Solved Examples

JEE Main/Boards

Example 1: Evaluate the following

(a) $\tan^{-1}(-1)$ (b) $\cot^{-1}(-1)$ (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Sol: Do it yourself.

(a) $\tan^{-1}(-1) = \frac{-\pi}{4}$ as $\tan\left(-\frac{\pi}{4}\right) = -1$

(b) $\cot^{-1}(-1) = \frac{3\pi}{4}$ as $\cot\left(\frac{3\pi}{4}\right) = -1$

(c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ as $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example 2: Find the angle $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

Sol: Write the angle $\frac{2\pi}{3}$ as $\pi - \frac{\pi}{3}$ and proceed.

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) \neq \frac{2\pi}{3}\left(\text{as } \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}.$$

Example 3: Find the value of

$\cos\ [2\sin^{-1}x + \cos^{-1}x]$ at $x = \frac{1}{5}$.

Sol: Use $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$$\cos[2\sin^{-1}x + \cos^{-1}x]$$

$$= \cos[\cos^{-1}x + \sin^{-1}x + \sin^{-1}x]$$

$$= \cos\left[\frac{\pi}{2} + \sin^{-1}x\right] = -\sin(\sin^{-1}(x))$$

$$= -x = -\frac{1}{5}$$

Example 4: Prove that $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$

Sol: Use substitution.

Let $2\sin^{-1}x = \theta$, where $\theta \in [-\pi, \pi]$;

then $x = \sin\frac{\theta}{2}$

$$\therefore \sin(2\sin^{-1}x) = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\sin\frac{\theta}{2}\sqrt{1-\sin^2\frac{\theta}{2}} \qquad = 2x\sqrt{1-x^2}$$

Example 5: Find the angle

(a) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$; (b) $\sin^{-1}(\sin 5)$

Sol: (a) Write $\frac{3\pi}{4}$ as $\pi - \frac{\pi}{4}$

(b) Write 5 as 5 - 2π

(a) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$$

(b) We know $\sin^{-1}(\sin\theta) = \theta$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = [-1.57, 1.57]$$

$5 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while $5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin 5 = \sin(5 - 2\pi + 2\pi) = \sin(5 - 2\pi)$$

$$\therefore \sin^{-1}\sin 5 = 5 - 2\pi$$

Example 6: If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Sol: Take one of the term to the R.H.S. and take cosine on both sides.

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$$

Taking cosine on both sides we get

$$xy-\sqrt{1-x^2}\,\sqrt{1-y^2}=-z$$

$$\Rightarrow \quad xy+z=\sqrt{1-x^2}\,\sqrt{1-y^2}$$

Squaring we get

$$x^2y^2+z^2+2xyz=1-x^2-y^2+x^2y^2$$

$$\Rightarrow \quad x^2+y^2+z^2+2xyz=1.$$

Example 7: Prove that $f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is a constant for all $x\ge 1$. Find this constant.

Sol: Convert the $\sin^{-1}$ function on the R.H.S. to $\tan^{-1}$ and proceed.

If $0\le x\le 1$ then $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x.$

Hence If $x\ge 1 (0<1/x\le 1)$ then $\sin^{-1}\frac{2x}{1+x^2}$

$$=\sin^{-1}\left(\frac{\frac{2}{x}}{1+\frac{1}{x^2}}\right)=2\tan^{-1}\frac{1}{x} \text{ For } x\ge 1$$

$$f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x+2\tan^{-1}\frac{1}{x}$$

$$=2[\tan^{-1}x+\cot^{-1}x]=2.\pi/2=\pi=\text{constant}.$$

Example 8: Solve the equation: $2\tan^{-1}(\cos x)$ $=\tan^{-1}(2\operatorname{cosec}x)$.

Sol: Substitute a variable in place of $\tan^{-1}(\cos x)$ and take tan on both sides.

If $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x):\sin x\ne 0$

$$\tan[2\tan^{-1}(\cos x)]=2\operatorname{cosec}x \qquad \text{....(i)}$$

Assume $\tan^{-1}(\cos x)=\theta:\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$\therefore$ L.H.S. $=\tan[2\tan^{-1}(\cos x)]$

$$=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\cos x}{1-\cos^2 x}=\frac{2\cos x}{\sin^2 x}$$

Substituting this value (i) we get

$$\frac{2\cos x}{\sin^2 x}=2\operatorname{cosec}x;\ \cos x=\sin x;\ \tan x=1;$$

$$\therefore \quad x=n\pi+\frac{\pi}{4};\ n\in I$$

Example 9: Solve $\cos^{-1}x+\cos^{-1}y=\pi/2$ and $\tan^{-1}x-\tan^{-1}y=0$

Sol: From the second of the given equations, we have

$x=y.\Rightarrow \quad \tan^{-1}-\tan^{-1}y=0\Rightarrow x=y$

Substituting $x=y$ in the first, we have

$2\cos^{-1}x=\pi/2$ or $\cos^{-1}x=\pi/4$

or $x=\cos\pi/4=1/\sqrt{2}=y$

It is clearly evident that these values satisfy the given equations. Hence the solution set of the given equations is $(x=1/\sqrt{2}, y=1/\sqrt{2})$

Example 10: If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$ prove that

$$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta.$$

Sol: Do it yourself.

Let $\cos^{-1}\frac{x}{2}=\alpha$ and $\cos^{-1}\frac{y}{3}=\beta$

$\therefore \cos\alpha=\frac{x}{2}$ and $\cos\beta=\frac{y}{3}$

Given $\alpha+\beta=\theta \quad \therefore \cos(\alpha+\beta)=\cos\theta$

or $\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos\theta$

$$\text{or } \frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}.\sqrt{1-\frac{y^2}{9}}=\cos\theta$$

$$\text{or, } \frac{xy}{6}-\frac{\sqrt{4-x^2}\,\sqrt{9-y^2}}{6}=\cos\theta$$

or $(xy-6\cos\theta)^2=(4-x^2)(9-y^2)$

or, $x^2y^2+36\cos^2\theta-12xy\cos\theta=36-9x^2-4y^2+x^2y^2$

or, $9x^2-12xy\cos\theta+4y^2=36(1-\cos\theta^2)$

JEE Advanced/Boards

Example 1: If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$

prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$

Sol: In the given equation take cosine on both sides and proceed.

$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$

$\cos\left(\cos^{-1}\frac{x}{a}\right)\cos\left(\cos^{-1}\frac{y}{b}\right)-\sin\left(\cos^{-1}\frac{x}{a}\right)\sin\left(\cos^{-1}\frac{y}{b}\right)$

$=\cos\alpha$

$\Rightarrow \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$

$\Rightarrow \frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}$

Squaring both sides

$\left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$

or $\frac{x^2y^2}{a^2b^2}+\cos^2\alpha-\frac{2xy}{ab}\cos\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=1-\cos^2\alpha$

or $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Example 2: Prove

$\tan^{-1}x=2\tan^{-1}(\text{cosec}\tan^{-1}x-\tan\cot^{-1}x)$.

Sol: Use substitution.

R.H.S. $=2\tan^{-1}(\text{cosec}\tan^{-1}x-\tan\cot^{-1}x)$

$=2\tan^{-1}\begin{bmatrix}\text{cosec}\,\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) \\ -\tan\tan^{-1}\left(\frac{1}{x}\right)\end{bmatrix}$ or

$2\tan^{-1}\left[\text{cosec}\,\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)-\tan\left\{\pi+\tan^{-1}\frac{1}{x}\right\}\right]$

depending on x > 0 or x < 0

$=2\tan^{-1}\left[\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Let $\tan^{-1}x=\theta:\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; then $x=\tan\theta$

R.H.S. $=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]=2\tan^{-1}\tan\frac{\theta}{2}$

$=2.\frac{\theta}{2}=\tan^{-1}x=$ L.H.S.

Example 3: Prove that

$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\begin{cases}0 & \text{if } \frac{\pi}{4}<A<\frac{\pi}{2} \\ \pi & \text{if } 0<A<\frac{\pi}{4}\end{cases}$

Sol: Divide the solution in two cases when $\frac{\pi}{4}<A<\frac{\pi}{2}$ and $0<A<\frac{\pi}{4}$ and use the definition accordingly.

Case I: $\frac{\pi}{4}<A<\frac{\pi}{2}$

$0<\cot A<1$ and $0<\cot^3 A<1$

$\therefore\ \tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\tan^{-1}\left[\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right]=\tan^{-1}\left[-\frac{\sin 2A}{2\cos 2A}\right]$

$=\tan^{-1}\left[-\frac{\sin 2A}{2\cos 2A}\right]=-\tan^{-1}\left[\frac{1}{2}\tan 2A\right]$

$\Rightarrow\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0$

Case II: $0<A<\frac{\pi}{4}$

$\cot A>1$ and $\cot^3 A>1\Rightarrow\cot A.\cot^3 A>1$

Hence, $\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=$

$\pi+\tan^{-1}\left(\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right)$

[As $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ If $x>0.\ y>0$ and $xy>1$]

$=\pi-\tan^{-1}\left(\frac{1}{2}\tan 2A\right)$ [From case 1]

$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=\pi$

Example 4: Find the sum

$\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+$ .... to Infinity.

Sol: Write the general term of the series and express it as a difference of two terms (telescopic series).

Let T_n denote the nth term of the series

$$\therefore \quad T_r=\cot^{-1}(2r^2)=\cot^{-1}\left(\frac{4r^2}{2}\right)$$

$$=\cot^{-1}\left(\frac{1+4r^2-1}{2}\right)=\cot^{-1}\left(\frac{1+(2r+1)(2r-1)}{(2r+1)-(2r-1)}\right)$$

$$=\tan^{-1}\left[\frac{(2r+1)-(2r-1)}{1+(2r+1)\,(2r-1)}\right]$$

$$=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$$

$\therefore \quad \cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\ldots\ldots+$

$\cot^{-1}2n^2=\tan^{-1}(2n+1)-\tan^{-1}(1)$ As

$n\to\infty \quad \tan^{-1}(2n+1)\to\pi/2$

Hence, required sum = $\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$.

Example 5: If X_1, X_2, X_3, X_4 are the roots of the equation

$$X^4-X^3\sin2\beta+X^2\cos2\beta-X\cos\beta-\sin\beta=0$$

where $\sin\beta\neq\frac{1}{2}$ prove that $\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4=n\pi+\frac{\pi}{2}-\beta$ for some $n\in I$.

Sol: Use theory of equations.

X_1, X_2, X_3, X_4 are the roots of the given equation

$$\therefore \sum X_1=\sin2\beta, \sum X_1X_2=\cos2\beta$$

$$\Sigma\, X_1X_2X_3=\cos\beta,\; X_1X_2X_3X_4=-\sin\beta$$

$$\tan\left[\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4\right]$$

$$=\frac{\Sigma X_1-\Sigma X_1X_2X_3}{1-\Sigma X_1X_2+X_1X_2X_3X_4}=\frac{\sin2\beta-\cos\beta}{1-\cos2\beta-\sin\beta}$$

$$\frac{\cos\beta(2\sin\beta-1)}{\sin\beta(2\sin\beta-1)}=\cos\beta$$

$$\tan\left[\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4\right]$$

$$=\cos\beta=\tan\left(\frac{\pi}{2}-\beta\right)$$

$$\therefore \tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4=$$

$n\pi+\frac{\pi}{2}-\beta$ for some $n\in I$.

Example 6: Find the value of

$$\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

$$-\cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

Where $\left(\frac{k}{2}<x<2k, k>0\right)$

Sol: We have

$$\text{Sin}^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

$$-\cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

$$=\sin^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\}-\cos^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\}$$

$$=\frac{\pi}{2}-2\cos^{-1}\left\{\frac{\sqrt{3}x}{\sqrt{(4x^2-4kx+4k^2)}}\right\}$$

$$=\frac{\pi}{2}-\cos^{-1}\left\{\frac{6x^2}{4x^2-4kx+4k^2}-1\right\}$$

$$=\sin^{-1}\left(\frac{2X^2+4kx-4k^2}{4X^2-4kx+4k^2}\right)=\sin^{-1}\left(\frac{X^2+2xk-2k^2}{2X^2-2xk+2k^2}\right)$$

Example 7: Find the number of real solutions of the equation $\sqrt{1+\cos2x}=\sqrt{2}\sin^{-1}(\sin x)-\pi\le x\le\pi$

Sol: Divide the solution into three cases when $-\frac{\pi}{2}\le X\le\frac{\pi}{2}$, $\frac{\pi}{2}<X\le\pi$ and $-\pi\le X<-\frac{\pi}{2}$ and proceed.

Here $|\cos x|=\sin^{-1}(\sin x)$.

If $-\frac{\pi}{2} \le X \le \frac{\pi}{2}$, then $\cos^{-1}\cos x = x$

In the case there is one solution obtained graphically.

If $\frac{\pi}{2} < X \le \pi$ then $-\cos x = \sin^{-1}\{\sin(\pi - x)\} = \pi - x$

$\therefore \cos x = x - \pi$

In the case there is one solution obtained graphically.

If $-\pi \le X < -\frac{\pi}{2}$ then

$-\cos x = \sin^{-1}\{\sin(-\pi - x)\} = -x - \pi$

i.e. $\cos x = x + \pi$

This gives no solution as can be seen from their graphs.

Example 8: Find the integral values of p at which the system of equations $\cos^{-1} x + (\sin^{-1} y)^2 = p\pi^2 / 4$; and $(\cos^{-1} x)(\sin^{-1} y)^2 = \pi^2 / 16$ possess solutions. Also find these solutions.

Sol: Start with the range of $\cos^{-1} x$ and $\sin^{-1} y$ and use it in the two given equations.

The given system of the equation is

$\cos^{-1} x + (\sin^{-1} y)^2 = p\pi^2 / 4$ ....(i)

$(\cos^{-1} x)(\sin^{-1} y)^2 = \pi^4 / 16$ ....(ii)

It is clear that

$0 < \cos^{-1} x \le \pi$; $-\pi / 2 \le \sin^{-1} y \le \pi / 2$.

So $0 < (\sin^{-1} y)^2 \le \pi^2 / 4 \sin^{-1} y \ne 0$ [From ii]

$\therefore 0 < \cos^{-1} x + (\sin^{-1} y)^2 \le \pi + \pi^2 / 4$

i.e. $0 < \frac{p\pi}{4} \le \pi + \frac{\pi^2}{4}$ ....(iii)

From (i) and (ii) we get $p \not\le 0$

$\cos^{-1} x + \frac{\pi^4}{16\cos^{-1} x} = \frac{p\pi^2}{4}$

Or $16\, (\cos^{-1} x)^2 - 4p\pi^2 \cos^{-1} x + \pi^4 = 0$ ....(iv)

As $\cos^{-1} x$ is real $16 p^2\pi^2 \ge 0$

Or $p^2 \ge 4$ i.e. $p \le -2$ ....(v)

From (iii) and (v)

$p^2 \le (\pi / 4) + 1, p \ge 20$

p is integer so p = 2 for p = 2 (4) gives

$16(\cos^{-1} x)^2 - 8\pi^2 \cos^{-1} x + \pi^4 = 0$ or

$\left[4\cos^{-1} x - \pi^2\right]^2 = 0$ or

$\cos^{-1} x - \pi^2 / 4$ i.e. $x = \cos(\pi^2 / 4)$ ....(vii)

Then (ii) gives $(\sin^{-1} y)^2 = \pi^2 / 4$

Or $\sin^{-1} y = \pm\pi^2 / 2$ i.e. ± 1 ....(viii)

Hence p=2 and (x,y)= $\left[\cos(\pi^2 / 4), \pm 1\right]$

JEE Main/Boards

Exercise 1

Q.1 Evaluate: $\sin^{-1}(\sin \pi / 4)$

Q.2 Evaluate: $\tan^{-1}(\tan(-6))$

Q.3 Evaluate: $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\right)$

Q.4 Prove that: $\tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$

Q.5 Evaluate: $\cos\left\{\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right\}$

Q.6 Evaluate: $\sin(\cos^{-1} 3 / 5)$

Q.7 Prove that: $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

Q.8 Prove that: $4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

Q.9 Solve for x: $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{5\pi^2}{8}$

Q.10 Solve for x: $\tan^{-1}\left(x+1\right)+\tan^{-1}x+\tan^{-1}$ $\left(x-1\right)=\tan^{-1}3$

Q.11 Find the value of

$\tan^{-1}\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

Q.12 Prove that: $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Q.13 Differentiate $\tan^{-1}\left[\frac{1-\cos x}{\sin x}\right]$ w.r.t.x.

