20. INVERSE TRIGONOMETRIC FUNCTIONS

1. INTRODUCTION TO INVERSE TRIGONOMETRY

The inverse trigonometric functions are the inverse functions of the trigonometric functions. They are sometimes referred to as cyclometric functions.

2. IMPORTANT DEFINITIONS

Given two non-empty sets X and Y, let $f:X \to Y$ be a function, such that $y = f(x)$. The set X is called as the domain of f while the set Y is called as the co-domain of f. The set $\{f(x): x \in X\}$ is called as range of f.A map $f: A \to B$ is said to be one-one or injective, if and only if, distinct elements of A have distinct images in B, i.e. if, and only if, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for all $x_1, x_2 \in A$

Onto map or Surjective map: A map $f: A \to B$ is said to be an onto map or Surjective map if, and only if, each element of B is the image of some element of A, i.e. if, and only if, Range of f = co-domain of f.

Objective map: A map $f : A \to B$ is an objective map if, and only if, it is both one – one and onto.

3. INVERSE FUNCTIONS

If $f : X \to Y$ is one-to-one and onto (i.e. f is objective), then, we can define a unique function $g : Y \to X$, such that $g(y) = x$, where $x \in X$ is such that $y = f(x)$. Thus, the domain of g = range of f and range of g = domain of f. The function is called the inverse of f and is denoted by f^{-1}.

(a) Trigonometric functions are many-one functions but these become one-one, onto, if we restrict the domain of trigonometric functions. Similarly, co-domain is equated to range to make it an onto function. We can say that the inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions.

(b) Inverse of sin (sine functions) is denoted by $\sin^{-1}$ (arc sine function). We also write it as $\sin^{-1} x$. Similarly, other inverse trigonometric functions are given by $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$ and $\text{cosec}^{-1} x$.

(c) Note that $\sin^{-1} x \neq \frac{1}{\sin x}$ and $(\sin^{-1} x)^2 \neq \sin^{-2} x$, Also $\sin^{-1} x \neq (\sin x)^{-1}$

(d) Domain and Range of Inverse Trigonometric Functions:

	Function	Domain	Range (Principal value branch)
(i)	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

	Function	Domain	Range (Principal value branch)
(ii)	$y=\cos^{-1} x$	$-1\le x\le 1$	$0 \le y \le \pi$
(iii)	$y=\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} \le y < \frac{\pi}{2}$
(iv)	$y=\text{cosec}^{-1} x$	$x\ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
(v)	$y=\sec^{-1} x$	$x\ge 1$ or $x \le -1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
(vi)	$y=\cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

(e) The principal value of an inverse trigonometric function is the value of that inverse trigonometric function which lies in the range of principal branch.

MASTERJEE CONCEPTS

If no branch of an inverse trigonometric function is mentioned, then it can be implied that the principal value branch of that function.

You can remember range as set of angles that have the smallest absolute values satisfying for all the values of domain.

Vaibhav Gupta (JEE 2009 AIR 54)

4. TRANSFORMATION OF TRIGONOMETRIC FUNCTIONS TO INVERSE TRIGONOMETRIC FUNCTIONS

4.1 sin x to $\sin^{-1}$ x

The graph of an inverse trigonometric function can be obtained from the graph of the original by interchanging x and y axes.

Note: It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as the mirror image in the line y = x.

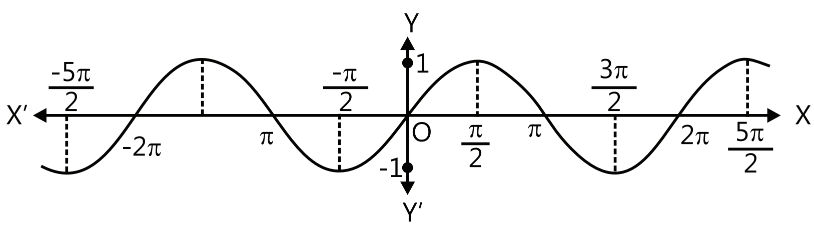


Figure 20.1

(a) $y = \sin x, x \in R$ and $|y| \le 1$; $y=\sin^{-1} x, |x| \le 1, y \in [-\pi/2, \pi/2]$

4.2 cos x to cos^{-1} x

(b) $y = \cos x, x \in R \text{ and } |y| \le 1$ $\quad y = \cos^{-1} x, x \in [-1, 1] \text{ and } y \in [0, \pi]$

