7. TRIGONOMETRIC RATIOS, IDENTITIES AND EQUATIONS

1. INTRODUCTION

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations e.g. $\cos\theta = 0, \cos^2\theta - 4\cos\theta = 1, \sin^2\theta + \sin\theta = 2\cos^2\theta - 4\sin\theta = 1$.

2. TRIGONOMETRIC FUNCTIONS (CIRCULAR FUNCTIONS)

Function	Domain	Range
sin A	R	[–1, 1]
cos A	R	[-1, 1]
tan A	$R - [(2n+1)\pi/2, n \in I]$	$R = (-\infty, \infty)$
cosec A	$R - [n\pi, n \in I]$	$(-\infty, -1] \cup [1, \infty)$
sec A	$R - \{(2n+1)\pi/2, n \in I\}$	$(-\infty, -1] \cup [1, \infty)$
cot A	$R - [n\pi, n \in I]$	$(-\infty, \infty)$

We find, $|\sin A| \le 1, |\cos A| \le 1, \sec A \ge 1$ or $\sec A \le -1$ and $\text{cosec}\,A \ge 1$ or $\text{cosec}\,A \le -1$

2.1 Some Basic Formulae of Trigonometric Functions

(a) $\sin^2 A + \cos^2 A = 1.$

(b) $\sec^2 A - \tan^2 A = 1$

(c) $\text{cosec}^2 A - \cot^2 A = 1$

(d) $\sin A\,\text{cosec}\,A = \tan A \cot A = \cos A \sec A = 1$

A system of rectangular coordinate axes divide a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The signs of the trigonometric ratios in the four quadrants are shown in Fig 7.1.

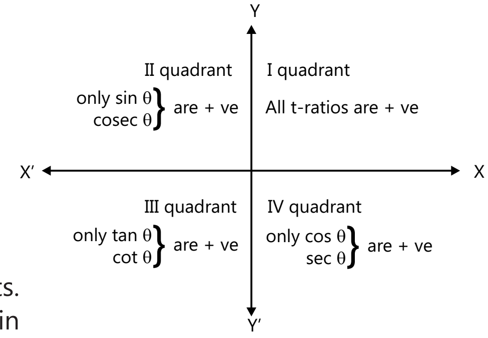

Figure 7.1

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A crude way to remember the sign is "Add Sugar to Coffee". This implies the 1st letter of each word gives you the trigonometric functions with a +ve sign.

Eg. Add-1st word $\Rightarrow$ 1st quadrant 1st letter=A $\Rightarrow$ All are positive to-3rd word $\Rightarrow$ 3rd quadrant 1st letter-t $\Rightarrow$ tan θ (cot θ) are positive.

Ravi Vooda (JEE 2009, AIR 71)

Sine, cosine and tangent of some angles less than 90°:

Trigonometric ratios	**0°**	**15°**	**18°**	**30°**	**36°**
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$
	37°	**45°**	**53°**	**60°**	**90°**
sin	$\approx 3/5$	$\frac{1}{\sqrt{2}}$	$\approx 4/5$	$\frac{\sqrt{3}}{2}$	1
cos	$\approx 4/5$	$\frac{1}{\sqrt{2}}$	$\approx 3/5$	$\frac{1}{2}$	0
tan	$\approx 3/4$	1	$\approx 4/3$	$\sqrt{3}$	Not defined

Illustration 1: Prove the following identities:

(i) $\left(1+\tan^2 A\right)+\left(1+\frac{1}{\tan^2 A}\right)=\frac{1}{\sin^2 A-\sin^4 A}$

(ii) $\frac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2$ **(JEE MAIN)**

Sol: (i) Simply by using Pythagorean and product identities, we can solve these problems.

(i) L.H.S. = $(1 + \tan^2 A) + \left(1+\frac{1}{\tan^2 A}\right) = \sec^2 A + (1 + \cot^2 A)$

$=\sec^2 A + \text{cosec}^2 A = \frac{1}{\cos^2 A}+\frac{1}{\sin^2 A}=\frac{\sin^2 A+\cos^2 A}{\sin^2 A.\cos^2 A}\left[\because \sin^2\theta+\cos^2\theta=1\right]$

$$= \frac{1}{\sin^2 A\left(1-\sin^2 A\right)} = \frac{1}{\sin^2 A - \sin^4 A} = \text{R.H.S.}\left[\because \cos^2\theta = 1-\sin^2\theta\right]$$

Hence proved.

(ii) L.H.S. = $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \frac{\sec^2\theta}{\operatorname{cosec}^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$ ... (i)

Now, R.H.S. = $\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \left(\frac{1-\tan\theta}{1-\frac{1}{\tan\theta}}\right)^2 = \left(\frac{1-\tan\theta}{\frac{\tan\theta-1}{\tan\theta}}\right)^2 = \left(\frac{1-\tan\theta}{-(1-\tan\theta)}.\tan\theta\right)^2 = \tan^2\theta$ ... (ii)

From (i) and (ii), clearly, L.H.S. = R.H.S. **Proved.**

Illustration 2: Prove the following identities: **(JEE MAIN)**

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A\cos^2 A} - 2 = \sec^2 A\operatorname{cosec}^2 A - 2$

(ii) $\sec^4 A\,(1 - \sin^4 A) - 2\tan^2 A = 1$

Sol: Use algebra and appropriate identities to solve these problems.

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{\sin^4 A + \cos^4 A}{\sin^2 A\cos^2 A} = \frac{\left(\sin^2 A\right)^2 + \left(\cos^2 A\right)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A.\cos^2 A}{\sin^2 A\cos^2 A}$

$$= \frac{\left(\sin^2 A + \cos^2 A\right)^2 - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A} = \frac{1-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$= \frac{1}{\sin^2 A\cos^2 A} - \frac{2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A} = \sec^2 A\ \operatorname{cosec}^2 A - 2$ = R.H.S. **Proved.**

(ii) L.H.S. = $\sec^4 A\,(1 - \sin^4 A) - 2\tan^2 A = \sec^4 A - \frac{\sin^4 A}{\cos^4 A} - 2\tan^2 A = \sec^4 A - \tan^4 A - 2\tan^2 A$

$= \left(1+\tan^2 A\right)^2 - \tan^4 A - 2\tan^2 A = 1 + 2\tan^2 A + \tan^4 A - \tan^4 A - 2\tan^2 A = 1$ = R.H.S. **Proved.**

Illustration 3: Prove the following identities:

(i) $\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \operatorname{cosec}\alpha + \cot\alpha$ (ii) $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \sec\alpha + \tan\alpha$ **(JEE MAIN)**

Sol: By rationalizing L.H.S. we will get required result.

(i) L.H.S. = $\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}\times\frac{1+\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{(1+\cos\alpha)^2}{1-\cos^2\alpha}}$

$= \sqrt{\frac{(1+\cos\alpha)^2}{\sin^2\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{1}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha} = \operatorname{cosec}\alpha + \cot\alpha = \text{R.H.S.}$ **Proved.**

(ii) L.H.S. = $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \sqrt{\frac{(1+\sin\alpha)(1+\sin\alpha)}{(1-\sin\alpha)(1+\sin\alpha)}} = \sqrt{\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}} = \sqrt{\frac{(1+\sin\alpha)^2}{\cos^2\alpha}}$

$$= \frac{1+\sin\alpha}{\cos\alpha} = \frac{1}{\cos\alpha} + \frac{\sin\alpha}{\cos\alpha} = \sec\alpha + \tan\alpha = \text{R.H.S.}$$ **Proved.**

Illustration 4: In each of the following identities, show that:

(i) $\frac{\cot A + \tan B}{\cot B + \tan A} = \cot A.\tan B$ (ii) $\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$ **(JEE ADVANCED)**

Sol: Apply tangent and cotangent identity.

