1-t^2)+(1-t^2)^2]}{t^2+t^4}dt$$

$$\Rightarrow I=\int\frac{1-t^2+1-2t^2+t^4}{t^2+t^4}dt$$

$$\Rightarrow I=\int\frac{2-3t^2+t^4}{t^2(t^2+1)}dt. \qquad ...(i)$$

Using partial fraction for,

$$\frac{y^2-3y+2}{y(y+1)}=1+\frac{A}{y}+\frac{B}{y+1}\ (\text{ where } y=t^2)$$

$\Rightarrow$ A = 2, B = –6

$$\therefore \frac{y^2-3y+2}{y(y+1)}=1+\frac{2}{y}-\frac{6}{y+1}$$

$\therefore$ Eq. (i) reduces to,

$$I=\int\left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)dt = t-\frac{2}{t}-6\tan^{-1}(t)+c$$

$$=\sin x-\frac{2}{\sin x}-6\tan^{-1}(\sin x)+c$$

Sol 2: (A) Given, $f(x)=\dfrac{x}{(1+x^n)^{1/n}}$ for $n\ge 2$

$$\therefore ff(x)=\frac{f(x)}{[1+f(x)^n]^{1/n}}=\frac{x}{(1+2x^n)^{1/n}}$$

and $fff(x)=\dfrac{x}{(1+3x^n)^{1/n}}$

$$\therefore g(x)=\underbrace{fofo...of}_{n\text{ times}}(x)=\frac{x}{(1+nx^n)^{1/n}}$$

$$\text{Let } I=\int x^{n-2}g(x)dx=\int\frac{x^{n-1}dx}{(1+nx^n)^{1/n}}=\frac{1}{n^2}\int\frac{n^2x^{n-1}dx}{(1+nx^n)^{1/n}}$$

$$=\frac{1}{n^2}\int\frac{\frac{d}{dx}(1+nx^n)}{(1+nx^n)^{1/n}}dx$$

$$\therefore I=\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+c$$

Sol 3: (C) Since, $I=\displaystyle\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$

$$J=\int\frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$$

$$\therefore J-I=\int\frac{(e^{3x}-e^x)}{1+e^{2x}+e^{4x}}dx$$

Put $e^x=u \Rightarrow e^x dx=du$

$$\therefore J-I=\int\frac{(u^2-1)}{1+u^2+u^4}du$$

$$=\int\frac{\left(1-\frac{1}{u^2}\right)}{1+\frac{1}{u^2}+u^2}du=\int\frac{\left(1-\frac{1}{u^2}\right)}{\left(u+\frac{1}{u}\right)^2-1}du$$

Put $u+\dfrac{1}{u}=t$

$$\Rightarrow\left(1-\frac{1}{u^2}\right)du=dt=\int\frac{dt}{t^2-1}$$

$$=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+c=\frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right|+c$$

$$=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+c$$

Sol 4: Given, $f(x)=\displaystyle\int\left(\frac{2\sin x-\sin 2x}{x^3}\right)dx$

On differentiating w.r.t. x, we get

$$f'(x)=\frac{2\sin x-\sin 2x}{x^3}=\frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)$$

$$\lim_{x\to 0}f'(x)=\lim_{x\to 0}2\left(\frac{\sin x}{x}\right)\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$

$$=4\cdot 1\cdot\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{4\times\left(\frac{x^2}{2}\right)}\right)=1$$

Sol 5: (i) Let $I=\displaystyle\int\sqrt{1+\sin\frac{x}{2}}dx$

$$\Rightarrow I=\int\sqrt{\cos^2\frac{x}{4}+\sin^2\frac{x}{4}+2\sin\frac{x}{4}\cos\frac{x}{4}}dx$$

$$\Rightarrow I=\int\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)dx$$

$$=4\sin\frac{x}{4}-4\cos\frac{x}{4}+c$$

(ii) Let $I = \int \frac{x^2}{\sqrt{1-x}} dx$

Put $1 - x = t^2 \Rightarrow -dx = 2t\, dt$

$\therefore I = \int \frac{(1-t^2)^2 \cdot (-2t)}{t} dt$

$= -2\int (1 - 2t^2 + t^4)\, dt = -2\left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right) + c$

