9. CIRCLE

1. INTRODUCTION

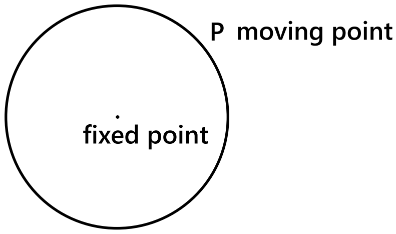

Figure 9.1

Definition: The locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same (i.e., constant) is known as a circle.

The fixed point is called the centre of the circle and the distance between the fixed point and moving point is called the radius of the circle.

2. DIFFERENT FORM OF EQUATION OF CIRCLE

2.1 General Form

The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f, and c are constants.

(a) Centre of the circle ≡ $(-g, -f)$. i.e., $(-\frac{1}{2}$ coefficient of x, $-\frac{1}{2}$ coefficient of y).

(b) Radius of the circle = $\sqrt{g^2 + f^2 - c}$.

Nature of the circle:

(a) If $g^2 + f^2 - c > 0$, then the radius of the circle will be real. Hence, it is possible to draw a circle on a plane.

(b) If $g^2 + f^2 - c = 0$, then the radius of the circle will be zero. Such a circle is known as point circle.

(c) If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2 + f^2 - c}$ of the circle will be an imaginary. Hence, it is not possible to draw a circle.

The condition for the general second degree equation to represent a circle:

The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

(a) $a = b \neq 0$ i.e. the coefficient of x^2 = the coefficient of $y^2 \neq 0$.

(b) $h = 0$ i.e. the coefficient of xy is 0.

(c) $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$, it implies that the general equation is non degenerate (i.e. equation cannot be written into two linear factors)

(d) $g^2 + f^2 - c \geq 0$

MASTERJEE CONCEPTS

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Degeneracy condition depends on determinant of 3x3 matrix and the type of conic depends on determinant of 2x2 matrix.
- Also the equation can be taken as intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

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2.2 Standard Form

The equation of circle with center (0,0) and radius 'a' is $x^2 + y^2 = a^2$.

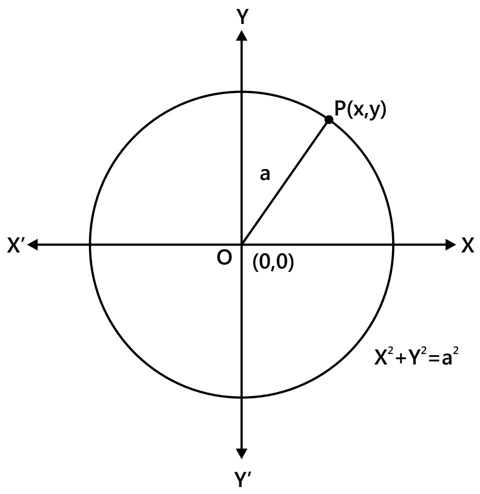

Figure 9.2: Standard Form

2.3 Central Form

The equation of the circle with centre (h, k) and radius 'r' is $(x - y)^2 + (y - k)^2 = r^2$.

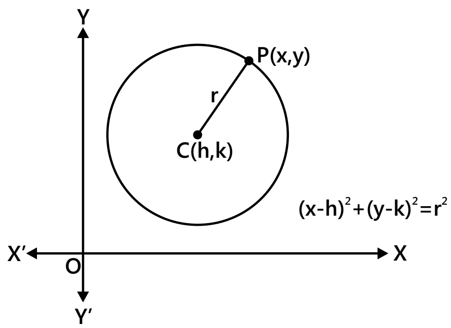

Figure 9.3: Central form

2.4 Diametric Form

Circle on a given diameter: The equation of the circle with (x_1, y_1) and (x_2, y_2) as the end points of the diameter is given by

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Centre $\equiv \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

and, Radius = $\sqrt{\left(\dfrac{x_2 - x_1}{2}\right)^2 + \left(\dfrac{y_2 - y_1}{2}\right)^2}$.

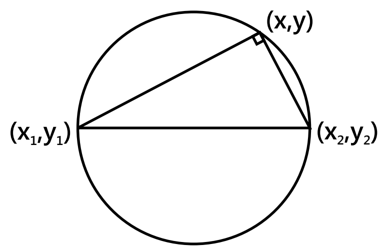

Figure 9.4: Diametric Form

2.5 Parametric Form

For $x^2 + y^2 = r^2$, parametric co-ordinates of any point on the circle is given by $(r\cos\theta, r\sin\theta)$, $(0 \le \theta < 2\pi)$.

(a) The parametric co-ordinates of a point on the circle $(x - h)^2 + (y - k)^2 = r^2$ is given by $(h + r\cos\theta, k + r\sin\theta)$, $(0 \le \theta < 2\pi)$.

(b) The parametric co-ordinates of any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + r\cos\theta$ and $y = -f + r\sin\theta$, (where $r = \sqrt{g^2 + f^2 - c}$, and $0 \le \theta < 2\pi$)

2.6 Equation of Circle under Special Conditions

(a) The equation of the circle through three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

The concept of family of circles can be used to derive this form.

Taking any two of the given three points as extremities of diameter, we get the equation of a circle S = 0 and the equation of straight line passing through these two points L = 0. Then the equation of circle passing through the intersection of circle and line is $S + \lambda L = 0$, where λ is a parameter. The value of λ can be found by substituting the third point in the equation as it also lies on the circle.

(b) Equation of circle circumscribing the triangle formed by the lines $a_r x + b_r y + c_r = 0$ where r = 1, 2, 3 is

$$\begin{vmatrix} \dfrac{a_1^2 + b_1^2}{a_1x + b_1y + c_1} & a_1 & b_1 \\ \dfrac{a_2^2 + b_2^2}{a_2x + b_2y + c_2} & a_2 & b_2 \\ \dfrac{a_3^2 + b_3^2}{a_3x + b_3y + c_3} & a_3 & b_3 \end{vmatrix} = 0$$

MASTERJEE CONCEPTS

Whenever the problem seems to be very complicated using formulas and geometrical approach, then try to apply trigonometric approach as well. Like the given circle may be in circle or ex-circle of some triangle. May be using properties of triangle we can solve it.

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Illustration 1: Find the equation of the circle which passes through the point of intersection of the lines x – 4y – 1 = 0 and 4x + y – 21 = 0 and whose centre is (2, –3). **(JEE MAIN)**

Sol: By solving given equation of lines simultaneously we will get point of intersection of lines i.e. P. Therefore using distance formula we will get radius of circle and using centre and radius we will get required equation.

Let P be the point of intersection of the lines

$x - 4y - 1 = 0$ .... (i)

and $4x + y - 21 = 0$ .... (ii)

From (i) and (ii), we get $x = 5, y = 1$. So, coordinates of P are $(5,1)$.Let $C(2,-3)$ be the centre of the circle.

Since the circle passes though P, therefore

CP= radius $\Rightarrow \sqrt{(5-2)^2 + (1+3)^2}$ = radius $\Rightarrow$ radius $= 5$

Hence the equation of the required circle is $(x - 2)^2 + (y + 3)^2 = 25$.

Illustration 2: Find the equation of a circle of radius 10 whose centre lies on x-axis and passes through the point (4, 6). **(JEE MAIN)**

Sol: Consider coordinates of centre of circle as (a, 0). Now by using distance formula, we can calculate the value of 'a' and then by using central form, we get required equation.

Here centre C lies on x-axis, and the circle passes through A (4,6).

Let C be (a,0)

$\therefore$ CA = radius

$\Rightarrow$ CA =10 $\Rightarrow \sqrt{(a-4)^2 + (0-6)^2} = 10$

$\Rightarrow (a-4)^2 + 36 = 100 \Rightarrow (a-4)^2 = 64$

$\Rightarrow a - 4 = \pm 8 \Rightarrow a = 12$ or $a = -4$.

Thus, the coordinates of the centre are (12, 0) or $(-4,0)$.

Hence, the equations of the required circles are

$(x-12)^2 + (y-0)^2 = 10^2$ and $(x+4)^2 + (y-0)^2 = 10^2$

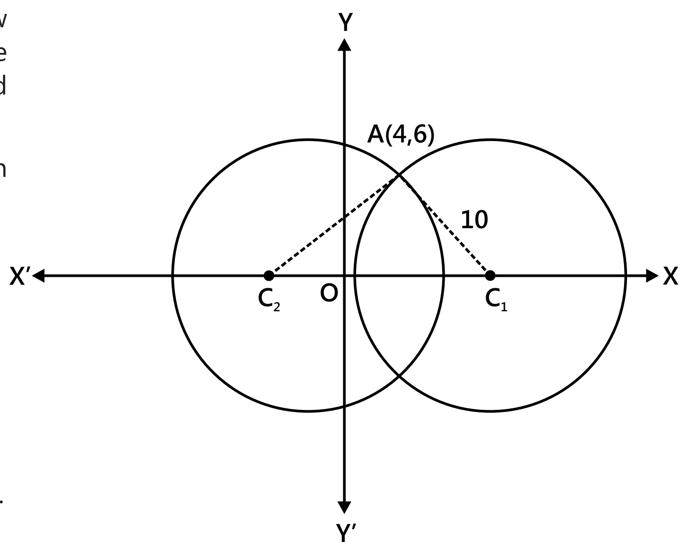


Figure 9.5

Illustration 3: Find the equation of the circle concentric with the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ and passing through the point $(5,4)$. **(JEE MAIN)**

Sol: Since both circle are concentric therefore their centre should be same. Hence equation of a required circle can be written as $x^2 + y^2 + 4x + 6y$ + (constant term) = 0.

Let the equation of the concentric circle be

$x^2 + y^2 + 4x + 6y + k = 0$

Since the point (5,4) lies on this circle,

$\therefore (5)^2 + (4)^2 + 4(5) + 6(4) + k = 0$

$\Rightarrow 25 + 16 + 20 + 24 + k = 0 \Rightarrow k = -85$

Therefore, the equation of the required circle is

$x^2 + y^2 + 4x + 6y - 85 = 0$

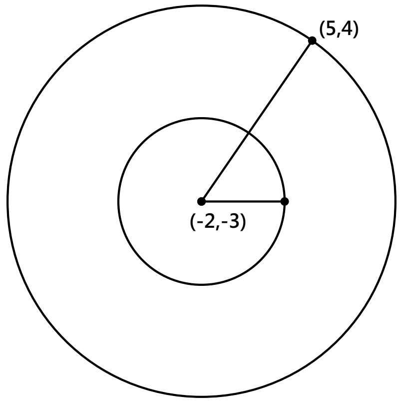


Figure 9.6

Illustration 4: Find the equation of a circle passing through the origin and making intercepts 4 and 3 on the y and x axis respectively. **(JEE MAIN)**

Sol: By observing the problem we conclude that given intercepts are end points of diameter of this circle. Therefore by using diametric form we can obtain the equation of circle.

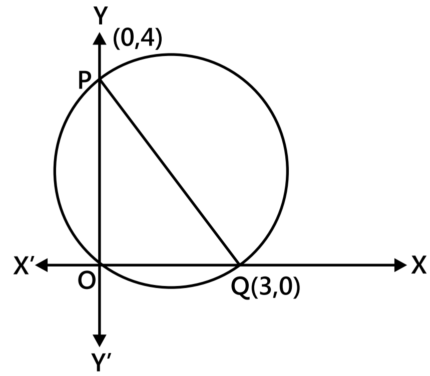


Figure 9.7

Let the intercepts be OP = 4, OQ = 5

$\therefore$ The co-ordinates of P and Q are (0, 4) and (3, 0) respectively.

Since POQ = 90°, hence PQ is a diameter.

$\therefore$ The required equation of the circle is

$(x-3)(x-0)+(y-0)(y-4)=0 \Rightarrow \quad x^2+y^2-3x-4y=0$

Illustration 5: Find the equation of the circle which is circumscribed about the triangle whose vertices $(-2,3)$, $(5,2)$ and $(6,-1)$. **(JEE ADVANCED)**

Sol: Consider (a, b) as the centre of circle and r as the radius. As circle passes from given vertices, therefore their distance from the centre are same. Therefore by using distance formula, we will get the value of a, b and r.

Since the circle passes through the points $(-2,3)$, $(5,2)$ and $(6,-1)$.

$\therefore \quad (-2-a)^2 + (3-b)^2 = r^2 \Rightarrow a^2 + 4a + 4 + b^2 - 6b + 9 = r^2$ .... (i)

$(5-a)^2 + (2-b)^2 = r^2 \Rightarrow a^2 - 10a + 25 + b^2 - 4b + 4 = r^2$ .... (ii)

$(6-a)^2 + (-1-b)^2 = r^2 \Rightarrow a^2 - 12a + 36 + b^2 + 2b + 1 = r^2$ .... (iii)

Subtracting (ii) from (i), we have

$14a - 21 - 2b + 5 = 0$ i.e., $14a - 2b = 16$ .... (iv)

Subtracting (iii) from (ii), we get

$2a - 11 - 6b + 3 = 0 \quad \Rightarrow 2a - 6b = 8$ .... (v)

Solving (iv) and (v), we get

$a = 1$ and $b = -1$

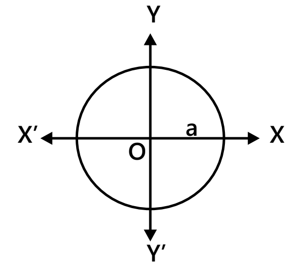
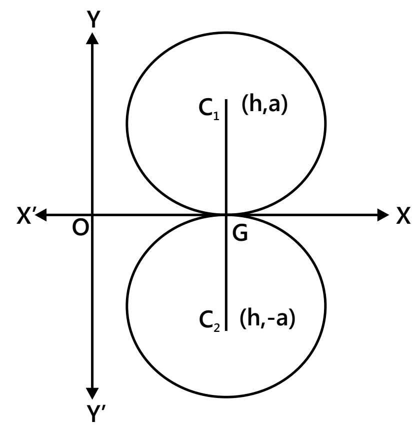
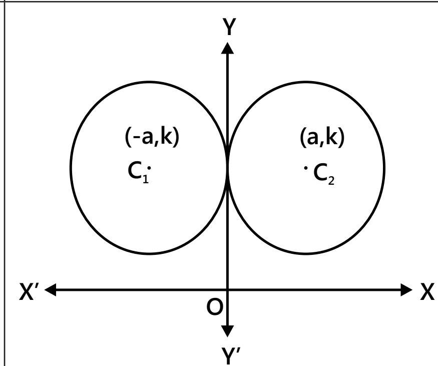
Putting the values of $a=1$ and $b=-1$ in (i), we get

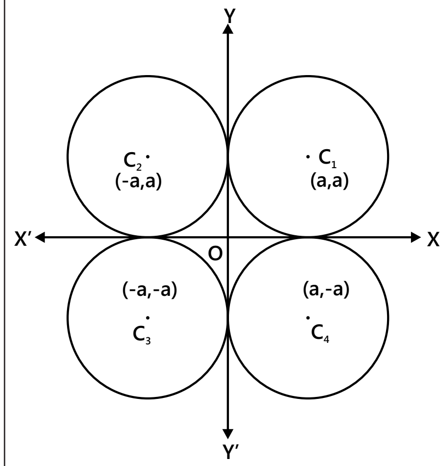
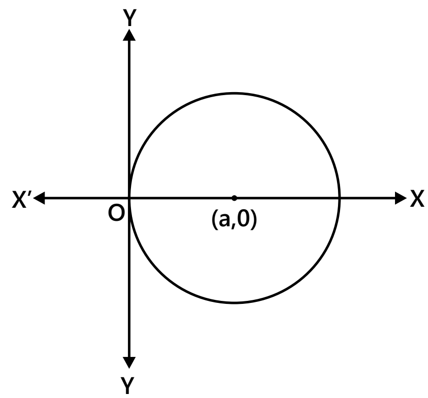
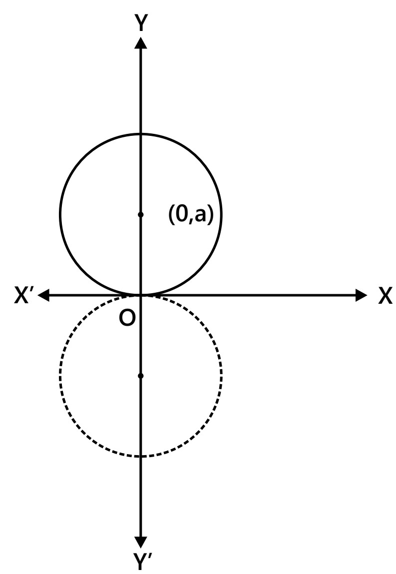
$1 + 4 + 4 + 1 + 6 + 9 = r^2 \quad \Rightarrow 25 = r^2 \quad \Rightarrow r = 5$