Q.14 Express $\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)\frac{\pi}{2}<x<\frac{\pi}{2}$ in the simplest form.

Q.15 Find the principle value $\cos^{-1}\left(-\frac{1}{2}\right)$

Q.16 Write the following functions in the simplest form: $\cot^{-1}\left(\sqrt{1+x^2}-x\right)$.

Q.17 Find the principle value of $\cot^{-1}\left(-\sqrt{3}\right)$.

Q.18 Prove that $3\cos^{-1}x=\cos^{-1}\left(4\times 3-3x\right)\left(-\sqrt{3}\right)$ $x\in\left[\frac{1}{2},1\right]$

Q.19 Write the following function in the simplest form:

$\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right], x<\pi$

Q.20 If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ then find the value of x.

Q.21 Write the following function in the simplest form:

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$.

Q.22 Prove that: $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Q.23 Solve $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

Q24 Prove that: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Q25 If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\theta$ prove that

$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta=\frac{y^2}{b^2}=\sin^2\theta$

Q.26 Find the value of the following:

$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Q.27 Prove that: $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(2\frac{\sqrt{2}}{3}\right)$

Q.28 The value of

$\cos^{-1}\left(\sqrt{\frac{1}{3}}\right)-\cos^{-1}\left(\sqrt{\frac{1}{6}}\right)+\cos^{-1}\left(\frac{\sqrt{10}-1}{3\sqrt{2}}\right)$ Is__________.

Q.29 The number of roots of the equation $\sqrt{\sin x}=\cos^{-1}\left(\cos x\right)$ is

Exercise 2

Single Correct Choice Type

Q.1 $\tan\,\cos^{-1}x$ is equal to

(A) $\frac{\sqrt{1-x^2}}{x}$ (B) $\frac{x}{\sqrt{1+x^2}}$

(C) $\frac{\sqrt{1+x^2}}{x}$ (D) $x\sqrt{1+x^2}$

Q.2 $\left|\sin^{-1}x\right|^2+\left|\sin^{-1}y\right|^2+2\left|\sin^{-1}x\right|\left|\sin^{-1}y\right|=\pi^2$

then x^2+y^2 is equal to

(A) 1 (B) 3/2 (C) 2 (D) ½

Q.3 Number of solution(s) of the equation

$\cot^{-1}\sqrt{x^2+3x+2}+\cos^{-1}\sqrt{x^2-3x-3}=\frac{\pi}{2}$ is

(A) 2 (B) 1

(C) 4 (D) Infinite

Q.4 If $\cos\left(\tan^{-1}x\right)=x$, then x^2 is equal to

(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}-1}{2}$

(C) $\frac{\sqrt{5}+1}{2}$ (D) $\frac{\sqrt{5}-1}{4}$

Q.5 The value of a for which $x^2+ax+\sin^{-1}\left(x^2-4x+5\right)+\cos^{-1}\left(x^2-4x+5\right)=0$, is

(A) $\frac{\pi}{4}+2$ (B) $\frac{\pi}{4}+1$

(C) $-\left|\frac{\pi}{4}+1\right|$ (D) $-(\frac{\pi}{4}+2)$

Q.6 Domain of the function

$f(x)=\sqrt{\sin^{-1}(\sin x)}+\sqrt{\cos^{-1}(\cos x)}$ is

(A) $\left|2n\pi, 2n\pi+\frac{\pi}{2}\right|, n\in I$

(B) $\left[(2n+1)\pi, (2n+2)\pi\right], n\in I$

(C) $\left[2n\pi, (2n+1)\pi\right], n\in I$

(D) $\left|2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}\right|, n\in I$

Q.7 If $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x, x\in[0,1]$

Then the interval in which θ lies is given by

(A) $\left[0,\frac{\pi}{3}\right]$ (B) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

(C) $\left[0,\frac{\pi}{4}\right]$ (D) $\left[\frac{\pi}{2},\frac{3\pi}{4}\right]$

Q.8 $\tan^{-1}2+\tan^{-1}3=\operatorname{cosec}^{-1}x$ then x is equal to

(A) 4 (B) $\sqrt{2}$ (C) $-\sqrt{2}$ (D) None of these

Q. 9 The function

$f(x)=\cot^{-1}\sqrt{(x+3)x}+\cos^{-1}\sqrt{x^2+3x+1}$

is defined on the set S where S=

(A) $\{0,3\}$ (B) $(0,3)$ (C) $\{0,-3\}$ (D) $(-3,0)$

Q.10 $\alpha=\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)$ and

$\beta=\cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)$ then

(A) $\tan\alpha=\cot\beta$ (B) $\tan\alpha=-\cot\beta$

(C) $\tan\alpha=\tan\beta$ (D) $\tan\alpha=-\tan\beta$

Q.11 If $x=2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\sqrt{3}\right)$

and $y=\cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right)$ then which of

the following statements hold good?

(A) $y=\cos\frac{3\pi}{16}$ (B) $y=\cos\frac{5\pi}{16}$

(C) $x=4\cos^{-1}y$ (D) None of these

Q.12 The set values of x satisfying the equation $\tan^2\left(\sin^{-1}x\right)>1$ is

(A) $[-1,1]$ (B) $\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$

(C) $(-1,1)-\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$ (D $[-1,1]-\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

Q.13 The equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

(A) All real values of a (B) a<-1

(C) a > 1 (D) $\frac{-1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}$

Q.14 If $\left(\sin^{-1}x+\sin^{-1}w\right)\left(\sin^{-1}y+\sin^{-1}z\right)=\pi^2$

then $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$; $(N_1, N_2, N_3, N_4 \text{ E } N)-$

(A) Has a maximum value 2.

(B) Has a minimum value 2.

(C) Is independent of N_1, N_2, N_3, N_4

(D) None of these

Q.15 If $\theta = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$ then $\cot\theta$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Q.16 Which of the following function(s) is/are periodic?

(A) f(x)=x-[x], [x] denotes integral part of x

(B) g(x)= sin(1/x) $x \neq 0$ and g(0) =0

(C) h(x)=x cosx

(D) sin ($\sin^{-1}x$)

Q.17 $\cos\left(2\tan^{-1}\left(\frac{1}{7}\right)\right)$ equals

(A) $\sin(4\cot^{-1}3)$ (B) $\sin(3\cot^{-1}4)$

(C) $\cos(3\cot^{-1}4)$ (D) $\cos(4\cot^{-1}4)$

Q.18 $\sin^{-1}\left(2\times\sqrt{1-x^2}\right) = 2\sin^{-1}x$ is true if: $x \in$

(A) [0,1] (B) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

Q.19 If the sum $\sum_{k=1}^{k=n} \tan^{-1}\frac{2k}{2+k^2+k^4} = \tan^{-1}\frac{6}{7}$ then the value of n is equal to :

(A) 2 (B) 3 (C) 4 (D) 5

Q.20 The domain of definition of the function

$f(x) = \text{arc}\cos\left[\frac{3x^2-7x+8}{1+x^2}\right]$ where [x] denotes the greatest integer function is:

(A) (1, 6) (B) [1, 6]

(C) [0, 1] (D) (-2, 5)

Q.21 Consider two geometric progressions $a_1, a_2, a_3 ....a_n$ & $b_1, b_2, b_3 ....b_n$ with $a_r = \frac{1}{b_r} = 2^{r-1}$ and another sequence $t_1, t_2, t_3 ....t_n$ such that $t_r = \cot^{-1}(2a_r + b_r)$. Then $\lim_{n\to\infty}\sum_{r=1}^{n} t_r$ is :

(A) 0 (B) $\pi/4$ (C) $\tan^{-1}2$ (D) $\pi/2$

Q.22 Number of point(s) where f(x) $=\sin^{-1}(3x-4x^3)$ is not differentiable is

(A) 1 (B) 2 (C) 3 (D) 4

Q.23 Solution of the equation

$\sec^{-1}x = \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(\frac{5}{3\sqrt{3}}\right)$ is

(A) $\frac{18}{3-\sqrt{6}}$ (B) $\frac{18}{\sqrt{6}-3}$

(C) $\frac{\sqrt{6}+3}{8}$ (D) None of these

Q.24 The value of

$\left[\tan\left\{\frac{\pi}{4} + \sin^{-1}\left(\frac{a}{b}\right)\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}\right]^{-1}$

Where (0<a<b) is

(A) $\frac{b}{2a}$ (B) $\frac{a}{2b}$

(C) $\frac{\sqrt{b^2-a^2}}{2b}$ (D) $\frac{\sqrt{b^2-a^2}}{2a}$

Q.25 If $x = \tan^{-1}1 - \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{1}{2}$;

$y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ then:

(A) $x = \pi y$ (B) $y = \pi x$

(C) $\tan x = -(4/3)y$ (D) $\tan x = (4/3)y$

Q.26 Which of the following satisfy the equation?

$2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{\sqrt{4x^2-4x^2}}\right)$

(A) (-1, 0) (B) (0, 1)

(C) $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (D) [-1, 1]

Q.27 Find values of x if $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x-2)$?

(A) $\left\{\frac{1}{2}, 1\right\}$ (B) $\left[\frac{1}{2}, 1\right]$

(C) $\left[\frac{1}{3}, 1\right]$ (D) $\left\{\frac{1}{3}, 1\right\}$

Q.28 $f(x)=\sin^{-1}\left|\frac{1-x^2}{1+x^2}\right|$ and

$g(x)=\cot^{-1}x-\tan^{-1}x$ are identical for:

(A) $x\in[0,1]$ (B) $x\in(-\infty,0]$

(C) $x\in[-1,1]$ (D) $x\in(-\infty,-1]\cup[1,\infty)$

Q.29 $\tan\left[\cos^{-1}\left\{\sin\left(2\tan^{-1}2\right)\right\}\right]$ is equal to

(A) $\frac{4}{3}$ (B) $\frac{4}{5}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$

Q.30 $\sum_{n=1}^{\infty}\left|\frac{\sin^{-1}x+\cos^{-1}x}{\pi r}^{n}\right|$ is finite.

$x\in[-1,1]$ and r>0.then the possible values of 'r' is.

(A) $\left|\frac{1}{2},\infty\right|$ (B) $(2,\infty)$

(C) $(1,\infty)$ (D) $(0,\infty)$

Q.31 $y=\sin^{-1}(\sin x)$, x is the element of [0 π] divides the region bounded by coordinate axes

$x=\pi$ and $y=\frac{\pi}{2}$ into 3 region whose areas are

A_1,A_2,A_3 with $A_1\le A_2\le A_3$ then

(A) $A_1+A_2+2A_3=\pi^2$

(B) $A_1+A_3-A_2=\frac{\pi^2}{2}$

(C) $A_1+A_2-A_3=0$

(D) $2(A_1+A_2)-A_3=0$

Q.32 The sum $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4-2n^2+2}$ is equal to:

(A) $\tan^{-1}2+\tan^{-1}3$ (B) $4\tan^{-1}1$

(C) $\frac{\pi}{2}$ (D) $\sec^{-1}\left(1-\sqrt{2}\right)$

Previous Years' Questions

Q.1 The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is ***(1983)***

(A) $\frac{6}{17}$ (B) $\frac{17}{6}$

(C) $\frac{16}{7}$ (D) None of above

Q.2 The principle value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is ***(1986)***

(A) $-\frac{2\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{3}$

Q.3 The number of real solutions of

$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is ***(1999)***

(A) Zero (B) One

(C) Two (D) Infinite

Q.4 If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\right)$

$=\frac{\pi}{2}$, for $0<|x|<\sqrt{2}$, then x equals ***(2001)***

(A) 1/2 (B) 1 (C) -1/2 (D) -1

Q.5 The value of x for which

$\sin\left[\cos^{-1}(1+x)\right]=\cos\left(\tan^{-1}x\right)$ is ***(2004)***

(A) $\frac{1}{2}$ (B) 1 (C) 0 (D) $-\frac{1}{2}$

Q.6 If 0<x<1, then

$\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}$ is equal to ***(2008)***

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x

(C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

Q.7 Let a, b, c be positive real number

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c+)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c+)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c+)}{ab}}.$$

Then $\tan\theta$ equal .... ***(1981)***

Q.8 The numerical value of

$\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$ is equal to.... ***(1984)***

Q.9 The greater of the two angle A = $2\tan^{-1}\left(2\sqrt{2}-1\right)$

B= $3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$ is .... ***(1989)***

Q.10 AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point c on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7m. From D the angle of elevation of the point A is 45°. Then the height of the pole is ***(2008)***

(A) $\frac{7\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}-1}$m (B) $\frac{7\sqrt{3}}{2}\cdot(\sqrt{3}+1)$m

(C) $\frac{7\sqrt{3}}{2}\cdot(\sqrt{3}-1)$m (D) $\frac{7\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}-1}$

Q.11 The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$ is ***(2008)***

(A) $\frac{6}{17}$ (B) $\frac{3}{17}$ (C) $\frac{4}{17}$ (D) $\frac{5}{17}$

Q.12 Let $\cos(\alpha+\beta)=\frac{4}{5}$ and let $\sin(\alpha-\beta)=\frac{5}{13}$, where $0\le\alpha,\beta\le\frac{\pi}{4}$, then $\tan 2\alpha =$ ***(2010)***

(A) $\frac{56}{33}$ (B) $\frac{19}{12}$ (C) $\frac{20}{7}$ (D) $\frac{25}{16}$

Q.13 For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is ***(2010)***

(A) There is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$

(B) There is a regular polygon with $\frac{r}{R}=\frac{2}{3}$

(C) There is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$

(D) There is a regular polygon with $\frac{r}{R}=\frac{1}{2}$

Q.14 A line AB in three-dimensional space makes angle 45° and 120° with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals ***(2010)***

(A) 45° (B) 60° (C) 75° (D) 30°

Q.15 Let $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x|<\frac{1}{\sqrt{3}}$. Then the value of y is ***(2015)***

(A) $\frac{3x-x^3}{1-3x^2}$ (B) $\frac{3x+x^3}{1-3x^2}$

(C) $\frac{3x-x^3}{1+3x^2}$ (D) $\frac{3x+x^3}{1+3x^2}$

Q.16 If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation cosx + cos2x + cos3x + cos4x = 0, is ***(2016)***

(A) 5 (B) 7 (C) 9 (D) 3

Q.17 Consider $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right)$.

A normal to $y=f(x)=\frac{\pi}{6}$ also passes through the point: ***(2016)***

(A) $\left(0,\frac{2\pi}{3}\right)$ (B) $\left(\frac{\pi}{6},0\right)$

(C) $\left(\frac{\pi}{4},0\right)$ (D) (0, 0)

JEE Advanced/Boards

Exercise 1

Q.1 If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

For 0<x<1 then prove that $\alpha+\beta=\pi$ what is the value of $\alpha+\beta$ will be if x>1?

Q.2 If $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ prove that

x^2= sin 2 y.

Q.3 Find the sum of following series upto n terms where x > 0.

(i) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+.....+\tan^{-1}\frac{2^{n-1}}{1+2^{n-1+}}...0$

(ii) $\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}$

$+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}$

Q.4 If $x \in \left[-1,-\frac{1}{2}\right]$ then express the function

$f(x)=\sin^{-1}(3x-4x^3)+\cos^{-1}(4x^3-3x)$ in the form of a

$\cos^{-1}x+b\pi$ where a and b are rational numbers.

Q.5 Solve the following equations:

(i) $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

(ii) $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}(3x)$

(iii) $\tan^{-1}\frac{1-x}{1+x}+\tan^{-1}\frac{2x-x}{2x+x}=\tan^{-1}\frac{23}{36}$

(iv) $\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$

Q.6 Find all the positive integral solution of

$\tan^{-1}x+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\sin^{-1}\frac{3}{\sqrt{10}}$

Q.7 If α and β are the roots of the equation $x^2-4x+1=0\,(\alpha>\beta)$ then find the value of

$f(\alpha,\beta)=\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^2\frac{\beta}{\alpha}\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^2\frac{\alpha}{\beta}\right)$

Q.8 Consider the functions $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $h(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(i) If x ∈ (-1, 1) then the find the solution of the Equation $f(x)+g(x)+h(x)=\pi/2$

(ii) Find the value of f(2) +g(2)=h(2).

Q.9 Solve the following inequalities

(i) arc $\cot^2 x - 5\,\text{arc}\cot x + 6 > 0$

(ii) arc sin x>arc cos x

(iii) $\tan^2$ (arc sin x)>1

Q.10 Show that roots r, s and t of the cubic x(x-2)(3x-7)=2 are real and positive.

