6. COMPLEX NUMBER

1. INTRODUCTION

The number system can be briefly summarized as $N \subset W \subset I \subset Q \subset R \subset C$, where $N$, $W$, $I$, $Q$, $R$ and $C$ are the standard notations for the various subsets of the numbers belong to it.

- $N$ - Natural numbers = \{1, 2, 3 ..... \}
- $W$ - Whole numbers = \{0, 1, 2, 3 ..... \}
- $I$ - Integers = \{....2, -1, 0, 1, 2 .....\}
- $Q$ – Rational numbers = \{1/2, 3/2, .... \}
- $IR$ – Irrational numbers = \{\sqrt{2}, \sqrt{3}, \pi \}
- $C$ – Complex numbers

A complex number is generally represented by the letter "z". Every complex number $z$, can be written as, $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$.

$x$ is called the real part of complex number, and $y$ is the imaginary part of complex number.

Note that the sign $+$ does not indicate addition as normally understood, nor does the symbol "i" denote a number. These are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers $(x, y)$ to form a single complex number.

Swiss-born mathematician Jean Robert Argand, after a systematic study on complex numbers, represented every complex number as a set of ordered pair $(x, y)$ on a plane called complex plane.

All complex numbers lying on the real axis were called purely real and those lying on imaginary axis as purely imaginary.

Hence, the complex number $0 + 0i$ is purely real as well as purely imaginary but it is not imaginary.
Complex Number

Note

The symbol i combines itself with real number as per the rule of algebra together with
\[ i^2 = -1; \quad i^3 = -i; \quad i^4 = 1; \quad i^{2014} = -1; \quad i^{2015} = -i \]
and so on.

In general, \( i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \) \( n \in \mathbb{I} \) and \( i^4 + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0 \)

Hence, \( 1 + i^1 + i^2 + \ldots + i^{2014} + i^{2015} = 0 \)

The imaginary part of every real number can be treated as zero. Hence, there is one-one mapping between
the set of complex numbers and the set of points on the complex plane.

MASTERJEE CONCEPTS

Complex number as an ordered pair: A complex number may also be defined as an ordered pair of real
numbers and may be denoted by the symbol \((a, b)\). For a complex number to be uniquely specified, we
need two real numbers in a particular order.

Vaibhav Gupta (JEE 2009, AIR 54)

2. ALGEBRA OF COMPLEX NUMBERS

(a) Addition: \((a + ib) + (c + id) = (a + c) + i(b + d)\)

(b) Subtraction: \((a + ib) - (c + id) = (a - c) + i(b - d)\)

(c) Multiplication: \((a + ib)(c + id) = (ac - bd) + i(ad + bc)\)

(d) Reciprocal: If at least one of \(a, b\) is non-zero, then the reciprocal of \(a + ib\) is given by
\[
\frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
\]

(e) Quotient: If at least one of \(c, d\) is non-zero, then quotient of \(a + ib\) and \(c + id\) is given by
\[
\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}
\]

(f) Inequality in complex numbers is not discussed/defined. If \(a + ib > c + id\) is meaningful only if \(b = d = 0\).
However, equalities in complex numbers are meaningful. Two complex numbers \(z_1\) and \(z_2\) are said to be equal
if \(\text{Re}(z_1) = \text{Re}(z_2)\) and \(\text{Im}(z_1) = \text{Im}(z_2)\). (Geometrically, the position of complex number \(z_1\) on complex plane)

(g) In real number system if \(p^2 + q^2 = 0\) implies, \(p = 0 = q\). But if \(z_1\) and \(z_2\) are complex numbers then \(z_1^2 + z_2^2 = 0\)
does not imply \(z_1 = z_2 = 0\). For e.g. \(z_1 = i\) and \(z_2 = 1\).

However if the product of two complex numbers is zero then at least one of them must be zero, same as in
case of real numbers.

(h) In case \(x\) is real, then \(|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}\)
but in case of complex number \(z\), \(|z|\) means the distance of the
point \(z\) from the origin.
Mathematics | 6.3

MASTERJEE CONCEPTS

- The additive inverse of a complex number \( z = a + ib \) is \(-z\) (i.e. \(-a - ib\)).
- For every non-zero complex number \( z \), the multiplicative inverse of \( z \) is \( \frac{1}{z} \).
- \( |z| \geq |\text{Re}(z)| \geq \text{Re}(z) \) and \( |z| \geq |\text{Im}(z)| \geq \text{Im}(z) \).
- \( \frac{z}{|z|} \) is always a uni-modular complex number if \( z \neq 0 \).