Thus, the required equation of the circle is

$(x-1)^2 + (y+1)^2 = 25 \quad \Rightarrow x^2 + 2x + 1 + y^2 + 1 + 2y = 25 \Rightarrow x^2 + y^2 - 2x + 2y - 23 = 0$

3. EQUATION OF CIRCLE IN SOME SPECIAL CASES

Equation	**Centre/Radius**	**Properties**	**Figures**
(a) $x^2 + y^2 = a^2$	$(0, 0)$; a	When the centre of the circle coincides with the origin center = (0,0)	 **Figure 9.8**
(b) $(x - h)^2 + (y \pm a)^2 = a^2$	$(h, \pm a)$; a	Touches x axis only, y coordinate of centre = ±a	 **Figure 9.9**
(c) $(x \pm a)^2 + (y - k)^2 = a^2$	$(\pm a, k)$; a	Touches y–axis only, x coordinate of centre = ±a	 **Figure 9.10**

Equation	Centre/Radius	Properties	Figures
(d) $(x \pm a)^2 + (y \pm a)^2 = a^2$	$(\pm a, \pm a)$; a	Touches both the axes depending on the quadrant center = $(\pm a, \pm a)$	 Figure 9.11
(e) $x^2 + y^2 - 2ax = 0$	C (a, 0) ; a	When the circle passes through the origin and centre lies on x axis	 Figure 9.12
(f) $x^2 + y^2 - 2ay = 0$	C (0, a); a	When the circle passes through the origin and centre lies on y axis.	 Figure 9.13

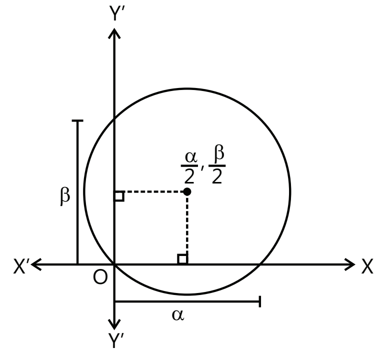
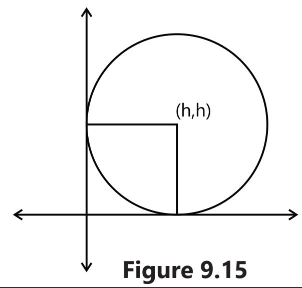
Equation	Centre/Radius	Properties	Figures
(g) $x^2 + y^2 - \alpha x - \beta y = 0$	$\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$; $\frac{1}{2}\sqrt{\alpha^2 + \beta^2}$	Passes through (0, 0) and has intercepts α and β on x & y axes respectively.	 Figure 9.14
(h) $(x-h)^2 + (y-h)^2 = h^2$ or $x^2 + y^2 - 2hx - 2hy + h^2 = 0$	(h, h); h	When circle touches both the axes	 Figure 9.15

Illustration 6: Find the equation of the circle which passes through two points on the x–axis which are at distances 12 from the origin and whose radius is 13.

Sol: There are two circles which passes through two points A and A′ on x–axis which are at a distance 12 from the origin and whose radius is 13. The centre of these circles lie on y–axis (perpendicular bisector of chord AA′)

In ΔAOC, $AC^2 = OA^2 + OC^2$

$\Rightarrow 13^2 = 12^2 + OC^2 \Rightarrow OC = 5$

So the coordinates of the centre of the required circles are (0, 5) and C′ (0, –5). Hence the equations of the required circles are

$(x - 0)^2 + (y \pm 5)^2 = 13^2 \Rightarrow x^2 + y^2 \pm 10y - 144 = 0.$

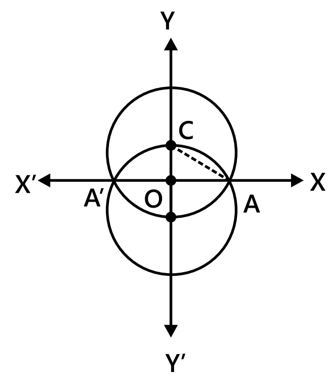


Figure 9.16

4. INTERCEPTS ON THE AXES

The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

Therefore,

(a) The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x–axis in real and distinct points, touches or does not meet in real points according as $g^2 > c$- Distinct points

$g^2 = c$ – Touches

$g^2 < c$ – Does not meet

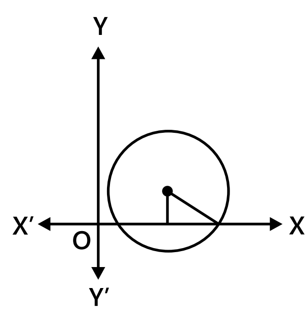


Figure 9.17: Intercept made by circle on x-axis

(b) Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y–axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

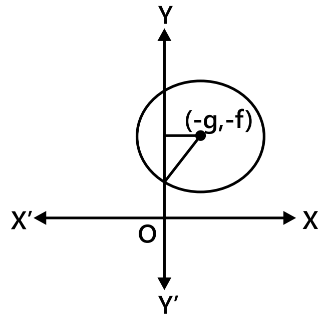


Figure 9.18: Intercept made by circle on y-axis

Illustration 7: Find the equation to the circle which touches the positive axis of y at a distance 4 from the origin and cuts off an intercept of 6 from the axis of x. **(JEE MAIN)**

Sol: As circle touches Y axis therefore Y coordinate of centre of circle is 4 so by using formula of intercept we will get the value of X coordinate of centre of circle and c.

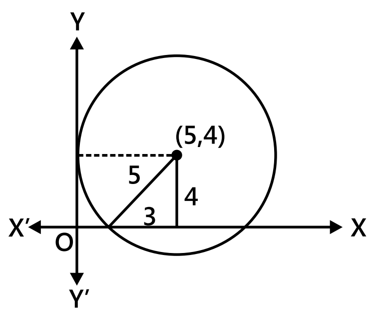


Figure 9.19

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

This meets the axis of y in points given by $y^2 + 2fy + c = 0$

The roots of this equation must be each equal to 4, so that it must be equivalent to $(y - 4)^2 = 0 \Rightarrow 2f = -8$ & $c = 16$

$\because$ Intercept made on the x-axis = 6

$\Rightarrow 6 = 2\sqrt{g^2 - 16} \quad \Rightarrow g = \pm 5$.

Hence, the required equation is $x^2 + y^2 \pm 10x - 8y + 16 = 0$.

5. POSITION OF A POINT W.R.T A CIRCLE

(a) If CP < radius, then the point P lies inside the circle. (Refer Fig. 9.20 (i))

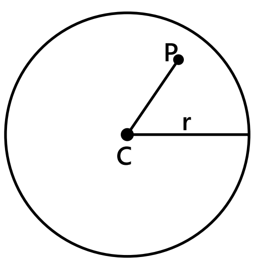


Figure 9.20 (i)

(b) If CP = radius, then the point P lies on the circumference.

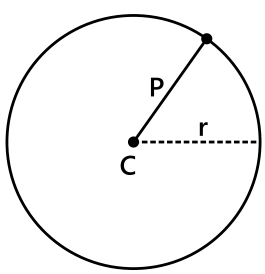


Figure 9.20 (ii)

(c) CP > radius, then the point P lies outside the circle.

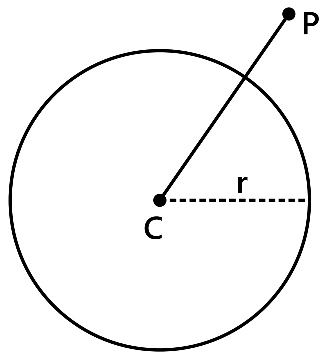


Figure 9.20 (iii)

Hence, any point (x, y) lies outside, on or inside if

$\sqrt{(x_1+g)^2+(y_1+f)^2}$ > = < $\sqrt{g^2+f^2-c}$ $\Rightarrow (x_1 + g^2) + (y_1 + f)^2$ >

= < $(g^2 + f^2 - c)$

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ > = < 0

Or, S_1 > = < 0 where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Therefore, a point (x_1, y_1) lies outside, on or inside a circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

5.1 Power of a Point w.r.t. a circle

Let P (x_1, y_1) be a point and a secant (a line which cuts the curve in two point) PAB is drawn.

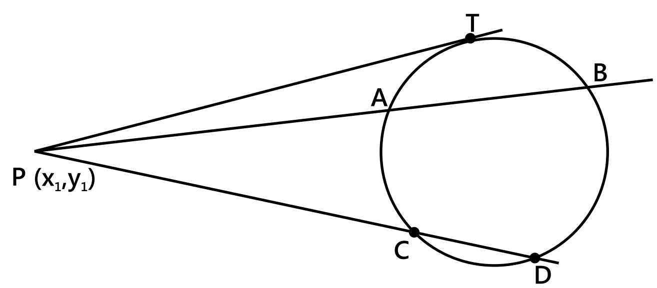


Figure 9.21

The power of $P(x_1, y_1)$ w.r.t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA.PB which is S_1, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Power remains constant for the circle i.e. independent of A and B

$PA \times PB = PC \times PD = PT^2$ = square of the length of a tangent

5.2 The Least and Greatest distance of a Point from a Circle

Let S = 0 be a circle and P (x_1, y_1) be a point. If the diameter of the circle through P intersect the circle at Q and R,

then QP = $|PC - r|$ = least distance; and

PR = PC + r = greatest distance

where 'r' is the radius and C is the centre of the circle.

(Refer Fig. 9.23)

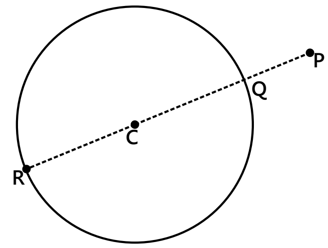


Figure 9.22

Illustration 8: The coordinates of the point on the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ farthest from the origin are **(JEE MAIN)**

(A) $\left(2+\frac{8}{\sqrt{5}}, 1+\frac{4}{\sqrt{5}}\right)$ (B) $\left(1+\frac{4}{\sqrt{5}}, 2+\frac{8}{\sqrt{5}}\right)$ (C) $\left(1+\frac{8}{\sqrt{5}}, 2+\frac{4}{\sqrt{5}}\right)$ (D) None of these

Sol: (B) The required point lies on the normal to circle through the origin, i.e. on the line 2x = y. Therefore by substituting y = 2x in above equation of circle we will get coordinates of required point.

$x^2 + 4x^2 - 2x - 8x - 11 = 0 \Rightarrow 5x^2 - 10x - 11 = 0 \Rightarrow x = 1 \pm \frac{4}{\sqrt{5}}$ and $y = 2\left(1 \pm \frac{4}{\sqrt{5}}\right)$

and the required point farthest from the origin is $\left(1+\frac{4}{\sqrt{5}}, 2+\frac{8}{\sqrt{5}}\right)$.

Illustration 9: The point (1, 3) is inside the circle S whose equation is of the form $x^2 + y^2 - 6x - 10y + k = 0$, k being an arbitrary constant. Find the possible values of k if the circle S neither touches the axes nor cuts them. **(JEE ADVANCED)**

Sol: As (1, 3) lies inside the circle S therefore $S_1 < 0$ and it does not touches x and y axes. On the basis of this we can solve the problem and will get range of k.

$1^2 + 3^2 - 6\times1 - 10\times3 + k < 0; \quad \therefore k < 26$ ...(i)

Solving y = 0 and $x^2 + y^2 - 6x - 10y + k = 0$, we get $x^2 - 6x + k = 0$

Since the circle S does not intersect with the x-axis,

$\Rightarrow$ discriminant < 0 i.e., $36 - 4k < 0 \quad \Rightarrow k > 9$ ...(ii)

Solving x = 0 and $x^2 + y^2 - 6x - 10y + k = 0$, we get $y^2 - 10y + k = 0$

Since the circle S does not intersect with the y-axis,

$\Rightarrow$ discriminant < 0 i.e., $100 - 4k < 0 \quad \Rightarrow k > 25$ ...(iii)

From (i), (ii) and (iii), we get $25 < k < 26$, i.e., $k \in (25, 26)$.

6. LINE AND A CIRCLE

The length of the intercept cut off from the line y = mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$.

(a) If $a^2(1 + m^2) > c^2$, or, $|c| < a\sqrt{1+m^2}$

i.e., the line will intersects the circle at two real and different points.

(b) If $a^2(1 + m^2) = c^2$, or, $|c| = a\sqrt{1+m^2}$

i.e., the line will touch the circle at only one point i.e. the line will be a tangent.

(c) If $a^2(1 + m^2) < c^2$, or, $|c| > a\sqrt{1+m^2}$

i.e., the line will meet the circle at two imaginary points.

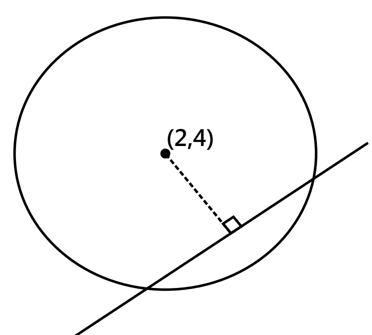


Figure 9.23

Illustration 10: Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-40 < c < 20$. **(JEE MAIN)**

Sol: Since the line cuts the circle so length of perpendicular from centre of circle upon line is less than the radius of circle.

$\left|\dfrac{3\times2-4\times4-c}{\sqrt{9+16}}\right| < 6 \qquad \Rightarrow |10 + c| < 30$

$\Rightarrow \quad -30 < 10 + c < 30 \qquad \Rightarrow -40 < c < 20$

Illustration 11: If $4l^2 - 5m^2 + 6l + 1 = 0$ then show that the line $lx + my + 1 = 0$ touches a fixed circle. Find the radius and centre of the circle. **(JEE ADVANCED)**

Sol: If line touches the circle then perpendicular distance from centre of circle to the line is equal to the radius of circle so by using distance formula of point to line we will get one equation and other is given $4l^2 - 5m^2 + 6l + 1 = 0$. hence by solving these two equation we will get required answer.

Let the circle be $(x-\alpha)^2 + (y-\beta)^2 = a^2$

The line $lx + my + 1 = 0$ touches the circle if

$$a = \left|\frac{l\alpha + m\beta + 1}{\sqrt{l^2 + m^2}}\right| \quad \text{or} \quad a^2(l^2 + m^2) = (l\alpha + m\beta + 1)^2$$

or $\quad (a^2 - \alpha^2)\, l^2 + (a^2 - \beta^2)\, m^2 - 2\, l\beta\alpha\, m - 2\, l\, \alpha - 2m\beta - 1 = 0$ ... (i)

But $\quad 4l^2 - 5\, m^2 + 6l + 1 = 0$ ... (ii)

It is possible to find α,β, a if (i) and (ii) are identical.