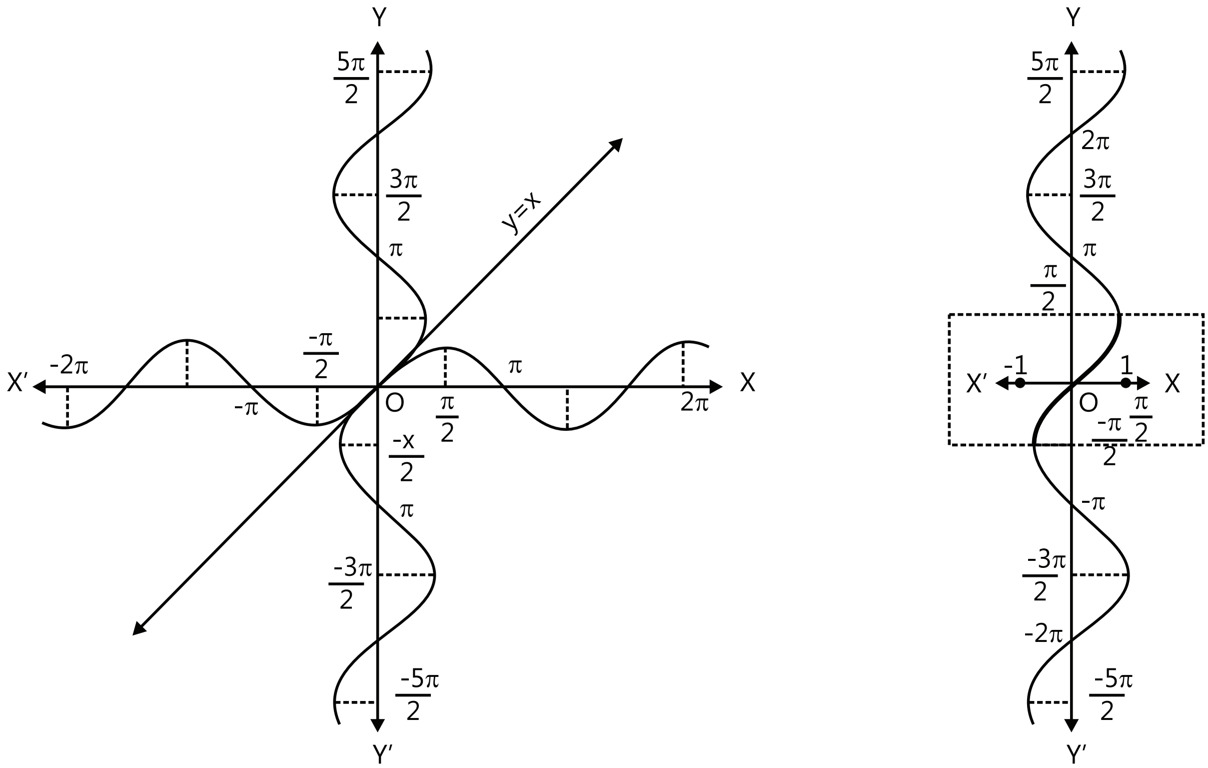


Figure 20.2

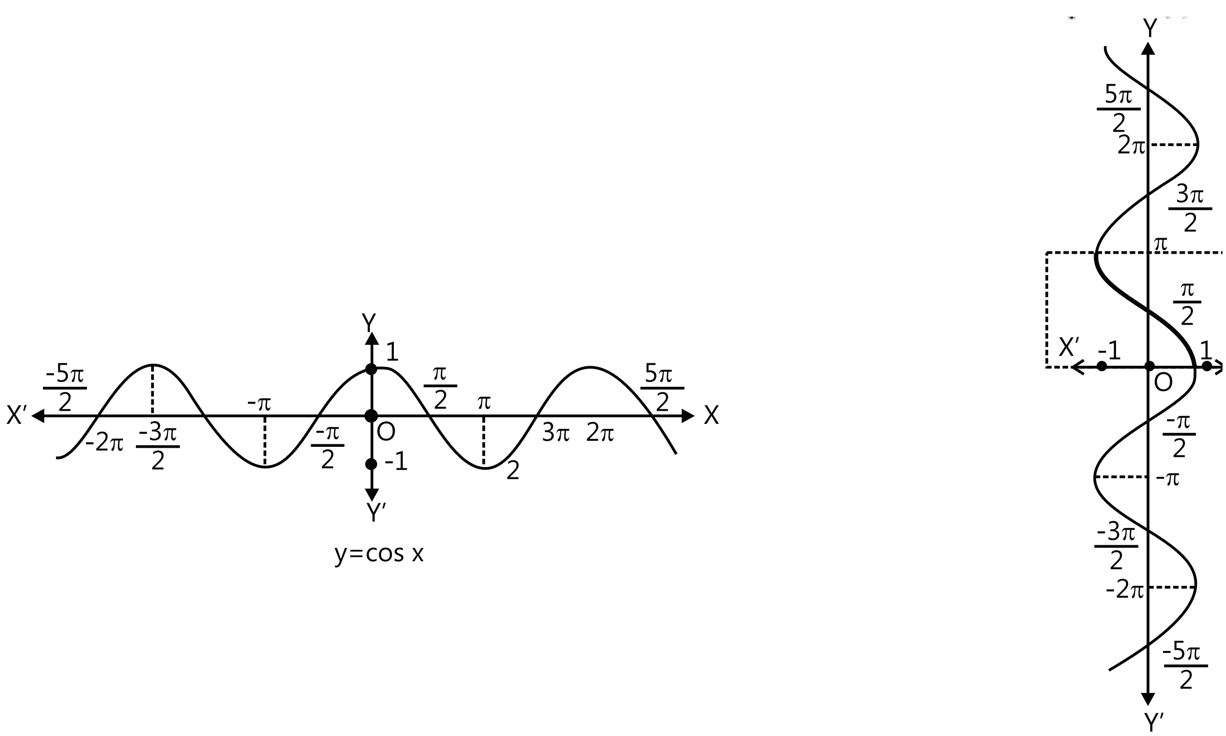
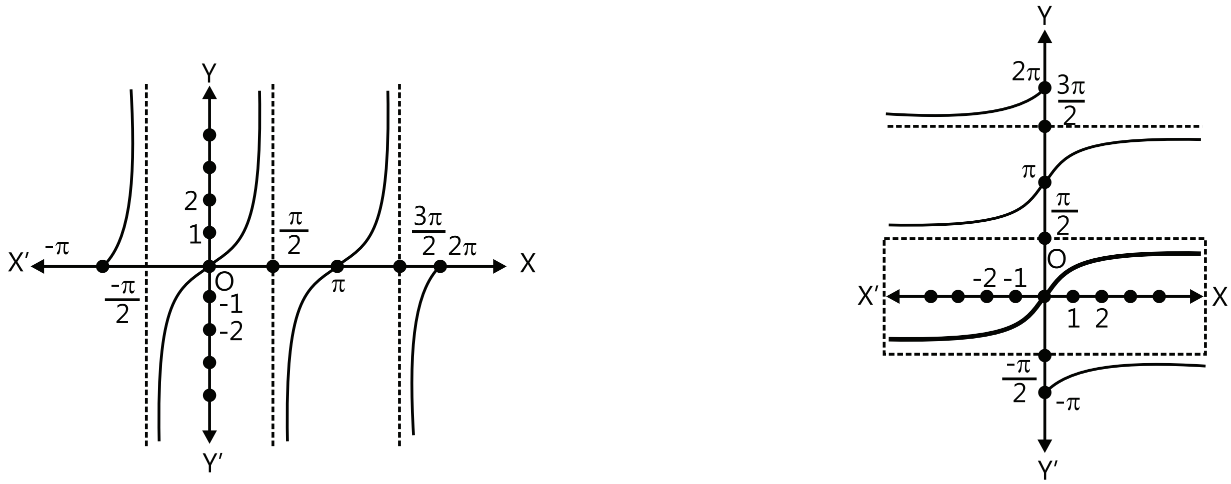


Figure 20.3

4.3 tan x to tan⁻¹ x

(c) $y=\tan x, x\in R-\left\{x: x=(2\pi+1)\frac{\pi}{2}, n\in Z\right\}$ and $y\in R$ $y=\tan^{-1}x, x\in R$ and $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

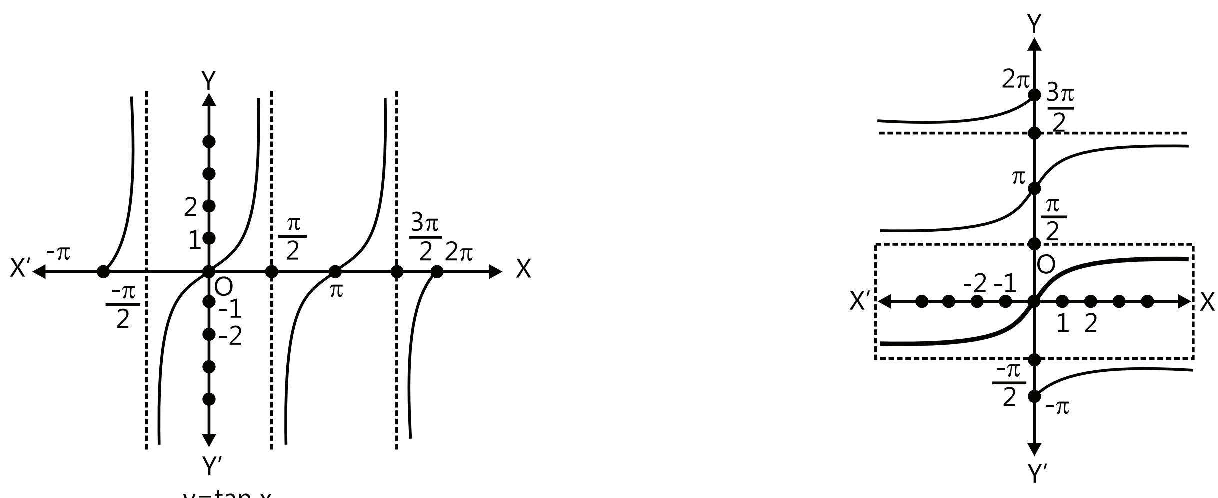


y=tan x

Figure 20.4

4.4 cot x to cot⁻¹ x

(d) $y=\cot x, x\in R-\{x:x=n\pi,\ n\in Z\}$ and $y\in R$ $y=\cot^{-1}x, x\in R$ and $y\in(0,\pi)$



y=tan x

y=tan⁻¹ x

Figure 20.5

4.5 sec x to sec⁻¹ x

(e) $y=\sec x, x\in R-\left\{x: x=(2n+1)\frac{\pi}{2}, n\in z\right\}$ and $y\in R-(-1,1)$ $y=\sec^{-1}x, x\in R-(-1,1)$ and $y\in[0,\pi]\left\{\frac{\pi}{2}\right\}$