(i) L.H.S. $= \frac{\cot A + \tan B}{\cot B + \tan A} = \frac{\frac{\cos A}{\sin A} + \frac{\sin B}{\cos B}}{\frac{\cos B}{\sin B} + \frac{\sin A}{\cos A}} = \frac{\frac{\cos A\cos B + \sin A\sin B}{\sin A\cos B}}{\frac{\cos A\cos B + \sin A\sin B}{\sin B\cos A}}$

$$= \frac{\sin B\cos A}{\sin A\cos B} = \left(\frac{\cos A}{\sin A}\right)\left(\frac{\sin B}{\cos B}\right) = \cot A\tan B = \text{R.H.S.}$$ **Proved.**

(ii) L.H.S. $= \tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B} = \frac{\sin^2 A\cos^2 B - \cos^2 A\sin^2 B}{\cos^2 A\cos^2 B}$

$$= \frac{\sin^2 A\left(1-\sin^2 B\right) - \left(1-\sin^2 A\right)\sin^2 B}{\cos^2 A\cos^2 B} = \frac{\sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B}{\cos^2 A\cos^2 B} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B} = \text{R.H.S.}$$ **Proved.**

Illustration 5: Prove the following identities: $\frac{1}{\operatorname{cosec}\theta - \cot\theta} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta + \cot\theta}$ **(JEE ADVANCED)**

Sol: By rearranging terms we will get $\frac{1}{\operatorname{cosec}\theta - \cot\theta} + \frac{1}{\operatorname{cosec}\theta + \cot\theta} = \frac{2}{\sin\theta}$, and then using Pythagorean identity we can solve this problem.

We have, $\frac{1}{\operatorname{cosec}\theta - \cot\theta} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta + \cot\theta}$

$$\Rightarrow \frac{1}{\operatorname{cosec}\theta - \cot\theta} + \frac{1}{\operatorname{cosec}\theta + \cot\theta} = \frac{1}{\sin\theta} + \frac{1}{\sin\theta} \Rightarrow \frac{1}{\operatorname{cosec}\theta - \cot\theta} + \frac{1}{\operatorname{cosec}\theta + \cot\theta} = \frac{2}{\sin\theta}$$

Now, L.H.S. $= \frac{1}{\operatorname{cosec}\theta - \cot\theta} + \frac{1}{\operatorname{cosec}\theta + \cot\theta} = \frac{\operatorname{cosec}\theta + \cot\theta + \operatorname{cosec}\theta - \cot\theta}{(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta)}$

$$= \frac{2\operatorname{cosec}\theta}{\left(\operatorname{cosec}^2\theta - \cot^2\theta\right)} \quad \left[\because \operatorname{cosec}^2\theta - \cot^2\theta = 1\right]$$

$$= \frac{2\operatorname{cosec}\theta}{1} = \frac{2}{\sin\theta} = \text{R.H.S.} \quad \left[\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$ **Proved.**

Alternative Method

R.H.S $= \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta + \cot\theta} = \operatorname{cosec}\theta - \frac{(\operatorname{cosec}\theta - \cot\theta)}{\operatorname{cosec}^2\theta - \cot^2\theta}$

$= \operatorname{cosec}\theta - \operatorname{cosec}\theta + \cot\theta$

$= \cot\theta$ **Proved.**

Illustration 6: Prove that:

(i) $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}=\sin^2 A.\cos^2 A$

(ii) $\dfrac{\sin A}{\sec A+\tan A-1}+\dfrac{\cos A}{\operatorname{cosec} A+\cot A-1}=1$ **(JEE ADVANCED)**

Sol: Using algebra and appropriate identities, we can prove this.

(i) L.H.S. = $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$

$$=\frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\operatorname{cosec} A)(\sec^2 A+\sec A\operatorname{cosec} A+\operatorname{cosec}^2 A)}\qquad\left[\because a^3-b^3=(a-b)(a^2+ab+b^2)\right]$$

$$=\frac{\frac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\operatorname{cosec} A)\left(\frac{1}{\cos^2 A}+\frac{1}{\cos A\sin A}+\frac{1}{\sin^2 A}\right)}=\frac{(\sin A\cos A+1)\left(\frac{\sin A}{\sin A\cos A}-\frac{\cos A}{\sin A\cos A}\right)}{(\sec A-\operatorname{cosec} A)\left(\frac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)(\sec A-\operatorname{cosec} A)}{(\sec A-\operatorname{cosec} A)(1+\sin A\cos A)}\times\sin^2 A\cos^2 A\qquad\left[\because \sin^2\theta+\cos^2\theta=1\right]=\sin^2 A\cos^2 A=\text{R.H.S.}$$ **Proved.**

(ii) L.H.S. = $\dfrac{\sin A}{\sec A+\tan A-1}+\dfrac{\cos A}{\operatorname{cosec} A+\cot A-1}$

$$=\frac{\sin A\operatorname{cosec} A+\sin A\cot A-\sin A+\cos A\sec A+\cos A\tan A-\cos A}{(\sec A+\tan A-1)(\operatorname{cosec} A+\cot A-1)}$$

$$=\frac{1+\cos A-\sin A+1+\sin A-\cos A}{\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}-1\right)\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}-1\right)}=\frac{2}{\left(\frac{1+\sin A-\cos A}{\cos A}\right)\frac{(1+\cos A-\sin A)}{\sin A}}$$

$$=\frac{2\sin A\cos A}{[1+(\sin A-\cos A)][1-(\sin A-\cos A)]}=\frac{2\sin A\cos A}{1-(\sin A-\cos A)^2}$$

$$=\frac{2\sin A\cos A}{1-(\sin^2 A+\cos^2 A-2\sin A\cos A)}\qquad\left[\because (a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}=\frac{2\sin A\cos A}{1-1+2\sin A\cos A}\qquad\left[\because \sin^2\theta+\cos^2\theta=1\right]=\frac{2\sin A\cos A}{2\sin A\cos A}=1=\text{R.H.S.}$$ **Proved.**

Illustration 7: Prove that:

$$\left(\frac{1}{\sec^2\theta-\cos^2\theta}+\frac{1}{\operatorname{cosec}^2\theta-\sin^2\theta}\right)\sin^2\theta.\cos^2\theta=\frac{1-\sin^2\theta.\cos^2\theta}{2+\sin^2\theta.\cos^2\theta}$$ **(JEE ADVANCED)**

Sol: Write L.H.S. in terms of cosine and sine functions.

$$\text{L.H.S.} = \left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\operatorname{cosec}^2 - \theta\sec^2\theta}\right)\sin^2\theta.\cos^2\theta$$

$$= \left[\frac{1}{\frac{1}{\cos^2\theta} - \cos^2\theta} + \frac{1}{\frac{1}{\sin^2\theta} - \sin^2\theta}\right]\sin^2\theta\cos^2\theta = \left[\frac{\cos^2\theta}{1-\cos^4\theta} + \frac{\sin^2\theta}{1-\sin^4\theta}\right]\sin^2\theta\cos^2\theta$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)(1-\cos^2\theta)} + \frac{\sin^2\theta}{(1-\sin^2\theta)(1+\sin^2\theta)}\right]\sin^2\theta\cos^2\theta \qquad \left[\because a^2 - b^2 = (a-b)(a+b)\right]$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)\sin^2\theta} + \frac{\sin^2\theta}{\cos^2\theta(1+\sin^2\theta)}\right]\sin^2\theta\cos^2\theta$$

$$= \frac{\cos^4\theta}{1+\cos^2\theta} + \frac{\sin^4\theta}{1+\sin^2\theta} = \frac{\left[\cos^4\theta(1+\sin^2\theta) + \sin^4\theta(1+\cos^2\theta)\right]}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{\cos^4\theta + \sin^2\theta\cos^4\theta + \sin^4\theta + \sin^4\theta\cos^2\theta}{(1+\cos^2\theta)(1+\sin^2\theta)} = \frac{\sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{(\sin^2\theta + \cos^2\theta) - \sin^2\theta\cos^2\theta}{1+\sin^2\theta+\cos^2\theta+\sin^2\theta\cos^2\theta} = \frac{1-\sin^2\theta\cos^2\theta}{1+1+\sin^2\theta\cos^2\theta} = \frac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\cos^2\theta} = \text{R.H.S.}$$

Proved.

3. TRANSFORMATIONS

3.1 Compound, Multiple and Sub-Multiple Angles

Circular functions of the algebraic sum of two angles can be expressed as circular functions of separate angles.

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$; $\qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A.\tan B}\ ; \qquad \cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$$

Circular functions of multiples of an angle can be expressed as circular functions of the angle.