Also compute the value of $\tan^{-1}(r)+\tan^{-1}(s)\tan^{-1}(t)$.

Q.11 Let $f(x)=\frac{\pi}{4}+\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)-\tan^{-1}x$

and $a_i\,(a_i<a_{i+1}\,\forall i=1,2,3,....n)$ be the positive integral values of x for which $\operatorname{sgn}(f(x))=1$, where sgn (y) denotes signum function of y. Find $\sum_{i=1}^{n}a_i^2$.

Q.12 Solve for x: $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right)<\pi-3$.

Q.13 Let $f(x)=\tan^{-1}(\cot x-2\cot 2x)$ and $\sum_{r=1}^{5}f(r)=a-b\pi$ where a, b E N. Find the value of (a+b).

Q.14 Let $f(x) = (2a+b)\cos^{-1}x + (a+2b)\sin^{-1}x$

Where a, b $\in$ R and a>b.

If domain of and range of f are the same

set then find the value of $\pi(a-b)$.

Q.15 Identify the pairs(s) of functions which are identical. Also plot the graphs in each case.

(i) $y = \tan(\cos^{-1}x);\ y = \dfrac{\sqrt{1-x^2}}{x}$

(ii) $y = \tan(\cot^{-1}x);\ y = \dfrac{1}{x}$

(iii) y = sin (arc tan x); $y = \dfrac{x}{\sqrt{1-x^2}}$

(iv) y = cos (arc tan x); y= sin (arc cot x)

Q.16 Find the domain and the following functions.

(i) $f(x) = \cot^{-1}\left(2x - x^2\right)$

(ii) $f(x) = \sec^{-1}\left(\log_3 \tan x + \log_{\tan x} 3\right)$

(iii) $f(x) = \cos^{-1}\left(\dfrac{\left[2x^2+1\right]}{x^2+1}\right)$

(iv) $f(x) = \tan^{-1}\left(\log_4\left(5x^2 - 8x + 4\right)\right)$

Q.17 Let $y = \sin^{-1}(\sin 8)^5 - \tan^{-1}(\tan 1) + \cos^{-1}$

(cos 12) – $\sec(\sec 9) + \cot^{-1}(\cot 6) - \operatorname{cosec}^{-1}$ – (cosec 7).
If y simplifies to $a\pi + b$ then find (a-b).

Q.18 Let $\alpha = \sin^{-1}\left(\dfrac{36}{85}\right)$ $\beta = \cos^{-1}\left(\dfrac{4}{5}\right)$ and

$\gamma = \tan^{-1}\left(\dfrac{8}{15}\right)$ find $(\alpha+\beta+\gamma)$ and hence

Prove that (i) $\Sigma \cot\alpha = \prod \cot\alpha$ (ii) $\Sigma \tan\alpha.\tan\beta = 1$

Q.19 Show that:

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right) = \frac{13\pi}{7}$$

Q.20 Prove that:

(i) $\cos^{-1}\left(\dfrac{5}{13}\right) + \cos^{-1}\left(-\dfrac{7}{25}\right) + \sin^{-1}\dfrac{36}{325} = \pi$

(ii) arc $\cos\sqrt{\dfrac{2}{3}}$ – arc $\cos\dfrac{\sqrt{6}+1}{2\sqrt{3}} = \dfrac{\pi}{6}$

Q.21 If a>b>c>0 then find the value of:

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

Q.22 If α and β are the roots of the equation

$x^2 + 5x - 49 = 0$ then find the value of

$\cot\left(\cot^{-1}\alpha + \cot^{-1}\beta\right)$.

Q.23 Find all value of k for which there is a triangle

Whose angles have measure $\tan^{-1}\left(\dfrac{1}{2}\right)$

$\tan^{-1}\left(\dfrac{1}{2} + k\right)$ and $\tan^{-1}\left(\dfrac{1}{2} + 2k\right)$

Q.24 In a ΔABC if $\angle A = \angle B$

$= \dfrac{1}{2}\left(\sin^{-1}\left(\dfrac{\sqrt{6}+1}{2\sqrt{3}}\right) + \sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\right)$ and

$C = 6.3^{\frac{1}{4}}$ then find the area of ΔABC.

Q.25 Find the simplest value of

(i) f (x) =arc cos x +arc

$\cos\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right), x \in \left(\dfrac{1}{2}, 1\right)$

(ii) $f(x) = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right), x \in R - \{0\}$

Q.26 Let $f(x) = \cot^{-1}\left(x^2 + 4x + \alpha^2 - \alpha\right)$ be a function defined $R \to (0, \pi/2]$ then find the complete set of real values of α for which $f(x)$ is onto.

Q.27 Prove that: $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \dfrac{1}{2}\cos^{-1}\dfrac{3}{5}$

$= \dfrac{1}{2}\sin^{-1}\dfrac{4}{5}$.

Q.28 Prove that

$$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right\}=\frac{x}{2}<x<\frac{\pi}{2}$$

Q.29 Find the domain of definition the following functions.

(Real the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x)=\arc\cos\frac{2x}{1+x}$

(ii) $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\frac{1+x^2}{2x}$

(iii) $f(x)=\sin^{-1}\left(\frac{x-3}{2}\right)-\log_{10}(4-x)$

(iv) $f(x)=\sin^{-1}\left(2x+x^2\right)$

(v) $f(x)=\frac{\sqrt{1-\sin^{-1}x}}{\log_5\left(1-4x^2\right)}+\cos^{-1}\left(1-\{x\}\right)$

where $\{x\}$ is the fractional part of x.

(vi) $f(x)=\sqrt{3-x}+\cos^{-1}\left(\frac{3-2x}{5}\right)$

$+\log_6\left(2|x|-3\right)+\sin^{-1}\left(\log_2 x\right)$

(vii) $f(x)=\log_{10}\left(1-\log_7\left(x^2-5x+13\right)\right)$

$+\cos^{-1}\left(\frac{3}{2+\sin\frac{9\pi x}{2}}\right)$

(viii) $f(x)=e^{\sin^{-1}\left(\frac{x}{2}\right)}+\tan^{-1}\left(\frac{x}{2}-1\right)+\ln\left(\sqrt{x-[x]}\right)$

Exercise 2

Single Correct Choice Type

Q.1 Solution set of the inequality x^2-4x+5 $>\sin^{-1}(\sin 3)+\cos^{-1}(\cos 2)-\pi$ is.

(A) R (B) R-{1}

(C) R-{2} (D) R-{-2}

Q.2 If x_1, x_2, x_3, x_4 are roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$ then $\sum_{i=1}^{4}$ tan-1 x_i is equal to

(A) $x - \beta$ (B) $\pi - 2\beta$

(C) $\left(\frac{\pi}{2}\right) - \beta$ (D) $\left(\frac{\pi}{2}\right) - 2\beta$

Q.3 Range of the function,

$f(x)=\cot^{-1}\left(\log_{4/5}\left(5x^2-8x+4\right)\right)$ is.

(A) $(0,\pi)$ (B) $\left[\frac{\pi}{4},\pi\right)$

(C) $\left(0,\frac{\pi}{4}\right]$ (D) $\left(0,\frac{\pi}{2}\right)$

Q.4 Domain of the explicit form of the function y represented implicitly by the equation.

$(1+x)\cos y - x^2 = 0$

(A) (-1, 1] (B) $\left(-1,\frac{1-\sqrt{5}}{2}\right]$

(C) $\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]$ (D) $\left[0,\frac{1+\sqrt{5}}{2}\right]$

Q.5 Number of integral value(s) of x satisfying

$4\left(\tan^{-1}x\right)^2-\left(\tan^{-1}x\right)-3\le 0$ is –

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 The area of the region bounded by the curves $y=x^2$ and $\sec^{-1}[-\sin^{-2} x]$ (where [.] denotes greatest integer function) is

(A) $\pi\sqrt{\pi}$ (B) $\frac{4}{3}\pi\sqrt{\pi}$

(C) $\frac{2}{3}\pi\sqrt{\pi}$ (D) $\frac{1}{3}\pi\sqrt{\pi}$

Q.7 If $\tan^{-1}\frac{a}{x}+\tan^{-1}\frac{b}{x}+\tan^{-1}\frac{c}{x}+\tan^{-1}\frac{d}{x}=\frac{\pi}{2}$

Then $x^4 - x^2\sum ab + abcd$ is equal to

(A) – 1 (B) 0 (C) 1 (D) 2

Q.8 The solution set of the equation

$$\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)-\sin^{-1}x$$

(A) [-1, 1] –{0} (B) (0, 1] U {-1}

(C) [-1, 0) U {1} (D) [-1, 1]

Q.9 The domain and range of the function

$f(x)=\operatorname{cosec}^{-1}\sqrt{\log\frac{3-4\sec x^2}{1-2\sec x}}$ are respectively

(A) $R;\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(B) $R^+;\left(0,\frac{\pi}{2}\right)$

(C) $\left(2\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}\right)-\{2n\pi\};\left(0,\frac{\pi}{2}\right)$

(D) $\left(2\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}\right)-\{2n\pi\};\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

Q.10 Solution set of equation [$\sin^{-1}$ x] = [$\cos^{-1}$ x] where [*] represents integral part function is

(A) (cos 1sin1) (B) [cos 1sin1]

(C) (sin 1 cos 1) (D) [sin 1 cos 1]

Multiple Correct Choice Type

Q.11 Which of the following statement (s) is/ are meaningless?

(A) $\cos^{-1}\left(\ln\left(\frac{2e+4}{3}\right)\right)$ (B) $\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$

(C) $\cot^{-1}\left(\frac{\pi}{2}\right)$ (D) $\sec^{-1}(\pi)$

Q.12 If the numerical value of tan

$\left(\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{2}\right)\right)$ is $\frac{a}{b}$ then

(A) a + b = 23 (B) a – b = 11

(C) 3b = a + 1 (D) 2a = 3b

Q.13 Which of the following equation represents a circle

(A) $y^2=\sin\left(\cos^{-1}x\right)$ (B) $y=\sin\left(\cos^{-1}(1-x)\right)$

(C) $y^2=\sin^2\left(\cos^{-1}x\right)$ (D) $y=\sin^{-1}\left(\cos^2 x\right)$]

Q.14 If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ then

(A) $x^2+y^2+z^2+2xyz=1$

(B) $2\left(\sin^{-1}x+\sin^{-1}y+\sin^{-1}z\right)$

$=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$

(C) $xy+yz+zx=x+y+z-1$

(D) $\left(x+\frac{1}{x}\right)+\left(x+\frac{1}{y}\right)+\left(x+\frac{1}{z}\right)\geq 6$

Match the Columns

Q.15 Column I contains functions and column II contains their range. Match the entries of column I with the entries of column II.

	Column I		Column II
(A)	$f(x)=\sin^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(p)	$(0,\pi)$
(B)	$g(x)=\cos^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(q)	$\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$
(C)	$h(x)=\tan^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(r)	$\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$
(D)	$k(x)=\cot^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(s)	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Previous Years' Questions

Q.1 Match the conditions/expressions in column I with statement in column II.

Let (x, y) be such that $\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$

	Column I		Column II
(A)	If a=1 and b=0, then (x, y)	(P)	lies on the circle $x^2+y^2=1$
(B)	If a=1 and b=1, then (x, y)	(q)	lies on $(x^2-1)(y^2-1)=0$
(C)	If a=1 and b=2, then (x, y)	(r)	lies on y=x
(D)	If a=2 and b=2, then (x, y)	(s)	lies on $(4x^2-1)(y^2-1)=0$

Q.2 Solve the following equation for x

$\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ ***(1978, 3M)***

Q.3 Find the value of $\cos\left(2\cos^{-1}x+\sin^{-1}x\right)$ at $x=\frac{1}{5}$ where $0\le\cos^{-1}x\le\pi$ and $-\pi/2\le\sin^{-1}x\le\pi/2$. ***(1981)***

Q.4 Prove that $\cos\tan^{-1}[\sin(\cot^{-1}x)]=\sqrt{\frac{x^2+1}{x^2+2}}$ ***(2002)***

Q.5 If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the length of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is ***(2010)***

Q.6 If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) ***(2015)***

(A) $\cos\beta>0$ (B) $\sin\beta>0$

(C) $\cos(\alpha+\beta)>0$ (D) $\cos\alpha<0$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.5 Q.10 Q.13

Q.16 Q.22 Q.29

Exercise 2

Q.3 Q.10 Q.18

Q.21 Q.24 Q.29

Q.30 Q. 32

Previous Years' Questions

Q.3 Q.4 Q.5

Q.7

JEE Advanced/Boards

Exercise 1

Q.3 Q.11 Q.15

Q.17 Q.21 Q.24

Q.29 Q.30

Exercise 2

Q.3 Q.4 Q.6

Q.9 Q.10 Q.12

Previous Years' Questions

Q.1 Q.3

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\pi/4$	**Q.2** $2\pi-6$	**Q.3** $\frac{\sqrt{3}}{2}$
Q.5 -1	**Q.6** $\frac{4}{5}$	**Q.9** -1
Q.10 $-1;\ 5\pm\sqrt{19}$	**Q.11** $z=\frac{x+y}{1-xy}$	**Q.13** 1/2
Q.14 $y=\frac{\pi}{4}-\frac{x}{2}$	**Q.15** $\frac{2\pi}{3}$	**Q.16** $y=\frac{\pi}{4}+\frac{1}{2}\tan^{-1}x$
Q.17 $\frac{5\pi}{6}$	**Q.19** $\frac{\pi}{4}-x$	**Q.20** $\pm\frac{1}{\sqrt{2}}$
Q.21 $3\tan^{-1}\frac{x}{a}$	**Q.23** $x = 1/6$	**Q.26** $\frac{3\pi}{4}$
Q.28 $=\cos^{-1}\left(\frac{1}{3}\right)$	**Q.29** ∞	

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** D	**Q.5** D	**Q.6** C
Q.7 B	**Q.8** D	**Q.9** C	**Q.10** A	**Q.11** A	**Q.12** C
Q.13 D	**Q.14** C	**Q.15** C	**Q.16** A	**Q.17** A	**Q.18** B
Q.19 B	**Q.20** A	**Q.21** B	**Q.22** B	**Q.23** D	**Q.24** C
Q.25 C	**Q.26** B	**Q.27** A	**Q.28** A	**Q.29** D	**Q.30** A
Q.31 C	**Q.32** A				

Previous Years' Questions

Q.1 B	**Q.2** C	**Q.3** C	**Q.4** B	**Q.5** D	**Q.6** C
Q.7 0	**Q.8** $-\frac{7}{17}$	**Q.9** A	**Q.10** B	**Q.11** A	**Q.12** A
Q.13 B	**Q.14** B	**Q.15** A	**Q.16** B	**Q.17** A	

JEE Advanced/Boards

Exercise 1

Q.1 0

Q.2 $x^2 = \frac{2\tan y}{1+\tan^2 y} = \sin 2y$

Q.3 (i) $\frac{\pi}{4}$ (ii) $\text{arc}\tan(x+n) - \text{arc}\tan x$

Q.4 $6\cos^{-1}x - \frac{9\pi}{2}$ so $a = 6$ $b = -\frac{9}{2}$

Q.5 (i) $x = \frac{1}{2}\sqrt{\frac{3}{7}}$; (ii) $x = 0, \frac{1}{2}, -\frac{1}{2}$; (iii) $x = \frac{4}{3}$; (iv) $x = 2-\sqrt{3}$ or $\sqrt{3}$

Q.6 x=1; y=2 & x=2 ; y=7

Q.7 56

Q.8 (i) $2-\sqrt{3}$; (ii) $\cot^{-1}\left(\frac{-3}{4}\right)$

Q.9 (i) $(\cot 2, \infty) \cup (-\infty, \cot 3)$ (ii) $\left|\frac{\sqrt{2}}{2}, 1\right|$ (iii) $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, \frac{\sqrt{2}}{2}\right)$

Q.10 $\frac{3\pi}{4}$

Q.11 5

Q.12 $x \in (-1, 1)$

Q.13 20

Q.14 –2

Q.15 (i), (ii), (iii) and (iv) all are identical

Q.16 (i) D:$x \in R$ R :$[\pi/4, \pi)$ (ii) D:$\in \left(n\pi . n\pi + \frac{\pi}{2}\right) - \left\{x \middle| x + \frac{\pi}{2}\right\} n \in I$: R:$\left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left\{\frac{\pi}{2}\right\}$