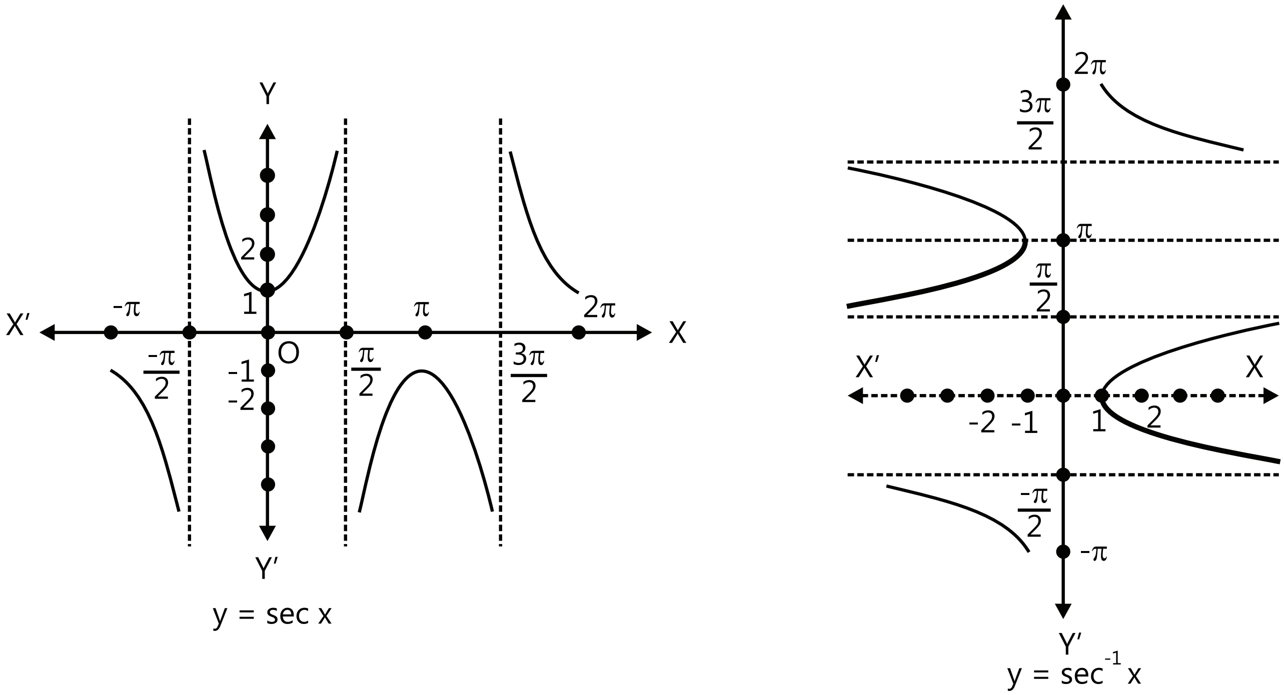


Figure 20.6

4.6 cosec x to cosec^{-1} x

(f) $y = \text{cosec}\,x, x \in \{x : x = n\pi, n \in Z\}$ and $y \in R - (-1, 1)$ $y = \text{cosec}^{-1} x, x \in R - (-1,1)$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

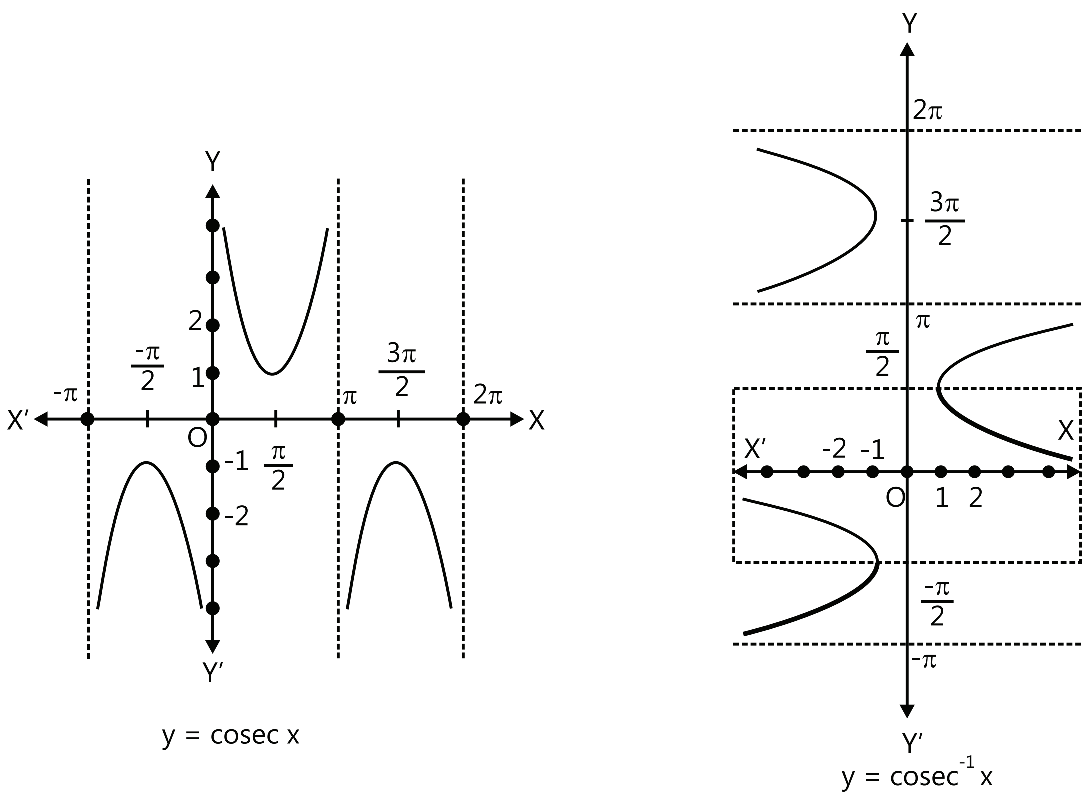


Figure 20.7

Illustration 1: Find the domain of definition of the function f(x) = $\sqrt{3\cos^{-1}(4x)-\pi}$. **(JEE MAIN)**

Sol: Use the condition that the expression inside the square root is ≥ zero.

For domain of f(x) = $\sqrt{3\cos^{-1}(4x)-\pi}$, we must have

$$4x \ge \cos\left(\frac{\pi}{3}\right) \Rightarrow 4x \ge \frac{1}{2} \quad \Rightarrow x \ge \frac{1}{8} \quad .....(i)$$

$$\text{Also } -1 \le 4x \le 1 \Rightarrow \frac{-1}{4} \le x \le \frac{1}{4} \quad .....(ii)$$

$\therefore$ From (i) and (ii), we get $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

MASTERJEE CONCEPTS

In case of confusion, try solving problems by replacing inverse functions with angles and applying trigonometric identities.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 2: If $0 < \cos^{-1}x < 1$ and $1 + \sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + \sin^3(\cos^{-1}x) + ...\infty = 2$, then find the value of x. **(JEE MAIN)**

Sol: Use summation of infinite GP series.

We have $1+\sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + ...\infty = 2$

$$\Rightarrow \frac{1}{1-\sin(\cos^{-1}x)} = 2 \Rightarrow \frac{1}{2} = 1-\sin(\cos^{-1}x) \Rightarrow \sin(\cos^{-1}x) = \frac{1}{2} \Rightarrow \cos^{-1}x = \frac{\pi}{6} \quad \Rightarrow x = \frac{\sqrt{3}}{2}$$

Illustration 3: Let $f(x) = \frac{2}{\pi}(\sin^{-1}[x] + \tan^{-1}[x] + \cot^{-1}[x])$, where [x] denotes the greatest integer less than or equal to x. If A and B denote the domain and range of f(x) respectively, find the number of integers in A∪B. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}[x] + \cot^{-1}[x] = \frac{\pi}{2}$ and proceed.

For domain of f(x), we must have $-1 \le [x] \le 1 \Rightarrow -1 \le x < 2$, so set A = [-1,2)

$$f(x) = \frac{2}{\pi}\left(\sin^{-1}[x] + \frac{\pi}{2}\right) \qquad \left(\text{As} \quad \tan^{-1}[x] + \cot^{-1}[x] = \frac{\pi}{2}, \forall x \in A\right)$$

So, set B = {0, 1, 2} = Range of f(x). Now, $A \cup B = [-1,2) \cup \{0,1,2\} = [-1,2]$

Hence, number of integers in $(A \cup B) = 4$

5.PROPERTIES/IDENTITIES OF INVERSE TRIGONOMETRIC FUNCTIONS

5.1 Complementary Angles

(a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1,1]$

(b) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\forall x\in R$

(c) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2},\forall x\in(-\infty,-1]\cup[1,\infty)$

5.2 Negative Arguments

(a) $\sin^{-1}(-x)=-\sin^{-1}x,\ \forall x\in[-1,1]$

(b) $\cos^{-1}(-x)=\pi-\cos^{-1}x,\ \forall x\in[-1,1]$

(c) $\tan^{-1}(-x)=-\tan^{-1}x,\forall x\in R$

(d) $\cot^{-1}(-x)=\pi-\cot^{-1}x,\forall x\in R$

(e) $\sec^{-1}(-x)=\pi-\sec^{-1}x,\forall x\in(-\infty,-1]\cup[1,\infty)$

(f) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x\forall x\in(-\infty,-1)\cup[1,\infty)$

5.3 Reciprocal Arguments

(a) $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x};|x|\ge 1$ (Both the functions are identical)

and $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x};|x|\le 1, x\ne 0$ (Both the functions are not identical)