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A} = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}\ ; \qquad \sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A\ ; \qquad \tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Circular functions of half of an angle can be expressed as circular functions of the complete angle.

$$\sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}};\ \cos\frac{A}{2}=\sqrt{\frac{1+\cos A}{2}};\quad \tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{1-\cos A}{\sin A}=\frac{\sin A}{1+\cos A}$$

3.2 Complementary and Supplementary Angles

$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=\cos\theta$

$\tan(-\theta)=-\tan\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$

$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$

$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$

$\sin(\pi-\theta)=\sin\theta$

$\cos(\pi-\theta)=-\cos\theta$

$\tan(\pi-\theta)=-\tan\theta$

$\sin(\pi+\theta)=-\sin\theta$

$\cos(\pi+\theta)=-\cos\theta$

$\tan(\pi+\theta)=\tan\theta$

3.3 Product to Sum and Sum to Product

$$\sin C+\sin D=2\sin\frac{C+D}{2}.\cos\frac{C-D}{2};\qquad \sin C-\sin D=2\cos\frac{C+D}{2}.\sin\frac{C-D}{2}$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}.\cos\frac{C-D}{2};\qquad \cos C-\cos D=2\sin\frac{C+D}{2}.\sin\frac{D-C}{2}$$

Note: $\sin C+\cos D=\sin C+\sin\left(\frac{\pi}{2}-D\right)=2\sin\frac{C+\frac{\pi}{2}-D}{2}.\cos\frac{C-\frac{\pi}{2}+D}{2}$

$$\tan C+\tan D=\frac{\sin C}{\cos C}+\frac{\sin D}{\cos D}=\frac{\sin(C+D)}{\cos C.\cos D};\qquad \sin A.\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$$

$$\sin A.\sin B=\frac{1}{2}\{\cos(A-B)-\cos(A+B)\};\quad \cos A.\cos B=\frac{1}{2}\{\cos(A-B)+\cos(A+B)\}$$

$$\sin(A+B).\sin(A-B)=\sin^2 A-\sin^2 B;\qquad \cos(A+B).\cos(A-B)=\cos^2 A-\sin^2 B$$

3.4 Power Reduction

$$\sin^2 A=\frac{1}{2}(1-\cos 2A)\qquad \cos^2 A=\frac{1}{2}(1+\cos 2A)$$

$$\tan^2 A=\frac{1-\cos 2A}{1+\cos 2A};\quad \sin^3 A=\frac{3\sin A-\sin 3A}{4};\ \cos^3 A=\frac{3\cos A+\cos 3A}{4}$$

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- $\cos A.\cos 2A.\cos 2^2.A\cos 2^3 A......\cos 2^{n-1}A = \begin{cases} \dfrac{\sin 2^n A}{2^n \sin A} & \text{if} \quad A \neq n\pi \\ 1 & \text{if} \quad A = 2n\pi \\ -1 & \text{if} \quad A = (2n+1)\pi \end{cases}$
- $\sin(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 .... \cos A_n (S_1 - S_3 + S_5 - S_7 + ...)$
- $\cos(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 .... \cos A_n (1 - S_2 + S_4 - S_6 ....)$
- $\tan(A_1 + A_2 + ... + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$

Where,

S_1 = tan A_1 + tan A_2 + ... + tan A_n = The sum of the tangents of the separate angles.

S_2 = tan A_1 tan A_2 + tan A_1 tan A_3 + ... = The sum of the tangents taken two at a time.

S_3 = tan A_1 tan A_2 tan A_3 + tan A_2 tan A_3 tan A_4 + ... = Sum of tangents three at a time, and so on.

If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$. $S_3 = {}^nC_3 \tan^3 A$, ...

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4. TRIGONOMETRIC IDENTITY

A trigonometric equation is said to be an identity if it is true for all values of the angle or angles involved. A given identity may be established by (i) Reducing either side to the other one, or (ii) Reducing each side to the same expression, or (iii) Any convenient, modification of the methods given in (i) and (ii).

4.1 Conditional Identity

When the angles, A, B and C satisfy a given relation, we can establish many interesting identities connecting the trigonometric functions of these angles. To prove these identities, we require the properties of complementary and supplementary angles. For example, if A + B + C = π, then

1. $\sin(B+C) = \sin A, \cos B = -\cos(C+A)$
2. $\cos(A+B) = -\cos C, \sin C = \sin(A+B)$
3. $\tan(C+A) = -\tan B, \cot A = -\cot(B+C)$
4. $\cos\dfrac{A+B}{2} = \sin\dfrac{C}{2}, \cos\dfrac{C}{2} = \sin\dfrac{A+B}{2}$
5. $\sin\dfrac{C+A}{2} = \cos\dfrac{B}{2}, \sin\dfrac{A}{2} = \cos\dfrac{B+C}{2}$
6. $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}, \tan\dfrac{B}{2} = \cot\dfrac{C+A}{2}$

Some Important Identities: If $A+B+C=\pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
3. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$
4. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

5. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\ \sin B\ \sin C$

6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$

8. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

9. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 8: Show that:

(i) $\sin(40^\circ + \theta)\cos(10^\circ + \theta) - \cos(40^\circ + \theta)\sin(10^\circ + \theta) = \frac{1}{2}$

(ii) $\cos\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \phi\right) - \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} - \phi\right) = \sin(\theta + \phi)$ **(JEE MAIN)**

Sol: Use sum and difference formulae of sine and cosine functions.

(i) L.H.S. = $\sin(40^\circ + \theta)\cos(10^\circ + \theta) - \cos(40^\circ + \theta)\sin(10^\circ + \theta)$

$= \sin\{(40^\circ + \theta) - (10^\circ + \theta)\}$ $[\because \sin(A - B) = \sin A\cos B - \cos A\sin B]$ $= \sin 30^\circ = \frac{1}{2} = \text{R.H.S.}$ **Proved.**

(ii) L.H.S. = $\cos\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \phi\right) - \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} - \phi\right)$

$= \cos\left\{\left(\frac{\pi}{4} - \theta\right) + \left(\frac{\pi}{4} - \phi\right)\right\}$ $[\because \cos(A + B) = \cos A\cos B - \sin A\sin B]$ $= \cos\left\{\frac{\pi}{2} - (\theta + \phi)\right\}$

$= \sin(\theta + \phi) = \text{R.H.S.}$ $\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$ **Proved**

Illustration 9: Find the value of $\tan(\alpha + \beta)$, given that $\cot\alpha = \frac{1}{2}, \alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}, \beta \in \left(\frac{\pi}{2}, \pi\right)$.

(JEE MAIN)

Sol: As we know, $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha . \tan\beta}$, therefore by using product and Pythagorean identities we can obtain the values of $\tan\alpha$ and $\tan\beta$.

Given, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$

Also, $\sec\beta = -\frac{5}{3}$. Then $\tan\beta = \sqrt{\sec^2\beta - 1} = \pm\sqrt{\frac{25}{9} - 1} = \pm\frac{4}{3}$

But $\beta \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \tan\beta = -\frac{4}{3}$ $[\because \tan\beta \text{ is} - \text{ve in II quadrant}]$

Substituting $\tan\alpha = 2$ and $\tan\beta = -\frac{4}{3}$ in (1), we get $\tan(\alpha + \beta) = \frac{2 + \left(-\frac{4}{3}\right)}{1 - (2)\left(-\frac{4}{3}\right)} = \frac{+\frac{2}{3}}{\frac{11}{3}} = +\frac{2}{11}$

Illustration 10: Prove that: $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$ **(JEE MAIN)**

Sol: Here we can write $\tan 3A$ as $\tan(2A + A)$, and then by using $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ we can solve this problem.

We have: 3A = 2A + A $\Rightarrow \tan 3A = \tan(2A + A)$ $\Rightarrow \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

$\Rightarrow \tan 3A(1-\tan 2A\tan A)=\tan 2A+\tan A \;\; \Rightarrow \tan 3A-\tan 3A\tan 2A\tan A=\tan 2A+\tan A$

$\Rightarrow \tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A$ **Proved.**

Illustration 11: Prove that: $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}=2\cos\theta$ **(JEE MAIN)**

Sol: Use $1+\cos 2\theta=2\cos^2\theta$, to solve this problem.

