Complex Number

PROBLEM-SOLVING TACTICS

(a) On a complex plane, a complex number represents a point.

(b) In case of division and modulus of a complex number, the conjugates are very useful.

(c) For questions related to locus and for equations, use the algebraic form of the complex number.

(d) Polar form of a complex number is particularly useful in multiplication and division of complex numbers. It directly gives the modulus and the argument of the complex number.

(e) Translate unfamiliar statements by changing z into x+iy.

(f) Multiplying by $\cos\theta$ corresponds to rotation by angle θ about O in the positive sense.

(g) To put the complex number $\frac{a+ib}{c+id}$ in the form A + iB we should multiply the numerator and the denominator by the conjugate of the denominator.

(h) Care should be taken while calculating the argument of a complex number. If z = a + ib, then arg(z) is not always equal to $\tan^{-1}\left(\frac{b}{a}\right)$. To find the argument of a complex number, first determine the quadrant in which it lies, and then proceed to find the angle it makes with the positive x-axis.

For example, if z = – 1 – i, the formula $\tan^{-1}\left(\frac{b}{a}\right)$ gives the argument as $\frac{\pi}{4}$, while the actual argument is $\frac{-3\pi}{4}$.

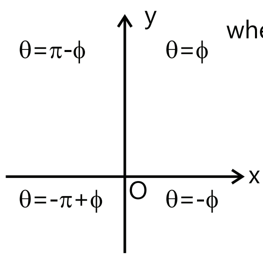
FORMULAE SHEET

(a) Complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.

(b) If z = x + iy then its conjugate $\bar{z} = x - iy$.

(c) Modulus of z, i.e. $|z| = \sqrt{x^2 + y^2}$

(d) Argument of z, i.e. $\theta = \begin{cases} \tan^{-1}\left|\dfrac{y}{x}\right| & x > 0, \quad y > 0 \\ \pi - \tan^{-1}\left|\dfrac{y}{x}\right| & x < 0, \quad y > 0 \\ -\pi + \tan^{-1}\left|\dfrac{y}{x}\right| & x < 0, \quad y < 0 \\ -\tan^{-1}\left|\dfrac{y}{x}\right| & x > 0, \quad y < 0 \end{cases}$



where, $\phi = \tan^{-1}\left|\dfrac{y}{x}\right|$

(e) If y=0, then argument of z, i.e. $\theta = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$

(f) If x=0, then argument of z, i.e. $\theta = \begin{cases} \dfrac{\pi}{2}, & \text{if } y > 0 \\ \dfrac{3\pi}{2}, & \text{if } y < 0 \end{cases}$

(g) In polar form $x = r\cos\theta$ and $y = r\sin\theta$, therefore $z = r(\cos\theta + i\sin\theta)$

(h) In exponential form complex number $z = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.

(i) $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

(j) Important properties of conjugate

(i) $z+\bar{z} = 2\text{Re}(z)$ and $z-\bar{z} = 2\,\text{Im}(z)$

(ii) $z=\bar{z} \Leftrightarrow z$ is purely real

(iii) $z+\bar{z}=0 \Leftrightarrow z$ is purely imaginary

(iv) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(v) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(vi) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(vii) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(k) Important properties of modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z=0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \le \text{Re}(z) \le |z|$

(iv) $-|z| \le \text{Im}(z) \le |z|$

(v) $z\bar{z} = |z|^2$

If z_1, z_2 are two complex numbers, then

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(iii) $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+\bar{z}_1 z_2 + z_1\bar{z}_2 = |z_1|^2+|z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(iv) $|z_1-z_2|^2 = |z_1|^2+|z_2|^2-\bar{z}_1 z_2 - z_1\bar{z}_2 = |z_1|^2+|z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

(l) Important properties of argument

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1)+\arg(z_2)$

In fact $\arg(z_1 z_2) = \arg(z_1)+\arg(z_2)+2k\pi$

$$\text{where, } k = \begin{cases} 0, & \text{if } -\pi < \arg(z_1)+\arg(z_2) \le \pi \\ 1, & \text{if } -2\pi < \arg(z_1)+\arg(z_2) \le -\pi \\ -1, & \text{if } \pi < \arg(z_1)+\arg(z_2) \le 2\pi \end{cases}$$

(iii) $\arg(z_1\bar{z}_2) = \arg(z_1)-\arg(z_2)$

(iv) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)$

(v) $|z_1 + z_2| = |z_1 - z_2| \quad \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(vi) $|z_1 + z_2| = |z_1| + |z_2| \quad \Leftrightarrow \arg(z_1) = \arg(z_2)$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

(m) Triangle on complex plane

(i) Centroid (G), $z_G = \frac{z_1 + z_2 + z_3}{3}$

(ii) Incentre (I), $z_I = \frac{az_1 + bz_2 + cz_3}{a + b + c}$

(iii) Orthocentre (H), $z_H \; \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\sum \tan A}$

(iv) Circumcentre (S), $z_S \; \frac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$

(n) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(o) $\sqrt{z} = \sqrt{x + iy} = \pm\left[\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right]$ for $y > 0$

(p) Distance between $A(z_1)$ and $B(z_2)$ is given by $|z_2 - z_1|$

(q) Section formula: The point P (z) which divides the join of the segment AB in the ratio m : n is given by $z = \frac{mz_2 + nz_1}{m + n}$.

(r) Midpoint formula: $z = \frac{1}{2}(z_1 + z_2)$.

(s) Equation of a straight line

(i) Non-parametric form: $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

(ii) Parametric form: $z = tz_1 + (1 - t)z_2$

(iii) General equation of straight line: $\bar{a}z + a\bar{z} + b = 0$

(t) Complex slope of a line, $\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$. Two lines with complex slopes μ_1 and μ_2 are

(i) Parallel, if $\mu_1 = \mu_2$

(ii) Perpendicular, if $\mu_1 + \mu_2 = 0$

(u) Equation of a circle: $|z - z_0| = r$