(iii) D:x $\in R$ R :$\left[0, \frac{\pi}{2}\right)$ (iv) D:x $\in R$ R :$\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Q.17 53

Q.21 0

Q.22 10

Q.23 $k = \frac{11}{4}$

Q.24 27

Q.25 (i) $\frac{\pi}{3}$; (ii) $\frac{\tan^{-1}x}{2}$

Q.26 $\frac{1 \pm \sqrt{17}}{2}$

Q.27 $\frac{1}{2}\sin^{-1}\frac{4}{5}$ = RHS

Q.28 $\frac{x}{2}$

Q.29 (i) $-1/3 \le x \le 1$; (ii) $\{1, -1\}$; (iii) $1 \le x < 4$; (iv) $[-(1+\sqrt{2}), (\sqrt{2}, -1)]$; (v) $x \in (-1/2, 1/2), x \neq 0$; (vi) $(3/2, 2]$; (vii) $\{7/3, 25/9\}$;(viii) (-2, 2)-{-1, 0,1}

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** C **Q.3** B **Q.4** C **Q.5** B **Q.6** B

Q.7 B **Q.8** C **Q.9** C **Q.10** A

Multiple Correct Choice Type

Q.11 A, B **Q.12** A, B, C **Q.13** B, C **Q.14** A, B

Match the Columns

Q.15 A → s; B → p; C → r; D → q

Previous Years' Questions

Q.1 A → p; B → q; C → p; D → s **Q.2** $x = \frac{1}{6}$ **Q.3** $-\frac{2\sqrt{6}}{5}$ **Q.5** $\sqrt{3}$ **Q.6** B, C, D

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Sol 2: $\tan^{-1}(\tan(-6)) = \tan^{-1}\tan(2\pi - 6) = 2\pi - 6$

Sol 3: $\sin\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{2}\right)$

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

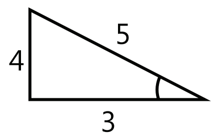
Sol 4: $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\frac{2+3}{1-6}$ xy> 1

$= \pi + \tan^{-1}\frac{5}{-5} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Sol 5: $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right\}$

$\cos\{\pi\} = -1$

Sol 6:



$= \sin\cos^{-1}\left(\frac{3}{5}\right) = \frac{4}{5}$

Sol 7: $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{13\times 7}}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

Sol 8: $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{70}\right) + \tan^{-1}\left(\frac{1}{99}\right)$

$= 2\tan^{-1}\left(\frac{5}{12}\right) - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\}$

$= \tan^{-1}\frac{2\times 5/2}{1-\left(\frac{5}{2}\right)^2} - \tan^{-1}\frac{\left(\frac{1}{70}-\frac{1}{99}\right)}{1+\frac{1}{70}\times\frac{1}{99}}$

$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{29}{6931}\right)$

$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$

$= \tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\} \approx \tan^{-1}(1)$

Sol 9: $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$(\tan^{-1}x)^2 + \frac{\pi^2}{4} + (\tan^{-1}x)^2 - 2\frac{\pi}{2}(\tan^{-1}x) = \frac{5\pi^2}{8}$

$2(\tan^{-1}x)^2 - \pi(\tan^{-1}x) - \frac{3\pi^2}{8} = 0$

$\tan^{-1}x = \frac{\pi \pm \sqrt{\pi^2 + 3\pi^2}}{4} = \frac{\pi \pm 2\pi}{4}$

$\tan^{-1}x = \frac{3\pi}{4}$ or $= \frac{-\pi}{4}$

$x = -1$

Sol 10: $\tan^{-1}(x + 1) + \tan^{-1}x + \tan^{-1}(x - 1)$

$= \tan^{-1}(3)$

$\tan^{-1}\left[\frac{(x+1)+x+(x-1)-x(x^2-1)}{1-x(x+1)-(x^2-1)-x(x-1)}\right] = \tan^{-1}(3)$

$\frac{3x-x^3+x}{1-x^2-x-x^2+1-x^2+x} = 3$

$\Rightarrow \frac{4x-x^3}{2-3x^2} = 3$

$\Rightarrow 4x - x^3 = 6 - 9x^2$

$\Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

$\Rightarrow (x + 1)(x^2 - 10x + 6) = 0$

$\Rightarrow x = -1, 5 \pm \sqrt{19}$

Sol 11: $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \sin^{-1}\frac{2y}{1+y^2}\right]$

$= \frac{\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}\right) + \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2y}{1+y^2}\right)\right)}{1-\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}\right)\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2y}{1+y^2}\right)\right)}$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$

$= \frac{x+y}{1-xy}$

Sol 12: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{14}{264}}\right)$

$= \tan^{-1}\left(\frac{48+77}{250}\right) = \tan^{-1}\frac{1}{2}$

Sol 13: $f(x) = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$

$f'(x) = \frac{1}{2}$

Sol 14: $\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{x}{2}}{1+\tan\frac{\pi}{4}.\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

$= \frac{\pi}{4}-\frac{x}{2}$

Sol 15: $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{2}+\frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

Sol 16: $\cot^{-1}(\sqrt{1+x^2} - x)$

Put $x = \tan y$

$= \cot^{-1}(\sec y - \tan y)$

$= \cot^{-1}\left(\frac{1-\sin y}{\cos y}\right) = \cot^{-1}\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$

$= \cot^{-1}\tan\left(\frac{\pi}{4}-\frac{y}{2}\right) = \frac{\pi}{4}+\frac{y}{2} = \frac{\pi}{4}+\frac{1}{2}\tan^{-1}x$

Sol 17: $\cot^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Sol 18: $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

put $x = \cos x$

L. H. S. = $3x$

R. H. S. = $\cos^{-1}(4\cos^3x - 3\cos x) = \cos^{-1}\cos 3x = 3x$

Sol 19: $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)=\tan^{-1}\left(\dfrac{1-\tan x}{1+\tan x}\right)$

$=\tan^{-1}\tan\left(\dfrac{\pi}{4}-x\right)=\dfrac{\pi}{4}-x$

Sol 20: $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x^2-1}{x^2-4}}\right)=\dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{-2+x-2-x+2x^2}{-4+1}\right)=\dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\dfrac{2x^2-4}{-3}=\dfrac{\pi}{4}$

$\Rightarrow \dfrac{2x^2-4}{-3}=1\Rightarrow 2x^2=1\Rightarrow x=\pm\dfrac{1}{\sqrt{2}}$

Sol 21: $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^2-3ax^2}\right)=\tan^{-1}\left[\dfrac{\dfrac{3x}{a}-\left(\dfrac{x}{a}\right)^3}{1-3\left(\dfrac{x}{a}\right)^2}\right]$

$\dfrac{x}{a}=\tan y$

$\Rightarrow \tan^{-1}\left(\dfrac{3\tan y-\tan^3 y}{1-3\tan^2 y}\right)=\tan^{-1}\tan 3y=3\tan^{-1}\dfrac{x}{a}$

Sol 22: $\sin^{-1}\dfrac{12}{13}+\cos^{-1}\dfrac{4}{5}+\tan^{-1}\dfrac{63}{16}$

$=\tan^{-1}\dfrac{12}{5}+\tan^{-1}\dfrac{3}{4}+\tan^{-1}\dfrac{63}{16}$

$=\pi+\tan^{-1}\left(\dfrac{\dfrac{12}{5}+\dfrac{3}{4}}{1-\dfrac{36}{20}}\right)+\tan^{-1}\dfrac{63}{16}$

$=\pi+\tan^{-1}\dfrac{63}{-16}+\tan^{-1}\dfrac{63}{16}=\pi$

Sol 23: $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{2x+3x}{1-6x^2}\right)=\dfrac{\pi}{4}$

$\dfrac{5x}{1-6x^2}=1\Rightarrow 6x^2-1+5x=0$

$(6x-1)(x+1)=0\Rightarrow x=1/6,-1$

(-1) does not satisfy,

so answer is $x=\dfrac{1}{6}$

Sol 24: $2\tan^{-1}\left(\dfrac{1}{2}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$

$=\tan^{-1}\dfrac{\dfrac{1}{2}+\dfrac{1}{2}}{1-\dfrac{1}{4}}+\tan^{-1}\left(\dfrac{1}{7}\right)$

$=\tan^{-1}\dfrac{\dfrac{4}{3}+\dfrac{1}{7}}{1-\dfrac{4}{21}}=\tan^{-1}\dfrac{31}{17}$

Sol 25: $\dfrac{xy}{ab}-\sqrt{1-\dfrac{x^2}{a^2}}\sqrt{1-\dfrac{y^2}{b^2}}=\cos\theta$

$\dfrac{xy-\sqrt{a^2-x^2}\sqrt{b^2-y^2}}{ab}=\cos\theta$

$1-\sin^2\theta=$

$\dfrac{x^2y^2+(a^2-x^2)(b^2-y^2)-2xy\sqrt{a^2-x^2}\sqrt{b^2-y^2}}{a^2b^2}$

$1-\sin^2\theta$

$=\dfrac{2x^2y^2-a^2y^2-b^2x^2-2xy(xy-ab\cos\theta)}{a^2b^2}+1$

$\Rightarrow \sin^2\theta=\dfrac{a^2y^2+b^2x^2-2xyab\cos\theta}{a^2b^2}$

$\Rightarrow \sin^2\theta=\dfrac{y^2}{b^2}+\dfrac{x^2}{a^2}-\dfrac{2xy}{ab}\cos\theta$

Sol 26: $\tan^{-1}(1)+\cos^{-1}\left(-\dfrac{1}{2}\right)+\sin^{-1}\left(-\dfrac{1}{2}\right)$

$=\dfrac{\pi}{4}+\dfrac{2\pi}{3}-\dfrac{\pi}{6}=\dfrac{\pi}{4}+\dfrac{\pi}{2}=\dfrac{3\pi}{4}$

Sol 27: L. H. S. $\Rightarrow \dfrac{9\pi}{8}-\dfrac{9}{4}\sin^{-1}\left(\dfrac{1}{3}\right)$

$=\dfrac{9}{4}\left[\sin^{-1}(1)-\sin^{-1}\left(\dfrac{1}{3}\right)\right]$

So L. H. S. $=\dfrac{9}{4}\left[\dfrac{\pi}{2}-\sin^{-1}\dfrac{1}{3}\right]$

$= \frac{9}{4}.\cos^{-1}\frac{1}{3} = \frac{9}{4}.\sin^{-1}\frac{2\sqrt{2}}{3}$

Sol 28: $\cos^{-1}\left(\sqrt{\frac{1}{3}}\right) - \cos^{-1}\left(\sqrt{\frac{1}{6}}\right) + \cos^{-1}\frac{\sqrt{10}-1}{3\sqrt{2}}$

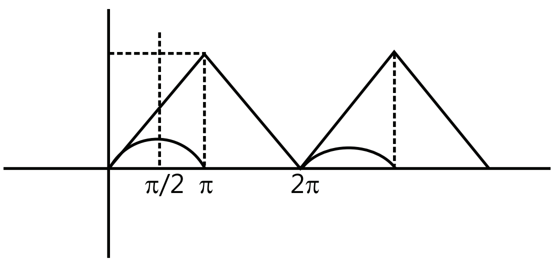
$= \cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{5}}{\sqrt{6}}\right) + \cos^{-1}\frac{\sqrt{10}-1}{\sqrt{18}}$

$= \cos^{-1}\frac{1+\sqrt{10}}{\sqrt{18}} + \cos^{-1}\frac{\sqrt{10}-1}{\sqrt{18}}$

$= \cos^{-1}\left[\frac{(\sqrt{10}-1)(\sqrt{10}+1)}{18} - \frac{\sqrt{(7-\sqrt{40})(7+\sqrt{40})}}{18}\right]$

$= \cos^{-1}\left(\frac{1}{3}\right)$

Sol 29: $\sqrt{\sin x} = \cos^{-1}\cos x \Rightarrow 0 \le \sqrt{\sin x} \le 1$



x = 2nπ always satisfy

so infinite roots.

Exercise 2

Single Correct Choice Type

Sol 1: (A) $= \tan\cos^{-1}x = \frac{\sqrt{1-x^2}}{x}$

Sol 2: (C) $(|\sin^{-1}x| + |\sin^{-1}y|^2) = \pi^2$

$\Rightarrow (|\sin^{-1}x| + |\sin^{-1}y|) = \pi$

$\Rightarrow |\sin^{-1}x| = \frac{\pi}{2} = |\sin^{-1}y|$

$\Rightarrow x = \pm y = \pm 1$

$\Rightarrow x^2 + y^2 = 2$

Sol 3: (A)

$\cot^{-1}\sqrt{(x-1)(x-2)} + \cos^{-1}\sqrt{\left(x-\frac{3}{4}\right)^2 + \frac{3}{4}} = \frac{\pi}{2}$

Domain for $\cot^{-1}\sqrt{(x-1)(x-2)}$ is

$x \in (-\infty, 1] \cup [2, \infty)$

while $\cos^{-1}\sqrt{\left(x-\frac{3}{2}\right)^2 + \frac{3}{4}}$ is defined for x = [1,2]

At x = 1

$\Rightarrow \cot^{-1}(0) + \cos^{-1}(1) = \frac{\pi}{2}$

At x = 2

$\Rightarrow \cot^{-1}(0) + \cot^{-1}(1) = \frac{\pi}{2}$

Hence two solutions.

Sol 4: (D) $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}} = x$

$\frac{1}{1+x^2} = x^2 \qquad \Rightarrow x^4 + x^2 - 1 = 0$

$x^2 = t \qquad \Rightarrow t^2 + t - 1 = 0$

$\Rightarrow \left(t + \frac{1}{2}\right)^2 = \frac{5}{4}$

$x^2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ = positive

$x^2 = \frac{\sqrt{5}-1}{2}$

Sol 5: (D) $x^2 - 4x + 5 = (x-2)^2 + 1$

x = 2, to define $\sin^{-1}(x^2-4x+5)$

So $4 + 2a + \frac{\pi}{2} + 0 = 0 \Rightarrow a = -\frac{\pi}{4} - 2$

Sol 6: (C) $f(x) = \sqrt{\sin^{-1}\sin x} + \sqrt{\cos^{-1}\cos x}$

sin x must not be negative to define f(x). So the domain is

$x \in [2n\pi, (2n+1)\pi]$, $n \in I$

Sol 7: (B) $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

$x \in [0, 1] \Rightarrow \frac{\pi}{4} \Rightarrow \theta \le \frac{\pi}{2}$

Sol 8: (D) $\tan^{-1}(2) + \tan^{-1}(3)$

$= \pi - \tan^{-1}\left(\frac{3+2}{1-6}\right) = \frac{3\pi}{4} = \text{cosec}^{-1}(x)$

$\frac{-\pi}{2} \le \text{cosec}^{-1}(x) \le \frac{\pi}{2}$

So none of these.