The condition is

$$\frac{a^2-\alpha^2}{4} = \frac{a^2-\beta^2}{-5} = \frac{-2\alpha}{6} = \frac{2\beta}{0} = \frac{-1}{1} \quad \text{... (iii)}$$

$\therefore\ \beta = 0,\ \alpha = 3$ and $\dfrac{a^2-\alpha^2}{4} = -1$ which implies $a^2 - 3^2 = -4$, i.e., $a = \sqrt{5}$ .Also $\alpha = 3,\ \beta = 0,\ a = \sqrt{5}$ satisfies equation (iii) and hence the line touches the fixed circle $(x-3)^2 + (y-0)^2 = \left(\sqrt{5}\right)^2$

or $x^2 + y^2 - 6x + 4 = 0$, whose centre $= (\alpha, \beta) = (3, 0)$ and radius $= a = \sqrt{5}$.

Illustration 12: Find equation of a line with slope gradient 1 and such that $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it ? **(JEE ADVANCED)**

Sol: As given slope of line is 1, therefore its equation will be $y = x + c$ Hence by using perpendicular distance formula we will get distance of line from centre of respective circle and then by using Pythagoras we can obtain length of intercepts made by line to these circles and which are equal. Therefore we can obtain value of c and required equation of circle.

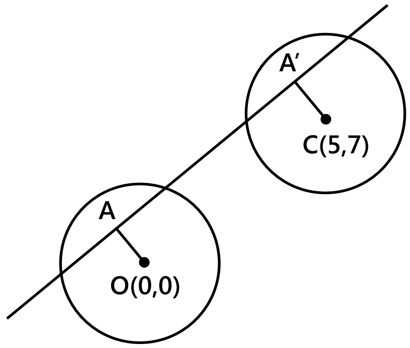


Figure: 9.24

Let 2ℓ be the length of the intercept made by the two circle.

For $x^2 + y^2 = 4$, centre ≡ (0,0) and radius = 2, and

For $x^2 + y^2 - 10x - 14y + 65 = 0$, centre ≡ (5,7) and

radius = 3.

$\therefore\ OA = \left|\dfrac{c}{\sqrt{2}}\right| \quad$ and $CA' = \left|\dfrac{5-7+c}{\sqrt{2}}\right| \Rightarrow CA' = \left|\dfrac{c-2}{\sqrt{2}}\right|$

Now, from the diagram we get

$$4 - OA^2 = \ell^2 \quad \text{...(i)}$$

and $9 - CA'^2 = \ell^2$ ...(ii)

$\Rightarrow 4 - \frac{c^2}{2} = 9 - \left(\frac{c-2}{2}\right)^2$ [from (i) and (ii)]

$\Rightarrow c = -\frac{3}{2}$

The equation of line is 2x - 2y = 3

Illustration 13: Find the values of α for which the point $(2\alpha, \alpha + 1)$ is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord whose equation is $x - y + 1 = 0$. **(JEE ADVANCED)**

Sol: As point $(2\alpha, \alpha + 1)$ lies inside the circle S, therefore $S_1 < 0$. Hence by substituting the point in the equation, we will get the range of α and as it lies in larger segment made by line $x - y + 1 = 0$. The centre of circle i.e. (1, 1) and $(2\alpha, \alpha + 1)$ will have the same sign.

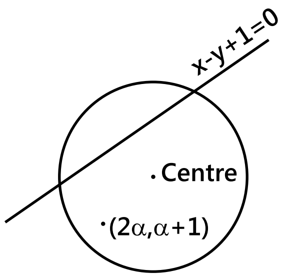


Figure 9.25

$\therefore (2\alpha)^2 + (\alpha + 1)^2 - 2\times2\alpha - 2(\alpha + 1) - 8 < 0$

$\Rightarrow 5\alpha^2 - 4\alpha - 9 < 0$ or $(5\alpha - 9)(\alpha + 1) < 0$

$\Rightarrow -1 < \alpha < 9/5$ ... (i)

Also, as the point lies in the larger segment, the centre (1, 1) and the point $(2\alpha, \alpha + 1)$ must be on the same side of the line $x - y + 1 = 0$.

Clearly, $1 - 1 + 1 > 0$; So, $2\alpha - (\alpha + 1) + 1 > 0$;

$\therefore \alpha > 0$ ... (ii)

$\therefore$ The set of values of α satisfying (i) and (ii) is $\left(0, \frac{9}{5}\right)$.

7. TANGENTS

7.1. Point Form

(a) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$.

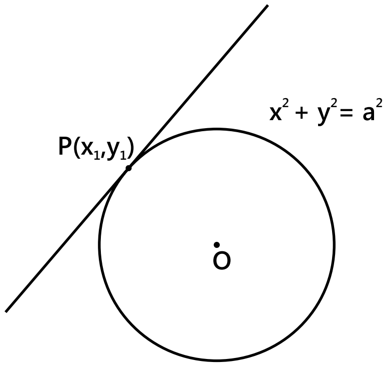


Figure 9.26

(b) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

7.2. Parametric Form

Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a \cos\theta, a \sin\theta)$, then equation of tangent at $(a \cos\theta, a \sin\theta)$ is $x\, a \cos\theta + y\, a \sin\theta = a^2$ or, $x \cos\theta + y \sin\theta = a$.

7.3. Condition for Tangency

A line L = 0 touches the circle S = 0. If length of perpendicular (*p*) drawn from the centre of the circle to the line is equal to radius of the circle i.e. $p = r$. This is the condition of tangency for the line L = 0. Circle $x^2 + y^2 = a^2$ will touches the line $y = mx + c$ if $c = \pm a\sqrt{1+m^2}$

Illustration 14: For what value of c will the line y = 2x + c be a tangent to the circle $x^2 + y^2 = 5$. **(JEE MAIN)**

Sol: The equation of the tangent to the circle $x^2 + y^2 = a^2$ in slope form is $y = mx + a\sqrt{1+m^2}$.

On comparison, we get $a = \sqrt{5}$ and m = 2.

$\therefore c = \pm \sqrt{5}\times\sqrt{1+2^2}$ $\Rightarrow c = \pm 5$

The required equation is $y = 2x \pm 5$

7.4. Slope Form

The straight line y = mx + c touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Therefore, the equation of the tangent in the slope form is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(\frac{\mp ma}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$.

7.5 Length of Tangent

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is $PT_1 = PT_2 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$\therefore$ Length of Tangent = $\sqrt{S_1}$

Note:

(i) PT^2 is called the power of the point with respect to a given circle, where PT is the tangent from a point P to a given circle.

(ii) Area of quadrilateral PT_1CT_2 = 2 × (Area of triangle PT_1C), and

(iii) The angle between tangents PT_1 and PT_2 is equal to $2\tan^{-1}\left(\frac{r}{\sqrt{S_1}}\right)$.

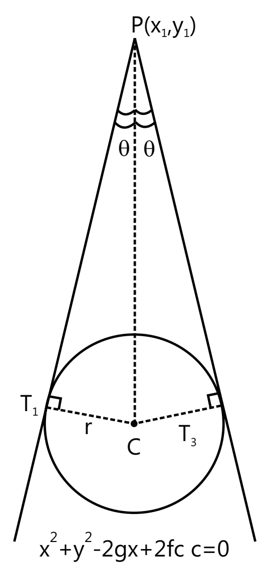


Figure 9.27

Illustration 15: If OA and OB are tangents from the origin O, to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, c > 0 and C is the centre of the circle, then area of the quadrilateral OACB is **(JEE MAIN)**

(A) $\frac{1}{2}\sqrt{c(g^2+f^2-c)}$ (B) $\sqrt{c(g^2+f^2-c)}$

(C) $c\sqrt{g^2+f^2-c}$ (D) $\sqrt{\frac{g^2+f^2-c}{c}}$

Sol: (B) As we know quadrilateral OACB is formed by two right angle triangle OAC and triangle OBC. Line OA and OB are tangent to the circle from common point O. Therefore OA = OB and (AC = CB) radius of circle, hence both triangle are equal. Therefore Area of the quadrilateral OACB = 2 Area of the triangle OAC.

$OA = OB = \sqrt{S_1} = \sqrt{c}$ (Length of the tangent from the origin) and, CA = CB = $\sqrt{g^2 + f^2 - c}$ (Radius of the circle)

∴ Area of the quadrilateral OACB = 2 Area of the triangle OAC

$= 2 \times \left(\frac{1}{2}\right) OA \times CA \qquad = \sqrt{c}\sqrt{g^2 + f^2 - c}$

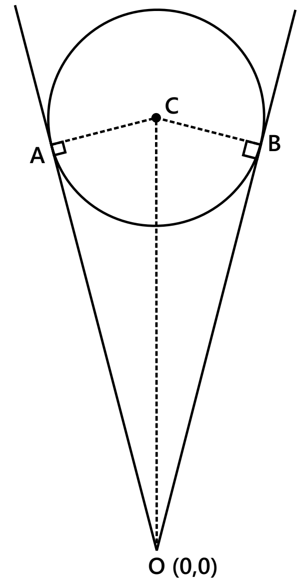


Figure 9.28

Illustration 16: The locus of a point which moves such that the tangents from it to the two circles $x^2 + y^2 - 5x - 3 = 0$ and $3x^2 + 3y^2 + 2x + 4y - 6 = 0$ are equal, is **(JEE MAIN)**

(A) $7x + 4y - 3 = 0$ (B) $17x + 4y + 3 = 0$ (C) $3x - 4y + 9 = 0$ (D) $13x - 4y + 15 = 0$

Sol: (B) Use the formula for length of tangent. Let P(h, k) be any point on the locus.

The length of the tangent from P to the first circle is $\sqrt{h^2 + k^2 - 5h - 3}$, ...(i)

Similarly, the length of the tangent to the other circle is $\sqrt{h^2 + k^2 + \frac{2}{3}h + \frac{4}{3}k - \frac{6}{3}}$. ...(ii)

On equating (i) and (ii), we get $17h + 4k + 3 = 0$,

Therefore, the required locus is $17x + 4y + 3 = 0$.

7.6 Pair of Tangents

From a given point $P(x_1, y_1)$ two tangents PA and PB can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$.

The combined equation of the pair of tangents is

$SS_1 = T^2$, where

S = 0 is the equation of circle,

T = 0 is the equation of tangent at (x_1, y_1), and

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

(S_1 is obtained by replacing x by x_1 and y by y_1 in S)

Pair of tangents from point (0, 0) to the circle are at right angles if $g^2 + f^2 = 2c$.

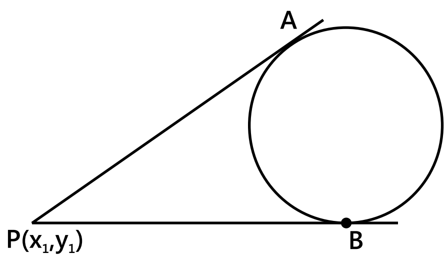


Figure 9.29

Illustration 17: Find the equation of the pair of the tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1). **(JEE MAIN)**

Sol: Here $(x_1, y_1) = (0, 1)$. So by using formula $SS_1 = T^2$ we can get required equation, where $S = x^2 + y^2 - 2x + 4y = 0$, $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T = xx_1 + yy_1 - (x+x_1) + 2(y+x_1)$.

Given circle is $S \equiv x^2 + y^2 - 2x + 4y = 0$ ...(i)

Let P be the point (0, 1).

$\therefore S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \qquad \Rightarrow S_1 \equiv 0^2 + 1^2 - 2.0 + 4.1 = 5$

And, $T \equiv xx_1 + yy_1 - (x+x_1) + 2(y+x_1) \qquad \Rightarrow T \equiv x(0) + y(1) - (x + 0) + 2(y + 1)$

i.e., $T \equiv -x + 3y + 2$.

Hence, the equation of pair of tangents from P (0, 1) to the given circle is $SS_1 = T^2$

i.e. $5\left(x^2 + y^2 - 2x + 4y\right) = \left(-x + 3y + 2\right)^2$

$\Rightarrow 5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

$\Rightarrow 4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

$\Rightarrow 2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ...(ii)

Note: From (ii), we have $2x^2 + 3(y - 1)x - (2y^2 - 4y + 2) = 0$.

This is a quadratic equation in x, hence by using quadratic formula we get

$x = \dfrac{3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$ or, $4x - 3y + 3 = \pm\sqrt{25y^2 - 50y + 25}$

or, $4x - 3y + 3 = \pm 5(y - 1)$.

$\therefore$ Separate equations of tangents are $x - 2y + 2 = 0$ and $2x + y - 1 = 0$.

Illustration 18: From a point on the line $4x - 3y = 6$ tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ which make an angle of $\tan^{-1}\dfrac{24}{7}$ between them. Find the coordinates of all such points and the equations of tangents.

(JEE ADVANCED)

Sol: Consider a point P on the line $4x - 3y = 6$ and use the formula.

Let $P(x_1, y_1)$ be a point on the line $4x - 3y = 6$.

If θ is the angle between the tangents, then $\tan\theta = \dfrac{24}{7}$.

For the given circle, Centre C = (3, 2) and

Radius = CA = $\sqrt{3^2 + 2^2 - 4} = 3$ for $\tan\theta$

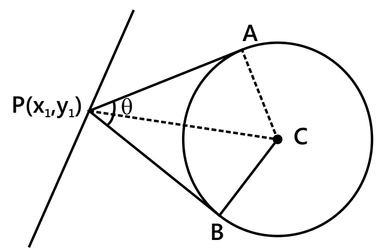


Figure 9.30

$\therefore$ The length of tangent, PA = $\sqrt{S_1} = \sqrt{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4}$

$\therefore \tan\dfrac{\theta}{2} = \dfrac{AC}{PA} = \dfrac{3}{\sqrt{S_1}}$

$\Rightarrow \tan^2\dfrac{\theta}{2} = \dfrac{9}{S_1}$ or, $\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4 - 9}{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4 + 9}$ or $\dfrac{1 - \frac{9}{-S_1}}{1 + \frac{9}{S_1}} = \dfrac{S_1 - 9}{S_1 + 9} = \dfrac{7}{25} \Rightarrow S_1 = 16$

$\therefore \dfrac{7}{25} = \dfrac{x_1^2 + y_1^2 - 6x_1 - 4y_1 - 5}{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 13}$ $\left(\because \tan\theta = \dfrac{24}{7}\right)$

or, $x_1^2 + y_1^2 - 6x_1 - 4y_1 - 12 = 0$ ... (i)

As (x_1, y_1) is on the line $4x - 3y = 6$, we get $4x_1 - 3y_1 = 6$ ... (ii)

Solving (i) and (ii), we get

$x_1^2 + \left(\dfrac{4x_1 - 6}{3}\right)^2 - 6x_1 - 4\left(\dfrac{4x_1 - 6}{3}\right) - 12 = 0$

$\Rightarrow 9x_1^2 + (4x_1 - 6)^2 - 54x_1 - 12(4x_1 - 6) - 108 = 0$

$\Rightarrow 25x_1^2 - 150x_1 = 0$ $\Rightarrow x_1(x_1 - 6) = 0$

$\Rightarrow x_1 = 0, 6$ and, $y_1 = \dfrac{4x_1 - 6}{3} = -\dfrac{6}{3}, \dfrac{18}{3} = -2, 6$.

$\therefore (x_1, y_1) \equiv (0, -2)$ and $(6, 6)$.

The equation of the pair of tangents is given by $SS_1 = T^2$

where $S \equiv x^2 + y^2 - 6x - 4y + 4$,

$S_1 = x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4$, and

$T = xx_1 + yy_1 - 3(x + x_1) - 2(y + y_1) + 4$

$\therefore$ The pair of tangents from (0, –2) is

$(x^2 + y^2 - 6x - 4y + 4)\cdot(0 + 4 - 0 + 8 + 4) = (0 + y(-2) - 3(x) - 2(y - 2) + 4)^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = (-3x - 4y + 8)^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = 9x^2 + 16y^2 + 64 + 24xy - 48x - 64y$

$\Rightarrow 7x^2 - 24xy - 48x = 0 \qquad \Rightarrow x(7x - 24y - 48) = 0$

$\therefore$ The tangents from (0, –2) are x = 0, and 7x –24y –48 =0.