L.H.S. = $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}=\sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$=\sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$ $[\because 1+\cos 2\theta=2\cos^2\theta]$

$=\sqrt{2+\sqrt{2+2\cos 4\theta}}=\sqrt{2+\sqrt{2(1+\cos 4\theta)}}=\sqrt{2+\sqrt{2(2\cos^2 2\theta)}}=\sqrt{2+2\cos 2\theta}$

$=\sqrt{2(1+\cos 2\theta)}=\sqrt{2.2\cos^2\theta}=\sqrt{4\cos^2\theta}=2\cos\theta=$ R.H.S. **Proved.**

Illustration 12: If tan A = $\frac{m}{m-1}$ and tan B = $\frac{1}{2m-1}$, prove that A – B = $\frac{\pi}{4}$ **(JEE ADVANCED)**

Sol: Simply using tan (A – B) = $\frac{\tan A-\tan B}{1+\tan A.\tan B}$, we can prove above equation.

We have, tan A = $\frac{m}{m-1}$ and tan B = $\frac{1}{2m-1}$

Now, tan (A – B) = $\frac{\tan A-\tan B}{1+\tan A.\tan B}$ ... (i)

Substituting the values of tan A and tan B in (i), we get

$$\tan(A-B)=\frac{\frac{m}{m-1}-\frac{1}{2m-1}}{1+\left(\frac{m}{m-1}\right)\left(\frac{1}{2m-1}\right)}=\frac{2m^2-m-m+1}{(m-1)(2m-1)}\times\frac{(m-1)(2m-1)}{2m^2-3m+1+m}=1$$

$\Rightarrow \tan(A-B)=\tan\frac{\pi}{4}$ $\left[\because \tan\frac{\pi}{4}=1\right]$ $\Rightarrow A-B=\frac{\pi}{4}$ **Proved.**

Illustration 13: If $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$; prove that $\tan(\alpha-\beta)=(1-n)\tan\alpha$ **(JEE ADVANCED)**

Sol: Same as above problem $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$, therefore by substituting $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, we can prove given equation.

L.H.S. = $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ ... (i)

Substituting $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$ in (i), we get L.H.S. = $\frac{\frac{\sin\alpha}{\cos\alpha}-\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\frac{\sin\alpha}{\cos\alpha}.\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$ $\left[\because \tan\alpha=\frac{\sin\alpha}{\cos\alpha}\right]$

$$= \frac{\sin\alpha\left(1-n\sin^2\alpha\right)-n\sin\alpha\cos^2\alpha}{\cos\alpha\left(1-n\sin^2\alpha\right)+n\sin^2\alpha\cos\alpha} = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha\cos^2\alpha}{\cos\alpha - n\sin^2\alpha\cos\alpha + n\sin^2\alpha\cos\alpha}$$

$$= \frac{\sin\alpha - n\sin\alpha\left(\sin^2\alpha+\cos^2\alpha\right)}{\cos\alpha} = \frac{\sin\alpha - n\sin\alpha}{\cos\alpha}$$

$\left[\because \sin^2\alpha+\cos^2\alpha=1\right]$ $= \dfrac{(1-n)\sin\alpha}{\cos\alpha} = (1-n)\tan\alpha = $ R.H.S. **Proved.**

Illustration 14: If $\theta+\phi=\alpha$ and $\sin\theta = k\sin\phi$, prove that $\tan\theta = \dfrac{k\sin\alpha}{1+k\cos\alpha}, \tan\phi = \dfrac{\sin\alpha}{k+\cos\alpha}$ **(JEE ADVANCED)**

Sol: Here $\phi = \alpha-\theta$, substitute this in $\sin\theta = k\sin\phi$ and then use compound angle formula to obtain required result.

We have, $\theta+\phi=\alpha \Rightarrow \phi = \alpha-\theta$ ... (i)

and $\sin\theta = k\sin\phi$ ... (ii)

$\Rightarrow \sin\theta = k\sin(\alpha-\theta)$ [Using (i)] $= k\left[\sin\alpha\cos\theta - \cos\alpha\sin\theta\right]$

$\Rightarrow \sin\theta = k\sin\alpha\cos\theta - k\cos\alpha\sin\theta$ ... (iii)

Dividing both sides of (iii) by $\cos\theta$, we get $\tan\theta = k\sin\alpha - k\cos\alpha.\tan\theta$

$\Rightarrow \tan\theta + k\cos\alpha.\tan\theta = k\sin\alpha \Rightarrow \tan\theta(1+k\cos\alpha) = k\sin\alpha \Rightarrow \tan\theta = \dfrac{k\sin\alpha}{1+k\cos\alpha}$ **Proved.**

Again, $\sin\theta = k\sin\phi$ $\Rightarrow \sin(\alpha-\phi) = k\sin\phi$ $\left[\theta+\phi=\alpha \Rightarrow \theta = \alpha-\phi\right]$

$\Rightarrow \sin\alpha\cos\phi - \cos\alpha\sin\phi = k\sin\phi$ ... (iv)

Dividing both side of (iv) by $\cos\phi$, we get

$\Rightarrow \sin\alpha - \cos\alpha\tan\phi = k\tan\phi \Rightarrow (k+\cos\alpha)\tan\phi = \sin\alpha \Rightarrow \tan\phi = \dfrac{\sin\alpha}{k+\cos\alpha}$ **Proved.**

Illustration 15: Prove that: $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$ **(JEE ADVANCED)**

Sol: Use $\cos\alpha+\cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$, to solve this problem.

L.H.S = $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma) = (\cos\alpha+\cos\beta)+\left[\cos\gamma+\cos(\alpha+\beta+\gamma)\right]$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right).\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{R.H.S.}$ **Proved.**

Illustration 16: If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then show that xy + yz + zx = 0. **(JEE ADVANCED)**

Sol: Consider $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$, obtain the value of x , y and z in terms of k, and solve L.H.S. of given equation.

Let $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$ ... (i)

$\Rightarrow \frac{1}{x} = \frac{\cos\theta}{k}, \frac{1}{y} = \frac{\cos\left(\theta+\frac{2\pi}{3}\right)}{k}, \frac{1}{z} = \frac{\cos\left(\theta+\frac{4\pi}{3}\right)}{k}$

Now, L.H.S. = xy + yz + zx = $\frac{xyz}{z}+\frac{xyz}{x}+\frac{xyz}{y} = xyz\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)$

$= xyz\left[\frac{\cos\left(\theta+\frac{4\pi}{3}\right)}{k}+\frac{\cos\theta}{k}+\frac{\cos\left(\theta+\frac{2\pi}{3}\right)}{k}\right]$ [Using (i)] $= \frac{xyz}{k}\left[\cos\left(\theta+\frac{4\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\theta\right]$

$= \frac{xyz}{k}\left[2\cos\frac{2\theta+2\pi}{2}\cos\frac{\pi}{3}+\cos\theta\right] = \frac{xyz}{k}\left[2\cos(\pi+\theta)\cos\frac{\pi}{3}+\cos\theta\right] = \frac{xyz}{k}\left[-2\cos\theta.\left(\frac{1}{2}\right)+\cos\theta\right]$

$= \frac{xyz}{k}\left[-\cos\theta+\cos\theta\right] = \frac{xyz}{k}[0] = 0 \Rightarrow xy+zy+zx = 0$ **Proved.**

Illustration 17: Prove that: $\cos\theta\cos2\theta\cos4\theta......\cos2^{n-1}\theta = \frac{\sin\left(2^n\theta\right)}{2^n\left(\sin\theta\right)}$ **(JEE ADVANCED)**

Sol: Multiply and divide L.H.S. by $2\sin\theta$ and apply $\sin(2\theta) = 2\sin\theta\cos\theta$.

Here, we observe that each angle in L.H.S. is double of the preceding angle.

L.H.S. = $\cos\theta\cos2\theta\cos4\theta......\cos2^{n-1}\theta$

$= \frac{1}{2\sin\theta}(2\sin\theta.\cos\theta)\cos2\theta.\cos4\theta.....\cos2^{n-1}\theta = \frac{1}{2^2\sin\theta}(2\sin2\theta.\cos2\theta)\left(\cos4\theta......\cos2^{n-1}\theta\right)$

$= \frac{1}{2^3\sin\theta}(2\sin4\theta.\cos4\theta)\left[\cos8\theta\cos16\theta......\cos2^{n-1}\theta\right]$ $[\because \sin2n\theta = 2\sin n\theta\cos n\theta]$

$= \frac{1}{2^4\sin\theta}(2\sin8\theta.\cos8\theta)\left[\cos16\theta......\cos2^{n-1}\theta\right] = \frac{1}{2^n\sin\theta}\left[2\sin2^{n-1}\theta\cos2^{n-1}\theta\right] = \frac{\sin\left(2^n\theta\right)}{2^n\sin\theta} = \text{R.H.S.}$ **Proved.**

Illustration 18: If $\cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$, prove that $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$ **(JEE ADVANCED)**

Sol: Substitute $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ and $\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$ in given equation i.e. $\cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$.

