PROBLEM-SOLVING TACTICS

(a) Let S = 0, S′ = 0 be two circles with centers C_1, C_2 and radii R_1, R_2 respectively.

(i) If $C_1C_2 > r_1 + r_2$ then each circle lies completely outside the other circle.

(ii) If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally. (Trick) the point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally.

(iii) If $|r_1 - r_2| < C_1C_2 < r1 + r2$ then the two circles intersect at two points P and Q.

(iv) If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally. (Trick) The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ externally.

(v) If $C_1C_2 < |r_1 - r_2|$ then one circle lies completely inside the other circle.

(b) Two intersecting circles are said to cut each other orthogonally if the angle between the circles is a right angle. Let the circles be $S = x^2 + y^2 + 2gx + 2fy + c = 0$, $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$.

And let d be the distance between the centers of two intersecting circles with radii r1, r2. The two circles will intersect orthogonally if and only if

(i) $D^2 =$ and

(ii) $2g\,g' + 2f\,f' = c + c'$.

FORMULAE SHEET

1. General equation of a circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) Centre of the circle = $(-g, -f)$.

$g = \frac{1}{2}$ coefficient of x, and $f = \frac{1}{2}$ coefficient of y.

(ii) $r = \sqrt{g^2 + f^2 - c}$

2 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

(i) $a = b \neq 0$ (ii) $h = 0$ (iii) $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$ (iv) $g^2 + f^2 - c \geq 0$

3. if centre of circle is (h, k) and radius 'r' then equation of circle is: $(x - h)^2 + (y - k)^2 = r^2$

4. The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is : $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Centre : $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. $r = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$

5. (i) In parametric form:

$x = -g + \sqrt{(g^2 + f^2 - c)}\cos\theta$ and $y = -f + \sqrt{(g^2 + f^2 - c)}\sin\theta$, $(0 \leq \theta < 2\pi)$

6. (i) Circle passing through three non-collinear points

$A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is represented by $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

7. Circle circumscribing the triangle formed by the lines

$a_i x + b_i y + c_i = 0 \ (i = 1,2,3)$: $\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1 x + b_1 y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2 x + b_2 y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3 x + b_3 y + c_3} & a_3 & b_3 \end{vmatrix} = 0$

8. Intercepts length made by the circle On X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

9. Position of point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

When $S_1 \equiv x_1^2 + y_1^2 + 2gx + 2fy + c > \ = \ < 0$ respectively.

10. The power of $P(x_1, y_1)$ w.r.t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA. PB which is $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_2 + c$.

PA. PB = PC.PD = PT^2 = square of the length of a tangent

11. Intercept length cut off from the line y =mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2\left(1+m^2\right)-c^2}{1+m^2}}$

12. The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

13. The equation of tangent at $(a \cos\theta, a \sin\theta)$ is $x \cos\theta + y \sin\theta = a$

14. Condition for tangency:

line y = mx + c is tangent of the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

and the point of contact of tangent $y = mx \pm a\sqrt{1+m^2}$ is $\left(\frac{\mp ma}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$

15. The length of the tangent from a point $P(x_1,y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

16. Pair of tangent from point (0, 0) to the circle are at right angles if $g^2 + f^2 = 2c$.

17. Equation of director circle of the circle $x^2 + y^2 = a^2$ is equal to $x^2 + y^2 = 2a^2$.

18. Equation of Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

19. The equation of normal at any point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - x_1y = 0$ or $\frac{x}{x_1} = \frac{y}{y_1}$.

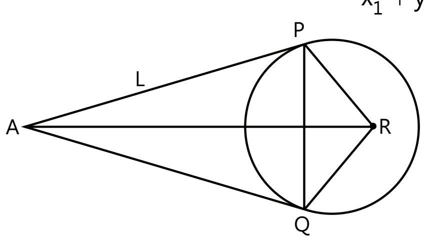
20. Equation of normal at $(a \cos\theta, a \sin\theta)$ is $y = x \tan\theta$ or $y = mx$.

21. The equation of the chord of contact of tangents drawn from a point (x_1, y_1)

to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_2 = a^2$. And

to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

22. Area of Δ APQ is given by $\frac{a\left(x_1^2+y_1^2-a^2\right)^{\frac{3}{2}}}{x_1^2+y_1^2} = \frac{RL^3}{R^2+L^2}$. Where L & R are length of tangent and radius of circle.



23. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Bisected at the point (x_1, y_1) is given $T = S_1$.

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

24. The equation of the common chord of two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is equal to $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ i.e., $S_1 - S_2 = 0$.

25. Length of the common chord : PQ = 2(PM) = $2\sqrt{C_1P^2 - C_1M^2}$. Where,

C_1P = radius of the circle $S_1 = 0$

C_1M =perpendicular length from the centre C_1 to the common chord PQ.

26. Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 = a^2$

w.r.t. (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ and $xx_1 + yy_1 - a^2 = 0$. Respectively.

27. The pole of the line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$: $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

28. P (x_1, y_1) and $Q(x_2, y_2)$ are conjugate points of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

When $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

If P and Q are conjugate points w.r.t. a circle with centre at O and radius r then $PQ^2 = OP^2 + OQ - 2r^2$.

29. The points P and T are a intersection point of direct common tangents and transverse. Common tangents respectively, and it divide line joining the centres of the circles externally and internally respectively in the ratio of their radii.

$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally)}$$

$$\frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of P and T are.

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \text{ and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

30. If two circles $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ of r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos(180 - \theta) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \text{or} \quad \cos = (180 - \theta) = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_1^2 + f_1^2 - c_2}}.$$

31. Condition for orthogonality: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

32. $S_1 - S_2 = 0$ the equation of the radical axis of the two circle. i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ which is a straight line.

33. The two limiting points of the given co-axial system are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

34. If two limiting points of a coaxial system of circles is (a, b) and (α, β).

then $S_1 + \lambda S_2 = 0, \lambda \neq -1$. or, $\{(x - a)^2 + (y - b)^2\} + \lambda\{(x - \alpha)^2 + (y - \beta)^2\} = 0, \lambda \neq -1$ is the

Coaxial system of circle.

35. If origin is a limiting point of the coaxial system containing the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2 + f^2}, \frac{-fc}{g^2 + f^2}\right)$.