Illustration 46: Find out the shortest distance between the line y = x – 2 and the parabola $y = x^2 + 3x + 2$.

(JEE MAIN)

Sol: The distance would be minimum at the point on the parabola where the slope of the tangent is equal to the slope of the given line.

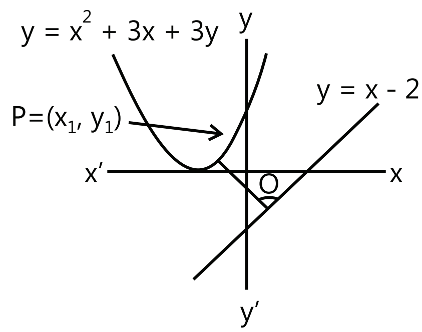

Figure 21.8

Let P(x_1, y_1) is the point closest to the line y = x – 2

Then, $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of the line

$\Rightarrow \quad 2x_1 + 3 = 1 \Rightarrow x_1 = -1$ and $y_1 = 0$

Therefore, point (–1, 0) is the closest and its perpendicular distance from the line y = x – 2 gives the shortest distance.

$\Rightarrow$ Shortest distance = $\frac{3}{\sqrt{2}}$ units

Illustration 47: Which of the following points of the curve $y = x^2$ is closest to (4, – ½)? **(JEE MAIN)**

(A) (1, 1) (B) (2, 4) (C) (2/3, 4/9) (D) (4/3, 16/9)

Sol:(A) Using distance formula find the distance of the given point from the curve and find the minima.

Let the required point be (x, y) on the curve.

Hence, d = $\sqrt{(x-4)^2 + (y+1/2)^2}$ should be minimum, which is enough to consider.

$D = (x-4)^2 + (y + 1/2)^2 = (x-4)^2 + (x^2 + 1/2)^2$

$D' = 4x^3 + 4x - 8$

Now for critical points

$D' = 0$ so $x^3 + x - 2 = 0 \Rightarrow x = 1$

Clearly D″ at x = 1 is 16 > 0.

Thus, D is minimum when x = 1. Hence the required point is (1, 1).

PROBLEM-SOLVING TACTICS

- Reduce any fractions to be as basic as possible.
- Recognise when we can use the chain rule. it enables us to differentiate functions that often seem impossible to differentiate. Whenever you see a nested function, try to assess if the chain rule is needed (it usually is).
- We always want to start a long chain of differentiation by differentiating the last part of the function to touch the input - in short, the outermost part of the function.

FORMULAE SHEET

$\frac{dc}{dx} = 0$	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
$\frac{d}{dx}a^x = (\ln a)\, a^x$	$\frac{d}{dx}a^u = (\ln a)\, a^u\frac{du}{dx}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^u = e^u\frac{du}{dx}$
$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$	$\frac{d}{dx}\log_a u = \frac{1}{(\ln a)u}\frac{du}{dx}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$
$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$	$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u\frac{du}{dx}$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

* Equation of tangent to the curve y = f(x) at $A(x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

* Equation of normal at (x_1, y_1) to the curve y = f(x) is $(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$

*** Length of Tangent, Normal, Subtangent and Subnormal**

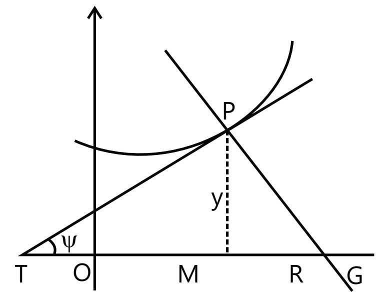

Figure 21.9

Tangent: $PT = MP \operatorname{cosec} \Psi = y\sqrt{1+\cot^2 \psi} = \left| \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right|$

Subtangent: $TM = MP \cot \Psi = \left| \frac{y}{(dy/dx)} \right|$

Normal: $GP = MP \sec \Psi = y\sqrt{1+\tan^2 \psi} = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$

Subnormal: $MG = MP \tan \Psi = \left| y\left(\frac{dy}{dx}\right) \right|$

*** Angle of Intersection of Two Curves**

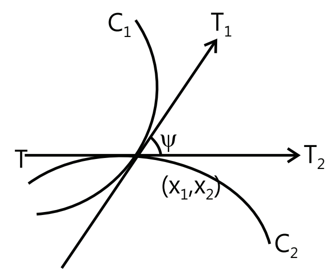

Figure 21.10

$\tan \Psi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$,

where m_1 and m_2 are the slopes of the tangents T_1 and T_2 at the intersection point (x_1, y_1).

Solved Examples

JEE Main/Boards

Example 1: Show that the function f(x) = | x | is continuous at x = 0, but not differentiable at x = 0.

Sol: Evaluate $f'(0^+)$ and $f'(0^-)$.

We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 = f(0)$

The function is continuous at x = 0

We also have

$f'(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$

$f'(0^-) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{-(-x) - 0}{-x} = -1$

Since, $f'(0^+) \neq f'(0^-)$, the function is not differentiable at x = 0

Example 2: Find the derivative of the function f(x), defined by f(x) = sin x by 1st principle.

Sol: Use the first principle to find the derivative of the given function.

Let dy be the increment in y corresponding to an increment dx in x. We have

y = sin x

y + dy = sin (x + dx)