Now, $\cos\theta = \dfrac{a\cos\phi + b}{a + b\cos\phi}$ $\Rightarrow \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{a.\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)+b}{a+b\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)}$ [Using (i)]

$$\Rightarrow \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{a\left[1-\tan^2\frac{\phi}{2}\right]+b\left[1+\tan^2\frac{\phi}{2}\right]}{a\left[1+\tan^2\frac{\phi}{2}\right]+b\left[1-\tan^2\frac{\phi}{2}\right]} = \frac{a-a\tan^2\frac{\phi}{2}+b+b\tan^2\frac{\phi}{2}}{a+a\tan^2\frac{\phi}{2}+b-b\tan^2\frac{\phi}{2}}$$

Applying componendo and dividendo, we get

$$\frac{2\tan^2\frac{\theta}{2}}{2} = \frac{2a\tan^2\left(\frac{\phi}{2}\right)-2b\tan^2\left(\frac{\phi}{2}\right)}{2a+2b} = \frac{(a-b)\tan^2\frac{\phi}{2}}{a+b} \Rightarrow \tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$$ **Proved**

5. SOLUTION OF TRIGONOMETRIC EQUATION

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

Eg.: if $\sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}, ...$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature). These solutions can be classified as:

(i) Principal solution (ii) General solution

5.1 Principal Solution

The solutions of a trigonometric equation which lie in the interval $[-\pi, \pi)$ are called principal solutions.

Methods for Finding Principal Value

Suppose we have to find the principal value of θ satisfying the equation $\sin\theta = -\dfrac{1}{2}$. Since $\sin\theta$ is negative, θ will be in 3rd or 4th quadrant. We can approach the 3rd and the 4th quadrant from two directions. Following the anticlockwise direction, the numerical value of the angle will be greater than π. The clockwise approach would result in the angles being numerically less than π. To find the principal value, we have to take the angle which is numerically smallest.

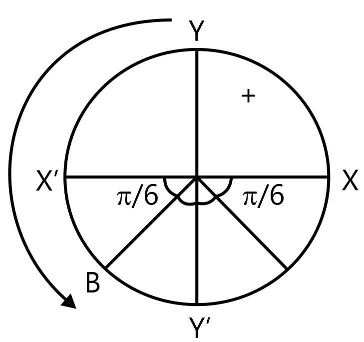

Figure 7.2

For Principal Value

(a) If the angle is in the 1st or 2nd quadrant, we must select the anticlockwise direction and if the angles are in the 3rd or 4th quadrant, we must select the clockwise direction.

(b) Principal value is never numerically greater than π.

(c) Principal value always lies in the first circle (i.e. in first rotation)

On the above criteria, θ will be $-\frac{\pi}{6}$ or $-\frac{5\pi}{6}$. Among these two $-\frac{\pi}{6}$ has the least numerical value. Hence $-\frac{\pi}{6}$ is the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$

From the above discussion, the method for finding principal value can be summed up as follows:

(a) First identify the quadrants in which the angle lies.

(b) Select the anticlockwise direction for the 1st and 2nd quadrants and select clockwise direction for the 3rd and 4th quadrants.

(c) Find the angle in the first rotation.

(d) Select the numerically least value among these two values. The angle thus found will be the principal value.

(e) In case, two angles, one with a positive sign and the other with a negative sign have the same numerical value, then it is the convention to select the angle with the positive sign as the principal value.

5.2 General Solution

The expression which gives all solutions of a trigonometric equation is called a General Solution.

General Solution of Trigonometric Equations

In this section we shall obtain the general solutions of trigonometric equations

$\sin\theta = 0, \cos\theta = 0, \tan\theta = 0$ and $\cot\theta = 0$.

General Solution of $\sin\theta = 0$

By Graphical approach:

The graph clearly shows that $\sin\theta = 0$ at

$\theta = 0, \pi, 2\pi, \ldots\ldots, -\pi, -2\pi \ldots\ldots$

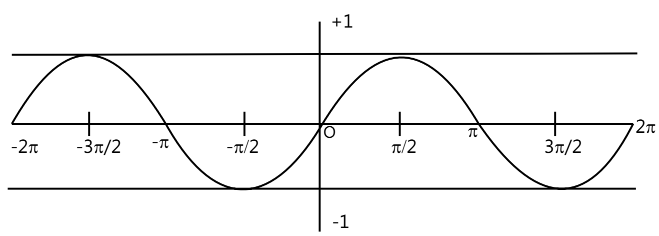

Figure 7.3

So the general solution of $\sin\theta = 0$ is $\theta = n\pi : n \in I$ where n = 0, ± 1, ± 2 ....

Note: Trigonometric functions are periodic functions. Therefore, solutions of trigonometric equations can be generalized with the help of periodicity of trigonometric functions.

MASTERJEE CONCEPTS

A trigonometric identity is satisfied by any value of an unknown angle while a trigonometric equation is satisfied by certain values of the unknown.

Vaibhav Krishnan (JEE 2009, AIR 22)

Method for Finding Principal Value

(a) First note the quadrants in which the angle lies.

(b) For the 1st and 2nd quadrants, consider the anticlockwise direction. For the 3rd and 4th quadrants, consider the clockwise direction.

(c) Find the angles in the 1st rotation.

(d) Select the numerically least value among these two values. The angle found will be the principal value.

Illustration 19: Principal value of $\tan\theta = -1$ is **(JEE MAIN)**

Sol: Solve it by using above mentioned method.

$\because \tan\theta$ is negative

$\therefore \theta$ will lie in 2nd or 4th quadrant

For the 2nd quadrant, we will choose the anticlockwise direction and for the 4th quadrant, we will select the clockwise direction.

In the first circle, two values $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are obtained.

Among these two, $-\frac{\pi}{4}$ is numerically least angle.

Hence, the principal value is $-\frac{\pi}{4}$

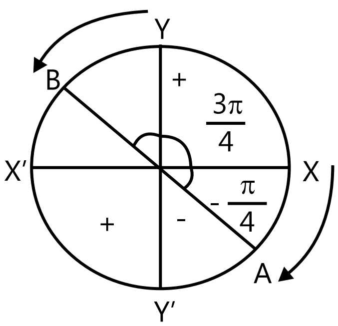

Figure 7.4

Illustration 20: Principal value of cos $\theta = \frac{1}{2}$ is: **(JEE MAIN)**

Sol: Here $\cos\theta$ is $(+)$ve hence θ will lie in 1st or 4th quadrant.

$\because \cos\theta$ is $(+)$ve $\therefore \theta$ will lie in the 1st or the 4th quadrant.

For the 1st quadrant, we will select the anticlockwise direction and for the 4th quadrant, we will select the clockwise direction.

As a result, in the first circle, two values $\frac{\pi}{3}$ and $\frac{-\pi}{3}$ are found.

Both $\frac{\pi}{3}$ and $-\frac{\pi}{3}$ have the same numerical value.

In such a case, $\frac{\pi}{3}$ will be selected as the principal value, as it has a positive sign.

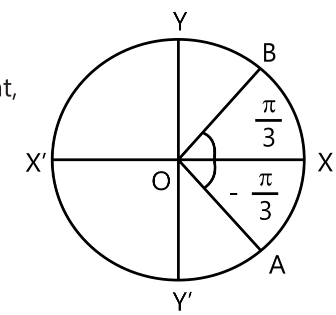

Figure 7.5

Illustration 21: Find the general solutions of the following equations:

(i) $\sin 2\theta = 0$ (ii) $\cos\left(\frac{3}{2}\theta\right) = 0$ (iii) $\tan^2 2\theta = 0$ **(JEE MAIN)**

Sol: By using above mentioned method of finding general solution we can solve these equation.