Sol 9: (C) $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$

$= \cot^{-1}\sqrt{x(x+3)} + \cos^{-1}\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{5}{4}}$

$x(x + 3) \ge 0 \Rightarrow x \in (-\infty, -3] \cup [0, \infty)$

$\left(x+\frac{3}{2}\right)^2 - \frac{5}{4} \Rightarrow 0$

$\Rightarrow x \in \left(-\infty, \frac{-3}{2} - \frac{\sqrt{5}}{2}\right] \cup \left[\frac{-3}{2} + \frac{\sqrt{5}}{2}, \infty\right)$

$\left(x+\frac{3}{2}\right)^2 - \frac{5}{4} \le 1 \Rightarrow x \in [-3, 0]$

So the answer $x \in \{0, -3\}$

Sol 10: (A) $\alpha = \sin^{-1}\cos\sin^{-1}x$

$\beta = \cos^{-1}\sin\cos^{-1}x$

$\tan\alpha = \tan\sin^{-1}\cos\sin^{-1}x$

$= \tan\sin^{-1}\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{x}$

$\tan\beta = \tan\cos^{-1}\sin\cos^{-1}x = \tan\cos^{-1}\sqrt{1-x^2} = \frac{x}{\sqrt{1-x^2}}$

$\cot\beta = \tan\alpha$

Sol 11: (A) $x = 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\sqrt{3}$

$x = \frac{2\pi}{3} + \frac{\pi}{4} + \frac{\pi}{3}; \qquad x = \frac{5\pi}{4}$

$y = \cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right) = \cos\frac{1}{2}\sin^{-1}\sin\frac{5\pi}{8}$

$= \cos\frac{1}{2}\left[\pi - \frac{5\pi}{8}\right] = \cos\frac{3\pi}{16}$

Sol 12: (C) $[\tan(\sin^{-1}x)]^2 = \left[\frac{x}{\sqrt{1-x^2}}\right]^2 > 1$

$\Rightarrow \frac{x^2}{1-x^2} > 1$

$\Rightarrow \frac{-1+2x^2}{1-x^2} > 0$

$\Rightarrow \frac{(\sqrt{2}x-1)(\sqrt{2}x+1)}{(x-1)(x+1)} > 0$

$x \in (-1, 1) - \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

Sol 13: (D) $\sin^{-1}x = 2\sin^{-1}a$

$\Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$

$\Rightarrow -\frac{1}{\sqrt{2}} \Rightarrow a \le \frac{1}{\sqrt{2}}$

Sol 14: (C) $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^z$ for this to satisfy

$x = w = y = z = 1$

or $x = w = y = z = -1$

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ or $\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} = 0$

independent of N_1, N_2, N_3, N_4

Sol 15: (C) $\theta = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$

$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{11}\right) + \tan^{-1}\frac{1}{18}$

$\Rightarrow \theta = \tan^{-1}\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11\times 8}} = \tan^{-1}\frac{65}{195} = \tan^{-1}\frac{1}{3}$

$\Rightarrow \cot\theta = 3$

Sol 16 : (A) (A) f(x) = {x} (Periodic)

(B) $g(x) = x\sin\frac{1}{x}$ (not periodic)

(C) h(x) = x cos x (not periodic)

(D) $\sin(\sin^{-1}x)$ (not periodic)

Sol 17: (A) $2\tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{\frac{1}{7}+\frac{1}{7}}{1-\frac{1}{49}} = \tan^{-1}\frac{7}{24}$

$4\cot^{-1}3 = 4\tan^{-1}\frac{1}{3}$

$= 2\tan^{-1}\frac{3}{4} = \tan^{-1}\frac{6/4}{1-\frac{9}{16}} = \tan^{-1}\frac{24}{7}$

$3\cot^{-1}4 = 3\tan^{-1}\frac{1}{4}$

$= \tan^{-1}\frac{8}{15}+\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{47}{52}$

$4\cot^{-1}4 = \tan^{-1}\frac{47}{52} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{240}{173}$

Checking all options one by one

Sol 18: (B) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$

Put x = sin y

$\sin^{-1}\sin 2y = 2\sin^{-1}\sin y\cos y$

$\Rightarrow -1 \le \sin 2y \le 1$

$\Rightarrow -\frac{\pi}{2} \le 2y \le \frac{\pi}{4}$

$\Rightarrow -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

So it is true if $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Sol 19: (B) $\sum_{k=1}^{k=n}\tan^{-1}\frac{2k}{2+k^2+k^4} = \tan^{-1}\frac{6}{7}$

L. H. S. $= \sum\tan^{-1}\frac{2k}{1+(k^4+k^2+1)}$

$= \sum\tan^{-1}\frac{k^2+k+1-(k^2-k+1)}{1+(k^2-k+1)(k^2+k+1)}$

$= \sum_{k=1}^{\infty}(\tan^{-1}(k^2+k+1)-\tan^{-1}(k^2-k+1)$

$= \tan^{-1}(n^2+n+1) - \tan^{-1}(1) = \tan^{-1}\frac{6}{7}$

$\Rightarrow \tan^{-1}\left[\left(n+\frac{1}{2}\right)^2+\frac{3}{4}\right] = \tan^{-1}\frac{13}{1}$

$\left(n+\frac{1}{2}\right)^2 = \frac{49}{4} \Rightarrow n = 3$

Sol 20: (A) $f(x) = \cos^{-1}\left[\left(\frac{3x^2-7x+8}{1+x^2}\right)\right]$

$1 + x^2 \ge 1$

$3x^2 - 7x + 8 = 3(x^2+1) - 7x + 5$

$= 3\left[\left(x-\frac{7}{6}\right)^2+8-\frac{49}{36}\right] = 3\left[\left(x-\frac{7}{6}\right)^2+\frac{23}{36}\right]$

$\Rightarrow \frac{3x^2-7x+8}{x+1} = 3-\frac{7x-5}{x^2+1}$

$\Rightarrow -1 \le 3-\frac{7x-5}{x^2+1} < 2$

$\Rightarrow -4 \le \frac{-(7x-5)}{(x^2+1)} < -1$

$\Rightarrow 4 \ge \frac{(7x-5)}{x^2+1} > 1$

$4x^2 - 7x + 9 \ge 0$ & $x^2 - 7x + 6 < 0$

always true & $(x-6)(x-1) < 0 \Rightarrow x \in (1, 6)$

Sol 21: (B) $a_r = 2^{r-1} = \frac{1}{b_r}$

$2a_r + \frac{1}{b_r} = 2^r + 2^{1-r} = 2^2 + \frac{2}{2^r}$

$t_r = \cot^{-1}(2a_r + b_r) = \tan^{-1}\frac{2^r}{2^{2r}+2}$

$= \tan^{-1}\frac{2^{r-1}}{1+2^{2r-1}} = \tan^{-1}\frac{2^r-2^{r-1}}{1+2^r\,2^{2r-1}}$

$= \tan^{-1}(2^r) - \tan^{-1}(2^{r-1})$

$\sum_{r=1}^{\infty}t_r = \tan^{-1}(2^{\infty}) - \tan^{-1}(2^{1-1}) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$

Sol 22: (B) $f(x) = \sin^{-1}(3x-4x^3)$

Let's put $x = \sin y = \sin^{-1}\sin 3y$

$-\frac{\pi}{2} \le 3y \le \frac{\pi}{2}$ $f(x) = 3\sin^{-1}x$

$\frac{3\pi}{2} \ge 3y \ge \frac{\pi}{2}$ $f(x) = \pi - 3\sin^{-1}x$

$-\frac{3\pi}{2} \Rightarrow 3y \le -\frac{\pi}{2}$ f(x) = $-\pi - 3\sin^{-1}x$

It's not differentiable 2 times.

Sol 23: (D) $\sec^{-1}x = \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{5}{3\sqrt{3}}$

$\sec^{-1}(x) = \pi - \cos^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\frac{\sqrt{2}}{3\sqrt{3}}$

$= \pi + \cos^{-1}\left(\frac{\sqrt{2}}{3\sqrt{3}}\frac{1}{2} + \frac{5}{3\sqrt{3}}\frac{\sqrt{3}}{2}\right)$

$\sec^{-1}x = \pi + \cos^{-1}\left(\frac{5\sqrt{3}+\sqrt{2}}{6\sqrt{3}}\right)$

$x = \sec\left(\pi + \cos^{-1}\frac{15+\sqrt{6}}{18}\right) = -\frac{18}{15+\sqrt{6}}$

Sol 24: (C) $\left[\tan\left(\frac{x}{4} + \frac{1}{2}\sin^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{a}{b}\right)\right]^{-1}$

$= \left[\frac{1+\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}{1-\tan\frac{1}{2}\sin^{-1}\frac{a}{b}} + \frac{1-\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}{1+\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}\right]^{-1}$

$= \left[\frac{2\left(1+\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)}{\left(1-\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)}\right]^{-1}$

$= \frac{1-\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}}{2\left(1+\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)} = \frac{1}{2}\cos\sin^{-1}\frac{a}{b} = \frac{1}{2}\frac{\sqrt{b^2-a^2}}{b}$

Sol 25: (C) $x = \tan^{-1}(1) - \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} - \frac{2\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{4}$

$y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \sqrt{\frac{1+\cos\cos^{-1}\left(\frac{1}{8}\right)}{2}}$

$= \sqrt{\frac{9}{16}} = \frac{3}{4}$

$y = -\frac{3}{4}\tan x.$

Sol 26: (B) RHS = $\cot^{-1}\frac{2x^2-1}{\sqrt{4x^2-4x^4}}$

$= \cos^{-1}\frac{\sqrt{4x^2-4x^4}}{4x^4+1-4x^2+4x^2-4x^4} = \cos^{-1}\sqrt{4x^2-4x^4}$

Put x = cos y

$f(x) = \cos^{-1}|\sin 2y|$

RHS = $\frac{\pi}{2} - \sin^{-1}|\sin 2y|$

Since $|\sin 2y| \ge 0$, so RHS will always be greater than zero.

Then x can be (0, 1)

Sol 27: (A) $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x-2)$

$x \in [-1, 1]$

$(3x-2) \in [-1, 1]$

$\Rightarrow x \in \left[\frac{1}{3}, 1\right]$

$\Rightarrow \sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$

$\Rightarrow 2\sin^{-1}x = \frac{\pi}{2} + \sin^{-1}(3x-2)$

Taking cosine of both sides

$\Rightarrow \cos(2\sin^{-1}x) = -(3x-2)$

$\Rightarrow 1 - 2x^2 = -3x + 2$

$\Rightarrow 2x^2 - 3x + 1 = 0$

$\Rightarrow (x-1)\left(x-\frac{1}{2}\right) = 0$

$x = 1, \frac{1}{2}$

Sol 28: (A) $f(x) = \sin^{-1}\left|\frac{1-x^2}{1+x^2}\right|$

$g(x) = \cot^{-1}x - \tan^{-1}x = \frac{\pi}{2} - 2\tan^{-1}x$

Put x = tan y in f(x)

$f(x) = \sin^{-1}|\cos 2y|$

$\frac{\pi}{2} - \cos^{-1}|\cos 2y|$

$f(x) = g(x)$ when $x \in [0, 1]$

Sol 29: (D) $\tan[\cos^{-1}\{\sin(2\tan^{-1}2)\}]$

$= \tan[\cos^{-1}\{2\sin(\tan^{-1}2)\cos(\tan^{-1}2)\}]$

$$= \tan\left[\cos^{-1}\left(\frac{2\times 2}{\sqrt{5}}\times\frac{1}{\sqrt{5}}\right)\right] = \tan\cos^{-1}\frac{4}{5} = \frac{3}{4}$$

Sol 30: (A) $\sum_{n=1}^{\infty}\left|\frac{\sin^{-1}x-\cos^{-1}x}{\pi r}\right|$ is finite

$$= \sum_{n=1}^{\infty}\left|\frac{\frac{\pi}{2}-2\cos^{-1}x}{\pi r}\right|^{n}$$

$$\frac{\pi}{2} \geq \frac{\pi}{2}-2\cos^{-1}x \geq -\frac{3\pi}{2}$$

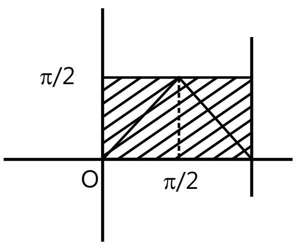
$$\pi r > \frac{\pi}{2}$$

$$\Rightarrow r > \frac{1}{2}$$

Sol 31: (C) $y = \sin^{-1}(\sin x)$, $x \in [0, \pi]$

$0 < x \leq \frac{\pi}{2}$ $y = x$

$\frac{\pi}{2} < x \Rightarrow \pi y = \pi - x$



$A_1 \leq A_2 \leq A_3$

$A_1 = A_2 = \frac{A_3}{2}$

$A_1 = \frac{1}{2}\times\frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{8}$

$A_3 = \frac{\pi^2}{4}$

Sol 32: (A) $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4-2n^2+2}$

$$= \sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{1+(n^2-1)(n^2-1)}$$

$$= \sum_{n=1}^{\infty}\tan^{-1}\frac{(n+1)^2-(n-1)^2}{1+(n-1)^2(n+1)^2}$$

$$= \sum_{n=1}^{\infty}\tan^{-1}(n+1)^2-\tan^{-1}(n-1)^2$$

$= 2[\tan^{-1}(\infty)] - \tan^{-1}(1) - \tan^{-1}0 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\tan^{-1}(2) + \tan^{-1}(3) = \pi + \tan^{-1}(-1) = \frac{3\pi}{4}$

$4\tan^{-1}(1) - \pi$

$\sec^{-1}(1-\sqrt{2}) = \cos^{-1}\left(\frac{1}{1-\sqrt{2}}\right) = -\cos^{-1}(\sqrt{2}+1)$

Previous Years' Questions

Sol 1: (B) $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$$= \tan\left[\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$\left[\because \cos^{-1}\left(\frac{4}{5}\right)=\tan^{-1}\left(\frac{3}{4}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right] = \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{17}{6}$$

Sol 2: (C) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Sol 3: (C) Given function is $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

Function is defined, if

(i) $x(x+1) \geq 0 \Rightarrow$ Domain of square root function.

(ii) $x^2 + x + 1 \geq 0 \Rightarrow$ Domain of square root function.

(iii) $\sqrt{x^2+x+1} \leq 1 \Rightarrow$ Domain of $\sin^{-1}$ function.

From (ii) and (iii)

$0 \leq x^2 + x + 1 \leq 1 \cap x^2 + x \geq 0$

$\Rightarrow 0 \leq x^2 + x + 1 \leq 1 \cap x^2 + x + 1 \geq 1$

$\Rightarrow x^2 + x + 1 = 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow x = 0, x = -1$

Sol 4: (B) We know that, $\sin^{-1}(\alpha) + \cos^{-1}(\alpha) = \frac{\pi}{2}$

Therefore, α should be equal in both functions.

$$\therefore x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots = x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots$$

$$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}}$$

$\Rightarrow \frac{\frac{x}{2+x}}{2} = \frac{\frac{x^2}{2+x^2}}{2}$

$\Rightarrow \frac{2x}{2+x} = \frac{2x^2}{2+x^2}$

$\Rightarrow 2x(2 + x^2) = 2x^2(2 + x)$

$\Rightarrow 4x + 2x^3 = 4x^2 + 2x^3$

$\Rightarrow x(4 + 2x^2 - 4x - 2x^2) = 0$

$\Rightarrow$ Either $x = 0$ or $4 - 4x = 0$

$\Rightarrow x = 0$ or $x = 1$

$\because 0 < |x| < \sqrt{2}$,

$\therefore x = 1$ and $x \neq 0$

Sol 5: (D) Given, $\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1}x)$...(i)

and we know,

$\cot^{-1}\theta = \sin^{-1}\left(\frac{1}{\sqrt{1+\theta^2}}\right)$,

and $\tan^{-1}\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+\theta^2}}\right)$

$\therefore$ From Eq. (i),

$$\sin\left(\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

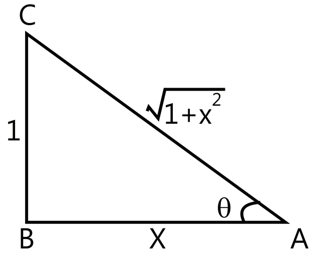
$$\Rightarrow \frac{1}{\sqrt{1+(1+x)^2}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow 1 + x^2 + 2x + 1 = x^2 + 1$

$\Rightarrow x = -\frac{1}{2}$

Sol 6: (C) We have, $0 < x < 1$

Let $\cot^{-1} x = \theta$



$\Rightarrow \cot\theta = x$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}} = \sin(\cot^{-1}x)$

and $\cos\theta = \frac{x}{\sqrt{1+x^2}} = \cos(\cot^{-1}x)$

Now

$$\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left(x\frac{x}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right)^2-1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left(\frac{1+x^2}{\sqrt{1+x^2}}\right)^2-1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\,[1+x^2-1]^{\frac{1}{2}} = x\sqrt{1+x^2}$$

Sol 7: Given,

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{a+b+c}\left(\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ca}}+\sqrt{\frac{c}{ab}}\right)-(a+b+c)\sqrt{\frac{a+b+c}{abc}}}{1-(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\frac{a+b+c}{abc}}(a+b+c)-(a+b+c)\sqrt{\frac{a+b+c}{abc}}}{1-\frac{(a+b+c)(ab+bc+ca)}{abc}}\right)$$

$\Rightarrow \theta = \tan^{-1}0 \Rightarrow \tan\theta = 0$

Sol 8 : $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right] = \tan\left[\tan^{-1}\left(\frac{2.\frac{1}{5}}{1-\frac{1}{25}}\right)-\frac{\pi}{4}\right]$

$= \tan\left[\tan^{-1}\left(\frac{5}{12}\right)-\frac{\pi}{4}\right]$

$$= \frac{\tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right]-\tan\left(\frac{\pi}{4}\right)}{1+\tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right]\tan\frac{\pi}{4}} = \frac{\frac{5}{12}-1}{1+\frac{5}{12}\cdot 1} = -\frac{7}{17}$$

Sol 9: (A) Given, $A = 2\tan^{-1}(2\sqrt{2}-1)$ and

$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

Here, $A = 2\tan^{-1}(2\sqrt{2}-1)$

$= 2\tan^{-1}(2 \times 1.414 - 1) = 2\tan^{-1}(1.828)$

$\therefore\ A > 2\tan^{-1}(\sqrt{3}) = 2.\frac{\pi}{3} = \frac{2\pi}{3}$

To find the value of B, we first say

$\sin^{-1}\frac{1}{3} < \sin^{-1}\cdot\frac{1}{2} = \frac{\pi}{6}$

So that, $0 < 3\sin^{-1}\frac{1}{3} < \frac{\pi}{2}$

Now, $3\sin^{-1}\frac{1}{3} = \sin^{-1}\left(3\cdot\frac{1}{3} - 4\cdot\frac{1}{27}\right) = \sin^{-1}\left(\frac{23}{27}\right)$

$= \sin^{-1}(0.851) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

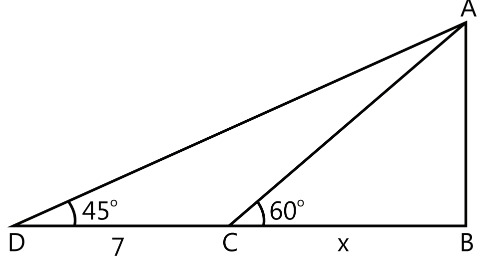
$\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{5}\right) = \frac{\pi}{3}$

$\therefore\ B < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Thus, $A > \frac{2\pi}{3}$ and $B < \frac{2\pi}{3}$

Hence, greater angle is A.