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 1: Find the square root of \( 5 + 12i \). (JEE MAIN)

Sol: \( z = 5 + 12i \)

Let the square root of the given complex number be \( a + ib \). Use algebra to simplify and get the value of \( a \) and \( b \).

Let its square root = \( a + ib \) \( \Rightarrow 5 + 12i = a^2 - b^2 + 2abi \)

\( \Rightarrow a^2 - b^2 = 5 \) \hspace{1cm} ... (i)

\( \Rightarrow 2ab = 12 \)

\( \Rightarrow (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 \) \( \Rightarrow (a^2 + b^2)^2 = 25 + 144 = 169 \) \( \Rightarrow a^2 + b^2 = 13 \) \hspace{1cm} ... (ii)

(i) + (iii) \( \Rightarrow 2a^2 = 18 \) \( \Rightarrow a^2 = 9 \) \( \Rightarrow a = \pm 3 \)

If \( a = 3 \) \( \Rightarrow b = 2 \)

If \( a = -3 \) \( \Rightarrow b = -2 \)

\( \therefore \) Square root = \( 3 + 2i, -3 - 2i \) \( \therefore \) Combined form \( \pm (3 + 2i) \)

Illustration 2: If \( z = (x, y) \in \mathbb{C} \). Find \( z \) satisfying \( z^2 \times (1 + i) = (-7 + 17i) \). (JEE MAIN)

Sol: Algebra of Complex Numbers.

\( (x + iy)^2 (1 + i) = -7 + 17i \)

\( \Rightarrow (x^2 - y^2 + 2xyi) (1 + i) = -7 + 17i \)

\( \Rightarrow (x^2 - y^2 + iy(x^2 - y^2) + 2xyi - 2xy) = -7 + 17i \)

\( \Rightarrow (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy) = -7 + 17i \) \( \Rightarrow x = 3, y = 2 \) \( \Rightarrow x = -3, y = -2 \)

\( \Rightarrow z = -3 + i(-2) = -3 - 2i \)

Illustration 3: If \( x^2 + 2(1 + 2i)x - (11 + 2i) = 0 \). Solve the equation. (JEE ADVANCED)

Sol: Use the quadratic formula to find the value of \( x \).

\( \therefore x = \frac{-2(1 + 2i) \pm \sqrt{4 - 16 + 16i + 44 + 8i}}{2} \)

\( \Rightarrow 2x = (-2)(1 + 2i) \pm \sqrt{32 + 24i} \)

\( \Rightarrow x = (\pm 1)(1 + 2i) \pm \sqrt{8 + 6i} = -1 - 2i \pm (3 + i) \)

\( x = 2 - i, -4 - 3i \)
Illustration 4: If \( f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44 \). Find \( f(3+2i) \).

(JEE ADVANCED)

Sol: Let \( x = 3 + 2i \), and square it to form a quadratic equation. Then try to represent \( f(x) \) in terms of this quadratic.

\[
x = 3 + 2i
\]

\[
\Rightarrow (x - 3)^2 = -4 \Rightarrow x^2 - 6x + 13 = 0
\]

\[
x^4 - 4x^3 + 4x^2 + 8x + 44 = x^2 (x^2 - 6x + 13) + 2x^3 - 9x^2 + 8x + 44
\]

\[
\Rightarrow f(x) = x^2 (x^2 - 6x + 13) + 2(x^3 - 6x^2 + 13x) + 3(x^2 - 6x + 13) + 5 \quad \Rightarrow f(x) = 5
\]

3. IMPORTANT TERMS ASSOCIATED WITH COMPLEX NUMBER

Three important terms associated with complex number are conjugate, modulus and argument.

(a) **Conjugate:** If \( z = x + iy \) then its complex conjugate is obtained by changing the sign of its imaginary part and denoted by \( \bar{z} \) i.e. \( \bar{z} = x - iy \) (see Fig 6.3).