(i) We have, $\sin 2\theta = 0 \Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2}$ where, $n = 0, \pm 1, \pm 2, \pm 3......$

Hence, the general solution of $\sin 2\theta = 0$ is $\theta = \frac{n\pi}{2}, n \in Z$

(ii) We know that, the general solution of the equation $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}, n \in Z$

Therefore, $\cos\left(\frac{3\theta}{2}\right) = 0 \Rightarrow \frac{3\theta}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \theta = (2n+1)\frac{\pi}{3}$, where n = 0, ± 1, ± 2 ....

Which is the general solution of $\cos\left(\frac{3\theta}{2}\right) = 0$

(iii) We know that the general solution of the equation $\tan\theta = 0$ is $\theta = n\pi$, $n \in Z$

Therefore, $\tan^2 2\theta = 0 \Rightarrow \tan 2\theta = 0 \Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2}$, where n = 0, ± 1, ± 2 ....

Which is the required solution.

6. PERIODIC FUNCTION

A function $f(x)$ is said to be periodic if there exists T > 0 such that $f(x + T) = f(x)$ for all x in the domain of definition of $f(x)$. If T is the smallest positive real number such that $f(x + T) = f(x)$, then it is called the period of $f(x)$.

We know that, $\sin(2n\pi + x) = \sin x, \cos(2n\pi + x) = \cos x$, $\tan(n\pi + x) = \tan x$ for all $n \in Z$

Therefore, sinx, cosx and tanx are periodic functions. The period of sinx and cosx is 2π and the period of tanx is π.

Function	Period
$\sin(ax+b), \cos(ax+b), \sec(ax+b), \operatorname{cosec}(ax+b)$	$2\pi / a$
$\tan(ax+b), \cot(ax+b)$	π / a
$\lvert\sin(ax+b)\rvert, \lvert\cos(ax+b)\rvert, \lvert\sec(ax+b)\rvert, \lvert\operatorname{cosec}(ax+b)\rvert$	π / a
$\lvert\tan(ax+b)\rvert, \lvert\cot(ax+b)\rvert$	$\pi / 2a$

(a) Trigonometric equations can be solved by different methods. The form of solutions obtained in different methods may be different. From these different forms of solutions, it is wrong to assume that the answer obtained by one method is wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two different methods, the simplest way is to put values of n = ... – 2, – 1, 0, 1, 2, 3, ... etc. and then to find the angles in [0, 2π]. If all the angles in both the solutions are same, the solutions are equivalent.

(b) While manipulating the trigonometric equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation tan x = 2 sin x. Here by dividing both sides by sin x, we get cos x = 1/2.

(c) While equating one of the factors to zero, we must take care to see that the other factor does not become infinite. For example, if we have the equation sin x = 0, which can be written as cos x tan x = 0. Here we cannot put cos x = 0, since for cos x = 0, tan x = sin x / cos x is infinite.

(d) Avoid squaring: When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions that do not satisfy the given equation.

For example: Consider the equation, $\sin\theta + \cos\theta = 1$ .... (i)

Squaring, we get $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .... (ii)

This gives $\theta = 0, \pi/2, \pi, 3\pi/2$ ....

Verification shows that π and $3\pi/2$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$ and $\sin 3\pi/2 + \cos 3\pi/2 = -1, \neq 1$.

The reason for this is simple.

The equation (ii) is not equivalent to (i) and (ii) contains two equations: $\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore, we get extra solutions.

Thus if squaring is a must, verify each of the solutions.

Some Necessary Restriction: If the equation involves tan x, sec x, take cos x $\neq$ 0. If cot x or cosec x appear, take sin x $\neq$ 0. If log appears in the equation, then number > 0 and base of log > 0, $\neq$ 1.

Also note that $\sqrt{[f(\theta)]}$ is always positive. For example, $\sqrt{\sin^2\theta} = \lvert\sin\theta\rvert$, not ± sin θ.

Verification: Students are advised to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

7. SOME TRIGONOMETRIC EQUATIONS WITH THEIR GENERAL SOLUTIONS

Trigonometric equation	General solution
If $\sin\theta = 0$	$\theta = n\pi$
If $\cos\theta = 0$	$\theta = (n\pi + \pi/2) = (2n+1)\pi/2$
If $\tan\theta = 0$	$\theta = n\pi$
If $\sin\theta = 1$	$\theta = 2n\pi + \pi/2 = (4n+1)\pi/2$
If $\cos\theta = 1$	$\theta = 2n\pi$
If $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha$ where $\alpha \in [-\pi/2, \pi/2]$
If $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$
If $\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$ where $\alpha \in [-\pi/2, \pi/2]$
If $\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
If $\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
If $\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
If $\sin\theta = \sin\alpha$ $\cos\theta = \cos\alpha$	$\theta = 2n\pi + \alpha$
If $\sin\theta = \sin\alpha$ $\tan\theta = \tan\alpha$	$\theta = 2n\pi + \alpha$
If $\tan\theta = \tan\alpha$ $\cos\theta = \cos\alpha$	$\theta = 2n\pi + \alpha$

Note: Everywhere in this chapter, "n" is taken as an integer.

Illustration 22: Solve: $\sin m\theta + \sin n\theta = 0$ **(JEE MAIN)**

Sol: By using $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, we can solve this problem.

We have, $\sin m\theta + \sin n\theta = 0$

$$\Rightarrow \sin\left(\frac{m+n}{2}\right)\theta.\cos\left(\frac{m-n}{2}\right)\theta = 0 \Rightarrow \sin\left(\frac{m+n}{2}\right)\theta = 0 \quad \text{or} \quad \cos\left(\frac{m-n}{2}\right)\theta = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)\theta = 0 \Rightarrow \left(\frac{m+n}{2}\right)\theta = r\pi, r \in Z$ ... (i)

And $\cos\left(\frac{m-n}{2}\right)\theta = 0 \qquad \Rightarrow \cos\left(\frac{m-n}{2}\right)\theta = \cos\frac{\pi}{2}$

$\Rightarrow \left(\frac{m-n}{2}\right)\theta = (2p+1)\frac{\pi}{2}$, $p \in Z \Rightarrow \theta = \left(\frac{2p+1}{m-n}\right)\pi,$ ... (ii)

From (i) and (ii), we have $\theta = \frac{2r\pi}{m+n}$ or $\theta = \left(\frac{2p+1}{m-1}\right)\pi$ where, $m,n \in Z$

Illustration 23: Solve: $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$ **(JEE ADVANCED)**

Sol: Simply using algebra and method of finding general equation, we can solve above equation.

We have, $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 2\sin x(2\cos x + 1) + 1(2\cos x + 1) = 0 \quad \Rightarrow (2\sin x + 1)(2\cos x + 1) = 0$

$\Rightarrow 2\sin x + 1 = 0$ or $2\cos x + 1 = 0 \quad \Rightarrow \sin x = -\frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$\sin x = -\frac{1}{2} \Rightarrow \sin x = \sin\left(-\frac{\pi}{6}\right) \quad \Rightarrow x = -\frac{\pi}{6}$ The general solution of this is

$x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right) \quad \Rightarrow x = \pi\left[n + \frac{(-1)^{n+1}}{6}\right]$ ... (i)

and $\cos x = -\frac{1}{2} \quad \Rightarrow \cos x = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \quad \Rightarrow x = \frac{2\pi}{3}$

The general solution of this is $x = 2n\pi \pm \frac{2\pi}{3}$ i.e. $x = 2\pi\left(n \pm \frac{1}{3}\right)$ ... (ii)

From (1) and (2), we have $\pi\left[n + \frac{(-1)^{n+1}}{6}\right]$ and $2\pi\left(n \pm \frac{1}{3}\right)$ are the required solutions

8. METHODS OF SOLVING TRIGONOMETRIC EQUATIONS

8.1 Factorization

Trigonometric equations can be solved by use of factorization.

Illustration 24: Solve: $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ **(JEE MAIN)**

Sol: Use factorization method to solve this illustration.

$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x \Rightarrow \qquad (2\sin x - \cos x)(1 + \cos x) - \sin^2 x = 0$

$(2\sin x - \cos x)(1 + \cos x) - (1 - \cos x)(1 + \cos x) = 0$; $(1 + \cos x)(2\sin x - 1) = 0$

$1 + \cos x = 0$ or $2\sin x - 1 = 0$

$\cos x = -1$ or $\sin x = \frac{1}{2}$

$\cos x = \cos \pi$ or $\sin x = \sin \pi/6$

$\Rightarrow x = (2n+1)\pi, n \in I$ or $x = n\pi + (-1)^n \pi/6, n \in I$

$\therefore$ The solution of given equation is $(2n+1)\pi, n \in I$ or $n\pi + (-1)^n \pi/6$, $n \in I$

8.2 Sum to Product

Trigonometric equations can be solved by transforming a sum or difference of trigonometric ratios into their product.