Sol 10: (B)



BD = AB = 7 + x

Also $AB = x\tan 60^\circ = x\sqrt{3}$

$\therefore\quad x\sqrt{3} = 7 + x$

$x = \frac{7}{\sqrt{3}-1}$

$AB = \frac{7\sqrt{3}}{2}(\sqrt{3}+1)$

Sol 11: (A) Let $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

$\Rightarrow\quad E = \cot\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$\Rightarrow\quad E = \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$

$\Rightarrow\ E = \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$

Sol 12: (A) $\cos(\alpha+\beta) = \frac{4}{5} \quad\Rightarrow\quad \tan(\alpha+\beta) = \frac{3}{4}$

$\sin(\alpha-\beta) = \frac{5}{13} \quad\Rightarrow\quad \tan(\alpha-\beta) = \frac{5}{12}$

$\tan 2\alpha = \tan(\alpha+\beta+\alpha-\beta) = \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\frac{5}{12}} = \frac{56}{33}$

Sol 13: (B) $r = \frac{a}{2}\cot\frac{\pi}{n}$

'a' is side of polygon.

$R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

$\frac{r}{R} = \frac{\cot\frac{\pi}{n}}{\operatorname{cosec}\frac{\pi}{n}} = \cos\frac{\pi}{n}$

$\cos\frac{\pi}{n} \neq \frac{2}{3}$ for any $n \in N$

Sol 14: (B) $\ell = \cos 45^\circ = \frac{1}{\sqrt{2}}$

$m = \cos 120^\circ = -\frac{1}{2}$

$n = \cos\theta$

Where θ is the angle which line makes with positive z-axis.

Now $\ell + m^2 + n^2 = 1 \quad\Rightarrow\quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4} \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$ (θ being acute)

$\Rightarrow\quad \theta = \frac{\pi}{3}.$

Sol 15: (A) $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$x = \tan\theta$

$\frac{-\pi}{6} < \theta < \frac{\pi}{6}$

$\tan^{-1} y = \theta + \tan^{-1}\tan 2\theta = \theta + 2\theta = 3\theta$

$y = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$y = \frac{3x - x^3}{1 - 3x^2}$

Sol 16: (B) $0 \le x < 2\pi$

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$(\cos x + \cos 4x) + (\cos 2x + \cos 3x) = 0$

$2\cos\frac{5x}{2}\cos\frac{3x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$

$2\cos\frac{5x}{2}\left[2\cos x\cos\frac{x}{2}\right] = 0$

$\cos\frac{5x}{2} = 0$ or $\cos x = 0$ or $\cos\frac{x}{2} = 0$

$x = \frac{(2n+1)\pi}{5}$ or $x = (2n+1)\frac{\pi}{2}$ or $x = (2n+1)\pi$

$x = \left\{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

Number of solution is 7

Sol 17: (A) At $x = \frac{\pi}{6} \Rightarrow y = \frac{\pi}{3}$

$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right) \quad \because x \in \left(0, \frac{\pi}{2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$f(x) = \frac{\pi}{4} + \frac{x}{2} \qquad f'(x) = \frac{1}{2}$

Slope of normal = -2

Equation of normal $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

$y = -2x + \frac{2\pi}{3}$

JEE Advanced/Boards

Exercise 1

Sol 1: $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$; $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$,

Put x = tan y

$\alpha = 2\tan^{-1}\tan\left(y + \frac{\pi}{4}\right)$

$\beta = \sin^{-1}\cos 2y = \frac{\pi}{2} - \cos^{-1}\cos 2y$

If $0 < \tan x < 1$; $0 < y < \frac{\pi}{4}$

$0 < 2y < \frac{\pi}{2} \Rightarrow \frac{\pi}{4} < y + \frac{\pi}{4} < \frac{\pi}{2}$

$\alpha = 2\left(y + \frac{\pi}{4}\right) = 2y + \frac{\pi}{2}$

$\beta = \frac{\pi}{2} - 2y$

$\Rightarrow \alpha + \beta = \pi$

If $x > 1 \Rightarrow \tan y > 1$

$\Rightarrow \frac{\pi}{2} > y > \frac{\pi}{4} \Rightarrow \pi > 2y > \frac{\pi}{2}$

$\Rightarrow \frac{3\pi}{4} > y + \frac{\pi}{4} > \frac{\pi}{2}$

$\alpha = 2\left[-\pi + \frac{\pi}{4} + \tan^{-1} x\right] = \frac{-3\pi}{2} + 2\tan^{-1}x$

$\beta = \frac{\pi}{2} - [2 + 2\tan^{-1}] = +\frac{3\pi}{2} - 2\tan^{-1}x$

$\Rightarrow \alpha + \beta = 0$

Sol 2: $y = \tan^{-1}\left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$

$\tan y = \left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$

$\Rightarrow \frac{1 - \tan y}{1 + \tan y} = \sqrt{\frac{1-x^2}{1+x^2}}$

$\Rightarrow \frac{1 + \tan^2 y - 2\tan y}{1 + \tan^2 y + 2\tan y} = \frac{1-x^2}{1+x^2}$

$\Rightarrow x^2 = \frac{2\tan y}{1 + \tan^2 y} = \sin 2y$

Sol 3: (i) n^{th} term = $\tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}}$

$= \tan^{-1}\dfrac{2^n - 2^{n-1}}{1+2^n(2^{2x-1})}$

n^{th} term = $\tan^{-1}(2n) - \tan^{-1}(2^{n-1})$

Sum of infinite series

$= \tan^{-1}(\infty) - \tan^{-1}(1) = \dfrac{\pi}{4}$

(ii) $\tan^{-1}\dfrac{(x+1)-x}{1+x(x+1)} + \tan^{-1}\dfrac{(x+2)-(x+1)}{1+(x+2)(x+1)} + \ldots .$

$=\tan^{-1}(x+1) - \tan^{-1}(x) + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \ldots$

$= \tan^{-1}(x+2) - \tan^{-1}(x) + \ldots.$

$= \tan^{-1}(x+n) - \tan^{-1}(x)$

Sol 4: $x \in \left[-1, \dfrac{-1}{2}\right]$

$f(x) = \sin^{-1}(3x - 4x^3) + \cos^{-1}(4x^3 - 3x)$

$f(x) = g(x) + h(x)$

$g(x) = \sin^{-1}\sin 3y$ where $y = \sin^{-1}x$

$h(x) = \cos^{-1}\cos 3z$ where $z = \cos^{-1}x$

$x \in \left[-1, \dfrac{-1}{2}\right]$

$y = \sin^{-1}x \in \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{6}\right] \Rightarrow 3y \in \left[-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right]$

$z = \cos^{-1}x \in \left[\dfrac{2\pi}{3}, \pi\right] \Rightarrow 3z \in [2\pi, 3\pi]$

$g(x) = -\pi - 3\sin^{-1}x$

$= -\pi - 3\left(\dfrac{\pi}{2} - \cos^{-1}x\right) = -\dfrac{5\pi}{2} + 3\cos^{-1}x$

$h(x) = -2\pi + 3\cos^{-1}x$

$f(x) = 6\cos^{-1}x - \dfrac{9\pi}{2}$

$\therefore a = 6, b = -\dfrac{9}{2}$

Sol 5: (i) $\sin^{-1}x + \sin^{-1}2x = \dfrac{\pi}{3}$

$\Rightarrow \sin^{-1}x = \dfrac{\pi}{3} - \sin^{-1}2x$

$\Rightarrow x = \sin\left(\dfrac{\pi}{3} - \sin^{-1}2x\right)$

$\Rightarrow x = \dfrac{\sqrt{3}}{2}\cos\sin^{-1}2x - \dfrac{1}{2}(2x)$

$\Rightarrow 2x = \dfrac{\sqrt{3}}{2}\sqrt{1-4x^2} \Rightarrow \dfrac{16x^2}{3} = 1 - 4x^2$

$\dfrac{28x^2}{3} = 1 \Rightarrow x = \dfrac{\sqrt{3}}{\sqrt{28}} = \dfrac{1}{2}\sqrt{\dfrac{3}{7}}$

(ii) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}(x)$

$\Rightarrow \dfrac{(x-1)+(x+1)}{1-(x^2-1)} = \dfrac{3x-x}{1+3x^2}$

$\Rightarrow \dfrac{2x}{2-x^2} = \dfrac{2x}{1+3x^2} \Rightarrow 4x^2 = 1\ ;\ x = 0$

(iii) $\tan^{-1}\left(\dfrac{x-1}{x+1}\right) + \tan^{-1}\left(\dfrac{2x-x}{2x+x}\right)$

$= \tan^{-1}\dfrac{\dfrac{x-1}{x+1} + \dfrac{2x-1}{2x+1}}{1 - \dfrac{(x-1)(2x-1)}{(x+1)(2x+1)}}$

$= \tan^{-1}\dfrac{2x^2 - 1 - x + 2x^2 - 1 + x}{6x}$

$= \tan^{-1}\left(\dfrac{4x^2-2}{6x}\right) = \tan^{-1}\left(\dfrac{2x^2-1}{3x}\right)$

$\Rightarrow \dfrac{2x^2-1}{x} = \dfrac{23}{12}$

$\Rightarrow 24x^2 - 12 = 23x$

$\Rightarrow 24x^2 - 23x - 12 = 0$

$x = \dfrac{23 \pm \sqrt{529+1152}}{48} \Rightarrow x = \dfrac{4}{3}$

(iv) $\cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right) + \tan^{-1}\dfrac{2x}{x^2-1} = \dfrac{2\pi}{3}$

LHS = $\tan^{-1}\dfrac{\sqrt{(x^2+1)^2-(x^2-1)^2}}{(x^2-1)} + \tan^{-1}\dfrac{2x}{x^2-1}$

$= \tan^{-1}\dfrac{|2x|}{x^2-1} + \tan^{-1}\dfrac{2x}{x^2-1}$

If $x < 0$ LHS = $= 2\tan^{-1}\dfrac{2x}{x^2-1}$

If $x > 0 = 2\tan^{-1}\dfrac{2x}{x^2-1}$

$\frac{2x}{x^2-1} = \tan\frac{\pi}{3} \Rightarrow 2x = \sqrt{3}(x^2-1)$

$4x^2 = 3(x^4 + 1 - 2x^2) \Rightarrow 3x^4 - 10x^2 + 3 = 0$

$\Rightarrow (x^2-3)(3x^2-1) = 0 \Rightarrow x = \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}$

$x = \pm\frac{1}{\sqrt{3}}$ does not satisfy.

Sol 6: $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

LHS $= \tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}\frac{\left(x+\frac{1}{y}\right)}{1-\frac{x}{y}} = \tan^{-1}(3)$

$\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3 \Rightarrow x+\frac{1}{y} = 3 - \frac{3x}{y}$

$\frac{1}{y}(1+3x) = 3 - x \Rightarrow y = \frac{1+3x}{3-x}$

At x = 1 ; y = 2

At x = 2 ; y = 7

Sol 7: $x^2 - 4x + 1 = 0$

$(x-2)^2 = 3$

$\alpha = 2+\sqrt{3}$; $\beta = 2-\sqrt{3}$; $\alpha+\beta = 4$; $\alpha\beta = 1$

$f(\alpha,\beta) = \frac{(2-\sqrt{3})^3}{2}\text{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)$

$+ \frac{(2+\sqrt{3})^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$

$= \frac{(2-\sqrt{3})^3}{2}\frac{1}{[\sin\theta_1/2]^2} + \frac{(2+\sqrt{3})^3}{2}\frac{1}{[\cos\theta_2/2]^2}$

$\frac{(2-\sqrt{3})^3}{1-\cos\tan^{-1}\left[\frac{(2-\sqrt{3})}{(2-\sqrt{3})}\right]} + \frac{(2+\sqrt{3})^3}{1-\cos\tan^{-1}\left[\frac{(2+\sqrt{3})}{(2-\sqrt{3})}\right]}$

$= \frac{(2-\sqrt{3})^3}{1-\frac{2+\sqrt{3}}{\sqrt{14}}} + \frac{(2+\sqrt{3})^3}{1+\frac{(2-\sqrt{3})}{2\sqrt{7}}}$

$= \frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}} + \frac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$

$= \sqrt{\alpha^2+\beta^2}\left[\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta}\right]$

By putting all value, we get = 56

Sol 8: $f(x) = \sin^{-1}\frac{2x}{1+x^2}$; $g(x) = \cos^{-1}\frac{1-x^2}{1+x^2}$

$h(x) = \tan^{-1}\frac{2x}{1-x^2}$

put x = tan y

$f(x) = \sin^{-1}\sin 2y$; $g(x) = \cos^{-1}\cos 2y$

$h(x) = \tan^{-1}\tan 2y$

(i) $x \in (-1, 1)$

$-\frac{\pi}{4} \le y \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2y \le \frac{\pi}{2}$

$f(x) = 2y = 2\tan^{-1}x$

$g(x) = \begin{cases} 2\tan^{-1}x & ; x \ge 0 \\ -2\tan^{-1}x & ; x \le 0 \end{cases}$

$h(x) = 2\tan^{-1}x$

$f(x) + g(x) + h(x) = \begin{cases} 2\tan^{-1}x & ; x \le 0 \\ 6\tan^{-1}x & ; x \ge 0 \end{cases}$

$x = \tan\frac{\pi}{12} = 2-\sqrt{3}$

(ii) f(2) + g(2) + h(2)

$f(2) = \sin^{-1}\left(\frac{4}{5}\right)$

$g(2) = \cos^{-1}\left(\frac{-3}{5}\right)$

$h(2) = \tan^{-1}\left(\frac{4}{-3}\right) = \cot^{-1}\left(\frac{-3}{4}\right)$

$f(2) = -g(2)$

$f(2) + g(2) + h(2) = \cot^{-1}\left(\frac{-3}{4}\right)$

Sol 9: (i) $(\cot^{-1}x)^2 - 5(\cot^{-1}x) + 6 > 0$

$(\cot^{-1}x - 3)(\cot^{-1}x - 2) > 0$

$\cot^{-1}x \in (-\infty, 2) \cup (3, \infty)$

$\cot^{-1}x \in (0, 2) \cup (3, \pi)$

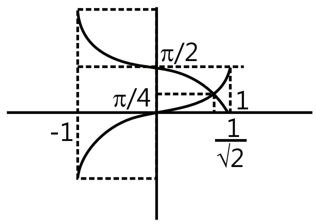
$x \in (\cot 2, \infty) \cup (-\infty, \cot 3)$

(ii) $\sin^{-1}x > \cos^{-1}x$

$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$0 \le \cos^{-1}x \Rightarrow \pi$

$x \in \left[\frac{1}{\sqrt{2}}, 1\right]$



(iii) $\tan^2(\sin^{-1}x) > 1$

$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \frac{x^2}{1-x^2} > 1$

$\Rightarrow \frac{2x^2-1}{x^2-1} < 0$

$x \in \left(-1, \frac{-1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, 1\right)$

Sol 10: $x(x-2)(3x-7) = 2$ are real and positive at $x = 0, +2, \frac{7}{3}$ it has (–2) value.