(b) $\sec^{-1}x=\cos^{-1}\frac{1}{x};|x|\ge 1$ (Both the functions are identical)

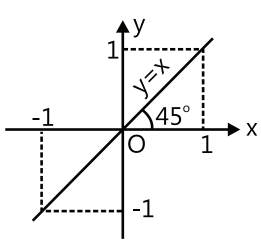
and $\cos^{-1}x=\sec^{-1}\frac{1}{x};|x|\le 1$ (Both the functions are not identical)

(c) $\tan^{-1}x=\cot^{-1}\left(\frac{1}{x}\right),\quad x\in(0,\infty)=-\pi+\cot^{-1}\left(\frac{1}{x}\right),\quad x\in(-\infty,0)$,

and $\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right),\ x\in(0,\infty)=\pi+\tan^{-1}\left(\frac{1}{x}\right),\quad x\in(-\infty,0)$

5.4 Forward Inverse Identities

(a) $y=\sin(\sin^{-1}x)=x, x\in[-1,1], y\in[-1,1]$, y is aperiodic



(b) $y=\cos(\cos^{-1}x)=x, x\in[-1,1], y\in[-1,1]$, y is aperiodic

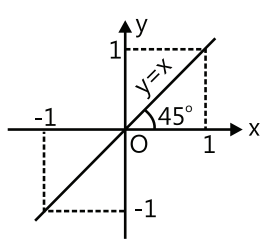


Figure 20.8

(c) $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

(d) $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

Figure 20.9

(e) $y = \text{cosec}(\text{cosec}^{-1} x) = x$, $|x| \geq 1, |y| \geq 1$, y is aperiodic

(f) $y = \sec(\sec^{-1} x) = x$, $|x| \geq 1, |y| \geq 1$, y is aperiodic

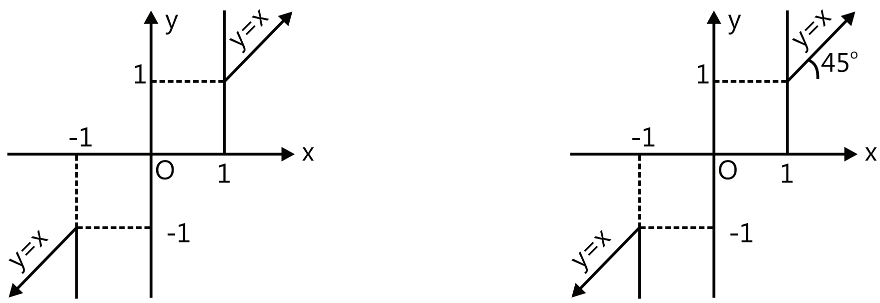


Figure 20.10

Also,

$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

$\sin(\cos^{-1} x) = \sqrt{1 - x^2}$

$\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{1 + x^2}}$

$\tan(\cos^{-1} x) = \dfrac{\sqrt{1 - x^2}}{x}$

$\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1 + x^2}}$

$\tan\left(\sin^{-1} x\right) = \dfrac{x}{\sqrt{1 - x^2}}$

5.5 Inverse Forward Identities

(a) $y = \sin^{-1}(\sin x) = x, \left[x \in R\right], y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, Periodic with period 2 π

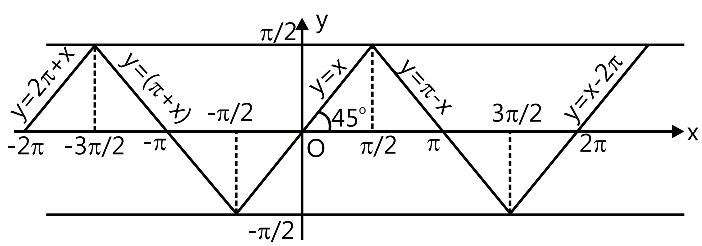


Figure 20.11

(b) $y = \cos^{-1}(\cos x) = x, x \in R, y \in [0, \pi]$, Periodic with period 2 π

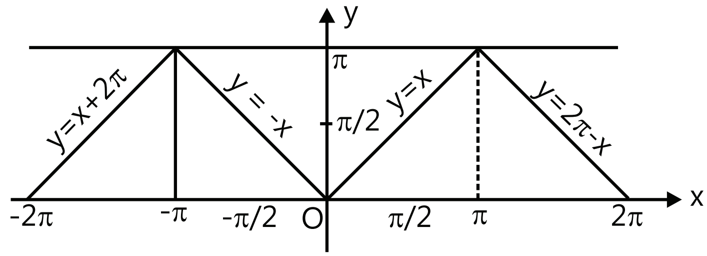


Figure 20.12

(c) $y = \tan^{-1}(\tan x) = x, x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, Periodic with period π

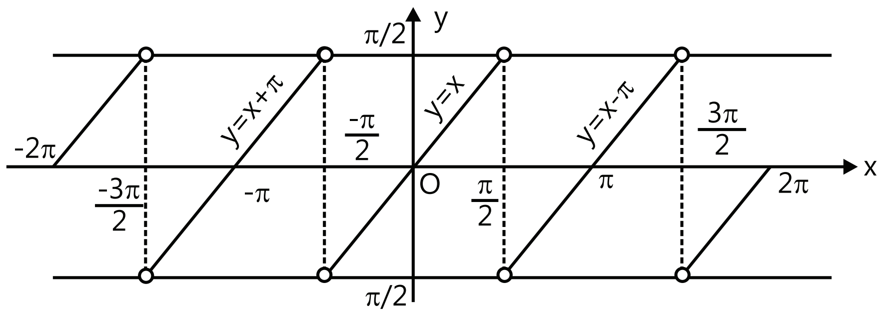


Figure 20.13

(d) $y = \cot^{-1}(\cot x) = x, x \in R - \{n\pi\}, y \in (0, \pi)$, periodic with π

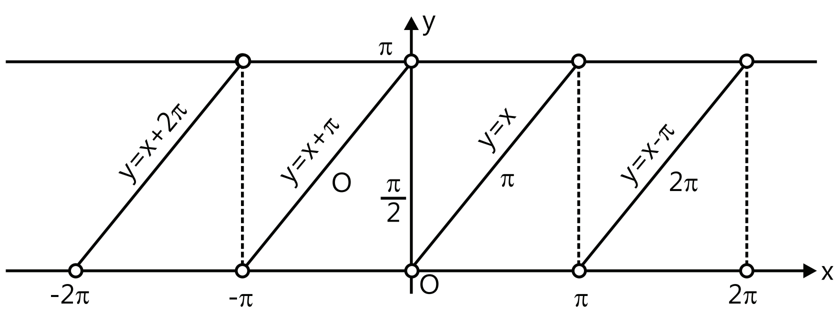


Figure 20.14

(e) $y = \text{cosec}^{-1}(\text{cosec}\, x) = x \in R - \{n\pi, n \in I\}, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ y is periodic with period 2π

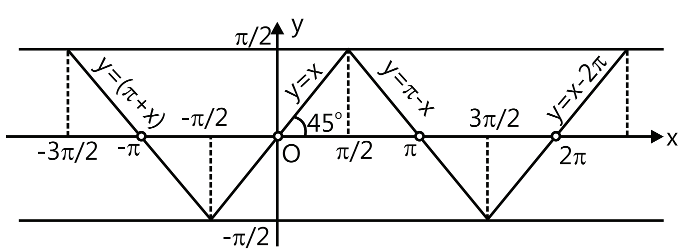


Figure 20.15

(f) $y = \sec^{-1}(\sec x) = x$, y is periodic,

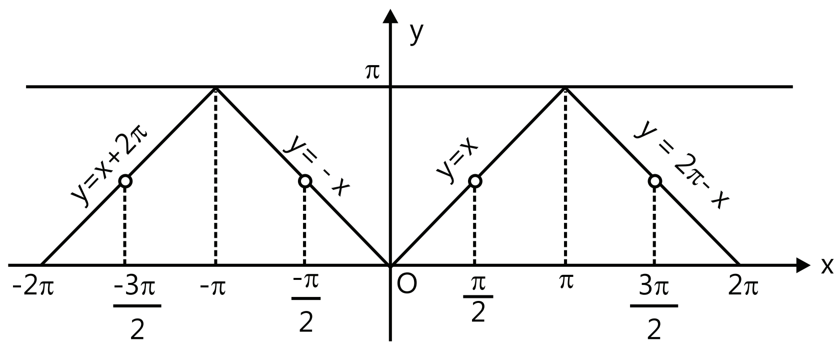


Figure 20.16

$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with period 2π

(i) $\tan^{-1}(\cot x) = \frac{1}{2}\pi - x$ for $x \in [0, \pi]$

(ii) $\sin^{-1}(\operatorname{cosec} x) = \frac{1}{2}\pi - x$ for $x \in [0, \pi]$

(iii) $\sec^{-1}(\cos x) = \frac{1}{2}\pi - x$ for $x \in \left[0, \frac{1}{2}\pi\right]$.