Illustration 25: If sin 5x + sin 3x + sin x = 0 and $0 \le x \le \pi/2$, then x is equal to. **(JEE MAIN)**

Sol: By using sum to product formula i.e. $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

sin 5x + sinx = – sin 3x $\Rightarrow 2\sin 3x\cos 2x + \sin 3x = 0 \;\Rightarrow \sin 3x(2\cos 2x + 1) = 0$

$\Rightarrow \sin 3x = 0, \cos 2x = -1/2 \;\Rightarrow x = n\pi, x = n\pi \pm (\pi/3)$

Illustration 26: Solve $\cos 3x + \sin 2x - \sin 4x = 0$ **(JEE MAIN)**

Sol: Same as above illustration, by using formula

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ We can solve this illustration.

cos 3x + sin 2x – sin 4x = 0 $\Rightarrow \cos 3x + 2\cos 3x.\sin(-x) = 0$

$\Rightarrow \cos 3x - 2\cos 3x.\sin x = 0 \quad \Rightarrow \cos 3x(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0$ or 1 – 2 sin x = 0 $\Rightarrow 3x = (2n+1)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = (2n+1)\frac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is $(2n+1)\frac{\pi}{6}, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

8.3 Product to Sum

Trigonometric equations can also be solved by transforming product into a sum or difference of trigonometric ratios.

Illustration 27: The number of solutions of the equation $\sin 5x\cos 3x = \sin 6x\cos 2x$, in the interval $[0, \pi]$, is: **(JEE MAIN)**

Sol: Simply by using product to sum method.

The given equation can be written as $\frac{1}{2}(\sin 8x + \sin 2x) = \frac{1}{2}(\sin 8x + \sin 4x)$

$\Rightarrow$ sin 2x – sin 4x = 0 $\Rightarrow$ – 2 sin x cos 3x = 0

Hence sin x = 0 or cos 3x = 0. That is, x = $n\pi(n \in I)$, or 3x = $k\pi + \frac{\pi}{2}(k \in I)$.

Therefore, since $x \in [0, \pi]$, the given equation is satisfied if x = 0, $\pi, \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$.

Hence, no. of solutions is 5.

8.4 Parametric Methods

General solution of trigonometric equation $a\cos\theta + b\sin\theta = c$

To solve the equation $a\cos\theta + b\sin\theta = c$, put $a = r\cos\phi, b = r\sin\phi$ such that $r = \sqrt{a^2 + b^2}, \phi = \tan^{-1}\frac{b}{a}$

Substituting these values in the equation, we have, $r\cos\phi\cos\theta + r\sin\phi\sin\theta = c$

$\cos(\theta - \phi) = \frac{c}{r}$ $\Rightarrow$ $\cos(\theta - \phi) = \frac{c}{\sqrt{a^2 + b^2}}$

If $|c| > \sqrt{a^2 + b^2}$, then the equation $a\cos\theta + b\sin\theta = c$ has no solution.

If $|c| \le \sqrt{a^2 + b^2}$, then put $\frac{|c|}{\sqrt{a^2 + b^2}} = \cos\alpha$, so that $\cos(\theta - \phi) = \cos\alpha$

$\Rightarrow (\theta - \phi) = 2n\pi \pm \alpha$ $\Rightarrow \theta = 2n\pi \pm \alpha + \phi$

Illustration 28: Solve: $\sin x + \sqrt{3}\cos x = \sqrt{2}$ **(JEE MAIN)**

Sol: Solve by using above mentioned parametric method.

Given, $\sqrt{3}\cos x + \sin x = \sqrt{2}$, dividing both sides by $\sqrt{a^2 + b^2}$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4} \qquad \Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6} \Rightarrow x = 2n\pi + \frac{5\pi}{12}, 2n\pi - \frac{\pi}{12} \text{ where } n \in I$$

Note: Trigonometric equations of the form a sin x + b cos x = c can also be solved by changing sin x and cos x into their corresponding tangent of half the angle. i.e t=tan x/2. The following example gives you insight.

Illustration 29: Solve: 3 cos x + 4 sin x = 5 **(JEE MAIN)**

Sol: As we know, $\cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ and $\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$. Therefore by substituting these values and solving we will be get the result.

3 cos x + 4 sin x = 5 ... (i)

$\because \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ & $\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ $\therefore$ Equation (i) becomes

$$\Rightarrow 3\left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 4\left(\frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) = 5 \qquad \text{... (ii)}$$

Let $\tan\frac{x}{2} = t$ $\therefore$ Equation (ii) becomes

$$3\left(\frac{1 - t^2}{1 + t^2}\right) + 4\left(\frac{2t}{1 + t^2}\right) = 5 \Rightarrow 4t^2 - 4t + 1 = 0 \Rightarrow (2t - 1)^2 = 0 \Rightarrow t = 1/2 \because t = \tan x/2$$

$\Rightarrow \tan\frac{x}{2} = \frac{1}{2} \Rightarrow \tan\frac{x}{2} = \tan\alpha$, where $\tan\alpha = \frac{1}{2}$ $\Rightarrow \frac{x}{2} = n\pi + \alpha$ $\Rightarrow x = 2n\pi + 2\alpha$ where, $\alpha = \tan^{-1}\left(\frac{1}{2}\right), n \in I$

8.5 Functions of sin x and cos x

Trigonometric equations of the form P (sin x ± cos x, sin x cos x) = 0, where P (y, z) is a polynomial, can be solved by using the substitution sin x ± cos x = C.

Illustration 30: Solve: sin x + cos x = 1 + sin x. cos x **(JEE MAIN)**

Sol: Consider sin x + cos x = t, and solve it by using parametric method.

∵ sin x + cos x = 1 + sin x. cos x ... (i)

Let sin x + cos x = t

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$ $\Rightarrow \sin x.\cos x = \frac{t^2 - 1}{2}$

Now, put sin x + cos x = t and sin x. cos x = $\frac{t^2 - 1}{2}$ in (i), we get t = 1 + $\frac{t^2 - 1}{2}$

$\Rightarrow t^2 - 2t + 1 = 0$ $\Rightarrow t = 1$ ∵ t = sin x + cos x $\Rightarrow \sin x + \cos x = 1$ ... (ii)

Dividing both sides of equation (ii) by $\sqrt{2}$, we get:

$\Rightarrow \sin x\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ $\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

If we take the positive sign, we get x = $2n\pi + \frac{\pi}{2}$, $n \in I$

If we take the negative sign, we get x = $2n\pi$, $n \in I$

8.6 Using Boundaries of sin x and cos x

Trigonometric equations can be solved by the use of boundness of the trigonometric ratios sinx and cos x.

MASTERJEE CONCEPTS

(i) The answer should not contain such values of angles which make any of the terms undefined or infinite.

(ii) Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of the general solution.

Suppose the equation is sin x = (tan x)/2. Now, cancelling sinx on both the sides, we get only $\cos x = \frac{1}{2}$, sin x = 0 is not counted.

(iii) Check that the denominator is not zero at any stage while solving equations.

(iv) While solving a trigonometric equation, squaring the equation at any step must be avoided if possible. If squaring is necessary, check the solution for extraneous values.

Suppose the equation is sin x = – sin x. We know that the only solution of this is sin x = 0 but on squaring, we get $(\sin x)^2 = (\sin x)^2$ which is always true.

(v) Domain should not change, it if changes, necessary corrections must be made.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 31: Solve: sin 3x + cos 2x = – 2 **(JEE ADVANCED)**

Sol: By using boundary condition of sin x and cos x.

Since $\sin 3x \geq -1$ and $\cos 2x \geq -1$, we have, $\sin 3x + \cos 2x \geq -2$

Thus, the equality holds true if and only if $\sin 3x = -1$ and $\cos 2x = -1$

$\Rightarrow 3x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ and $2x = 2n\pi \pm \pi$ i.e. $x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)$ and $x = n\pi \pm \frac{\pi}{2}$, $n \in I$

$\therefore$ Solution set is, $\left\{x \mid x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)\right\} \cap \left\{x \mid x = n\pi \pm \left(\frac{\pi}{2}\right)\right\}$

Note: Here, unlike all other problems, the solution set consists of the intersection of two solution sets and not the union of the solution sets.