$= -2\left\{\sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2}\right\}$

Sol 6: Let $I = \int (e^{\log x} + \sin x)\cos x\, dx$

$= \int (x + \sin x)\cos x\, dx$

$\therefore I = \int x\cos x\, dx + \frac{1}{2}\int (\sin 2x)\, dx$

$= (x \cdot \sin x - \int 1 \cdot \sin x\, dx) - \frac{\cos 2x}{4} + c$

$= x\sin x + \cos x - \frac{\cos 2x}{4} + c$

Sol 7: Let $I = \int \frac{(x-1)e^x}{(x+1)^3} dx$

$\Rightarrow I = \int \left\{\frac{x+1-2}{(x+1)^3}\right\} e^x\, dx$

$= \int \left\{\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right\} e^x\, dx$

$= \int e^x \cdot \frac{1}{(x+1)^2} dx - 2\int e^x \cdot \frac{1}{(x+1)^3} dx$

Applying integration by parts,

$= \left\{\frac{1}{(x+1)^2} \cdot e^x - \int e^x \cdot \frac{-2}{(x+1)^3} dx\right\}$

$-2\int e^x \cdot \frac{1}{(x+1)^3} dx = \frac{e^x}{(x+1)^2} + c$

Sol 8: Let $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^2 \cdot x^3\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + x^{-4} = t \Rightarrow -\frac{4}{x^5} dx = dt$

$\therefore I = -\frac{1}{4}\int \frac{dt}{t^{3/4}} = -\frac{1}{4} \cdot \frac{t^{1/4}}{1/4} + c$

$= -\left(1 + \frac{1}{x^4}\right)^{1/4} + c = -\frac{(x^4+1)^{1/4}}{x} + c$

Sol 9: Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$

$\therefore I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot (-2\sin\theta\cos\theta) d\theta$

$= -\int 2\tan\frac{\theta}{2} \cdot \sin\theta\cos\theta\, d\theta = -2\int 2\sin^2\frac{\theta}{2} \cdot \cos\theta\, d\theta$

$= -2\int (1 - \cos\theta)\cos\theta\, d\theta = -2\int (\cos\theta - \cos^2\theta)\, d\theta$

$= -2\int \cos\theta\, d\theta + \int (1 + \cos 2\theta)\, d\theta$

$= -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + c$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + c$

Sol 10: Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$

$= \frac{2}{\pi}\int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx$

$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + c$ ...(i)

Now, $\int \sin^{-1}\sqrt{x}\, dx$

Put $x = \sin^2\theta \Rightarrow dx = \sin 2\theta$

$= \int \theta \cdot \sin 2\theta\, d\theta$

$= -\frac{\theta\cos 2\theta}{2} + \int \frac{1}{2}\cos 2\theta\, d\theta$

$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta$

$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1 - \sin^2\theta}$

$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x}$ ...(ii)

From Eqs. (i) and (ii), we get

$$I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right] - x + c$$

$$= \frac{2}{\pi}\left[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right] - x + c$$

Sol 11: Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin^2 x}\,dx = \int \sqrt{\cot^2 x - 1}\,dx$$

Put $\cot x = \sec\theta \Rightarrow -\text{cosec}^2 x dx = \sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \sqrt{\sec^2\theta - 1}\cdot\frac{\sec\theta\cdot\tan\theta}{-(1+\sec^2\theta)}\,d\theta$$

$$= -\int \frac{\sec\theta\cdot\tan^2\theta}{1+\sec^2\theta}\,d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}\,d\theta$$

$$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int \frac{(1+\cos^2\theta) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}\,d\theta$$

$$= -\log|\sec\theta + \tan\theta| + 2\int \frac{\cos\theta}{2-\sin^2\theta}\,d\theta$$

$$= -\log|\sec\theta + \tan\theta| + \int \frac{dt}{2-t^2},\ \text{where } \sin\theta = t$$

$$= -\log|\sec\theta + \tan\theta| + 2\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c$$

$$= -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + c$$

Sol 12: Let $I = \int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$\therefore I = I_1 + I_2$,

where $I_1 = \int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}\right)dx$, $I_2 = \int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\,dx$