Similarly, the equation of the pair of tangents from (6, 6) is

$(x^2 + y^2 - 6x - 4y + 4)\cdot(36 + 36 - 6\cdot 6 - 4\cdot 6 + 4) = \{x\cdot 6 + y\cdot 6 - 3(x + 6) - 2(y + 6) + 4\}^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = (3x + 4y - 26)^2 = 9x^2 + 16y^2 + 676 + 24xy - 156x - 208y$

$\Rightarrow 7x^2 - 24xy + 60x + 144y - 612 = 0$

$\Rightarrow (7x - 24y + 102)(x - 6) = 0$

$\therefore$ The tangents from (6, 6)are x – 6 = 0, and 7x – 24y + 102 =0.

Illustration 19: Obtain the locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ which include an angle α. **(JEE ADVANCED)**

Sol: Consider (x_1, y_1) as the point of intersection of tangents to the given circle and then use $\tan\frac{\alpha}{2} = \frac{a}{\sqrt{S_1}}$ to get the desired result.

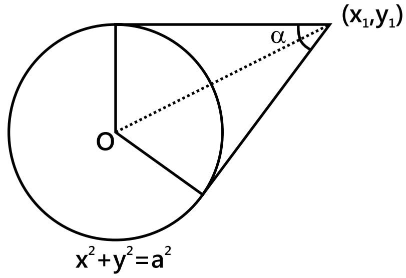


Figure 9.31

Let (x_1, y_1) be the point of intersection of a pair of tangents to the given circle.

If the pair of straight lines includes an angle α, then

$$\Rightarrow \tan\frac{\alpha}{2} = \frac{a}{\sqrt{S_1}} \qquad \Rightarrow \tan\alpha = \frac{2\frac{a}{\sqrt{S_1}}}{1 - \frac{a^2}{S_1}}$$

$$\Rightarrow \tan\alpha = \frac{2a\sqrt{x_1^2 + y_1^2 - a^2}}{y_1^2 + x_1^2 - 2a^2}$$

$$\Rightarrow \left(x_1^2 + y_1^2 - 2a^2\right)^2 \tan^2\alpha = 4a^2\left(x_1^2 + y_1^2 - a^2\right)$$

Hence, the required locus is $(x^2 + y^2 - 2a^2)^2 \tan^2\alpha = 4a^2(x^2 + y^2 - a^2)$.

7.7 Director Circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.

For the circle $x^2 + y^2 = a^2$, the equation of the director circle is $x^2 + y^2 = 2a^2$.

Hence, the centre of the director circle is same as the centre of the given circle, and the radius is $\sqrt{2}$ times the radius of the given circle.

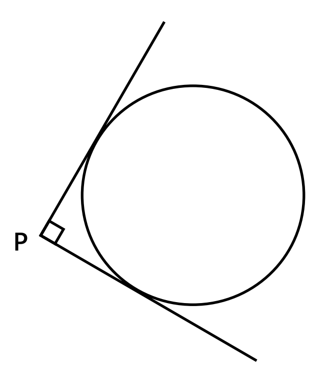


Figure 9.32

General Form: For the general form of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation of the director circle is given by $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

Illustration 20: Find the equation of the director circle of the circle $(x-2)^2 + (y+1)^2 = 2$. **(JEE MAIN)**

Sol: As we know, for the circle $x^2 + y^2 = a^2$, the equation of the director circle is $x^2 + y^2 = 2a^2$.

For the given circle, Centre $\equiv(2, -1)$ & Radius = $\sqrt{2}$.

$\therefore$ The centre of the director circle $\equiv$ (2, – 1), and the radius of the director circle = $\sqrt{2}\times\sqrt{2} = 2$.

$\therefore$ The required equation is $(x-2)^2 + (y+1)^2 = 4$.

8. NORMALS

The normal of a circle at any point is a straight line, perpendicular to the tangent and passing through the centre of the circle.

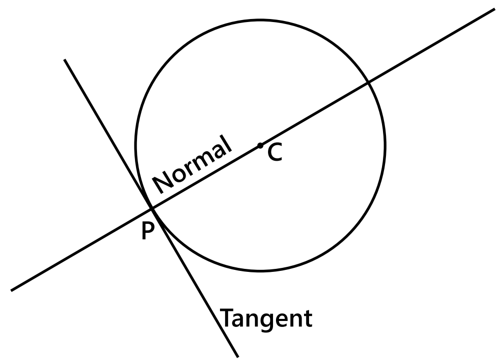


Figure 9.33

(a) Equation of normal: The equation of normal to the general form of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) on the circle is

$(y - y_1) = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$ or, $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

The equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xy_1 - xy_1 = 0$ or, $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

(b) Parametric Form: Equation of normal at $(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = a^2$ is $\dfrac{x}{a\cos\theta} = \dfrac{y}{a\sin\theta}$

or, $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta}$ or, $y = x\tan\theta$ or, $y = mx$ (where $m = \tan\theta$).

Illustration 21: Find the equation of the circle having the pair of lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having the size just sufficient to contain the circle $x(x-4) + y(y-3) = 0$. **(JEE ADVANCED)**

Sol: By solving equation $x^2 + 2xy + 3x + 6y = 0$ we will get point of intersection of normals i.e. centre of required circle. As given circle $x(x-4) + y(y-3) = 0$ lies inside the required circle hence distance between centres will be equal to the difference between their radius, therefore we can find out radius of required circle by using distance formula.

Given the equation of pair of normal is $x(x+3) + 2y(x+3) = 0$

$\Rightarrow (x+3)(x+2y) = 0$

$\therefore$ Either $(x+3)=0$ ...(i) or $(x+2y)=0$ ...(ii)

On solving (i) and (ii), we get $x = -3$ and $y = \dfrac{3}{2}$

$\therefore$ The centre $\equiv \left(-3, \dfrac{3}{2}\right)$ (The point of intersection of the normals).

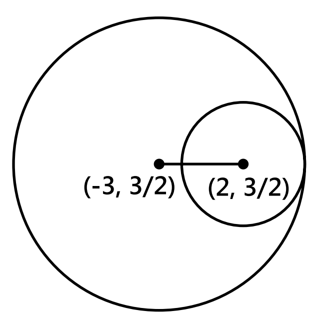


Figure 9.34

For the circle $x^2 + y^2 - 4x - 3y = 0$ ...(iii)

centre = $\left(2, \dfrac{3}{2}\right)$ and radius, $r = \sqrt{(-2)^2 + \left(\dfrac{-3}{2}\right)^2 - 0} = \dfrac{5}{2}$.

If the circle $x^2 + y^2 - 4x - 3y = 0$ lies inside another circle of radius 'a', then

$a - r$ = distance between the centres $\left(-3, \dfrac{3}{2}\right)$ and $\left(2, \dfrac{3}{2}\right)$

$\Rightarrow a - \dfrac{5}{2} = \sqrt{(-3-2)^2 + \left(\dfrac{3}{2} - \dfrac{3}{2}\right)^2}$ $\Rightarrow a = 5 + \dfrac{5}{2}$ $\therefore a = \dfrac{15}{2}$.

Hence, the equation of the circle is $(x + 3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$ or, $x^2 + y^2 + 6x - 3y = 45$.

9. CHORD OF CONTACT

Consider a point $P(x_1, y_1)$ lying outside the circle. Tangents are drawn to touch the given circle at Q and R respectively (as shown in the diagram). The chord joining the points of contact of the two tangents to a circle (or any conic) from the point P, outside it, is known as the chord of contact.

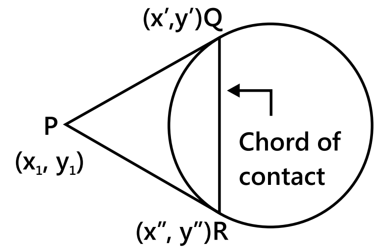


Figure 9.35

9.1 Equation of Chord of contact

The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$. Equation of chord of contact at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Clearly, the equation of the chord of contact coincides with the equation of the tangent.

Length of chord of contact

Consider a circle of radius 'r' and the length of perpendicular from the centre to the chord of contact be 'p', then the length of the chord, $QR = 2\sqrt{r^2 - p^2}$.

MASTERJEE CONCEPTS

- Area of $\Delta PQR = \frac{1}{2} \times PM \times QR = \frac{a\left(x_1^2 + y_1^2 - a^2\right)^{3/2}}{x_1^2 + y_1^2} = \frac{RL^3}{R^2 + L^2}$

  where, the length of the tangent, $L = \sqrt{x_1^2 + y_1^2 - a^2}$ and, radius of circle, R = a.

- Equation of circle circumscribing the triangle PQR is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

  Note: Circumscribing Circle also passes through centre of original Circle

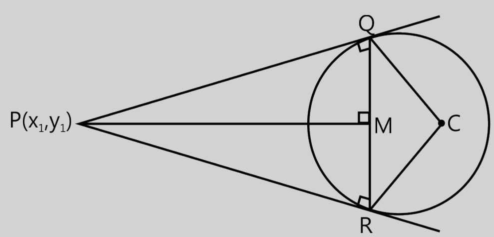


Figure 9.36

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 22: Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$. **(JEE MAIN)**

Sol: Equation of chord of contact is T = 0

Given circle is $S \equiv x^2 + y^2 - 2x + 4y + 7 = 0$ ...(i)

For point P ≡ (1, 2),

$S_1 > 0$, ⇒ the point P lies outside the circle. and, $T \equiv x(1) + y(2) - (x + 1) + 2(y + 2) + 7$ i.e. $T \equiv 4y + 10$

∴ The equation of the chord of contact is T = 0 i.e. 2y + 5 = 0.

Illustration 23: The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$ passes through the point **(JEE ADVANCED)**

(A) $\left(\frac{a}{2}, 0\right)$ (B) $\left(0, \frac{a}{2}\right)$ (C) (0, a) (D) (a, 0)

Sol: (A) and (C) Apply the condition of tangency to the equation of chord of contact.

Let P (h, k) be the point of intersection of the tangents at the extremities of the chord AB of the circle $x^2 + y^2 = a^2$.

$\therefore$ The equation of the chord of contact AB w.r.t. the point P is $hx + ky = a^2$.

The line $hx + ky = a^2$ touches the circle $x^2 + y^2 - 2ax = 0$ if $\left|\frac{h(a) + k(0) - a^2}{\sqrt{h^2 + k^2}}\right| = a$

$\Rightarrow (h - a)^2 = h^2 + k^2$

Therefore, the locus of (h, k) is $(x - a)^2 = x^2 + y^2$ or, $y^2 = a(a - 2x)$.

Clearly, points (A) and (C) satisfy the above equation.

9.2 Chord Bisected at a given Point

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

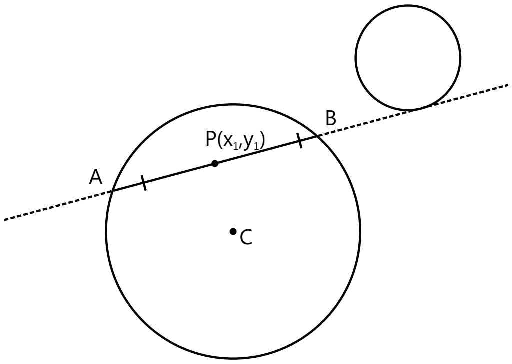


Figure 9.37: Chord bisected by point P

MASTERJEE CONCEPTS

The smallest chord of a circle passing through a point 'M' at a maximum distance from the centre is the one whose middle point is M.

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 24: Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is (1, –1). **(JEE MAIN)**

Sol: Use $T = S_1$

Given, $S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$

For point L (1, 1), $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$ and

$T = x.1 + y.(-1) + 3(x + 1) + 4(y - 1) - 11$ i.e. $T = 4x + 3y - 12$

Now equation of the chord having middle point, L(1, –1) is

$\therefore$ $4x + 3y - 12 = -11$ $\Rightarrow 4x + 3y - 1 = 0$

Second method:

Let C be the centre of the given circle, $C \equiv (-3, -4)$

$\therefore$ Slope of CL = $\frac{-4+1}{-3-1} = \frac{3}{4}$

$\therefore$ Equation of chord whose middle point is L, is

$\therefore \quad y + 1 = -\frac{4}{3}(x - 1)$ [$\because$ chord is perpendicular to CL]

Or, $4x + 3y - 1 = 0$

Illustration 25: Find the locus of the middle points of chords of given circle $x^2 + y^2 = a^2$ which subtends a right angle at the fixed point (p, q). **(JEE ADVANCED)**

Sol: As M(h, k)be the midpoint of the chord AB which subtends an angle of 90^0 at the point N (p , q) therefore a circle can be drawn with AB as the diameter and passing through the point N, hence AM = MN.

$\therefore$ AM = MN $\Rightarrow AM^2 = MN^2$ $\Rightarrow a^2 - (h^2 + k^2) = (h - p)^2 + (k - q)^2$

$\Rightarrow a^2 - h^2 - k^2 = h^2 + p^2 - 2hp + k^2 + q^2 - 2kq$

$\Rightarrow 2h^2 + 2k^2 - 2ph - 2qk + p^2 + q^2 - a^2 = 0$

$\Rightarrow h^2 + k^2 - ph - qk + \frac{1}{2}\left(p^2 + q^2 - a^2\right) = 0$

Hence, the required locus is $x^2 + y^2 - px - qy + \frac{1}{2}\left(p^2 + q^2 - a^2\right) = 0$.

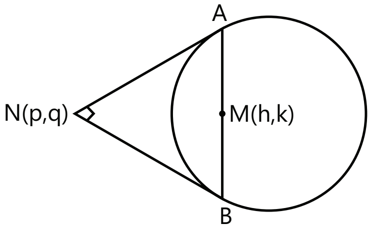


Figure 9.38

10. COMMON CHORD OF TWO CIRCLES

Definition: The chord joining the points of intersection of two given circles is called their common chord.

Equation of common chord: The equation of the common chord of two circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ .... (i)

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ .... (ii)

is given by $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$.

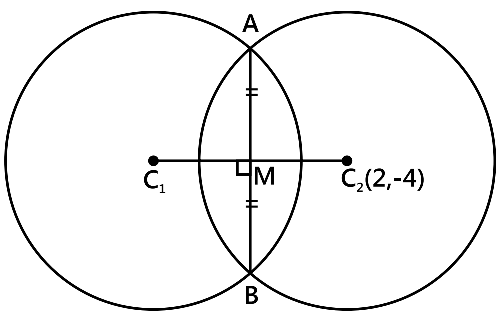


Figure 9.39: Common Chord

Length of the common chord: AB = 2(AM) = $2\sqrt{C_1A^2 - C_1M^2}$

Where, C_1A = radius of the circle $S_1 = 0$, and

C_1M = length of the perpendicular from the centre C_1 to the common chord AB.

Note: If the two circles touch each other, then the length of common chord is zero and the common chord is the common tangent to the two circles at the point of contact.

MASTERJEE CONCEPTS

The length of the common chord AB is maximum when it is the diameter of the smallest circle.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 26: Find the equation and the length of the common chord of two circles.

$2x^2 + 2y^2 + 7x - 5y + 2 = 0$ and $x^2 + y^2 - 4x + 8y - 18 = 0$ **(JEE MAIN)**

Sol: Use the formula for equation of common chord and length of common chord. Equation of common chord of circle is $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ and length of common chord is $2\sqrt{C_2A^2 - C_2M^2}$.