The conjugate satisfies following basic properties

(i) \( z + \bar{z} = 2\text{Re}(z) \)

(ii) \( z - \bar{z} = 2i\text{Im}(z) \)

(iii) \( z\bar{z} = x^2 + y^2 \)

(iv) If \( z \) lies in 1st quadrant then \( \bar{z} \) lies in 4th quadrant and \( -z \) in the 2nd quadrant.

(v) If \( x + iy = f(a + ib) \) then \( x - iy = f(a - ib) \)

For e.g. If \( (2 + 3i)^3 = x + iy \) then \( (2 - 3i)^3 = x - iy \)

and, \( \sin(\alpha + i\beta) = x + iy \Rightarrow \sin(\alpha - i\beta) = x - iy \)

(vi) \( z + \bar{z} = 0 \quad \Rightarrow z \) is purely imaginary

(vii) \( z - \bar{z} = 0 \quad \Rightarrow z \) is purely real

(b) **Modulus:** If \( P \) denotes a complex number \( z = x + iy \) then, \( OP = |z| = \sqrt{x^2 + y^2} \).

Geometrically, it is the distance of a complex number from the origin.

Hence, note that \( |z| \geq 0 , |i| = 1 \) i.e. \( |\sqrt{-1}| = 1 \).

All complex number satisfying \( |z| = r \) lie on the circle having centre at origin and radius equal to \( 'r' \).

(c) **Argument:** If OP makes an angle \( \theta \) (see Fig 6.4) with real axis in anticlockwise sense, then \( \theta \) is called the argument of \( z \). General values of argument of \( z \) are given by \( 2n\pi + \theta , n \in I \). Hence any two successive arguments differ by \( 2\pi \).

**Note:** A complex number is completely defined by specifying both modulus and argument. However for the complex number \( 0 + 0i \) the argument is not defined and this is the only complex number which is completely defined by its modulus only.

(i) **Amplitude (Principal value of argument):** The unique value of \( \theta \) such that \( -\pi < \theta \leq \pi \) is called principal value of argument. Unless otherwise stated, \( \text{amp} z \) refers to the principal value of argument.

(ii) **Least positive argument:** The value of \( \theta \) such that \( 0 < \theta \leq 2\pi \) is called the least positive argument.

\[
\text{If } \phi = \tan^{-1} \left( \frac{y}{x} \right).
\]
If \( x > 0, y > 0 \) (i.e. \( z \) is in first quadrant), then \( \arg z = \theta = \tan^{-1} \frac{|y|}{x} \).

If \( x < 0, y > 0 \) (i.e. \( z \) is in 2nd quadrant, then \( \arg z = \theta = \pi - \tan^{-1} \frac{|y|}{x} \).

If \( x < 0, y < 0 \) (i.e. \( z \) is in 3rd quadrant), then \( \arg z = \theta = -\pi + \tan^{-1} \frac{|y|}{x} \).

If \( x > 0, y < 0 \) (i.e. \( z \) is in 4th quadrant), then \( \arg z = \theta = -\tan^{-1} \frac{|y|}{x} \).

If \( y = 0 \) (i.e. \( z \) is on the X-axis), then \( \arg (x + i0) = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases} \).

If \( x = 0 \) (i.e. \( z \) is on the Y-axis), then \( \arg (0 + iy) = \begin{cases} 0, & \text{if } y > 0 \\ \frac{3\pi}{2}, & \text{if } y < 0 \end{cases} \).

**Illustration 5:** For what real values of \( x \) and \( y \), are \( -3 + ix^2y \) and \( x^2 + y + 4i \) complex conjugate to each other? (JEE MAIN)

**Sol:** As \( -3 + ix^2y \) and \( x^2 + y + 4i \) are complex conjugate of each other. Therefore \( -3 + ix^2y = x^2 + y + 4i \).