Now, $I_1 = \int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}\right)dx$

Put $x = t^{12} \Rightarrow dx = 12t^{11}dt$

$$\therefore I_1 = 12\int \frac{t^{11}}{t^4+t^3}\,dt = 12\int \frac{t^8 dt}{t+1}$$

$$= 12\int (t^7 - t^6 + t^5 - t^4 + t^3 - t^2 + t - 1)dt + 12\int \frac{dt}{t+1}$$

$$= 12\left(\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - 1\right) + 12\log(t+1)$$

And $I_2 = \int\left\{\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right\}dx$

Put $x = u^6 \Rightarrow dx = 6u^5\, du$

$$\therefore I_2 = \int \frac{\log(1+u)}{u^2+u^3}6u^5\, du = \int \frac{\log(1+u)}{u^2(1+u)}\cdot 6u^5\, du$$

$$6\int \frac{u^3}{(u+1)}\log(1+u)du = 6\int\left(\frac{u^3-1+1}{u+1}\right)\log(1+u)du$$

$$= 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)\log(1+u)du$$

$$= 6\int \underset{\text{II}}{(u^2 - u + 1)}\underset{\text{I}}{\log(1+u)}du - 6\int \frac{\log(1+u)}{(u+1)}du$$

$$= 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right)\log(1+u) - \int \frac{2u^3 - 3u^2 + 6u}{u+1}du - 6\frac{1}{2}[\log(1+u)]^2$$

$= (2u^3 - 3u^2 + 6u)\log(1+u)$

$$-\int\left(2u^2 - 5u + \frac{11u}{u+1}\right)du - 3[\log(1+u)]^2$$

$= (2u^3 - 3u^2 + 6u)\log(1+u)$

$$= -\left(\frac{2u^3}{3} - \frac{5}{2}u^2 + 11u - 11\log(u+1)\right) - 3[\log(1+u)]^2$$

$$\therefore I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} - 4x^{1/4}$$

$- 6x^{1/6} - 12x^{1/12} + 12\log(x^{1/12}+1)$

$+ (2x^{1/2} - 3x^{1/3} + 6x^{1/6}1^{111/1})\log(1+x^{1/6})$

$$-\left[\frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3}11x^{1/6} - 11\log(1+x^{1/6})\right]$$

$- 3 [\log (1 + x^{1/6})]^2 + c$

$$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12}$$

$$+\frac{1}{2}x^{1/3} - 4x^{1/4} - 7x^{1/6} - 12x^{1/12}$$

$$+(2x^{1/2} - 3x^{1/3} - 6x^{1/6} + 11)\log(1 + x^{1/6})$$

$$+ 12\log(1 + x^{1/2}) - 3[\log(1 + x^{1/6})]^2 + c$$

Sol 13: $\frac{x^3 + 3x + 2}{(x^2+1)^2(x+1)} = \frac{x^3 + 2x + x + 2}{(x^2+1)^2(x+1)}$

$$= \frac{x(x^2+1) + 2(x+1)}{(x^2+1)^2(x+1)} = \frac{x}{(x^2+1)(x+1)} + \frac{2}{(x^2+1)^2}$$

Again, $\frac{x}{(x^2+1)(x+1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+1)}$

$\Rightarrow x = (Ax + B)(x + 1) + C(x^2 + 1)$

Putting $x = -1$, we get $-1 = 2C \Rightarrow C = -1/2$

Equation coefficient of x^2, we get

$0 = A + C \Rightarrow A = -C = 1/2$

Putting $x = 0$, we obtain

$0 = B + C \Rightarrow B = -C = 1/2$

$$\frac{x^3 + 3x + 2}{(x^2+1)^2(x+1)} = \frac{x+1}{2(x^2+1)} - \frac{1}{2(x+1)} + \frac{2}{(x^2+1)^2}$$

$$\therefore I = \int \frac{x^3 + 3x + 2}{(x^2+1)^2(x+1)}\,dx$$

$$= -\frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{x+1}{x^2+1}dx + 2\int \frac{dx}{(x^2+1)^2}$$

$$\Rightarrow I = -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + 2I_1 \quad ...(i)$$

where $I_1 = \int \frac{dx}{(x^2+1)^2}$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore I_1 = \int \frac{\sec^2\theta d\theta}{(\tan^2\theta + 1)^2} = \int \cos^2\theta d\theta = \frac{1}{2}\int (1 + \cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{\tan\theta}{(1+\tan^2\theta)}$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{(1+x^2)}$$