At $x = 4 f(4) = 38$

At $x = \frac{1}{2} f\left(\frac{1}{2}\right) = \frac{17}{8}$

One root between 0 to $\frac{1}{2}$, one between $\frac{1}{2}$ to 2, one between $\frac{7}{3}$ to 4.

$\tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$

$= \tan^{-1}\left[\frac{r+s+t-rst}{1-(rs+st+tr)}\right]$ ..... (i)

equation is

$\Rightarrow 3x^2 - 13x^2 + 14x - 2 = 0$

$r + s + t = \frac{13}{3}$

$rst = \frac{-(-2)}{3} = \frac{2}{3}$

$rs + st + tr = \frac{+14}{3} = \tan^{-1}(-1) = \frac{-\pi}{4}$

since r, s, t are always positive so value will be $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Sol 11: $f(x) = \frac{\pi}{4} + \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) - \tan^{-1} x$

$\text{sgn}(f(x)) = 1$ when $f(x) > 0$

for any $x > 0$

$f(x) = \frac{\pi}{4} + \tan^{-1}\frac{1}{x} - \tan^{-1}x$

$f(x) = \frac{3\pi}{4} - 2\tan^{-1}x$

$\frac{3\pi}{4} - 2\tan^{-1}x > 0$

$0 < \tan^{-1}x < \frac{3\pi}{4}$

$0 \le x < \tan\left(\frac{3\pi/4}{2}\right)$

$\Rightarrow 0 < x < \sqrt{2} + 1$

$0 \le x < \sqrt{2} + 1$

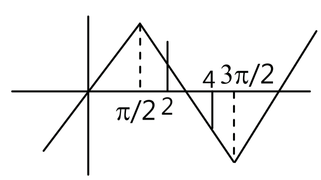
$x = 0, 1, 2$

Sol 12: $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$

$\frac{2x^4+4}{1+x^2} = 2 + \frac{2}{1+x^2}$

$4 \ge \frac{2x^2+4}{1+x^2} > 2$

$-\left(\frac{2x^2+4}{1+x^2}\right) + \pi < \pi - 3$



$\frac{2x^2+4}{1+x^2} > 3$

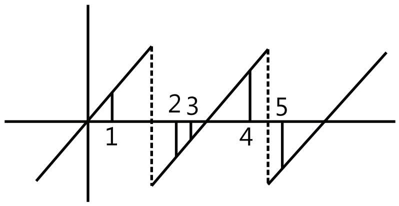
$\frac{1-x^2}{1+x^2} > 0$

$x \in (-1, 1)$

Sol 13: $f(x) = \tan^{-1}(\cot x - 2\cot 2x)$

$\sum_{r=1}^{5} f(r) = a - b\pi$

$f(x) = \tan^{-1}\left[\frac{1}{\tan x} - \frac{2(1-\tan^2 x)}{2\tan x}\right] = \tan^{-1}(\tan x)$



$\Sigma f(r) = 1 + (-\pi + 2) + (\pi + 3) + (\pi + 4) + (2\pi + 5)$

$= 15 - 5\pi$

$a = 15, b = 5, a + b = 20$

Sol 14: $f(x) = (2a + b)\cos^{-1}x + (a + 2b)\sin^{-1}x$

Domain $-1 \le x \le 1$

Then range should be $-1 \le f(x) \le 1$

$f(x) = a[2\cos^{-1}x + \sin^{-1}x] + b[\cos^{-1}x + 2\sin^{-1}x]$

$= a\left[\frac{\pi}{2} + \cos^{-1}x\right] + b\left[\frac{\pi}{2} + \sin^{-1}x\right]$

$= \frac{\pi}{2}(a + b) + (a\cos^{-1}x + b\sin^{-1}x)$

$= \frac{\pi}{2}(a + b) + a(\cos^{-1}x + \sin^{-1}x) + (b - a)\sin^{-1}x$

$= \frac{\pi}{2}(2a + b) + (b - a)\sin^{-1}x$

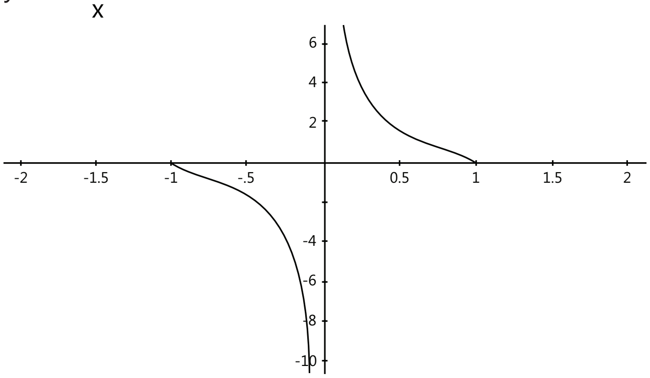
$\frac{\pi}{2}(3a) < f(x) < \frac{\pi}{2}(a + 2b)$

$\frac{\pi}{2}(3a) = -1 \Rightarrow a = \frac{-2}{3\pi}$; $b = \frac{4}{3\pi}$

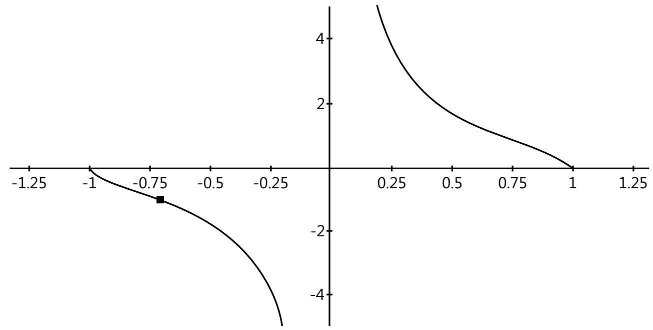
$\pi(a - b) = -2$

Sol 15: (i) $y = \tan(\cos^{-1}x) = \frac{\sqrt{1-x^2}}{x}$ except $x = 0$

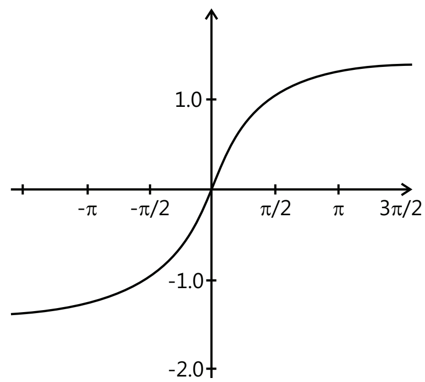
$y = \frac{\sqrt{1-x^2}}{x}$ identical



(ii) $y = \tan(\cot^{-1}x) = \frac{1}{x}$ except $x = 0$ identical



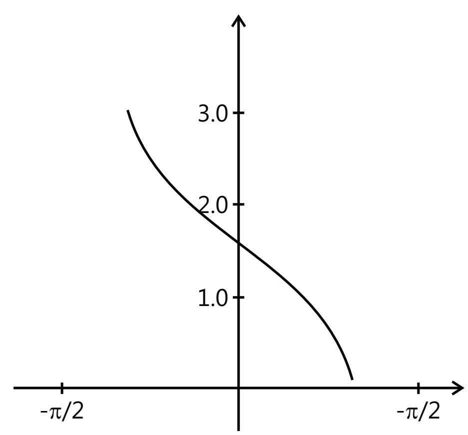
(iii) $y = \sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$ identical



(iv) $y = \cos(\tan^{-1}x)$

$= \cos\left(\frac{\pi}{2} - \cot^{-1}x\right) = -\sin(-\cot^{-1}x)$

$= \sin(\cot^{-1}x)$ identical



Sol 16: (i) $f(x) = \cot^{-1}(2x - x^2)$

$2x - x^2 = x(2 - x)$

Domain $x \in R$

Range $2x - x^2 < 1 \Rightarrow x \in \left[\frac{\pi}{4}, \pi\right)$

(ii) $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$

Domain $\tan x > 0$, $\tan x \neq 1$

$$\log_3 \tan x + \frac{1}{\log_3 \tan x} > 2$$

$$\text{or } \log_3 \tan x + \frac{1}{\log_3 \tan x} < -2$$

$$x \in \left[\left(n\pi, n\pi + \frac{\pi}{2}\right) - \left\{\pi + \frac{\pi}{4}\right\}\right]$$

$$\text{Range} \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left\{\frac{\pi}{2}\right\}$$

(iii) $f(x) = \cos^{-1}\frac{\sqrt{2x^2+1}}{x^2+1}$

Domain $\frac{\sqrt{2x^2+1}}{x^2+1} \leq 1$

$2x^2 + 1 \leq x^4 + 2x^2 + 1$

$x^4 \geq 0$

Always true $x \in R$

(iv) $f(x) = \tan^{-1}(\log_{4/5}(5x^2 - 8x + 4))$

$5x^2 - 8x + 4 > 0$

$$\left(x - \frac{8}{10}\right)^2 + \frac{4}{5} - \frac{64}{100} > 0$$

$$\left(x - \frac{8}{10}\right)^2 + \frac{16}{100} > 0$$

Domain $x \in R$

$$\text{Range } x \in \left(\frac{-\pi}{2}, \frac{\pi}{4}\right]$$

Sol 17: $y = \sin^{-1}\sin 8 - \tan^{-1}\tan 1$

$+ \cos^{-1}\cos 12 - \sec^{-1}\sec 9 + \cot^{-1}\cot 6 - \operatorname{cosec}^{-1}\operatorname{cosec} 7$

$8 \sim \frac{5\pi}{2} + h$; $12 \sim 4\pi - h$; $6 \sim 2\pi - h$

$1 \sim \frac{\pi}{2} - h$; $9 \sim 3\pi - h$; $7 \sim \frac{5\pi}{2} - h$

$y = (3\pi - 8) - 1 + 4\pi - 12 - (9 - 2\pi)$

$+ (6 - \pi) - (7 - 2\pi) = -31 + 10\pi$

Sol 18: $\alpha = \sin^{-1}\left(\frac{36}{85}\right)$ $\beta = \cos^{-1}\frac{4}{5}$

$$\gamma = \tan^{-1}\left(\frac{8}{15}\right)$$

$$\alpha + \beta + \gamma = \sin^{-1}\left(\frac{36}{85}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{8}{15}\right)$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{8}{15}\right)$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\frac{\frac{3}{4} + \frac{8}{15}}{1 - \frac{24}{60}}$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\left(\frac{77}{36}\right) = \frac{\pi}{2}$$

(i) $\Sigma\cot\alpha = \cot\alpha + \cot\beta + \cot\gamma$

since $\alpha + \beta + \gamma = \frac{\pi}{2}$

$1 = \tan\alpha.\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha$

$\cot\alpha\cot\beta\cot\gamma = \cot\alpha + \cot\beta + \cot\gamma$

Hence prove.

(ii) Since $\alpha + \beta + \gamma = \pi/2$

Hence, $\Sigma(\tan\alpha\tan\beta) = 1$

Sol 19: $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\cos\frac{46\pi}{7}$

$$+ \tan^{-1}\left(-\tan\frac{13\pi}{7}\right) + \cot^{-1}\cot\left(\frac{-19\pi}{8}\right)$$

$$\text{LHS} = \frac{33\pi}{7} - 5\pi + 7\pi - \frac{46\pi}{7}$$

$$- \left(\frac{13\pi}{8} - 2\pi\right) + \pi + \left(\frac{19\pi}{8} - 2\pi\right)$$

$$= \frac{-13\pi}{7} + 2\pi + \pi - \frac{13\pi}{4} = \frac{13\pi}{4}$$

Sol 20: (i) $\cos^{-1}\frac{5}{13} + \cos^{-1}\left(\frac{-7}{25}\right) + \sin^{-1}\left(\frac{36}{325}\right)$

$$= \tan^{-1}\left(\frac{12}{5}\right) - \tan^{-1}\frac{24}{7} + \sin^{-1}\left(\frac{36}{325}\right)$$

$$= \tan^{-1}\frac{\left(\frac{12}{5} - \frac{24}{7}\right)}{1 + \frac{12}{5} \times \frac{24}{7}} + \sin^{-1}\frac{36}{325}$$

$= \tan^{-1}\left(\frac{-36}{323}\right) + \tan^{-1}\left(\frac{36}{323}\right) = \pi$

(ii) LHS = $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$

$= \cos^{-1}\left[\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right) + \sqrt{\frac{1}{3}}\sqrt{1-\frac{7+2\sqrt{6}}{12}}\right]$

$= \cos^{-1}\left[\left(\frac{\sqrt{6}+1}{3\sqrt{2}}\right) + \frac{\sqrt{5-2\sqrt{6}}}{3\sqrt{2}\sqrt{2}}\right]$

$= \cos^{-1}\left(\frac{\sqrt{12}+\sqrt{2}+\sqrt{5-2\sqrt{6}}}{6}\right)$

$= \cos^{-1}\left(\frac{\sqrt{12}+\sqrt{3}}{6}\right) = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

Sol 21:

$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$

$= \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$

$= \tan^{-1}(a) - \tan^{-1}(b) + \tan^{-1}(b)$

$- \tan^{-1}(c) + \tan^{-1}(c) - \tan^{-1}(a) = 0$

Sol 22: $x^2 + 5x - 49 = 0 \Rightarrow \alpha, \beta$

$\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$

$= \frac{(\cot\cot^{-1}\alpha)(\cot\cot^{-1}\beta)-1}{(\cot\cot^{-1}\alpha)+(\cot\cot^{-1}\beta)}$

$= \frac{\alpha\beta-1}{\alpha+\beta} = \frac{-1-49}{-5} = 10$

Sol 23: $\theta_1 + \theta_2 + \theta_3 = \pi$

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}+k\right) + \tan^{-1}\left(\frac{1}{2}+2k\right) = \pi$

$\Rightarrow$ Use the formula

$\tan^{-1}x + \tan^{-1}y + \tan z = \tan^{-1}\left(\frac{x+y+z}{1-xy-yz-zx}\right)$

$\frac{1}{2} + \left(\frac{1}{2}+k\right) + \left(\frac{1}{2}+2k\right) = \frac{1}{2}\left(\frac{1}{2}+k\right)\left(\frac{1}{2}+2k\right)$

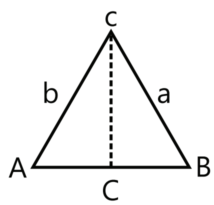
$\Rightarrow 1 + 2k + 1 + 4k + 1 = \frac{(2k+1)(4k+1)}{4}$

$\Rightarrow 24k + 12 = 8k^2 + 1 + 6k$

$\Rightarrow 8k^2 - 18k - 11 = 0$

$\Rightarrow k = \frac{11}{4}$

Sol 24 :



Area (ΔABC) = $\frac{1}{2} \times c \times (b \sin A)$

$\angle A = \angle B =$

$\frac{1}{2}\left[\sin^{-1}\left(\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{3}}\left(\frac{-1}{2}\right)\right) + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \frac{1}{2}\left[\sin^{-1}\sin\left(\frac{2\pi}{3} - \sin^{-1}\frac{1}{\sqrt{3}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$\angle A = \angle B = \frac{\pi}{3}, \quad \angle C = \frac{\pi}{3}$

Area ΔABC = $\frac{C}{2} \times \frac{C}{\sin C}\sin B \sin A$

$= \frac{C^2}{2} \times \frac{\sin^2 A}{\sin C} = \frac{C^2}{2} \times \frac{\sin^2 A}{\sin(180^o - 2A)} = \frac{C^2}{4}\tan A$

$= \frac{(6)^2(3)^{1/2}}{4}\sqrt{3} = 27$

Sol 25: (i) $f(x) = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right), x \in \left(\frac{1}{2}, 1\right)$

$f(x) = \cos^{-1}x + \cos^{-1}\left[\frac{1}{2}(x) + \frac{\sqrt{3}}{2}\sqrt{(1-x^2)}\right]$

$= \cos^{-1}x + \left|\cos^{-1}x - \cos^{-1}\frac{1}{2}\right|$

$x \in \left(\frac{1}{2}, 1\right) \cos^{-1}x < \cos^{-1}\frac{1}{2}$

$= \cos^{-1}x + \cos^{-1}\frac{1}{2} - \cos^{-1}x = \cos^{-1}\frac{1}{2} = \frac{\pi}{6}$

(ii) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put x = tan y

$f(x) = \tan^{-1}\frac{1-\cos y}{\sin y} = \tan^{-1}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$

$= \tan\frac{y}{2} = \frac{1}{2}\tan^{-1}x$

Sol 26 : $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$

⇒ f(x) is onto function so

$\Rightarrow x^2 + 4x + \alpha^2 - \alpha \geq 0$

$\Rightarrow (x + 2)^2 + \alpha^2 - \alpha - 4 \geq 0$

$\Rightarrow (\alpha^2 - \alpha - 4)$ should be zero

$\Rightarrow \alpha^2 - \alpha - 4 = 0$

$\Rightarrow \alpha = \frac{1\pm\sqrt{17}}{2}$

Sol 27 : LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$

$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{2}{36}}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{y}{2}\right)$

$= \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right) = \frac{1}{2}\tan^{-1}\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{4}}$

$= \frac{1}{2}\tan^{-1}\frac{4}{3} = \frac{1}{2}\cos^{-1}\frac{3}{5}$

$= \frac{1}{2}\sin^{-1}\frac{4}{5}$ = RHS

Sol 28 : LHS =

$$\cot^{-1}\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$= \cot^{-1}\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}$$

$= \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}$

Sol 29 : (i) $f(x) = \cos^{-1}\frac{2x}{1+x}$

We know that $-1 \leq \frac{2x}{1+x} \leq 1$

$\frac{2x}{1+x} + 1 \geq 0$ and $\frac{2x}{1+x} - 1 \leq 0$

$\frac{3x+1}{x+1} \geq 0$ and $\frac{x-1}{x+1} \leq 0$

$x \in (-\infty, -1) \cup \left(\frac{-1}{3}, \infty\right]$ and $x \in (-1, 1]$

So $x \in \left[\frac{-1}{3}, 1\right]$

(ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\frac{1+x^2}{2x}$

$\cos(1) \leq \cos(\sin x) \leq 1$

and $\frac{1+x^2}{2x} = \frac{1}{2}\left(\frac{1}{x}+x\right) \geq 1$

for x > 0

or $= \frac{1}{2}\left(\frac{1}{x}+x\right) \leq 1$ for x < 0

(iii) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

then x = 1, –1.