Given $S_1 = x^2 + y^2 + \frac{7}{2}x - \frac{5}{2}y + 1 = 0$ .... (i)

$S_2 = x^2 + y^2 - 4x + 8y - 18 = 0$ .... (ii)

Therefore, the equation of the common chord AB is $S_1 - S_2 = 0$

i.e. $\frac{15}{2}x - \frac{21}{2}y + 19 = 0 \Rightarrow 15x - 21y + 38 = 0$ .... (iii)

The length of the perpendicular from the centre $C_2(2, -4)$ to the common chord AB is $C_2M = \left|\frac{30+84+38}{\sqrt{15^2+21^2}}\right| = \frac{152}{\sqrt{666}}$

Radius of the circle $S_2 = 0$ is, $C_2A = \sqrt{38}$

$\therefore$ The length of the common chord = AB = 2AM

$= 2\sqrt{C_2A^2 - C_2M^2}$; $= 2\sqrt{38 - \left(\frac{152}{\sqrt{666}}\right)^2} = 2\sqrt{\frac{1102}{333}}$

Illustration 27: Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents. **(JEE ADVANCED)**

Sol: As we know that, if (x_1, y_1) is a point of intersection of tangents of circle $x^2 + y^2 = a^2$ then equation of chord of contact is $xx_1 + yy_1 = a^2$ and the equation of common chord of two circles are $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$. By using these two formulae we can solve the problem.

Given circles are $S_1 \equiv x^2 + y^2 - 12 = 0$ ... (i)

and $S_2 \equiv x^2 + y^2 - 5x + 3y - 2 = 0$ ... (ii)

The equation of common chord is $S_1 - S_2 = 0$ i.e. $5x - 3y - 10 = 0$ ... (iii)

Let this line meet circle (i) at A and B, and $P(\alpha, \beta)$ be the point of intersection of the tangents at A and B. Therefore, the equation of the chord of contact AB is $x\alpha + y\beta - 12 = 0$ ... (iv)

As (iii) and (iv) represent the same line, therefore on comparison, we get

$\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{6}{5}$ $\therefore$ $\alpha = 6$ and $\beta = -\frac{18}{5}$. Hence, $P \equiv \left(6, -\frac{18}{5}\right)$.

11. DIAMETER OF CIRCLE

The locus of the middle points of a system of parallel chords of a circle is known as the diameter of the circle.

Let the equation of parallel chords be

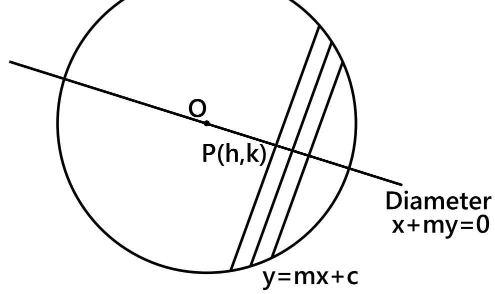


Figure 9.40

$y = mx + c$ (where, c is a parameter).

$\therefore$The equation of the diameter bisecting parallel chords

of the circle $x^2 + y^2 = a^2$ is given by $x + my = 0$.

12. POLE AND POLAR

Let P (x_1,y_1) be any point inside or outside the circle. Passing through the point P chords AB and A′ B′ are drawn. If the tangents at point A and point B intersect at Q (h, k) , then the locus of Q is a straight line and is called the polar of point P with respect to circle and P is called the pole. Similarly, if the tangents to the circle at A′ and B′ meet at Q′, then the locus of Q′ is the polar with P as its pole.

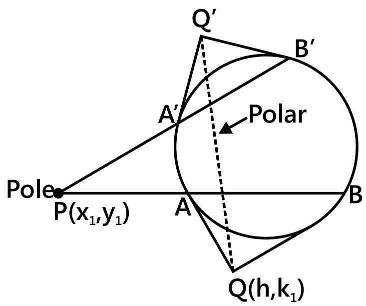


Figure 9.41(a): Polar of a point P outside the circle

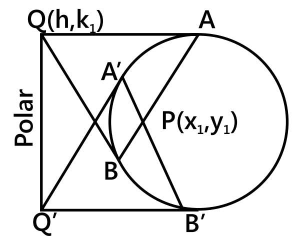


Figure 9.41(b): Polar of a point P inside the circle

Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ w.r.t. point P (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ i.e. T = 0.

If the circle is $x^2 + y^2 = a^2$, then its polar w.r.t. (x_1, y_1) is $xx_1 + yy_1 - a^2 = 0$ i.e. T = 0.

Pole of a line w.r.t. the circle $x^2 + y^2 = a^2$

Consider a line $lx + my + n = 0$ and let (x_1, y_1) be the pole of the line w.r.t. the circle $x^2 + y^2 = a^2$.

For the point (x_1, y_1),

The equation of polar w.r.t. the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$.

Since $lx + my + n = 0$ and $xx_1 + yy_1 - a^2 = 0$ represent the same line.

$$\therefore \frac{x_1}{l} = \frac{y_1}{m} = \frac{-a^2}{n} \quad \Rightarrow x_1 = -\frac{a^2 l}{n} \text{ and } y_1 = -\frac{a^2 m}{n}.$$

Hence, the pole of the line $lx + my + n = 0$ is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

Pole of a line w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Consider a line $lx + my + n = 0$.

If (x_1, y_1) is the pole, then the equation of polar is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Now, since $lx + my + n = 0$ and $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ represent the same line,

$$\therefore \frac{x_1 + g}{l} = \frac{y_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n}$$

On simplification, we get $\frac{x_1 + g}{l} = \frac{y_1 + f}{m} = \frac{g^2 + f^2 - c}{lg + fm - n}$

$$\Rightarrow \frac{x_1 + g}{l} = \frac{y_1 + f}{m} = \frac{r^2}{lg + mf - n}$$, where r is radius of the circle.

12.1 Conjugate Points and Conjugate Lines

(a) If the polar of point P (x_1, y_1) w.r.t. a circle $x^2 + y^2 = a^2$, passes through $Q(x_2, y_2)$, then the polar of Q will pass through P. Such points are called conjugate points and they satisfy the relation $x_1x_2 + y_1y_2 = a^2$

(b) If the pole of the line $l_1x + m_1y + n_1 = 0$ w.r.t. a circle lies on another line $l_2x + m_2y + n_2 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

Consider the circle $x^2 + y^2 = a^2$,

The pole P of the line $l_1x + m_1y + n_1 = 0$ w.r.t. the circle is given by $\left(-\frac{a^2 l_1}{n_1}, -\frac{a^2 l_1}{n_1}\right)$.

$\Rightarrow l_2\left(-\frac{a^2 l_1}{n_1}\right) + m_2\left(-\frac{a^2 m_1}{n_1}\right) + n_2 = 0 \qquad \therefore a^2(l_1l_2 + m_1m_2) = n_1n_2$

MASTERJEE CONCEPTS

Points P (x_1, y_1) and Q (x_2, y_2) are conjugate points w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ if $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

If P and Q are conjugate points w.r.t. a circle with centre at O and radius 'a' then $PQ^2 = OP^2 + OQ^2 - 2a^2$.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 28: Find the pole of the line $3x + 5y + 17 = 0$ with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$. **(JEE MAIN)**

Sol: If $P(\alpha, \beta)$ be the pole of line with respect to the given circle. Then the equation of polar of point $P(\alpha, \beta)$ w.r.t. the circle is $x\alpha + y\beta + 2(x + \alpha) + 3(y + \beta) + 9 = 0$. And this equation represent same line which is represented by equation $3x + 5y + 17 = 0$. By solving these two equation simultaneously we will get required pole.

Given circle is $\quad x^2 + y^2 + 4x + 6y + 9 = 0 \quad$ ... (i)

and, given line is $\quad 3x + 5y + 17 = 0 \quad$ ... (ii

$\Rightarrow (\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0 \quad$ ... (iii)

Since equation (ii) and (iii) represent the same line,

$\therefore \frac{\alpha+2}{3} = \frac{\beta+3}{5} = \frac{2\alpha+3\beta+9}{17} \Rightarrow 5\alpha + 10 = 3\beta + 9$

$\Rightarrow 5\alpha - 3\beta = -1 \quad$ ... (iv)

and, $17\alpha + 34 = 6\alpha + 9\beta + 27 \qquad \Rightarrow 11\alpha - 9\beta = -7 \quad$ ... (v)

From (iv) and (v), we get $\alpha = 1, \beta = 2$

Hence, the pole of the line $3x + 5y + 17 = 0$ w.r.t. the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is (1, 2).

Illustration 29: A variable circle is drawn to touch the axis of x at origin. Find locus of pole of straight line $lx + my + n = 0$ w.r.t. circle. **(JEE ADVANCED)**

Sol: As circle touches x-axis at origin therefore let $(0, \lambda)$ be its centre then equation of circle will be $x^2 + (y - \lambda)^2 = \lambda^2$. Hence by considering $P(x_1, y_1)$ be the pole and using polar equation we will get required result.

Let the centre of the circle be $(0, \lambda)$.

Then the equation of the circle is $x^2 + (y - \lambda)^2 = \lambda^2$

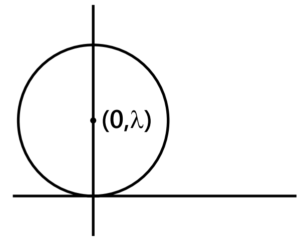


Figure 9.42

$\Rightarrow x^2 + y^2 - 2\lambda y = 0.$

Let $P(x_1, y_1)$ be the pole of the line $lx + my + n = 0$ w.r.t. the circle,

then, the equation of the polar is $xx_1 + yy_1 - \lambda(y + y_1) = 0$

$xx_1 + y(-\lambda + y_1) - \lambda y_1 = 0$

$\therefore$ On comparison, we get $\frac{x_1}{l} = \frac{-\lambda + y_1}{m} = \frac{-\lambda y_1}{n}$.

Hence, the locus of the pole is $ly^2 = mxy - xn$.

Illustration 30: Prove that if two lines at right angles are conjugate w.r.t. circle then one of them passes through centre. **(JEE ADVANCED)**

Sol: Let two perpendicular lines which are conjugate to each other be

$ax + by + c = 0$ ... (i)

$bx - ay + \lambda = 0$ ... (ii)

$\therefore$ The equation of the polar of a point (x_1, y_1) is $xx_1 + yy_1 - r^2 = 0$ ... (iii)

On comparing (i) and (iii), we get $\frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c}$.

From the definition of conjugate lines, we know that the point (x_1, y_1) should satisfy the equation $bx - ay + \lambda = 0$,

hence $\frac{-br^2a}{c} + \frac{ar^2b}{c} + \lambda = 0 \quad \Rightarrow \lambda = 0.$

Therefore, $bx - ay + \lambda = 0$ passes through (0, 0).

13. COMMON TANGENTS TO TWO CIRCLES

Different cases of intersection of two circles:

Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ ... (i) and, $(x - x_1)^2 + (y - y_2)^2 = r_2^2$ ... (ii)

Then following cases may arise:

Case I: When the distance between the centres is greater than the sum of radii. $\boxed{C_1C_2 > r_1 + r_2}$

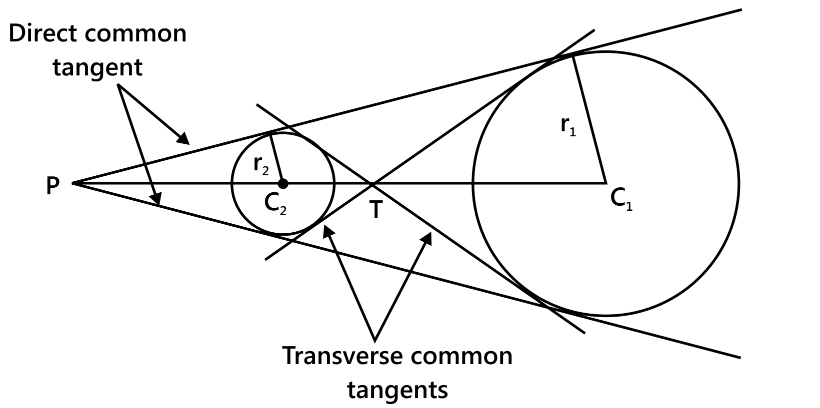


Figure 9.43: Common tangents for non-intersecting and non-overlapping circles

In this case four common tangents can be drawn, in which two are direct common tangents and the other two are transverse common tangents.

The points P and T , the point of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles. The point P and T divide the join of C_1 and C_2 externally and internally respectively in the ratio $r_1 : r_2$.

i.e. $\frac{C_1P}{C_2P} = \frac{r_1}{r_2}$ (externally) and $\frac{C_1T}{C_2T} = \frac{r_1}{r_2}$ (internally)

$$\therefore P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \text{ and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$

Steps to find Equations of Common Tangents

Let the equation of tangent of any circle in the slope form be $(y+f) = m(x+g) + a\sqrt{1+m^2}$ where, a is radius of circle and m is the slope of tangent.

The value of 'm' can be obtained by substituting the co-ordinates of the point P and T in the above equation.

Note: Length of an external (or direct) common tangent, $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$, and Length of an internal (or transverse) common tangent, $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$. where, d is the distance between the centres of the two circles, and r_1, r_2 are the radii of the two circles. Therefore, the length of internal common tangent is always less than the length of the external common tangent.

Case-II: When the distance between the centres is equal to the sum of radii (Circles touching externally)

$$\boxed{C_1C_2 = r_1 + r_2}$$

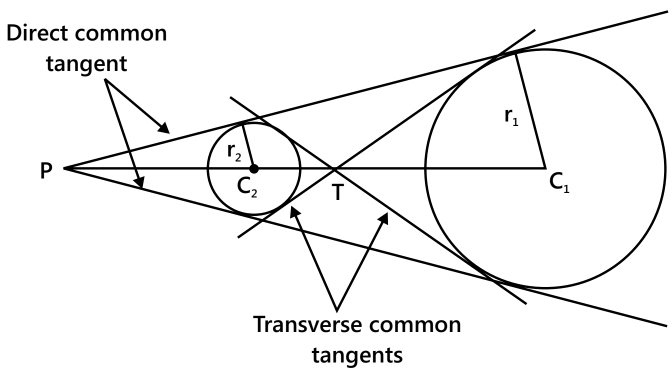


Figure 9.44: Common tangents of circles touching externally

In this case three common tangents can be drawn, two direct common tangents and one transverse common tangent.

Case III: When the distance between the centres is less than the sum of radii. (Intersecting circles)

$$\boxed{|r_1 - r_2| < C_1C_2 < r_1 + r_2}$$

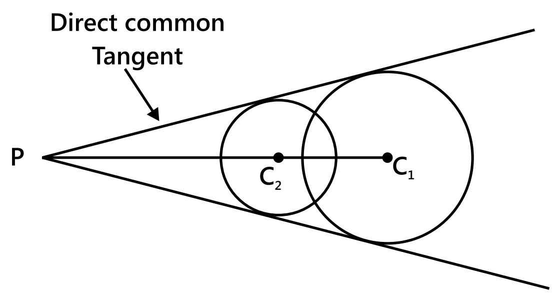


Figure 9.45: Common tangents for intersecting circles

In this case two direct common tangents can be drawn as shown in the diagram.

Case IV: When the distance between the centres is equal to the difference of the radii. (Circles touching each other internally), i.e. $|C_1C_2| = |r_1 - r_2|$.

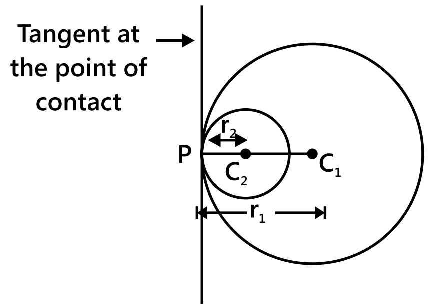


Figure 9.46: Common tangents for circles touching each other internally

In this case the total number of common tangents is one.