Illustration 32: $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)(\cos x) = 0$. Find the general solution. **(JEE ADVANCED)**

Sol: Open all brackets of given equation and then by using sum to product formula and method of finding general solution we will get the result.

$$\sin x\cos\frac{x}{4} - 2\sin^2 x + \cos x + \sin\frac{x}{4}\cos x - 2\cos^2 x = 0$$

$$\sin\left(x + \frac{x}{4}\right) + \cos x = 2 \Rightarrow \sin\frac{5x}{4} + \cos x = 2 \qquad \Rightarrow \sin\frac{5x}{4} = 1 \text{ and } \cos x = 1$$

$$\sin\frac{5x}{4} = 1 \Rightarrow \frac{5x}{4} = 2n\pi + \frac{\pi}{2} \qquad \Rightarrow x = 2(4n+1)\frac{\pi}{5}\ ; \ \cos x = 1 \Rightarrow x = 2m\pi$$

$$\Rightarrow x = 2\pi, 10\pi, 18\pi \ \ \ldots\ldots\text{AP} \qquad \Rightarrow x = 2\pi + (m-1)8\pi$$

$$\Rightarrow x = 2\pi(4m-3) \qquad m \in I$$

Illustration 33: Find the general solution of $2\sin\left(3x + \frac{\pi}{4}\right) = \sqrt{1 + 8\sin 2x\cos^2 2x}$ **(JEE ADVANCED)**

Sol: First square on both side and then using sum and difference formula we can solve this illustration.

$$4\sin^2\left(3x + \frac{\pi}{4}\right)^2 = 1 + 8\sin 2x\cos^2 2x \Rightarrow 4\left(\frac{\sin 3x}{\sqrt{2}} + \frac{\cos 3x}{\sqrt{2}}\right)^2 = 1 + 8\sin 2x\cos^2 2x$$

$$\Rightarrow \frac{4\sin^2 3x}{2} + \frac{4\cos^2 3x}{2} + 4\sin 3x\cos 3x = 1 + 8\sin 2x\cos^2 2x$$

$$\Rightarrow 2\sin^2 3x + 2\cos^2 3x + 2\sin 6x = 1 + 8\sin 2x\cos^2 2x$$

$\Rightarrow$ $1 + 2\sin 6x = 8\sin 2x\cos^2 2x$ $\qquad \Rightarrow$ $1 + 2\sin 6x = 4\sin 4x\cos 2x$

$\Rightarrow$ $1 + 2\sin 6x = 2(\sin 6x + \sin 2x)$ $\qquad \Rightarrow$ $1 = 2\sin 2x \Rightarrow \sin 2x = ½$

$\Rightarrow$ $x = \frac{\pi}{12} + 2n\pi$ $\qquad x = \frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$ $\qquad n \in I$

9. SIMULTANEOUS EQUATIONS

Two equations are given and we have to find the value of variable θ which may satisfy both the given equations, like $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$

So, the common solution is $\theta = 2n\pi + \alpha, n \in I$

Similarly, $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$

So, the common solution is $\theta = 2n\pi + \alpha,\ n \in I$

Illustration 34: The most general value of θ satisfying the equations $\cos\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = -1$ is: **(JEE MAIN)**

Sol: As above mentioned method we can find out the general value of θ.

$\cos\theta = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{4}$; $n \in I$ Put n = 1 $\theta = \frac{9\pi}{4}, \frac{7\pi}{4}$

$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right)$ $\Rightarrow \theta = n\pi - \pi/4,$ $n \in I$ Put $n = 1, \theta = \frac{3\pi}{4}$; Put $n = 2, \theta = \frac{7\pi}{4}$

The common value which satisfies both these equation is $\left(\frac{7\pi}{4}\right)$.

Hence, the general value is $2n\pi + \frac{7\pi}{4}$.

Illustration 35: The most general value of θ satisfying equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = 1/\sqrt{3}$ are: **(JEE MAIN)**

Sol: Similar to above illustration.

We shall first consider values of θ between 0 and 2π

$\sin\theta = -\frac{1}{2} = -\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right)$ or $\sin(2\pi - \pi/6)$

$\therefore \theta = 7\pi/6, 11\pi/6$; $\tan\theta = 1/\sqrt{3} = \tan(\pi/6) = \tan(\pi + \pi/6)$

$\therefore \theta = \pi/6, 7\pi/6$

Thus, the value of θ between 0 and 2π which satisfies both the equations is $7\pi/6$.

Hence, the general value of θ is $2n\pi + 7\pi/6$ where $n \in I$

PROBLEM SOLVING TACTICS

(a) Any formula that gives the value of $\sin\frac{A}{2}$ in terms of sin A shall also give the value of sin $\frac{n\pi + (-1)^n A}{2}$.

(b) Any formula that gives the value of $\cos\frac{A}{2}$ in terms of cos A shall also give the value of cos $\frac{2n\pi \pm A}{2}$.

(c) Any formula that gives the value of $\tan\frac{A}{2}$ in terms of tan A shall also give the value of tan $\frac{n\pi \pm A}{2}$.

(d) If α is the least positive value of θ which satisfies two given trigonometric equations, then the general value of θ will be $2n\pi+\alpha$. For example, $\sin\theta=\sin\alpha$ and $\cos\theta=\cos\alpha$, then, $\theta=2n\pi+\alpha, n\in I$

(i) $\sin(n\pi+\theta)=(-1)^n\sin\theta,\ n\in I$

(ii) $\cos(n\pi+\theta)=(-1)^n\cos\theta,\ n\in I$

(iii) $\sin(n\pi-\theta)=(-1)^{n-1}\sin\theta,\ n\in I$

FORMULAE SHEET

Tangent and cotangent Identities	$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$
Product Identities	$\sin\theta\times\operatorname{cosec}\theta=1$, $\cos\theta\times\sec\theta=1$, $\tan\theta\times\cot\theta=1$
Pythagorean Identities	$\sin^2\theta+\cos^2\theta=1$, $\tan^2\theta+1=\sec^2\theta$, $1+\cot^2\theta=\csc^2\theta$
Even/Odd Formulas	$\sin(-\theta)=-\sin\theta$, $\cos(-\theta)=\cos\theta$, $\tan(-\theta)=-\tan\theta$, $\cot(-\theta)=-\cot\theta$, $\sec(-\theta)=\sec\theta$, $\operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta$
Periodic Formulas (If n is an integer)	$\sin(2n\pi+\theta)=\sin\theta$, $\cos(2n\pi+\theta)=\cos\theta$, $\tan(n\pi+\theta)=\tan\theta$, $\cot(n\pi+\theta)=\cot\theta$, $\sec(2n\pi+\theta)=\sec\theta$, $\operatorname{cosec}(2n\pi+\theta)=\operatorname{cosec}\theta$
Double and Triple Angle Formulas	$\sin(2\theta)=2\sin\theta\cos\theta$, $\sin3\theta=3\sin\theta-4\sin^3\theta$ $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $\cos3\theta=4\cos^3\theta-3\cos\theta$ $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$, $\tan3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$
Complementary angles	$\sin\left(\dfrac{\pi}{2}\pm\theta\right)=\cos\theta$, $\cos\left(\dfrac{\pi}{2}\pm\theta\right)=\mp\sin\theta$, $\tan\left(\dfrac{\pi}{2}\pm\theta\right)=\mp\cot\theta$, $\cot\left(\dfrac{\pi}{2}-\theta\right)=\tan\theta$, $\sec\left(\dfrac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$, $\operatorname{cosec}\left(\dfrac{\pi}{2}-\theta\right)=\sec\theta$
Half Angle	$\sin^2\theta=\dfrac{1}{2}[1-\cos(2\theta)]$, $\cos^2\theta=\dfrac{1}{2}[1+\cos(2\theta)]$, $\tan^2\theta=\dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Sum and Difference	$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$, $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$, $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$,

Product to Sum	$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$, $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right]$, $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right]$, $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right)-\sin\left(\alpha-\beta\right)\right]$,
Sum to Product	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, $\sin\alpha-\sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$