5.6 Sum of Angles

(a) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \; x > 0; y > 0$

(c) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y >0 and $x^2 + y^2 \le 1$

(d) $\cos^{-1} x \pm \cos^{-1} y = \pi - \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y > 0 and $x^2+y^2 \le 1$

(e) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} \; x > 0 \; y > 0 \text{ and } xy < 1 \Rightarrow 0 < \tan^{-1} x + \tan^{-1} y < \frac{\pi}{2} \\ \pi - \tan^{-1}\frac{x+y}{1-xy} \; x > 0 \; y > 0 \text{ and } xy > 1 \Rightarrow \frac{\pi}{2} < \tan^{-1} x + \tan^{-1} y < \pi \end{cases}$

(f) x > 0 & y > 0 then $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$ (with no other restriction)

(g) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$; $\tan^{-1} - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

MASTERJEE CONCEPTS

The above results can be generalized as follows:

$$\tan^{-1}x_1+\tan^{-1}x_2+...+\tan^{-1}x_n=\tan^{-1}\left[\frac{S_1-S_3+S_5-.............}{1-S_2+S_4-S_6+.......}\right]$$

where S_k denotes the sum of products of $x_1,x_2,\ldots,x_n$ taken k at a time

Rohit Kumar (JEE 2012 AIR 78)

Illustration 4: Evaluate: $\sin\left(\tan^{-1}\frac{15}{8}\right)$ **(JEE MAIN)**

Sol: Convert $\tan^{-1}\frac{15}{8}$ to $\sin^{-1}$.

We know that $\sin(\sin^{-1}x)=x$, for all x $\in$ [-1, 1], So, will convert each expression in the form sin $(\sin^{-1}x)$ by using

$$\cos^{-1}\frac{b}{h}=\sin^{-1}\frac{p}{h},\ \tan^{-1}\frac{p}{b}=\sin^{-1}\frac{p}{h},\cot^{-1}\frac{p}{b}=\sin^{-1}\frac{b}{h}\text{ etc.}$$

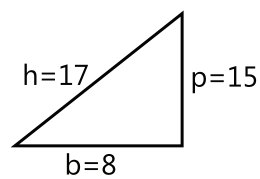


Figure 20.17

Where b, p and h denote the base, perpendicular and hypotenuse of a right triangle.

$$\sin\left(\tan^{-1}\frac{15}{8}\right)=\sin\left(\sin^{-1}\frac{15}{17}\right)=\frac{15}{17}$$

Illustration 5: Evaluate: $\cos\left(\text{cosec}^{-1}\frac{13}{12}\right)$ **(JEE MAIN)**

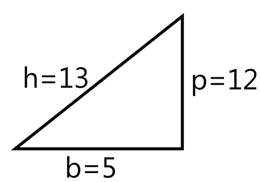


Figure 20.18

Sol: Write cosec^{-1} in terms of $\cos^{-1}$.

$$\cos\left(\text{cosec}^{-1}\frac{13}{12}\right)=\cos\left(\cos^{-1}\frac{5}{13}\right)=\frac{5}{13}\left[\because \text{cosec}^{-1}\frac{13}{12}=\cos^{-1}\frac{5}{13}\right]$$

Illustration 6: Find the principal value of $\cot^{-1}(-\sqrt{3})$ **(JEE MAIN)**

Sol: The principal value of $\cot^{-1}x$ lies in between 0 to π.

Let $\cot^{-1}(-\sqrt{3})=\theta$

Then $\cot\theta=-\sqrt{3}=-\cot\frac{\pi}{6}$

Since principal value branch of $\cot^{-1}x$ is $0<\theta<\pi$. Therefore, we want to find the value of θ such that $0<\theta<\pi$.

Now, $\cot\theta=-\cot\frac{\pi}{6}=\cot\left(\pi-\frac{\pi}{6}\right)=\cot\frac{5\pi}{6}$

Therefore, principal value of $\cot^{-1}(-\sqrt{3})=\frac{5\pi}{6}$.

Illustration 7: $\sin^{-1}\left(\sin\frac{10\pi}{7}\right)=\frac{10\pi}{7}$ **(JEE MAIN)**

Sol: Write $\frac{10\pi}{7}$ as $\pi+\frac{3\pi}{7}$ and expand.

$$=\sin^{-1}\left(\sin\frac{10\pi}{7}\right)=\sin^{-1}\left(-\sin\left(\frac{3\pi}{7}\right)\right)=\sin^{-1}\left(\sin\left(-\frac{3\pi}{7}\right)\right)=-\frac{3\pi}{7}$$

Illustration 8: $\cos^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right)$ **(JEE MAIN)**

Sol: $=\cos^{-1}\left(\cos\left(\frac{\pi}{2}+\frac{\pi}{9}\right)\right)=\cos^{-1}\left(\cos\left(\frac{11\pi}{18}\right)\right)=\frac{11\pi}{18}$

Illustration 9: $\sin^{-1}\left(\cos\frac{13\pi}{10}\right)$ **(JEE MAIN)**

Sol: Similar to previous example.

$$=\sin^{-1}\cos\frac{13\pi}{10}=\sin^{-1}\left(-\cos\frac{3\pi}{10}\right)=\sin^{-1}\left(-\sin\left(\frac{5\pi}{10}-\frac{2\pi}{10}\right)\right)=\sin^{-1}\left(-\sin\frac{\pi}{5}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right)=-\frac{\pi}{5}$$

Illustration 10: Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. **(JEE MAIN)**

Sol: Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=y$. Then $\sin y=\frac{1}{\sqrt{2}} \Rightarrow y=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

We know that, the range of the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$.

Therefore, principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

Illustration 11: Find the integral solution of the inequality $3x^2+8x<2\sin^{-1}(\sin 4)-\cos^{-1}(\cos 4)$.
(JEE ADVANCED)

Sol: Use inverse forward identities to simplify the equation.

$3x^2+8x<-4 \quad\Rightarrow\quad 3x^2+8x+4<0$

$\Rightarrow\quad 3x^2+6x+2x+4<0 \Rightarrow\quad 3x(x+2)+2(x+2)<0$

$(x+2)(3x+2)<0 \qquad x=-1$

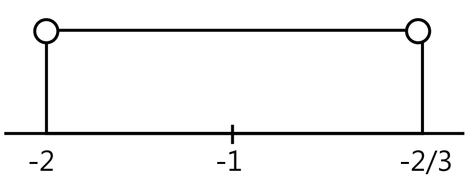


Figure 20.19

Illustration 12: Find the largest integral value of k, for which $(k-2)x^2+8x+k+4>\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12)$, for all $x\in R$. **(JEE ADVANCED)**

Sol: Use inverse forward identities.