Case V: When the distance between the centres is less than the difference of the radii. (Circles neither touch each other nor intersect), i.e. $|C_1C_2| < |r_1 - r_2|$.

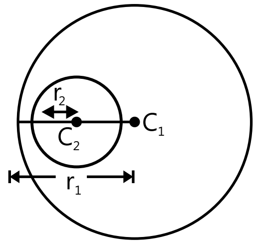


Figure 9.47

In this case, the number of common tangents is zero.

Illustration 31: Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally. **(JEE MAIN)**

Sol: When distance between centre of circle is equal to the sum of their radius then they touches eachother externally and when it is equal to the difference of their radius then circle touches eachother internally.

Let C_1 and C_2 be the centres and r_1 and r_2 the radii of $S_1 \equiv x^2 + y^2 - 2x - 4y = 0$ and $S_1 \equiv x^2 + y^2 - 8y - 4 = 0$ respectively.

$\therefore C_1 \equiv (1, 2), C_2 \equiv (0, 4), r_1 = \sqrt{5}, r_2 = 2\sqrt{5}$

Now, $\quad C_1C_2 = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and

$r_1 + r_2 = 3\sqrt{5}, |r_1 - r_2| = \sqrt{5}$

Thus, $C_1C_2 = |r_1 - r_2|$, hence the two circles touch each other internally,

Illustration 32: Prove that the two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$ **(JEE ADVANCED)**

Sol: Two circles touch each other if distance between centres of these two circles are equal to the sum or difference of their radius.

Let centres of given circles be C_1 and C_2 and their radii be r_1 and r_2 respectively.

$\therefore C_1 \equiv (-a, 0);\ r_1 = \sqrt{a^2 - c}$ and $C_2 \equiv (0, -b);\ r_2 = \sqrt{b^2 - c}$

Two circles touch each other, if $C_1C_2 = r_1 \pm r_2$

$\Rightarrow \sqrt{a^2 + b^2} = \sqrt{a^2 - c} \pm \sqrt{b^2 - c} \Rightarrow a^2 + b^2 = a^2 - c + b^2 - c \pm 2\sqrt{(a^2 - c)(b^2 - c)}$

$\Rightarrow c = \pm\sqrt{a^2b^2 - a^2c - b^2c + c^2} \Rightarrow c^2 = a^2b^2 - a^2c - b^2c + c^2$

$\Rightarrow a^2c + b^2c = a^2b^2 \qquad \therefore \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

Illustration 33: An equation of a common tangent to the circles $x^2 + y^2 + 14x - 4y + 28 = 0$ and $x^2 + y^2 - 14x + 4y - 28 = 0$ is **(JEE ADVANCED)**

(A) $x - 7 = 0$ (B) $y + 7 = 0$ (C) $28x + 45y + 371 = 0$ (D) None of these

Sol: (C) Calculate the distance between the centres and use the different cases of two circles.

Let $S_1 \equiv x^2 + y^2 + 14x - 4y + 28 = 0$ ...(i)

$\Rightarrow C_1 = (-7, 2)$ and $r_1 = 5$

and, $S_2 \equiv x^2 + y^2 - 14x + 4y - 28 = 0$ ...(ii)

$\Rightarrow C_2 = (7, -2)$ and $r_2 = 9$

$\therefore C_1C_2 = \sqrt{(7+7)^2 + (-2-2)^2} > r_1 + r_2$.

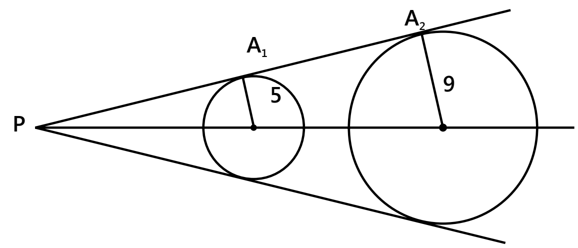


Figure 9.48

Hence, four common tangents are possible.

For $x - 7 = 0$, ... (iii)

Clearly, C_2 lies on the (iii).

For $y + 7 = 0$ ... (iv)

Length of perpendicular from $C_1 = 9 > r_1$.

For $28x + 45y + 371 = 0$ ... (v)

Length of perpendicular from $C_1 = \left|\frac{28(-7) + 45(2) + 371}{\sqrt{28^2 + 45^2}}\right| = \frac{265}{53} = r_1$.

Length of perpendicular from $C_2 = \left|\frac{28(7) + 45(-2) + 371}{\sqrt{28^2 + 45^2}}\right| = \frac{477}{53} = r_2$.

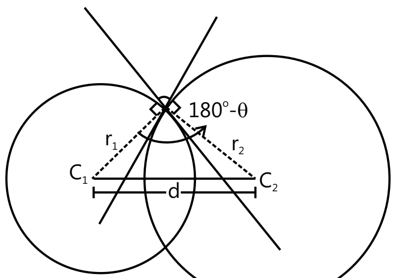


Figure 9.49

14. ANGLE OF INTERSECTION OF TWO CIRCLES

The angle of intersection between two circles S = 0 and S′ = 0 is defined as the angle between their tangents at their point of intersection.

If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$;

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by $\cos(180 - \theta) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

or, $\cos(180 - \theta) = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_1}}$.

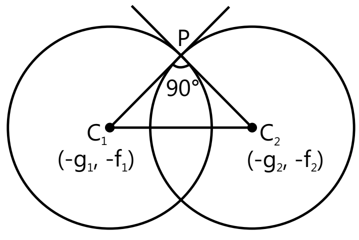


Figure 9.50: Angle of intersection

Condition of Orthogonality: Two circles are said to be orthogonal to each other if the angle of intersection of the two circles is 90°.

$\Rightarrow 2(g_1g_2 + f_1f_2) = c_1 + c_2$.

MASTERJEE CONCEPTS

If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes though the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polar w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal all the three circles.

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Illustration 34: If a circle passes through the point (3, 4) and cuts the circle $x^2 + y^2 = a^2$ orthogonally, the equation of the locus of its centre is **(JEE MAIN)**

(A) $3x + 4y - a^2 = 0$ (B) $6x + 8y = a^2 + 25$

(C) $6x + 8y + a^2 + 25 = 0$ (D) $3x + 4y = a^2 + 25$

Sol : (B)

As we know Two circle are said to be orthogonal if $2(g_1g_2 + f_1f_2) = c_1+c_2$. So by considering required equation of circle as $x^2 + y^2 + 2gx + 2fy + c = 0$ and As point (3, 4) satisfies this equation so by solving these two equation we will get required equation of the locus of its centre.

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

As the point (3, 4) lies on (i), we have $9 + 16 + 6g + 8f + c = 0$

$\Rightarrow 6g + 8f + c = -25$ ... (ii)

$\Rightarrow 2g\times 0 + 2f\times 0 = c - a^2$ $\Rightarrow c = a^2$.

$\therefore$ From equation (ii), we have $6g + 8f + a^2 + 25 = 0$.

Hence locus of the centre (–g, –f) is $6x + 8y - (a^2 + 25) = 0$.

Illustration 35: Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 56 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4x + 1 = 0$. **(JEE ADVANCED)**

Sol: By considering the required circle to be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and using orthogonality formula $2(g_1g_2 + f_1f_2) = c_1+c_2$ we will get a relation between g and f. Also as the centre lies on the line $3x + 4x + 1 = 0$, by solving these equation we will get required result.

Let the required circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

Given $S_1 \equiv x^2 + y^2 + 3x - 5y + 56 = 0$ ... (ii)

and, $S_2 \equiv x^2 + y^2 - 7x + \dfrac{29}{4} = 0$. ... (iii)

Since (i) is orthogonal to (ii) and (iii)

$\therefore\ 2g\left(\dfrac{3}{2}\right) + 2f\left(-\dfrac{5}{2}\right) = c + 6\ + 2f$ $\Rightarrow 3g - 5f = c + 6$ ... (iv) and $2g\left(-\frac{7}{2}\right) + 2f.0 \Rightarrow c + \frac{29}{4}$ $\Rightarrow -7g = c + \frac{29}{4}$ ...(v)

From (iv) and (v), we get $40g - 20f = -5$ ... (vi)

Given line is $3x + 4y = -1$ ... (vii)

$(-g, -f)$ also lies on the line (vii). $\Rightarrow -3g - 4f = -1$ ... (viii)

$\therefore g = 0, f = \frac{1}{4}$ and $c = -\frac{29}{4}$ [From (vi) and (viii)]

$\therefore$ The equation of the circle is $x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$ or, $4(x^2 + y^2) + 2y - 29 = 0$

15. FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the point of intersection of two given circle S = 0 and S′ = 0 is given by $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$)

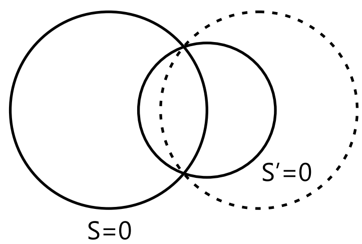


Figure 9.51

(b) The equation of the family of circles passing through the point of intersection of circle S = 0 and a line L = 0 is given by $S + \lambda L = 0$, (where λ is a parameter)

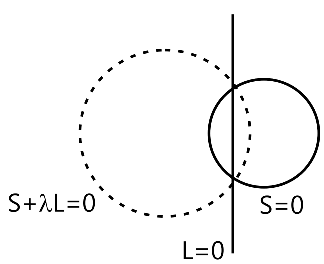


Figure 9.52

(c) The equation of the family of circles touching the circle S = 0 and the line L = 0 at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)

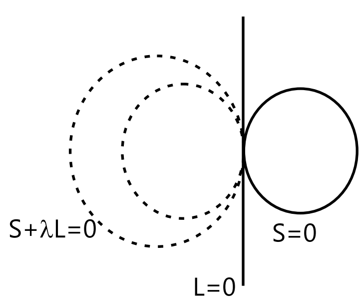


Figure 9.53

(d) The equation of a family of circles passing through two given points $P(x_1, x_1)$ and $Q(x_2, x_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{ (where } \lambda \text{ is a parameter)}$$

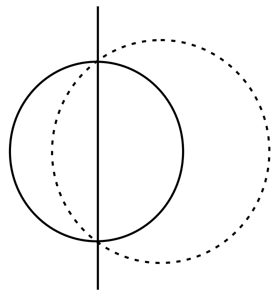


Figure 9.54

In this equation, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the equation of the circle with P and Q as the end points of the diameter and $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of the line through P and Q.

(e) The equation of the family of circles touching the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is $x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$ or, $S + \lambda L = 0$, where, $L = 0$ is the equation of the tangent to the circle at $P(x_1, y_1)$ and $\lambda \in R$.

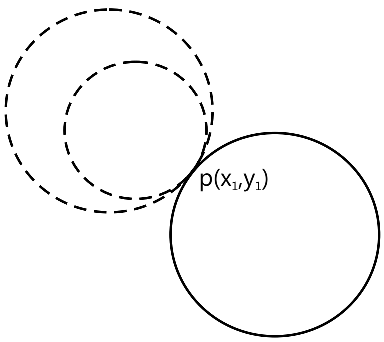


Figure 9.55

(f) The equation of family of circle, which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is

$(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$. And if m is infinite, the family of circle is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$, (where λ is a parameter)

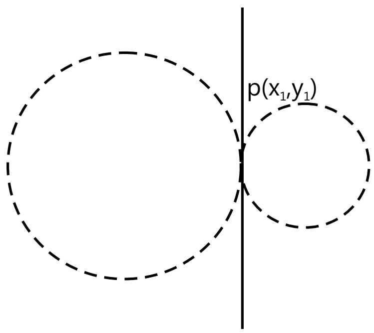


Figure 9.56

Note that $(x - x_1)^2 + (y - y_1)^2 = 0$ represents the equation of a point circle with centre at (x_1, y_1)

(g) Equation of the circles given in diagram is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\theta \{(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\} = 0$

(h) Family of circles circumscribing a triangle whose side are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by $L_1 L_2 + \lambda L_2 L_3 + \mu L_3 L_1 = 0$ provided coefficient of xy = 0 and co-coefficient of x^2 = co-efficient of y^2.

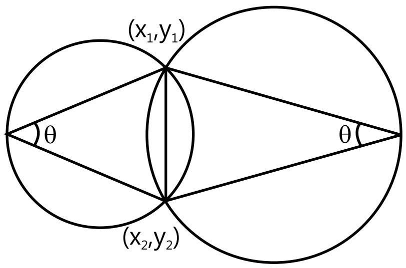


Figure 9.57

(i) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1 L_3 + \lambda L_2 L_4 = 0$ where value of λ can be found out by using condition that co-efficient of $x^2 = y^2$ and co-efficient of xy = 0.

Illustration 36: Find the equation of circle through the points A(1, 1) & B(2, 2) and whose radius is 1. **(JEE MAIN)**

Sol: As we know that, equation of family of circle passing through (x_1, y_1) and

(x_2, y_2) is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda(x - y) = 0$

Equation of AB is $x - y = 0$

∴ Equation of the family of circle passing through A and B is

$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$ or $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

∴ Radius = $\sqrt{\frac{(\lambda - 3)^2}{4} + \frac{(\lambda + 3)^2}{4} - 4}$.

According to the question, $\sqrt{\frac{(\lambda - 3)^2}{4} + \frac{(\lambda + 3)^2}{4} - 4} = 1$

or $(\lambda - 3)^2 + (\lambda + 3)^2 - 16 = 4$ or $2\lambda^2 = 2$ or $\lambda = \pm 1$

∴ Equation of circle is $x^2 + y^2 - 2x - 4y + 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$

Illustration 37: Find the equations of circles which touches $2x - y + 3 = 0$ and pass through the points of intersection of the line $x + 2y - 1 = 0$ and the circle $x^2 + y^2 - 2x + 1 = 0$. **(JEE MAIN)**

Sol: Here in this problem the equation of family of circle will be $S + \lambda L = 0$ by solving this equation we will get centre and radius of required circle in the form of λ and as this circle touches the line $2x - y + 3 = 0$ hence perpendicular distance from centre of circle to the line is equal to the radius of circle.

Let the equation of the family of circles be $S + \lambda L = 0$.

∴ $x^2 + y^2 - 2x + 1 + \lambda(x + 2y - 1) = 0$ or, $x^2 + y^2 - x(2 - \lambda) + 2\lambda y + (1 - \lambda) = 0$ ... (i)

⇒ Centre $(-g, -f)$ is $(\{2 - \lambda\}/2, -\lambda)$

$\Rightarrow r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{(2 - \lambda)^2}{4} + \lambda^2 - (1 - \lambda)} = \frac{1}{2}\sqrt{5\lambda^2} = \frac{\lambda}{2}\sqrt{5}$.

Since the circle touches the line 2x – y + 3 = 0,

$$\therefore \left|\frac{2.\left[(2-\lambda)/2\right]-(-\lambda)+3}{\pm\sqrt{5}}\right| = \frac{\lambda}{2}\sqrt{5} \text{ or, } 5 = \pm\frac{\lambda}{2}5 \quad \Rightarrow \lambda = \pm 2 \qquad \text{... (ii)}$$

Hence, the required circles are $x^2 + y^2 + 4y - 1 = 0$ and $x^2 + y^2 - 4x - 4y + 3 = 0$.

Illustration 38: If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y + p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for **(JEE MAIN)**

(A) All except two values of p (B) Exactly one value of p

(C) All values of p (D) All except one value of p.

Sol: (D) Here in this problem the equation of family of circle will be $S + \lambda L = 0$. and as the circle passes through (1, 1), we can find the values of P such that λ is any real no. except – 1.