$-1 \leq \frac{x-3}{2} \leq 1 \Rightarrow 1 \leq x \leq 5;\ 4 - x > 0$

$\Rightarrow x < 4$ so $x \in [1, 4]$

(iv) $f(x) = \sin^{-1}[x(x + 2)]$

We can write here that $-1 \leq x^2 + 2x \leq 1$

$x \in [-(1 + \sqrt{2}), (\sqrt{2} - 1)]$

(v) $f(x) = \frac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}[1 - \{x\}]$

$1 - \sin x \geq 0 \Rightarrow \sin x \leq 1$

and $1 - 4x^2 \neq 1$. Also $x \neq 0$ and $1 - 4x^2 > 0$

$\Rightarrow \left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) < 0 \Rightarrow x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

So domain $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$

(vi) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right)$

$+ \log_6(2|x| - 3) + \sin^{-1}(\log_2 x)$

$\Rightarrow -1 \le \log_2 x \le 1 \Rightarrow \frac{1}{2} \le x \le 2$

and $2|x| - 3 > 0 \Rightarrow |x| > \frac{3}{2}$

So now $x \in \left(-\infty, \frac{-3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and $-1 \le \frac{3-2x}{5} \le 1$

$\Rightarrow -8 \le -2x \le 2 \Rightarrow 4 \ge x \ge 1$

Now we have $x \in [1, 4]$ and $3 - x \ge 0$

$\Rightarrow x \le 3$

So domain will be $x \in \left(\frac{3}{2}, 2\right]$

(vii) $f(x) = \log_{10}(1 - \log_7(x^2 - 5x + 13) + \cos^{-1}\left(\frac{3}{2 + \sin\frac{9\pi}{2}x}\right)$

We can write here $-1 \le \frac{3}{2 + \sin\frac{9\pi}{2}x} \le +1$

$\Rightarrow \sin\frac{9\pi}{2}x = +1$

$\Rightarrow \frac{9\pi}{2}x = 2n\pi + \frac{\pi}{2} \Rightarrow x = \frac{4n+1}{9}$

$\Rightarrow x^2 - 5x + 13 = \left(x - \frac{5}{2}\right)^2 + 13 - \frac{25}{4} = \left(x - \frac{5}{2}\right)^2 + \frac{27}{4} < 7$

This gives $x \in (2, 3)$

So the domain would be $x = \frac{21}{9}, \frac{25}{9}$

(viii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left(\frac{x}{2} - 1\right) + \ln\sqrt{x - [x]}$

Now $x - [x] \ne 0 \Rightarrow x \notin I$

and $-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

So the Domain will be $(-2, 2) - \{-1, 0, 1\}$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $x^2 - 4x + 5 > \sin^{-1}(\sin 3) + 2\cos^{-1}(\cos 2) - \pi$

$(x-2)^2 + 1 > \pi - 3 + 4 - \pi$

$(x-2)^2 + 1 \ge 1$

Always true except {2}

Sol 2: (C) We have

$S_1 = \sum x_1 = \sin 2\beta$

$S_2 = \sum x_1 x_2 = \cos 2\beta$

$S_3 = \sum x_1 x_2 x_3 = \cos\beta$

$S_4 = x_1 x_2 x_3 x_4 = -\sin\beta$

So that $\sum_{i=1}^{4} \tan^{-1} x_i = \tan{-1} \frac{S_1 - S_3}{1 - S_2 + S_4}$

$= \tan^{-1}\frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \tan^{-1}\frac{\cos\beta\,(2\sin\beta - 1)}{\sin\beta\,(2\sin\beta - 1)}$

$= \tan^{-1}(\cot\beta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \beta\right)\right)$

$\Rightarrow \frac{\pi}{2} - \beta$

Sol 3: (B) $f(x) = \cot^{-1}\log_{4/5}(5x^2 - 8x + 4)$

$5x^2 - 8x + 4 \ge \frac{4}{5}$

$\log_{4/5}(5x^2 - 8x + 4) \le 1$

$f(x) \in \left[\frac{\pi}{4}, \pi\right)$

Sol 4: (C) $(1 + x)\cos y - x^2 = 0$

$y = \cos^{-1}\frac{x^2}{1+x}$

$\Rightarrow -1 \le \frac{x^2}{1+x} \le 1$

$\frac{x^2}{1+x} \le -1 \Rightarrow \frac{x^2 + x + 1}{1+x} \le 0$

$\frac{x^2 - x - 1}{1+x} \le 0$

$\frac{\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}}{x+1} \le 0$

$x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

Sol 5: (B) $4(\tan^{-1}x)^2 - (\tan^{-1}x) - 3 \le 0$

$(\tan^{-1}x)^2 - 2\left(\frac{1}{8}\tan^{-1}x\right) + \frac{1}{64} - \frac{1}{64} - \frac{3}{4} \Rightarrow 0$

$$\left[\tan^{-1}x - \frac{1}{8}\right]^2 - \frac{49}{64} \le 0$$

$$(\tan^{-1}x - 1)\left(\tan^{-1}x + \frac{3}{4}\right) \le 0$$

$$-\frac{3}{4} \le \tan^{-1}x \le 1$$

$$-\tan^{-1}\left(\frac{3}{4}\right) \le x \le \frac{\pi}{4}$$

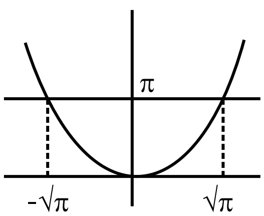
Sol 6: (B) $\sec^{-1}[-\sin^2x]$ is defined only if

$[-\sin^2x] = 1, -1$

$[-\sin^2x] = -1$ when $x \notin n\pi$

$\sec^{-1}(-1) = \pi$

So area bounded



$$= \int_{-\sqrt{\pi}}^{\pi} (\pi - x^2)dx = \left[\pi x - \frac{x^3}{3}\right]_{-\sqrt{\pi}}^{\sqrt{\pi}}$$

$$= \pi\sqrt{\pi} - \frac{\pi\sqrt{\pi}}{3} + \pi\sqrt{\pi} - \frac{\pi\sqrt{\pi}}{3} = \frac{4}{3}\pi\sqrt{\pi}$$

Sol 7: (B) We have from the given equation

$$\tan^{-1}\frac{(a+b)x}{x^2 - ab} = \frac{\pi}{2} - \tan^{-1}\frac{(c+d)x}{x^2 - cd}$$

$$\Rightarrow \tan^{-1}\frac{(a+b)x}{x^2 - ab} = \cot^{-1}\frac{(c+d)x}{x^2 - cd}$$

$$= \tan^{-1}\frac{x^2 - cd}{(c+d)x}$$

$$\Rightarrow (x^2 - ab)(x^2 - cd) = (a+b)(c+d)x^2$$

$$\Rightarrow x^4 - x^2\sum ab + abcd = 0$$

Sol 8: (C) $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) - \sin^{-1}x$

$$\sin^{-1}\sqrt{1-x^2} + \frac{\pi}{2} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$x \in [-1, 1] - \{0\}$

$$\cos\left(\frac{\pi}{2} + \sin^{-1}\sqrt{1-x^2}\right) = \cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$-\sin\sin^{-1}\sqrt{1-x^2} = \cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$-\sqrt{1-x^2} = \cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

If $x > 0$ then it won't satisfy except 1.

If $x < 0$ then it will satisfy.

$x \in [-1, 0) \cup \{1\}$

Sol 9: (C) $f(x) = \operatorname{cosec}^{-1}\sqrt{\log_{\frac{3-4\sec x}{1-2\sec x}} 2}$

$$2 \ge \frac{3-4\sec x}{1-2\sec x} > 1$$

$$\Rightarrow x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right) - \{2n\pi\}$$

Range $\in \left(0, \frac{\pi}{2}\right)$

Sol 10: (A) $[\sin^{-1}x] = [\cos^{-1}x]$

$$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$$

$1 \sim \frac{\pi}{3}$; $\frac{\pi}{2} \sim 1.6$

$[\sin^{-1}x] = -2, -1, 0, 1$

$0 \le \cos^{-1}x \Rightarrow \pi \Rightarrow [\cos^{-1}x] = 0,1,2$

so$[\sin^{-1}x] = [\cos^{-1}x] = 0$ or 1

$x \in [\cos 1, \sin 1]$

Multiple Correct Choice Type

Sol 11: (A, B) (A) $\cos^{-1}\left(\ln\frac{2e+4}{3}\right)$

$\frac{2e+4}{3} \sim 3 > e \Rightarrow \ln\left(\frac{2e+4}{3}\right) > 1$

meaning less because

$\cos^{-1}\left(\ln\frac{2e+4}{3}\right)$ is not defined.

(B) In $\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$, $\frac{\pi}{4} < 1$

$\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$ not defined

(C) $\cot^{-1}\left(\frac{\pi}{2}\right)$ defined

(D) $\sec^{-1}(\pi)$ defined

Sol 12: (A, B, C) Let $\cos^{-1}\left(\frac{4}{5}\right) = \alpha$, that is, $\cos\alpha = \frac{4}{5}$,

so that $\tan\alpha = \sqrt{\left(\frac{5}{4}\right)^2 - 1} = \frac{3}{4}$ ($\because$ $0 < \alpha < \pi$ and $\cos\alpha > 0$)

And $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \dfrac{\tan\alpha + \frac{2}{3}}{1 - \tan\alpha\cdot\frac{2}{3}}$

$= \dfrac{\frac{3}{4}+\frac{2}{3}}{1-\frac{2}{3}\cdot\frac{3}{4}} = \frac{17}{6} = \frac{a}{b}$ (given)

So, a = 17, b = 6, a +b = 23, a - b = 11 and 3b = a + 1

Sol 13: (B, C) (A) $y^2 = \sqrt{1-x^2}$

$\Rightarrow y^4 + x^2 = 1$

Not circle

(B) $y = \sin(\cos^{-1}(1 - x))$

$y = \sqrt{1-(1-x)^2}$

Half circle for y > 0

(C) $y^2 = (\sin\cos^{-1}x)^2$

$y^2 = (1 - x^2) \Rightarrow y^2 + x^2 = 1$

Which is a circle

(D) $y = \sin^{-1}\cos^2 y$

Not a circle

Sol 14: (A, B) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \pi - \cos^{-1}z$

Taking cosine of both sides

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$(xy + z)^2 = (1 - x^2)(1 - y^2)$

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$x^2 + y^2 + z^2 = 1 - 2xyz$

(B) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$

L. H. S.

$= \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y + \frac{\pi}{2} - \cos^{-1}z$

$= \frac{3\pi}{2} - (\cos^{-1}x + \cos^{-1}y + \cos^{-1}z) = \frac{\pi}{2}$

Match the Columns

Sol 15: A $\rightarrow$ s; B $\rightarrow$ p; C $\rightarrow$ r; D $\rightarrow$ q

(A) $f(x) = \sin^{-1}\left(\frac{x}{1+|x|}\right)$

Range $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(B) $g(x) = \cos^{-1}.\ \frac{x}{1+|x|}$

$\frac{x}{1+|x|}$ if $x \geq 0 \Rightarrow 0 \leq \frac{x}{1+x} < 1$

$x \leq 0 \Rightarrow 0 \geq \frac{x}{1-x} > -1$

the Range $f(x) \in (0, x)$

(C) $h(x) = \tan^{-1}\frac{x}{1+|x|}$

Range $f(x) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(D) $k(x) = \cot^{-1}\left(\frac{x}{1+|x|}\right)$

Range $f(x) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

Previous Years' Questions

Sol 1: A $\rightarrow$ p; B $\rightarrow$ q; C $\rightarrow$ p; D $\rightarrow$ s

(A) If a = 1, b = 0, then $\sin^{-1}x + \cos^{-1}y = 0$

$\Rightarrow \sin^{-1}x = -\cos^{-1}y$

$\Rightarrow x^2 + y^2 = 1$

(B) If a = 1 and b = 1, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}xy = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x - \cos^{-1}y = \cos^{-1}xy$

$\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = xy$

$\Rightarrow (x^2 - 1)(y^2 - 1) = 0$

(C) If a = 1, b = 2, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x - \cos^{-1}y = \cos^{-1}(2xy)$

$\Rightarrow xy + \sqrt{1-x^2}\,\sqrt{1-y^2} = 2xy$

$\Rightarrow x^2 + y^2 = 1$

(D) If a = 2, b = 2, then

$\sin^{-1}(2x) + \cos^{-1}(y) + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}(2x) - \cos^{-1}(y) = \cos^{-1}(2xy)$

$\Rightarrow 2xy + \sqrt{1-4x^2}\,\sqrt{1-y^2} = 2xy$

$\Rightarrow (4x^2 - 1)(y^2 - 1) = 0$

Sol 2: Given than, $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = 1 \Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow (x + 1)(6x - 1) = 0 \Rightarrow x = -1 \text{ or } \frac{1}{6}$

But x = –1 does not satisfy the given equation.

$\therefore$ We take $x = \frac{1}{6}$

Sol 3: Let $f(x)\ \cos(2\cos^{-1}x + \sin^{-1}x)$

$= \cos\left(\cos^{-1}x + \frac{\pi}{2}\right)\left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$

$= -\sin(\cos^{-1}x)$

$\Rightarrow f(x) = -\sin\left(\sin^{-1}\sqrt{1-x^2}\right)$

$\Rightarrow f\left(\frac{1}{5}\right) = -\sin\left(\sin^{-1}\sqrt{1-\frac{1}{5^2}}\right)$

$= -\sin\left(\sin^{-1}\frac{2\sqrt{6}}{5}\right) = -\frac{2\sqrt{6}}{5}$

Sol 4: LHS $= \cos\tan^{-1}[\sin(\cot^{-1}x)]$

$= \cos\tan^{-1}\left[\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right]$

$= \cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \sqrt{\frac{x^2+1}{x^2+2}}$ = RHS

Sol 5: $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \frac{2}{2R}(a\cos C + c\cos A)$

$= \frac{b}{R} = 2\sin B = 2\sin 60^\circ = \sqrt{3}$

Sol 6: (B, C, D) $\frac{\pi}{2} < \alpha < \pi,\ \pi < \beta < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$

$\Rightarrow \sin\beta > 0;\ \cos\alpha < 0$

$\Rightarrow \cos(\alpha+\beta) > 0$