Equating real and imaginary parts of the above question, we get

\[-3 = x^2 + y \Rightarrow y = -3 - x^2 \quad \text{... (i)}\]

and \( x^2y = -4 \quad \text{... (ii)}\)

Putting the value of \( y = -3 - x^2 \) from (i) in (ii), we get

\[x^2(-3 - x^2) = -4 \Rightarrow x^4 + 3x^2 - 4 = 0 \Rightarrow x^2 = \frac{-3 \pm \sqrt{9 + 16}}{2} = \frac{-3 \pm 5}{2} = \frac{-8}{2} = 1, \frac{-2}{2} = 1, -4\]

\[\therefore x^2 = 1 \Rightarrow x = \pm 1\]

Putting value of \( x = \pm 1 \) in (i), we get \( y = -3 - (1)^2 = -3 - 1 = -4 \)

Hence, \( x = \pm 1 \) and \( y = -4 \).

**Illustration 6:** Find the modulus of \( \frac{1+i}{1-i} - \frac{1-i}{1+i} \). (JEE MAIN)

**Sol:** As \( |z| = \sqrt{x^2 + y^2} \), using algebra of complex number we will get the result.

Here, we have

\[\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} - \frac{(1-i)(1-i)}{(1+i)(1+i)}\]

\[= \frac{1+i^2 + 2i}{1+1} - \frac{1-i^2 - 2i}{1+1} = \frac{1-1+2i}{2} - \frac{1-1-2i}{2} = \frac{2i}{2} = \frac{-2i}{2} = i + i = 2i \]

\[\Rightarrow \left| \frac{1+i}{1-i} - \frac{1-i}{1+i} \right| = |2i| = 2.\]
Illustration 7: Find the locus of $z$ if $| z - 3 | = 3 | z + 3 |$. (JEE MAIN)

Sol: Simply substituting $z = x + iy$ and by using formula $| z | = \sqrt{x^2 + y^2}$ we will get the result.

Let $z = x + iy$

$$| x + iy - 3 | = 3 | x + iy + 3 | \quad | (x - 3) + iy | = 3 | (x + 3) + iy |$$

$$\sqrt{(x - 3)^2 + y^2} = 3\sqrt{(x + 3)^2 + y^2}; \quad (x - 3)^2 + y^2 = 9(x + 3)^2 + 9y^2.$$

Illustration 8: If $\alpha$ and $\beta$ are different complex numbers with $| \beta | = 1$, then find $\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}$. (JEE ADVANCED)

Sol: By using modulus and conjugate property, we can find out the value of $\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}$.

We have, $| \beta | = 1 \Rightarrow | \beta |^2 = 1 \Rightarrow \beta \overline{\beta} = 1$

Now, $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \left| \frac{\beta - \alpha}{\beta \overline{\beta} - \bar{\alpha}\beta} \right| = \left| \frac{\beta - \alpha}{| \beta | \bar{\beta} - | \beta | \alpha} \right| = \frac{1}{| \beta |} = 1$. (as $| x + iy | = | x + iy |$)

Illustration 9: Find the number of non-zero integral solution of the equation $| 1 - i |^x = 2^x$. (JEE ADVANCED)

Sol: As $| z | = \sqrt{x^2 + y^2}$, therefore by using this formula we can solve it.

We have, $| 1 - i |^x = 2^x$

$$\Rightarrow \left[ \sqrt{1^2 + 1^2} \right]^x = 2^x \quad \Rightarrow \left( \sqrt{2} \right)^x = 2^x \quad \Rightarrow 2^{\frac{x}{2}} = 2^x \quad \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

∴ The number of non zero integral solution is zero.

Illustration 10: If $\frac{a + ib}{c + id} = p + iq$. Prove that $\frac{a^2 + b^2}{c^2 + d^2} = p^2 + q^2$. (JEE MAIN)

Sol: Simply by obtaining modulus of both side of $\frac{a + ib}{c + id} = p + iq$.

We have, $\frac{a + ib}{c + id} = p + iq$

$$\left| \frac{a + ib}{c + id} \right|^2 = |p + iq|^2 \Rightarrow \frac{a^2 + b^2}{c^2 + d^2} = p^2 + q^2.$$

Illustration 11: If $(x + iy)^{1/3} = a + ib$. Prove that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$. (JEE ADVANCED)

Sol: By using algebra of complex number. We have, $(x + iy)^{1/3} = a + ib$

$$x + iy = (a + ib)^3 = a^3 + i^3b^3 + 3a^2ib + 3a(ib)^2 = a^3 - b^3i + 3a^2bi - 3ab^2$$

$$x + iy = (a^3 - 3ab^2) + (3a^2b - b^3)i; \quad x = a^3 - 3ab^2 = a(a^2 - 3b^2); \quad y = 3a^2b - b^3 \Rightarrow \frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2).$$
4. REPRESENTATION OF COMPLEX NUMBER