Equation of a circle passing through P and Q is

$x^2 + y^2 + 3x + 7y + 2p - 5 + \lambda(x^2 + y^2 + 2x + 2y - p^2) = 0$ ... (i)

Since (i) also passes through (1 , 1), we get $(7 + 2p) - \lambda(p^2 - 6) = 0$

$$\Rightarrow \lambda = \frac{7+2p}{p^2-6} \neq -1 \Rightarrow p \neq -1.$$

Illustration 39: C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_3 which does not pass through C_2 is **(JEE ADVANCED)**

(A) $x - \sqrt{3}y + 2 = 0$ (B) $\sqrt{3}x - y + 2 = 0$

(C) $\sqrt{3}x - y - 2 = 0$ (D) $x + \sqrt{3}y + 2 = 0$

Sol: (B) Equation of any circle passing through any two point (x_1, y_1) and (x_2, y_2) is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

Equation of any circle passing through the centre of C_1 and C_2 is

$$(x-0)(x-1) + (y-0)(y-0) + \lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow x^2 + y^2 - x + \lambda y = 0$. ... (i)

If (i) represents C_3, its radius = 1

$\Rightarrow 1 = (1/4) + (\lambda^2/4) \qquad \Rightarrow \lambda = -\sqrt{3}$ (as λ cannot be +ve)

Hence, the equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$.

Since the radius of C_1 and C_3 are equal, their common tangents will be parallel to the line joining their centres (0, 0) and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

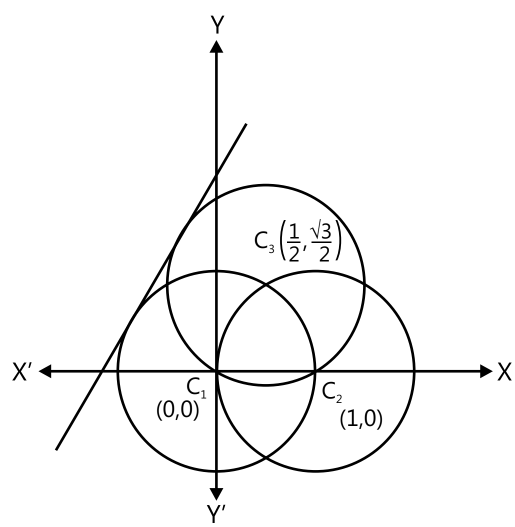


Figure 9.58

So, let the equation of a common tangent be $y = \sqrt{3}x + k$. ... (ii)

From the condition of tangency on C_1, we get $\left|\frac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow k = \pm 2$

Since the tangent does not pass through C_2, the equation of the required common tangent is $\sqrt{3}x - y + 2 = 0$.

Illustration 40: Find the equation of circle circumscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$. **(JEE ADVANCED)**

Sol: Given $L_1 \equiv 3x - y - 9 = 0$ $L_2 \equiv 5x - 3y - 23 = 0$ $L_3 \equiv x + y - 3 = 0$

Family of circles circumscribing a triangle whose side are $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is

$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & co-coefficient of x^2 = co-efficient of y^2.

$\therefore L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$\Rightarrow (3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$\Rightarrow (15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69)$

$+ \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$\Rightarrow (5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu)$

$+ y(50 - 14\lambda - 6\mu) + (207 + 69\lambda + 27\mu) = 0$ ... (i)

The equation (i) represents a circle if

coefficient of x^2 = coefficient of y^2

$\Rightarrow 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu \Rightarrow 8\lambda + 4\mu + 12 = 0$; $2\lambda + \mu + 3 = 0$ ... (ii)

and, coefficient of xy = 0

$\Rightarrow 2\lambda + 2\mu - 14 = 0 \Rightarrow \lambda + \mu - 7 = 0$ ... (iii)

From equation (ii) and (iii), we have $\lambda = -10$, $\mu = 17$

Putting these values of λ & μ in equation (i), we get $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

16. RADICAL AXIS AND RADICAL CENTRE

16.1 Radical Axis

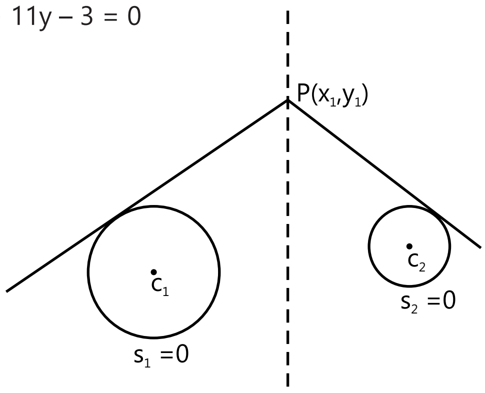


Figure 9.59

The radical axis of two circles is defined as the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal. The radical axis of two circles is a straight line.

$\sqrt{S_1} = \sqrt{S_2}$

$\Rightarrow S_1 = S_2 \Rightarrow S_1 - S_2 = 0$

Consider two circles given by $S_1 = 0$ and $S_2 = 0$. Then the equation of the radical axis of the two circle is $S_1 - S_2 = 0$

i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$,

which is a straight line.

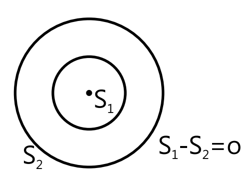


Figure 9.60

Properties of the radical axis

(a) For two intersecting circles the radical axis and common chord are identical. Also, the radical axis and the common tangent are same for two circles touching each other.

(b) The radical axis is perpendicular to the line joining the centres of the two circles.

(c) If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.

(d) Radical axis does not exist if circles are concentric.

(e) Radical axis does not always pass through the mid-point of the line joining the centre of the two circles.

(f) The radical axis of two circles bisects all common tangents of the two circles.

16.2 Radical Centre

The point of intersection of the radical axis of three circles, taken in pairs, is known as their radical centre.

Let the three circles be

$S_1 = 0$ .....(i), $S_2 = 0$ ...(ii), $S_3 = 0$ ...(iii)

Refer to the diagrams shown alongside.

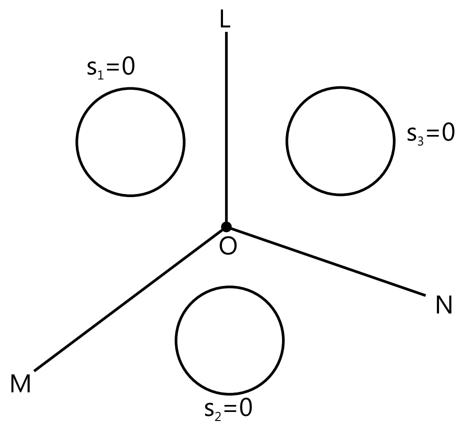


Figure 9.61

Let the straight line OL be the radical axis of the circles $S_1 = 0$ & $S_3 = 0$ and the straight line OM be the radical axis of the circles $S_1 = 0$ & $S_2 = 0$. The equation of any straight line passing through O is given by $(S_1 - S_2) + \lambda(S_3 - S_1) = 0$, where λ is any constant.

For $\lambda = 1$, this equation become $S_2 - S_3 = 0$, which is, equation of ON.

Clearly, the third radical axis also passes through the point where the straight lines OL and OM meet. Hence, the point of intersection of the three radical axis, O is the radical centre.

Properties of radical center

(a) Co-ordinates of radical centre can be found by solving the equation $S_1 = S_2 = S_3$.

(b) The radical centre does not exist if the centre of three circles are collinear.

(c) The circles with centre at radical centre and radius is equal to the length of tangents from radical centre to any of the circle will cut the three circle orthogonally.

(d) If circles are drawn on three sides of a triangle as diameter then radical centre of the three circles is the orthocenter of the triangle. Hence, in case of a right angled triangle, the radical centre of the three circles with the sides as diameter is the vertex with the right angle.

MASTERJEE CONCEPTS

Alternate approach to find the equation of the tangent of a circle passing through a point lying on a given circle.

Consider a point (x_1, y_1) on the given circle $S_1 = 0$. Then the equation of a point circle with (x_1, y_1) as the centre is $S_2 \equiv (x - x_1)^2 + (y - y_1)^2 = 0$. Now we have two circles - one given circle and another point circle. We now have to find the radical axis of those two circles, which is $S_1 - S_2 = 0$.

E.g.: Given a circle $x^2 + y^2 = 8$ and the point on circle is (2, 2), we need to find equation of a tangent to the circle at point (2, 2).

Point circle: $(x - 2)^2 + (y - 2)^2 = 0$ $\Rightarrow x^2 + y^2 - 4x - 4y + 8 = 0$

Hence, the radical axis is $S_1 - S_2 = 0$. $\Rightarrow x + y = 4$, which is also the tangent to the given circle at the point (2, 2).

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 41: Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal. $3x^2 + 3y^2 + 4x - 6y - 1 = 0$, $2x^2 + 2y^2 - 3x - 2y - 4 = 0$, $2x^2 + 2y^2 - x + y - 1 = 0$. **(JEE MAIN)**

Sol: Here by using formula $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, and $S_3 - S_1 = 0$ we will get equations of radical axis and solving these equations we will get required co-ordinate.

Reducing the equation of the circles to the standard form,

$S_1 \equiv x^2 + y^2 + \frac{4}{3}x - 2y - \frac{1}{3} = 0$

$S_2 \equiv x^2 + y^2 - \frac{3}{2}x - y - 2 = 0$

$S_3 \equiv x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2} = 0$

Hence, the equations of the three radical axis is given by

$L_1 \equiv \frac{17}{6}x - y + \frac{5}{3} = 0$ ... (i)

$L_2 \equiv -x - \frac{3}{2}y - \frac{3}{2} = 0,$ ... (ii)

and, $L_3 \equiv -\frac{11}{6}x + \frac{5}{2}y + \frac{1}{6} = 0.$ ... (iii)

Solving (i) and (ii), we get the point $\left(-\frac{16}{21}, \frac{31}{63}\right)$, which also satisfies the equation (iii).

This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

Illustration 42: Find the equation of the circle orthogonal to the three circles $x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9y + 29 = 0$ **(JEE ADVANCED)**

Sol: By using formula of radical axis we will get co- ordinate of radical centre which is also equal to the centre of required circle.

The given circles are

$S_1 \equiv x^2 + y^2 - 2x + 3y - 7 = 0$ ... (i)

$S_2 \equiv x^2 + y^2 + 5x - 5y + 9 = 0$ ... (ii)

and $S_3 \equiv x^2 + y^2 + 7x - 9y + 29 = 0$ ... (iii)

The radical axis of $S_1 = 0$ and $S_2 = 0$ is $7x - 8y + 16 = 0$ ... (iv)

The radical axis of $S_2 = 0$ and $S_3 = 0$ is $x - 2y + 10 = 0$ ...(v)

∴ The radical centre is (8, 9).

Therefore, the length of the tangent from (8, 9) to each of the given circles is $\sqrt{149}$.

∴ The required equation is $(x - 8)^2 + (y - 9)^2 = 149$ or $x^2 + y^2 - 16x - 18y - 4 = 0$.

Illustration 43: If two circles intersect a third circle orthogonally. Prove that their radical axis passes through the centre of the third circle. **(JEE ADVANCED)**

Sol: By considering equation of these circles as $S_r = x^2 + y^2 + 2g_rx + 2f_ry + c_r = 0$

(r = 1, 2, 3) and using radical axis formula we will prove given problem.

Let the given circles be $S_r = x^2 + y^2 + 2g_rx + 2f_ry + c_r = 0$ (r = 1, 2, 3)

Let S_1 and S_2 cut each other orthogonally, then we have

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$ ... (i)

Similarly, let S_2 and S_3 cut each other orthogonally, then we have

$2g_2g_3 + 2f_2f_3 = c_2 + c_3$ ... (ii)

Subtracting (ii) from (i), we get $2(g_1 - g_3)g_2 + 2(f_1 - f_3)f_2 = c_1 - c_3$ ... (iii)

Now the radical axis of S_1 and S_3 is $2(g_1 - g_3)x + 2(f_1 - f_3)y + c_1 - c_3 = 0$ ... (iv)

From (iii) and (iv), the point $(-g_2, -f_2)$ lies on the line (iv). Hence, proved.

Illustration 44: Prove that the square of the length of tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles, and the distance between their centres. **(JEE ADVANCED)**

Sol: Consider two circle as $S_1 \equiv x^2 + y^2 = a^2$ and $S_2 \equiv (x - h)^2 + y^2 = b^2$ and then by using radical axis formula and perpendicular distance formula we will prove given problem.

We have to prove that $PQ^2 = 2 \times PN \times C_1C_2$

Let the equation of the two circles be

$S_1 \equiv x^2 + y^2 = a^2$, and ... (i)

$S_2 \equiv (x - h)^2 + y^2 = b^2$ ... (ii)

Let $P \equiv (a\cos\theta, a\sin\theta)$ be a point on the circle $S_1 = 0$ $\quad \therefore PQ = \sqrt{a^2 - b^2 + h^2 - 2ah\cos\theta}$

and, Radical axis is $\{x^2 + y^2 - a^2\} - \{(x - h)^2 + y^2 - b^2\} = 0$

$\Rightarrow -2hx + h^2 + a^2 - b^2 = 0$ or $x = \dfrac{h^2 + a^2 - b^2}{2h}$

$\Rightarrow PN = \dfrac{h^2 + a^2 - b^2}{2h} - a\cos\theta = \dfrac{h^2 + a^2 - b^2 - 2ah\cos\theta}{2h}$

$\Rightarrow PN \times C_1C_2 = \dfrac{h^2 + a^2 - b^2 - 2ah\cos\theta}{2h} \times h \quad \Rightarrow PN \times C_1C_2 = \dfrac{PQ^2}{2}$

$\therefore PQ^2 = 2\,PN \times C_1C_2$

17. CO-AXIAL SYSTEM OF CIRCLES

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

(1) The equation of a system of co-axial circles, when the equation of the radical axis is $P \equiv lx + my + n = 0$ and, one circle of the system is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).

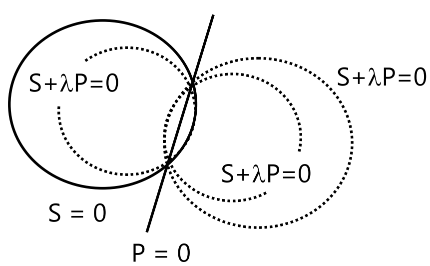


Figure 9.62

(2) The equation of a co-axial system of circles, when the equation of any two circles of the system are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively, is $S_1 + \lambda(S_1 - S_2) = 0$

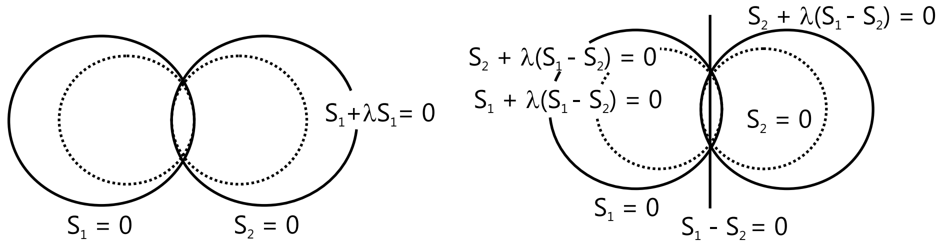


Figure 9.63

or $S_2 + \lambda_1(S_1 - S_2) = 0$

Other form

$S_1 + \lambda S_2 = 0, \quad (\lambda \neq -1)$

Properties of co-axial System of Circles

(a) Centres of all circles of a coaxial system lie on a straight line which is perpendicular to the common radical axis as the line joining the centres of two circles is perpendicular to their radical axis.