$\sin^{-1}(\sin 12)=\sin^{-1}(\sin(12-4\pi))=12-4\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

$\therefore\ (k-2)x^2 + 8x + k + 4 > 0, \quad \forall x \in R$

If k = 2, then 8x+4>0, (not possible)

and if $k \neq 2$, then k-2>0 $\Rightarrow$ k>2

and $64 - 4(k-2)(k+4) < 0 \Rightarrow 16 < k^2 + 2k - 8$

$\Rightarrow k^2 + 2k - 24 > 0 \Rightarrow (k+6)(k-4) > 0$

K = 5

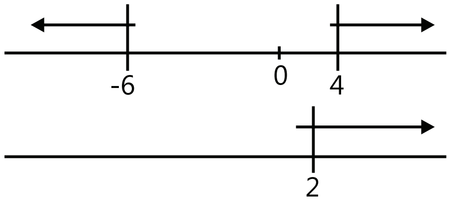


Figure 20.20

Illustration 13: Find domain of $f(x) = \dfrac{1}{\sqrt{\ln\left(\cot^{-1} x\right)}}$. **(JEE MAIN)**

Sol: Find the range of x for which $\ln(\cot^{-1}x) > 0 \Rightarrow \cot^{-1} x > 1 \Rightarrow x < \cot 1 \Rightarrow x \in (-\infty, \cot 1)$

Illustration 14: Evaluate the following: **(JEE MAIN)**

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)$ (iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ (iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\pi}{6}\right\}$

Sol: Recall that, $\sin^{-1}(\sin\theta) = \theta$, if $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$ and

$\tan^{-1}(\tan\theta) = \theta$, if $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Therefore,

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$

(iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) \neq \dfrac{7\pi}{6}$, because $\dfrac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \dfrac{5\pi}{6}\right)\right) \quad \left[\because \dfrac{7\pi}{6} = 2\pi - \dfrac{5\pi}{6}\right] = \cos^{-1}\left(\cos\dfrac{5\pi}{6}\right) \quad [\because \cos(2\pi - \theta) = \cos\theta] = \dfrac{5\pi}{6}$

(iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\pi}{6}\right\} = \cos\left(\dfrac{5\pi}{6} + \dfrac{\pi}{6}\right) \quad \left[\because \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}\right]$

Illustration 15: Evaluate the following:

(i) $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$ (ii) $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ (iii) $\sin(\cot^{-1} x)$ **(JEE MAIN)**

Sol: (i) Let $\cos^{-1}\dfrac{3}{5} = \theta$. Then, $\cos\theta = \dfrac{3}{5} \Rightarrow \sin\theta = \dfrac{4}{5}$

$\therefore\ \sin\left(\cos^{-1}\dfrac{3}{5}\right) = \sin\theta = \dfrac{4}{5}$

(ii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)=\sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

(iii) Let $\cot^{-1}x=\theta$, Then, $x=\cot\theta$

Now, $\cot\theta=x \Rightarrow \sin\theta=\frac{1}{\sqrt{1+x^2}}$ $\quad\therefore \sin(\cot^{-1}x)=\sin\theta=\frac{1}{\sqrt{1+x^2}}$

Illustration 16: Evaluate the following:

(i) $\sin^{-1}(\sin 5)$ (ii) $\cos^{-1}(\cos 10)$ **(JEE MAIN)**

Sol: Notice that the angle is in radians.

(i) Here, $\theta=5$ radians. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But

$2\pi-5$ and $5-2\pi$ both lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$\sin(5-2\pi)=\sin(-(2\pi-5))=-\sin(2\pi-5)=-(-\sin 5)=\sin 5$

$\Rightarrow \sin^{-1}(\sin 5)=\sin^{-1}(\sin(5-2\pi))=5-2\pi.$

(ii) We know that $\cos^{-1}(\cos\theta)=\theta$, if $0\le\theta\le\pi$. Here, $\theta=10$ radians. Clearly, it does not lie between 0 and π such that, $(4\pi-10)=\cos 10 \Rightarrow \cos^{-1}(\cos 10)=\cos^{-1}(\cos(4\pi-10))=4\pi-10$

Illustration 17: Evaluate the following:

(i) $\sin^{-1}(2\sin^{-1}0.8)$ (ii) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ **(JEE MAIN)**

Sol: Write the term inside the brackets in (i) and (ii) as $\sin^{-1}$ and $\tan^{-1}$ respectively.

(i) We know that: $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

$\therefore\ 2\sin^{-1}0.8=\sin^{-1}(2\times 0.8\times\sqrt{1-0.64})$

$\Rightarrow \sin^{-1}(2\sin^{-1}0.8)=\sin\{\sin^{-1}(0.96)\}=0.96$

(ii) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

$=\tan\left(\tan^{-1}\frac{5}{12}-\frac{\pi}{4}\right)$ $\quad\left[\text{From (ii) we have, } 2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{5}{12}\right]$

$=\tan\left(\tan^{-1}\frac{5}{12}-\tan^{-1}1\right)$ $\left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\text{ if } xy>-1\right]$ $=\tan\left[\tan^{-1}\left(\frac{-7}{17}\right)\right]=-\frac{7}{17}$

Illustration 18: Write the following in their simplest forms:

(i) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ (ii) $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$ **(JEE ADVANCED)**

Sol: (i) Use the formula $1-\cos x=2\sin^2 x/2$ and $1+\cos x=2\cos^2 x/2$

(ii) Write the term inside the square bracket in terms of $\sin^{-1}$.

(i) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\dfrac{2\sin^2 x/2}{2\cos^2 x/2}} = \tan^{-1}\left|\tan\dfrac{x}{2}\right| = \dfrac{|x|}{2}$

(ii) $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$

$= \sin\left[\cot^{-1}\left\{\cos\left(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)\right\}\right] = \sin\left(\cot^{-1}\dfrac{1}{\sqrt{1+x^2}}\right) = \sin\left\{\sin^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} \quad \left[\because \cot^{-1}x = \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right]$

$= \sqrt{\dfrac{1+x^2}{2+x^2}}$

Illustration 19: Express $\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right), -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ in the simplest form. **(JEE ADVANCED)**

Sol: Convert the term inside the bracket in terms of tan $\dfrac{x}{2}$ and proceed.

We write, $\Rightarrow \tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\dfrac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$

$= \left[\dfrac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right] = \tan^{-1}\left[\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right] = \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right] = \dfrac{\pi}{4}+\dfrac{x}{2}$

Alternatively,

$\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\dfrac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right] = \tan^{-1}\left[\dfrac{\sin\left(\frac{\pi-2x}{2}\right)}{1-\cos\left(\frac{\pi-2x}{2}\right)}\right]$

$= \tan^{-1}\left[\dfrac{2\sin\left(\frac{\pi-2x}{4}\right)\cos\left(\frac{\pi-2x}{4}\right)}{2\sin^2\left(\frac{\pi-2x}{4}\right)}\right] = \tan^{-1}\left[\cot\left(\dfrac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\dfrac{\pi}{2}-\dfrac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right] = \dfrac{\pi}{4}+\dfrac{x}{2}$

Illustration 20: If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right) = 1$, find the value of x. **(JEE MAIN)**

Sol: From the question, we have $\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right) = \dfrac{\pi}{2}$ and proceed.

We have $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right) = 1$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1} 1 \quad \Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5} \quad \Rightarrow \quad \cos^{-1}x = \cos^{-1}\frac{1}{5} \Rightarrow x = \frac{1}{5}$

Illustration 21: Find the value of cos $(\sec^{-1}x + \text{cosec}^{-1}x), |x| \geq 1.$ **(JEE MAIN)**

Sol: Use $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

We have $\cos\left(\sec^{-1}x + \text{cosec}^{-1}x\right) = \cos\left(\frac{\pi}{2}\right) = 0$

Illustration 22: Find maximum & minimum values of $\left(\sec^{-1}x\right)^2 + \left(\text{cosec}^{-1}x\right)^2$. **(JEE ADVANCED)**

Sol: Apply the identity $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ and then use suitable substitution to form a quadratic.

$y = (\sec^{-1}x)^2 + (\text{cosec}^{-1}x)^2$

$= (\sec^{-1}x + \text{cosec}^{-1}x)^2 - 2\sec^{-1}x \ \text{cosec}^{-1}x$

put $t = \sec^{-1}x$; $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

$y = \frac{\pi^2}{4} - 2t\left(\frac{\pi}{2} - t\right) = 2t^2 - \pi t + \frac{\pi^2}{4}$

$y = 2\left[t^2 - \frac{\pi}{2}t + \frac{\pi^2}{8}\right] = 2\left[\left(t - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right] = \frac{\pi^2}{8} + 2\left(t - \frac{\pi}{4}\right)^2 \therefore y_{min} = \frac{\pi^2}{8}; \quad y_{max} = \frac{5\pi^2}{4}$ at $t = \pi$

Illustration 23: Find the range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$. **(JEE MAIN)**

Sol: Find the domain of the given function and then find the range.

$f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

Here domain is only x = 1or –1;

So range will contain only 2 elements $\{3\pi/4, \pi/4\}$

Illustration 24: Find the number of solutions of the equation $\tan^{-1}x^3 + \cot^{-1}(e^x) = \frac{\pi}{2}$. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}A + \cot^{-1}A = \frac{\pi}{2}$ to simplify the given equation and then take the help of graph to find the number of solution.