4.1 Graphical Representation

Every complex number $x + iy$ can be represented in a plane as a point $P(x, y)$. $X$-coordinate of point $P$ represents the real part of the complex number and $y$-coordinate represents the imaginary part of the complex number. Complex number $x + 0i$ (real number) is represented by a point $(x, 0)$ lying on the $x$-axis. Therefore, $x$-axis is called the real axis. Similarly, a complex number $0 + iy$ (imaginary number) is represented by a point on $y$-axis. Therefore, $y$-axis is called the imaginary axis.

The plane on which a complex number is represented is called complex number plane or simply complex plane or Argand plane (see Fig 6.6). The figure represented by the complex numbers as points in a plane is known as Argand Diagram.

4.2 Algebraic Form

If $z = x + iy$; then $|z| = \sqrt{x^2 + y^2}$; $\bar{z} = x - iy$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Generally this form is useful in solving equations and in problems involving locus.

4.3 Polar Form

Figure 6.7 shows the components of a complex number along the $x$ and $y$-axes respectively. Then

$$z = x + iy = r (\cos \theta + i \sin \theta) = r \text{cis} \ \theta \ \text{where} \ |z| = r; \ \text{amp} \ z = \theta.$$

Alternative: $z = x + iy$

$$\Rightarrow z = \sqrt{x^2 + y^2} \left(\frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}}\right)$$

$$\Rightarrow z = |z| (\cos \theta + i \sin \theta) = r \text{cis} \ \theta$$

Note:
(a) $(\text{cis} \alpha) (\text{cis} \beta) = \text{cis}(\alpha + \beta)$
(b) $(\text{cis} \alpha) (\text{cis}(-\beta)) = \text{cis}(\alpha - \beta)$
(c) $\frac{1}{\text{cis} \alpha} = (\text{cis} \alpha)^{-1} = \text{cis}(-\alpha)$

MASTERJEE CONCEPTS

The unique value of $\theta$ such that $-\pi < \theta \leq \pi$ for which $x = r \cos \theta$ & $y = r \sin \theta$ is known as the principal value of the argument.

The general value of argument is $(2n\pi + \theta)$, where $n$ is an integer and $\theta$ is the principal value of arg $(z)$. While reducing a complex number to polar form, we always take the principal value.

The complex number $z = r \ (\cos \theta + i \sin \theta)$ can also be written as $r \text{cis} \ \theta$.

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4.4 Exponential Form

Euler’s formula, named after the famous mathematician Leonhard Euler, states that for any real number \( x \),
\[
e^{ix} = \cos x + i\sin x.
\]
Hence, for any complex number \( z = r (\cos \theta + i\sin \theta) \), \( z = re^{i\theta} \) is the exponential representation.

Note: (a) \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \) and \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \) are known as Euler’s identities.

(b) \( \cos ix = \frac{e^{x} + e^{-x}}{2} = \cosh x \) is always positive real \( \forall x \in \mathbb{R} \) and is \( > 1 \).
and, \( \sin ix = \frac{e^{x} - e^{-x}}{2} = i \sinh x \) is always purely imaginary.

4.5 Vector Representation

The knowledge of vectors can also be used to represent a complex number \( z = x + iy \). The vector \( \overrightarrow{OP} \), joining the origin \( O \) of the complex plane to the point \( P(x, y) \), is the vector representation of the complex number \( z=x+iy \), (see Fig 6.9). The length of the vector \( \overrightarrow{OP} \), that is, \( |\overrightarrow{OP}| \) is the modulus of \( z \).

The angle between the positive real axis and the vector \( \overrightarrow{OP} \), more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as \( \overrightarrow{OP} \) (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number \( z \).

Illustration 12: Find locus represented by \( \text{Re} \left( \frac{1}{x+iy} \right) < \frac{1}{2} \). (JEE MAIN)

Sol: Multiplying numerator and denominator by \( x-iy \).