(b) Circles passing through two fixed points P and Q form a coaxial system, because every pair of circles has the same common chord PQ and therefore, the same radical axis which is perpendicular bisector of PQ.

MASTERJEE CONCEPTS

The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant. This is the system with center on x-axis and y-axis as common radical axis.

Anvit Tawar (JEE 2009, AIR 9)

Illustration 45: Find the equation of the system of coaxial circles that are tangent at $\left(\sqrt{2},4\right)$ to the locus of point of intersection of mutually perpendicular tangents to the circle $x^2 + y^2 = 9$. **(JEE ADVANCED)**

Sol: The locus of point of intersection of mutually perpendicular tangents is known as the Director circle. Hence by using formula of director circle and co-axial system of circle we will get required result.

∴ The equation of the locus of point of intersection of perpendicular tangents is

$x^2 + y^2 = 18$ ... (i)

Since, $\left(\sqrt{2},4\right)$ satisfies the equation $x^2 + y^2 = 18$,

∴ The tangent at $\left(\sqrt{2},4\right)$ to the circle $x^2 + y^2 = 18$ is $x\cdot\sqrt{2} + y\cdot 4 = 18$ ... (ii)

The equation of the family of circles touching (i) at $\left(\sqrt{2},4\right)$ is

$x^2 + y^2 - 18 + \lambda(\sqrt{2}x + 4y - 18) = 0$ or $x^2 + y^2 + \sqrt{2}\lambda x + 4\lambda y - 18(\lambda + 1) = 0$ ... (iii)

Also any two circles of (iii) have the same radical axis $\sqrt{2}x + 4y - 18 = 0$

∴ The required equation of coaxial circles is (iii).

Illustration 46: Find equation of circle co-axial with $S_1=x^2+y^2+4x+2y+1=0$ and $S_2 = 2x^2 + 2y^2 - 2x - 4y - 3 = 0$ and centre of circle lies on radical axis of these 2 circles. **(JEE MAIN)**

Sol: By using $S_1 - S_2 = 0$ and $S_1 + \lambda L = 0$ we will get equation of radical axis and equation of co-axial system of circle respectively.

$$S_1 - S_2 = 0 \qquad \Rightarrow 5x + \frac{5}{2} + 4y = 0 \qquad \Rightarrow 10x + 8y + 5 = 0$$

$\therefore$ The equation of the radical axis is $10x + 8y + 5 = 0$

The equation of the coaxial system of circles is $x^2 + y^2 + 4x + 2y + 1 + \lambda(10x + 5 + 8y) = 0$

$\Rightarrow$ Centre $\equiv [-(2+5\lambda), -(1+4\lambda)]$ which lies on radical axis, after substituting we get $\Rightarrow \lambda = -\frac{23}{82}$

Illustration 47: For what values of l and m the circles $5(x^2 + y^2) + ly - m = 0$ belongs to the coaxial system determined by the circles $x^2 + y^2 + 2x + 4y - 6 = 0$ and $2(x^2 + y^2) - x = 0$? **(JEE ADVANCED)**

Sol: By using radical axis formula i.e. $S_1 - S_2 = 0$ we will get equations of radical axis and by solving them simultaneously we will get required value of l and m.

Let the circles be $S_1 \equiv x^2 + y^2 + 2x + 4y - 6 = 0$;

$S_2 \equiv x^2 + y^2 - \frac{1}{2}x = 0$;

$S_3 \equiv x^2 + y^2 + \frac{l}{5}y - \frac{m}{5} = 0$.

The equation of the radical axis of circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$,

i.e., $x^2 + y^2 + 2x + 4y - 6 - \left(x^2 + y^2 - \frac{1}{2}x\right) = 0$ or, $5x + 8y - 12 = 0$ ... (i)

The equation of the radical axis of circles $S_2 = 0$ and $S_3 = 0$ is $S_2 - S_3 = 0$,

i.e., $x^2 + y^2 - \frac{1}{2}x - \left(x^2 + y^2 + \frac{l}{5}y - \frac{m}{5}\right) = 0$ or, $5x + 2ly - 2m = 0$ ... (ii)

On comparing (i) and (ii), $\frac{5}{5} = \frac{8}{2l} = \frac{-12}{-2m} \Rightarrow 1 = \frac{4}{l} = \frac{6}{m}$ $\therefore l = 4, m = 6$.

18. LIMITING POINTS

Limiting point of system of co-axial circles are the centres of the point circles belonging to the family

Let the circle be $x^2 + y^2 + 2gx + c = 0$ where g is a variable and c is a constant.

$\therefore$ Centre $\equiv (-g, 0)$ and Radius $= \sqrt{g^2 - c}$.

A circle is said to be a point circle, if the radius is equal to 0, i.e. $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus, we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Depending on the sign of c, either the limiting points are real and distinct, real and coincident or imaginary.

18.1 System of Co-axial Circles when Limiting Points are given

Let (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the equation of the corresponding point circles are $S_1 \equiv (x - a)^2 + (y - b)^2 = 0$ and $S_2 \equiv (x - \alpha)^2 + (y - \beta)^2 = 0$.

$\therefore$ The coaxial system of circles is given by $S_1 + \lambda S_2 = 0, \lambda \neq -1$.

or, $(x - a)^2 + (y - b)^2 + \lambda\{(x - \alpha)^2 + (y - \beta)^2\} = 0, \lambda \neq -1$.

MASTERJEE CONCEPTS

If origin is a limiting point of the coaxial system containing the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2+f^2}, \frac{-fc}{g^2+f^2}\right)$.

A common tangent drawn to any two circles of a coaxial system subtends an angle of $\frac{\pi}{2}$ at the limiting points.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 48: Equation of a circles through the origin and belonging to the co-axial system, of which the limiting points are (1, 2), (4, 3) is **(JEE ADVANCED)**

(A) $x^2 + y^2 - 2x + 4y = 0$ (B) $x^2 + y^2 - 8x - 6y = 0$

(C) $2x^2 + 2y^2 - x - 7y = 0$ (D) $x^2 + y^2 - 6x - 10y = 0$

Sol: (C) As we know, if (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the equation of the corresponding point circles are $S_1 \equiv (x-a)^2 + (y-b)^2 = 0$ and $S_2 \equiv (x-\alpha)^2 + (y-\beta)^2 = 0$ so by using the formula of co-axial system i.e. $S_1 + \lambda S_2 = 0$ we will get required result.

Equations of the point circles having (1, 2) and (4,3) as centres is

$S_1 \equiv (x-1)^2 + (y-2)^2 = 0 \quad \Rightarrow x^2 + y^2 - 2x - 4y + 5 = 0$

and, $S_2 \equiv (x-4)^2 + (y-3)^2 = 0 \quad \Rightarrow x^2 + y^2 - 8x - 6y + 25 = 0$

$\therefore$ The co-axial system of circles is $S_1 + \lambda S_2 = 0$.

i.e. $x^2 + y^2 - 2x - 4y + 5 + \lambda(x^2 + y^2 - 8x - 6y + 25) = 0$ ... (i)

If (0, 0) lies on the circle given by equation (i), then

$0^2 + 0^2 - 2(0) - 4(0) + 5 + \lambda(0^2 + 0^2 - 8(0) - 6(0) + 25) = 0$

$\Rightarrow 5 + 25\lambda = 0 \quad$ or, $\lambda = -\left(\frac{1}{5}\right)$.

$\therefore$ The equation of the required circle is $5(x^2 + y^2 - 2x - 4y + 5) - (x^2 + y^2 - 8x - 6y + 25) = 0$

$\Rightarrow 4x^2 + 4y^2 - 2x - 14y = 0 \quad \Rightarrow 2x^2 + 2y^2 - x - 7y = 0$.

19. IMAGE OF THE CIRCLE BY LINE MIRROR

Here, let us consider a general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and a line $lx + my + n = 0$. If we take the image of the circle in the given line, then the radius of image circle remains unchanged and the centre lies on the opposite side of the line at an equal distance.

Let the centre of image circle be (x_1, y_1).

$\therefore$ Slope of $C_1C_2 \times$ Slope of $(lx + my + n = 0) = -1$ ..... (i)

And the mid-point of C_1 and C_2 lies on the line $lx + my + n = 0$

$$l\left(\frac{x_1-g}{2}\right) + m\left(\frac{y_1-f}{2}\right) + n = 0 \quad \text{..... (ii)}$$

From (i) and (ii), we get the centre of the image circle and the radius is $\sqrt{\left(g^2+f^2-c\right)}$ (same as the given circle), and hence the equation of the image.

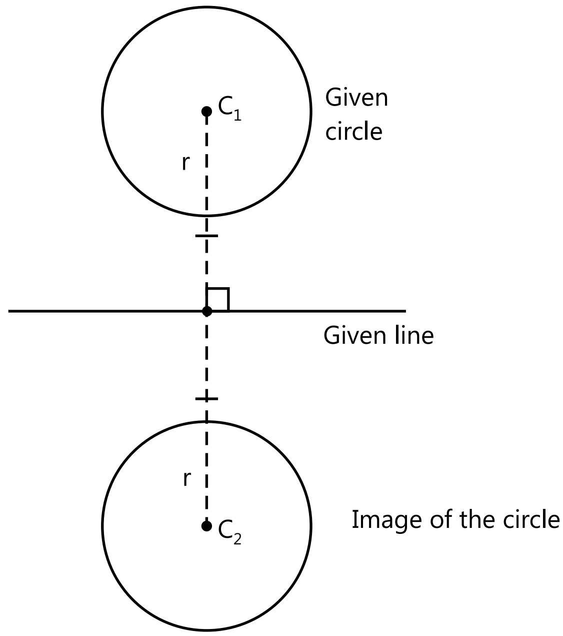


Figure 9.64: Image of a circle

PROBLEM-SOLVING TACTICS

(a) Let S = 0, S′ = 0 be two circles with centers C_1, C_2 and radii R_1, R_2 respectively.

(i) If $C_1C_2 > r_1 + r_2$ then each circle lies completely outside the other circle.

(ii) If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally. (Trick) the point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally.

(iii) If $|r_1 - r_2| < C_1C_2 < r1 + r2$ then the two circles intersect at two points P and Q.

(iv) If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally. (Trick) The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ externally.

(v) If $C_1C_2 < |r_1 - r_2|$ then one circle lies completely inside the other circle.

(b) Two intersecting circles are said to cut each other orthogonally if the angle between the circles is a right angle. Let the circles be $S = x^2 + y^2 + 2gx + 2fy + c = 0$, $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$.

And let d be the distance between the centers of two intersecting circles with radii r1, r2. The two circles will intersect orthogonally if and only if

(i) $D^2 =$ and

(ii) $2g\,g' + 2f\,f' = c + c'$.

FORMULAE SHEET

1. General equation of a circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) Centre of the circle = $(-g, -f)$.

$g = \frac{1}{2}$ coefficient of x, and $f = \frac{1}{2}$ coefficient of y.

(ii) $r = \sqrt{g^2 + f^2 - c}$

2 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

(i) $a = b \neq 0$ (ii) $h = 0$ (iii) $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$ (iv) $g^2 + f^2 - c \geq 0$

3. if centre of circle is (h, k) and radius 'r' then equation of circle is: $(x - h)^2 + (y - k)^2 = r^2$

4. The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is : $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Centre : $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. $r = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$

5. (i) In parametric form:

$x = -g + \sqrt{(g^2 + f^2 - c)}\cos\theta$ and $y = -f + \sqrt{(g^2 + f^2 - c)}\sin\theta$, $(0 \leq \theta < 2\pi)$

6. (i) Circle passing through three non-collinear points

$A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is represented by $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

7. Circle circumscribing the triangle formed by the lines

$a_i x + b_i y + c_i = 0 \ (i = 1,2,3)$: $\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1 x + b_1 y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2 x + b_2 y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3 x + b_3 y + c_3} & a_3 & b_3 \end{vmatrix} = 0$

8. Intercepts length made by the circle On X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

9. Position of point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

When $S_1 \equiv x_1^2 + y_1^2 + 2gx + 2fy + c >\ =\ < 0$ respectively.

10. The power of $P(x_1, y_1)$ w.r.t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA. PB which is $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_2 + c$.

PA. PB = PC.PD = PT^2 = square of the length of a tangent

11. Intercept length cut off from the line y =mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2\left(1+m^2\right)-c^2}{1+m^2}}$

12. The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

13. The equation of tangent at $(a \cos\theta, a \sin\theta)$ is $x \cos\theta + y \sin\theta = a$

14. Condition for tangency:

line y = mx + c is tangent of the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

and the point of contact of tangent $y = mx \pm a\sqrt{1+m^2}$ is $\left(\frac{\mp ma}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$

15. The length of the tangent from a point $P(x_1,y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

16. Pair of tangent from point (0, 0) to the circle are at right angles if $g^2 + f^2 = 2c$.

17. Equation of director circle of the circle $x^2 + y^2 = a^2$ is equal to $x^2 + y^2 = 2a^2$.

18. Equation of Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

19. The equation of normal at any point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - x_1y = 0$ or $\frac{x}{x_1} = \frac{y}{y_1}$.

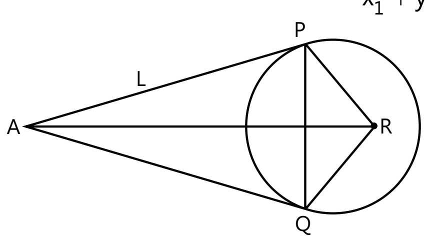
20. Equation of normal at $(a \cos\theta, a \sin\theta)$ is $y = x \tan\theta$ or $y = mx$.

21. The equation of the chord of contact of tangents drawn from a point (x_1, y_1)

to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_2 = a^2$. And

to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

22. Area of Δ APQ is given by $\frac{a\left(x_1^2 + y_1^2 - a^2\right)^{\frac{3}{2}}}{x_1^2 + y_1^2} = \frac{RL^3}{R^2 + L^2}$. Where L & R are length of tangent and radius of circle.



23. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Bisected at the point (x_1, y_1) is given $T = S_1$.

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

24. The equation of the common chord of two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is equal to $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ i.e., $S_1 - S_2 = 0$.

25. Length of the common chord : PQ = 2(PM) = $2\sqrt{C_1P^2 - C_1M^2}$. Where,

C_1P = radius of the circle $S_1 = 0$

C_1M =perpendicular length from the centre C_1 to the common chord PQ.

26. Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 = a^2$

w.r.t. (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ and $xx_1 + yy_1 - a^2 = 0$. Respectively.

27. The pole of the line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$: $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

28. P (x_1, y_1) and $Q(x_2, y_2)$ are conjugate points of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

When $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

If P and Q are conjugate points w.r.t. a circle with centre at O and radius r then $PQ^2 = OP^2 + OQ - 2r^2$.

29. The points P and T are a intersection point of direct common tangents and transverse. Common tangents respectively, and it divide line joining the centres of the circles externally and internally respectively in the ratio of their radii.

$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally)}$$
$$\frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of P and T are.

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \text{ and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

30. If two circles $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ of r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos(180 - \theta) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \text{or} \quad \cos = (180 - \theta) = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_1^2 + f_1^2 - c_2}}.$$

31. Condition for orthogonality: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

32. $S_1 - S_2 = 0$ the equation of the radical axis of the two circle. i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ which is a straight line.

33. The two limiting points of the given co-axial system are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

34. If two limiting points of a coaxial system of circles is (a, b) and (α, β).

then $S_1 + \lambda S_2 = 0, \lambda \neq -1$. or, $\{(x - a)^2 + (y - b)^2\} + \lambda\{(x - \alpha)^2 + (y - \beta)^2\} = 0, \lambda \neq -1$ is the

Coaxial system of circle.

35. If origin is a limiting point of the coaxial system containing the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2 + f^2}, \frac{-fc}{g^2 + f^2}\right)$.