$\cot^{-1}(e^x) = \frac{\pi}{2} - \tan^{-1}(x^3) = \cot^{-1}(x^3) \Rightarrow e^x = x^3 \Rightarrow x^3e^{-x} = 1$

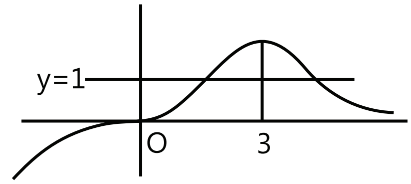


Figure 20.21

Plotting the graph of y = I and $y = x^3e^{-x}$ we can see that the line intersects the curve at two points. Hence there are 2 solutions for the above equation.

Illustration 25: Find the number of values of x satisfying the equation

$$\tan^{-1}\left(x-\frac{x^3}{4}+\frac{x^5}{16}-\ldots\ldots\right)+\cot^{-1}\left(x+\frac{x^2}{2}+\frac{x^3}{4}-\ldots\ldots\right)=\frac{\pi}{2} \text{ for } 0<|x|<2.$$ **(JEE ADVANCED)**

Sol: Use $\tan^{-1}A+\cot^{-1}A=\frac{\pi}{2}$.

We must have $x-\frac{x^3}{4}+\frac{x^5}{16}-\ldots\ldots=x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots\ldots$

$$\Rightarrow \frac{x}{1+\frac{x^2}{4}}=\frac{x}{1-\frac{x}{2}} \Rightarrow \frac{4x}{4+x^2}=\frac{2x}{2-x} \Rightarrow 2x^2(x+2)=0$$

$\therefore\ x=0,\ -2 \quad (\text{As } 0<|x|<2)$

Clearly no value of x satisfies given equation.

Illustration 26: Prove that $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

We have, $=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}x\frac{7}{24}}\right\}$ $\left[\because \tan^{-1}x+\tan^{-1}y =\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ If } xy<1\right]$

$$=\tan^{-1}\left\{\frac{48+77}{264-14}\right\}=\tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\left(\frac{1}{2}\right)$$

Illustration 27: If $\tan^{-1}4+\tan^{-1}5=\cot^{-1}(\lambda)$ then find λ. **(JEE MAIN)**

Sol: Write the L.H.S. in terms of $\cot^{-1}$ and compare.

We have $\tan^{-1}4+\tan^{-1}5=\tan^{-1}\frac{4+5}{1-20}=\pi-\tan^{-1}\frac{9}{19}=\pi-\cot^{-1}\frac{19}{9}$

$=\cot^{-1}\left(-\frac{19}{9}\right) \qquad \Rightarrow \lambda=-\frac{19}{9}$

Illustration 28: Prove that: $\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=\sin^{-1}\left(\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}x+\tan^{-1}y$.

We have, LHS $=\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=(\tan^{-1}1-\tan^{-1}x)-(\tan^{-1}1-\tan^{-1}y)=\tan^{-1}y-\tan^{-1}x$

$$= \tan^{-1}\left(\frac{y-x}{1+yx}\right) = \sin^{-1}\left(\frac{y-x}{\sqrt{(1+yx)^2+(y-x)^2}}\right) = \sin^{-1}\left\{\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right\} = \text{RHS}$$

Illustration 29: Prove that: (i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$

(iii) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$ **(JEE ADVANCED)**

Sol: Same as above.

(i) LHS $=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right] = \tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\frac{2}{9}=\text{R.H.S.}$$

(ii) L.H.S. $=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$

$$=\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right)-\tan^{-1}\frac{8}{19}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right)-\tan^{-1}\frac{8}{19}$$

$$=\tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\frac{8}{19}=\tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right)=\tan^{-1}\frac{425}{425}=\tan^{-1}1=\frac{\pi}{4}=\text{R.H.S.}$$

(iii) L.H.S. $=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$

$$=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right) \qquad =\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$=\tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23} \qquad =\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}=\frac{\pi}{4}=\text{R.H.S.}$$

Illustration 30: Show that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{3}{4}$. **(JEE MAIN)**

Sol: We have, L.H.S. $=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}=\tan^{-1}\frac{15}{20}=\tan^{-1}\frac{3}{4}=\text{R.H.S.}$

Illustration 31: Simplify $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$ if $\frac{a}{b}\tan x>-1$. **(JEE MAIN)**

Sol: Divide the numerator and denominator inside the bracket by bcos x and expand.

We have, $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x-a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\ \frac{a}{b}-x$

Illustration 32: Solve the following equations:

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ (ii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ **(JEE ADVANCED)**

Sol: Write $\frac{\pi}{4}$ as $\tan^{-1}1$ and simplify.

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\tan^{-1}1 \Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}1-\tan^{-1}\ \frac{x+1}{x+2}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right) \Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\ \frac{x+2-x-1}{x+2+x+1}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1}{2x+3} \Rightarrow \frac{x-1}{x-2}=\frac{1}{2x+3} \Rightarrow (2x+3)(x-1)=x-2$

$\Rightarrow 2x^2+x-3=x-2 \Rightarrow 2x^2-1=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

(ii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \frac{2\cos x}{\sin^2 x}=2\operatorname{cosec} x \Rightarrow \cos x=\sin x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

Illustration 33: Prove that: $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$ **(JEE MAIN)**

Sol: Covert the L.H.S. in terms of $\cos^{-1}$.

We have $\sin^{-1}\ \frac{3}{5}-\sin^{-1}\ \frac{8}{17}\ =\cos^{-1}\ \frac{4}{5}-\cos^{-1}\ \frac{15}{17}$ $\left[\because \sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5}\ \&\ \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}\right]$

$=\cos^{-1}\left\{\frac{4}{5}x\frac{15}{17}+\sqrt{1-\left(\frac{4}{5}\right)^2}\ x\sqrt{1-\left(\frac{15}{17}\right)^2}\right\}=\cos^{-1}\left\{\frac{4}{5}x\frac{15}{17}+\frac{3}{5}x\frac{8}{17}\right\}=\cos^{-1}\left\{\frac{60}{85}+\frac{24}{85}\right\}=\cos^{-1}\frac{84}{85}$

Illustration 34: Prove that: $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$ **(JEE MAIN)**

Sol: We have $\sin^{-1}\frac{4}{5}-\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\left\{\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}\right\}+\sin^{-1}\frac{16}{65}$

$$=\sin^{-1}\left\{\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right\}+\sin^{-1}\frac{16}{25}$$

$$=\sin^{-1}\left\{\frac{4}{5}x\frac{12}{13}+\frac{5}{13}x\frac{3}{5}\right\}+\sin^{-1}\frac{16}{25}=\sin^{-1}\frac{63}{65}+\sin^{-1}\frac{16}{25}$$

$$=\cos^{-1}\frac{16}{65}+\sin^{-1}\frac{16}{25}\left[\because \sin^{-1}\frac{63}{65}=\cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2}=\cos^{-1}\frac{16}{65}\right]$$

$$=\frac{\pi}{2}\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

6. SIMPLIFICATION OF INVERSE FUNCTIONS BY ELEMENTARY SUBSTITUTION

(a) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ if $-1\le x\le 1$

(b) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ if $-1\le x\le 1$

(c) $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right) & -1\le x\le 1\\ \sin^{-1}\left(\frac{2x}{1+x^2}\right) & 0\le x\le 1\\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & 0\le x<\infty\end{cases}$$

(d) $$\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x & -1\le x\le 1\\ \pi-2\tan^{-1}x & x\ge 1\\ -\pi-2\tan^{-1}x & x\le -1\end{cases}$$

(e) $$\cos^{-1}\frac{1-x^2}{1+x^2}=\left[\begin{array}{ll}2\tan^{-1}x & x\ge 0\\ -2\tan^{-1}x & x<0\end{array}\right.$$

(f) $$\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}\pi+2\tan^{-1}x & x<-1\\ 2\tan^{-1}x & -1<x<1\\ 2\tan^{-1}x-\pi & x>1\end{cases}$$

(g) $$\sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(h) $\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(i) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

(j) $f(x)=\sin^{-1}\frac{2x}{1+x^2}+2\tan^{-1}x = \pi$, if $x \ge 1$

(k) $f(x) = \sin^{-1}\frac{2x}{1+x^2}+2\tan^{-1}x=-\pi$ if $x \le -1$

(l) $$\sin^{-1}(3x-4x^3) = \begin{cases} -(\pi+3\sin^{-1}x) & -1 \le x \le 1/2 \\ 3\sin^{-1}x & -1/2 \le x \le 1/2 \\ \pi - 3\sin^{-1}x & 1/2 \le x \le 1 \end{cases}$$

(m) $$\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x - 2\pi & -1 \le x \le -1/2 \\ 2\pi - 3\cos^{-1}x & -1/2 \le x \le 1/2 \\ 3\cos^{-1}x & 1/2 \le x \le 1 \end{cases}$$

(n) $$\tan^{-1}\frac{3x-x^3}{1-3x^2} = \begin{cases} 3\tan^{-1}x & -\frac{1}{\sqrt{3}} < x\frac{1}{\sqrt{3}} \\ -\pi+3\tan^{-1}x & x > \frac{1}{\sqrt{3}} \\ \pi+3\tan^{-1}x & x < -\frac{1}{\sqrt{3}} \end{cases}$$

MASTERJEE CONCEPTS

While writing inverse trigonometric functions in their simplest forms, we use the following substitutions.