$\therefore$ From Eq. (i)

$$I = -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+1| + \frac{3}{2}\tan^{-1}x + \frac{x}{x^2+1} + c$$

Sol 14: Let $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)dx$$

Put $2x + 2 = 3\tan\theta \Rightarrow 2dx = 3\sec^2\theta d\theta$

$$\therefore I = \int \sin^{-1}\left(\frac{3\tan\theta}{\sqrt{9\tan^2\theta + 9}}\right)\cdot\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta\cdot\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{3}{2}\int \sin^{-1}(\sin\theta)\cdot\sec^2\theta\, d\theta$$

$$= \frac{3}{2}\int \theta\cdot\sec^2\theta\, d\theta = \frac{3}{2}[\theta\cdot\tan\theta - \int 1\cdot\tan\theta\, d\theta]$$

$$= \frac{3}{2}[\theta\tan\theta - \log\sec\theta] + c$$

$$= \frac{3}{2}\left[\tan^{-1}\left(\frac{2x+2}{3}\right)\cdot\left(\frac{2x+2}{3}\right) - \log\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right] + c_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log\left(1+\left(\frac{2x+2}{3}\right)^2\right) + c_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + c$$

$$\left(\text{let } \frac{3}{2}\log 3 + c_1 = c\right)$$

Sol 15: For any natural number m, the given integral can be written as,

$$I = \int (x^{3m} + x^{2m} + x^m)(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}\,dx$$

$$\Rightarrow I = \int (2x^{3m} + 3x^{2m} + 6x^m)^{1/m}(x^{3m-1} + x^{2m-1} + x^{m-1})\,dx$$

Put $2x^{3m} + 3x^{2m} + 6x^m = t$

$\Rightarrow (6mx^{3m-1} + 6mx^{2m-1} + 6mx^{m-1})dx = dt$

$$\therefore I = \int t^{1/m}\frac{dt}{6m} = \frac{1}{6m}\cdot\frac{t^{\frac{1}{m}+1}}{\left(\frac{1}{m}+1\right)}$$

$$= \frac{1}{6(m+1)}\cdot(2x^{3m}+3x^{2m}+6x^{m})^{\frac{(m+1)}{m}}+c$$

Sol 16: (C)

$$J-I = \int\frac{e^x\left(e^{2x}-1\right)}{e^{4x}+e^{2x}+1}dx = \int\frac{\left(z^2-1\right)}{z^4+z+1}dz \text{ where } z=e^x$$

$$J-I = \int\frac{\left(1-\frac{1}{z^2}\right)dx}{\left(z+\frac{1}{z}\right)-1} = \frac{1}{2}\log\left(\frac{e^x+e^{-x}-1}{e^x+e^{-x}+1}\right)$$

$$\therefore\ J-I = \frac{1}{2}\log\left(\frac{e^x+e^{-x}-1}{e^x+e^{-x}+1}\right)+c.$$

Sol 17: (C) $I = \int\frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$

Let $\sec x+\tan x = t$

$\Rightarrow \sec x-\tan x = 1/t$

$\sec x(\sec x+\tan x)dx = dt$ S

$\sec x\,dx = \frac{dt}{t}, \frac{1}{2}\left(t+\frac{1}{t}\right) = \sec x$

$$I = \frac{1}{2}\int\frac{\left(t+\frac{1}{t}\right)}{t^{9/2}}\frac{dt}{t} = \frac{1}{2}\int\left(t^{-9/2}+t^{-12/2}\right)dt$$

$$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1}\frac{t^{-13/2+1}}{-\frac{13}{2}2+1}\right]$$

$$= \frac{1}{2}\left[\frac{t^{-7/2}}{-\frac{7}{2}}+\frac{t^{-11/2}}{-\frac{11}{2}}\right]$$

$$= -\frac{1}{7}t^{-7/2}-\frac{1}{11}t^{-11/2}$$

$$= -\frac{1}{7}\frac{1}{t^{7/2}}-\frac{1}{11}\frac{1}{t^{11/2}}$$

$$= -\frac{1}{r^{11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right)$$

$$= -\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+k.$$