We have, \( \text{Re} \left( \frac{1}{x+iy} \right) < \frac{1}{2} \quad \text{and} \quad \text{Re} \left( \frac{x-iy}{x^2+y^2} \right) < \frac{1}{2} \)

\[
\Rightarrow \frac{x}{x^2+y^2} < \frac{1}{2} \quad \text{and} \quad x^2 + y^2 - 2x > 0
\]

Locus is the exterior of the circle with centre \((1, 0)\) and radius = 1.

Illustration 13: If \( z = 1 + \cos \frac{6\pi}{5} + i\sin \frac{6\pi}{5} \). Find \( r \) and amp \( z \). (JEE MAIN)

Sol: By using trigonometric formula we can reduce given equation in the form of \( z = r (\cos \theta + i\sin \theta) \).

\[
z = 2 \cos^2 \frac{3\pi}{5} + 2i \sin \frac{3\pi}{5} \cos \frac{3\pi}{5} = 2 \cos \frac{3\pi}{5} \left[ \cos \frac{3\pi}{5} + i\sin \frac{3\pi}{5} \right]
\]

\[
= -2 \cos \frac{2\pi}{5} \left[ -\cos \frac{2\pi}{5} + i\sin \frac{2\pi}{5} \right] = 2 \cos \frac{2\pi}{5} \left[ \cos \frac{2\pi}{5} \sin \frac{2\pi}{5} \right] \quad \text{Hence, } |z| = 2 \cos \frac{2\pi}{5} \quad \text{amp } z = -\frac{2\pi}{5}
\]
**Illustration 14:** Show that the locus of the point \( P(\omega) \) denoting the complex number \( z + \frac{1}{z} \) on the complex plane is a standard ellipse where \( |z| = a \), where \( a \neq 0, 1 \). (JEE ADVANCED)

**Sol:** Here consider \( w = x + iy \) and \( z = \alpha + i\beta \) and then solve this by using algebra of complex number.

Let \( w = z + \frac{1}{z} \) where \( z = \alpha + i\beta \) and \( \alpha^2 + \beta^2 = a^2 \) (as \( |z| = a \))

\[
x + iy = \alpha + i\beta + \frac{1}{\alpha + i\beta} = \alpha + i\beta + \frac{\alpha - i\beta}{\alpha^2 + \beta^2} = \left(\alpha + \frac{\alpha}{a^2}\right) + i\left(\beta - \frac{\beta}{a^2}\right) \Rightarrow x = \alpha \left(1 + \frac{1}{a^2}\right); \quad y = \beta \left(1 - \frac{1}{a^2}\right)
\]

\[
\therefore \frac{x^2}{\left(1 + \frac{1}{a^2}\right)^2} + \frac{y^2}{\left(1 - \frac{1}{a^2}\right)^2} = \alpha^2 + \beta^2 = a^2;
\]

\[
\therefore \frac{x^2}{\left(a + \frac{1}{a^2}\right)^2} + \frac{y^2}{\left(a - \frac{1}{a^2}\right)^2} = 1.
\]

**5. IMPORTANT PROPERTIES OF CONJUGATE, MODULUS AND ARGUMENT**

For \( z, z_1 \) and \( z_2 \in \mathbb{C} \),

(a) **Properties of Conjugate:**

(i) \( z + \overline{z} = 2\text{Re}(z) \)

(ii) \( z - \overline{z} = 2i\text{Im}(z) \)

(iii) \( \overline{\overline{z}} = z \)

(iv) \( z_1 + z_2 = \overline{z}_1 + \overline{z}_2 \)

(v) \( z_1 - z_2 = \overline{z}_1 - \overline{z}_2 \)

(vi) \( z_1 \cdot z_2 = \overline{z}_1 \cdot \overline{z}_2 \)

(vii) \( \frac{z_1}{z_2} = \frac{\overline{z}_1}{\overline{z}_2}; \quad z_2 \neq 0 \)

(b) **Properties of Modulus:**

(i) \( |z| \geq 0; |z| \geq \text{Re}(z); |z| \geq \text{Im}(z); |z| = |\overline{z}| = |-z| \)

(ii) \( z\overline{z} = |z|^2 \); if \( |z| = 1 \), then \( z = \frac{1}{\overline{z}} \)

(iii) \( |z_1 z_2| = |z_1| \cdot |z_2| \)