- For $\sqrt{a^2-x^2}$, we substitute $x = a\sin\theta$ or $x = a\cos\theta$
- For $\sqrt{a^2+x^2}$, we substitute $x = a\tan\theta$ or $x = a\cot\theta$
- For $\sqrt{x^2-a^2}$, we substitute $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
- For $\sqrt{a+x}$ and $\sqrt{a-x}$ occurring together or separately, we substitute $x = a\cos\theta$

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Illustration 35: Solve for x: $\sin\left(2\cos^{-1}\left(\cot(2\tan^{-1}x)\right)\right) = 0$ **(JEE ADVANCED)**

Sol: The R.H.S. is equal to zero implies $\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \frac{n\pi}{2}$ and proceed accordingly to find the value of x.

$$\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n=0 \\ \frac{\pi}{2} & \text{if } n=1 \\ \pi & \text{if } n=2 \end{cases} \Rightarrow \cot\left(2\tan^{-1}x\right) = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$$\Rightarrow 2\tan^{-1}x = \begin{cases} n\pi+\frac{\pi}{4} \\ n\pi+\frac{\pi}{2} \\ n\pi-\frac{\pi}{4} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \frac{n\pi}{2}+\frac{\pi}{8} \\ \frac{n\pi}{2}+\frac{\pi}{4} \\ \frac{n\pi}{2}-\frac{\pi}{8} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \frac{\pi}{8},-\frac{3\pi}{8} \\ \frac{\pi}{4},-\frac{\pi}{4} \\ -\frac{\pi}{8},\frac{3\pi}{8} \end{cases}$$

$$\Rightarrow x = \pm 1, \pm(\sqrt{2}-1), \pm(\sqrt{2}+1)$$

Illustration 36: Solve the system of inequalities involving inverse circular functions arc $\tan^2 x - 3$ arc $\tan x + 2 > 0$ and $[\sin^{-1}x] > [\cos^{-1}x]$ where [ ] denotes the greatest integer function. **(JEE ADVANCED)**

Sol: Substitute $\tan^{-1}x$ equal to t.

$\Rightarrow (t-2)(t-1) > 0$

$\Rightarrow t > 2$ or $t > 1$

$\Rightarrow \tan^{-1}x > 2$ or $\tan^{-1}x > 1$

$x \in (-\infty, \tan 1)$ $x > \tan 1$

Again $[\sin^{-1}x] > [\cos^{-1}x]$

$[\sin^{-1}x]$ can take the values $\{-2,-1,0,1\}$

And $[\cos^{-1}x]$ can take the values $\{0,1,2,3\}$

Hence $[\sin^{-1}x]$ can be greater than $[\cos^{-1}x]$ only

If $[\sin^{-1}x] = 1$ and $[\cos^{-1}x] = 1$

Now, $[\sin^{-1}x] = 1 \Rightarrow 1 \le \sin^{-1}x \le \pi/2$ $(1 \le \sin^{-1}x < 2)$

$\sin 1 \le x \le 1$

And $[\cos^{-1}x] = 0 \Rightarrow 0 \le \cos^{-1}x < 1$

$\cos 1 < x \le 1$

Now, x must satisfy

From this $x \in [\sin 1, 1]$

PROBLEM-SOLVING TACTICS

- Making a habit of writing angle values in radians rather in degrees makes the calculation of inverse trigonometric functions easier.
- Try to remember graphs of inverse trigonometric functions. Sometimes it is easier to approximate answers using graphical methods.
- Always verify whether the results are in the range or domain of the respective function.
- In some cases, constructing a right angled triangle for the given inverse function and then solving using properties of triangle is much helpful.
- In case of identities in inverse circular functions, principal values should be taken. As such signs of x, y, etc., will determine the quadrant in which the angles will fall. In order to bring the angles of both sides in the same quadrant, one should make an adjustment by π.

FORMULAE SHEET

1.	If y = sin x, then x = $\sin^{-1}$ y, similarly for other inverse T-functions.

2. Domain and Range of Inverse T-functions:		
Function	**Domain(D)**	**Range (R)**
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\cot^{-1} x$	$-\infty < x < \infty$	$0 < \theta < \pi$
$\sec^{-1} x$	$x \le -1, x \ge 1$	$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	$x \le -1, x \ge 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$

3.	Properties of Inverse T-functions:
	(i) $\sin^{-1}(\sin\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\cos^{-1}(\cos\theta) = \theta$ provided $0 \le \theta \le \pi$ $\tan^{-1}(\tan\theta) = \theta$ provided $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\cot^{-1}(\cot\theta) = \theta$ provided $0 < \theta < \pi$

	$\sec^{-1}(\sec\theta) = \theta$ provided $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$ $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$	
	(ii) $\sin(\sin^{-1}x) = x$ provided $-1 \le x \le 1$ $\cos(\cos^{-1}x) = x$ provided $-1 \le x \le 1$ $\tan(\tan^{-1}x) = x$ provided $-\infty < x < \infty$ $\cot(\cot^{-1}x) = x$ provided $-\infty < x < \infty$ $\sec(\sec^{-1}x) = x$ provided $-\infty < x \le -1$ or $1 \le x < \infty$ $\text{cosec}(\text{cosec}^{-1}x) = x$ provided $-\infty < x \le -1$ or $1 \le x < \infty$	
	(iii) $\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$ $\tan^{-1}(-x) = -\tan^{-1}x$ $\cot^{-1}(-x) = \pi - \cot^{-1}x$ $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$ $\sec^{-1}(-x) = \pi - \sec^{-1}x$	**(iv)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in [-1, 1]$ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in R$ $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in (-\infty,-1] \cup [1,\infty)$

4.	Value of one inverse function in terms of another inverse function:
	(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$ $= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}, \; 0 \le x \le 1$ **(ii)** $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$ $= \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}, \; 0 \le x \le 1$ **(iii)** $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}, \; x \ge 0$ **(iv)** $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x, \; \forall\, x \in (-\infty,1] \cup [1,\infty)$ **(v)** $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \; \forall\, x \in (-\infty,1] \cup [1,\infty)$ **(vi)** $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{for } x > 0 \\ -\pi + \cot^{-1}x & \text{for } x < 0 \end{cases}$

5.	Formulae for sum and difference of inverse trigonometric function:
	(i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$; if $x>0,\ y>0,\ xy<1$
	(ii) $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$; if $x>0,\ y>0,\ xy>1$
	(iii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1+xy}\right)$; if $xy>-1$
	(iv) $\tan^{-1}x-\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$; if $x>0,\ y<0,\ xy<-1$
	(v) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$
	(vi) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$; If $x,y,\geq 0\ \&\ x^2+y^2\leq 1$
	(vii) $\sin^{-1}x\pm\sin^{-1}y=\pi-\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$; If $x,y,\geq 0\ \&\ x^2+y^2>1$
	(viii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]$; If $x,y,>0\ \&\ x^2+y^2\leq 1$
	(ix) $\cos^{-1}x\pm\cos^{-1}y=\pi-\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]$; If $x,y,>0\ \&\ x^2+y^2>1$

6.	Inverse trigonometric ratios of multiple angles
	(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, if $-1\leq x\leq 1$
	(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$, if $-1\leq x\leq 1$
	(iii) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
	(iv) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$
	(v) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$
	(vi) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$