(iv) \( \left| \frac{z_1}{z_2} \right| = \left| \frac{|z_1|}{|z_2|} \right|, \quad z_2 \neq 0 \)

(v) \( |z^n| = |z|^n \)

(vi) \( |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 \left[ |z_1|^2 + |z_2|^2 \right] \)

(vii) \( \|z_1 - z_2\| \leq |z_1 + z_2| \leq |z_1| + |z_2| \) \quad \text{[Triangle Inequality]}
(c) Properties of Amplitude:

(i) \[ \text{amp} (z_1 \cdot z_2) = \text{amp} z_1 + \text{amp} z_2 + 2k\pi, \quad k \in \mathbb{I} \]

(ii) \[ \text{amp} \left( \frac{z_1}{z_2} \right) = \text{amp} z_1 - \text{amp} z_2 + 2k\pi, \quad k \in \mathbb{I} \]

(iii) \[ \text{amp} (z^n) = n \text{amp}(z) + 2k\pi \], where the value of \( k \) should be such that RHS lies in \((-\pi, \pi]\)

Based on the above information, we have the following

- \[ |\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2} |z| \]
- \[ |z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2| \]. Thus \( |z_1| + |z_2| \) is the greatest possible value of \( |z_1 + z_2| \) and \( |z_1| - |z_2| \) is the least possible value of \( |z_1 + z_2| \).
- If \( |z + \frac{1}{z}| = a \), the greatest and least values of \(|z|\) are respectively \( \frac{a + \sqrt{a^2 + 4}}{2} \) and \( \frac{-a + \sqrt{a^2 + 4}}{2} \).
- \[ |z_1 + \sqrt{z_1^2 - z_2^2}| + |z_2 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2| \]
- If \( z_1 = z_2 \Leftrightarrow |z_1| = |z_2| \) and \( \text{arg} z_1 = \text{arg} z_2 \)
- \[ |z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \text{arg} (z_1) = \text{arg} (z_2) \] i.e. \( z_1 \) and \( z_2 \) are parallel.
- \[ |z_1| = |z_2| \Leftrightarrow \text{arg} (z_1) = \text{arg} (z_2) \] where \( n \) is some integer.
- \[ |z_1 - z_2| = |z_1 - \sqrt{z_1^2 - z_2^2}| \Leftrightarrow \text{arg} (z_1) = \text{arg} (z_2) = 2n\pi, \] where \( n \) is some integer.
- \[ |z_1 - z_2| = |z_1 - \sqrt{z_1^2 - z_2^2}| \Leftrightarrow \text{arg} (z_1) = \text{arg} (z_2) = (2n + 1)\frac{\pi}{2}, \] where \( n \) is some integer.
- If \( |z_1| \leq 1, |z_2| \leq 1, \) then \( |z_1 + z_2|^2 \leq \left( |z_1| - |z_2| \right)^2 + \left( \text{arg} (z_1) - \text{arg} (z_2) \right)^2 \), and \( |z_1 + z_2|^2 \geq \left( |z_1| + |z_2| \right)^2 - \left( \text{arg} (z_1) - \text{arg} (z_2) \right)^2 \).

Illustration 15: If \( z_1 = 3 + 5i \) and \( z_2 = 2 - 3i \), then verify that \( \frac{z_1}{z_2} = \frac{3 + 5i}{2 - 3i} = \frac{(3 + 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{6 - 9i + 10i + 15i^2}{4 - 9i^2} = \frac{6 - 9i + 10i - 15}{4 + 9} = \frac{6 - 19i - 15}{13} = \frac{-9 + 19i}{13} \) \( \text{JEE MAIN} \)

Sol: Simply by using properties of conjugate.

\[
L.H.S. \quad \frac{z_1}{z_2} = \frac{3 + 5i}{2 - 3i} = \frac{(3 + 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{6 + 9i + 10i + 15i^2}{4 - 9i^2} = \frac{6 + 19i + 15(-1)}{4 + 9} = \frac{6 + 19i - 15}{13} = \frac{-9 + 19i}{13} = \frac{-9}{13} + \frac{19}{13}i
\]

\[
R.H.S. \quad \frac{z_1}{z_2} = \frac{3 + 5i}{2 - 3i} = \frac{3 - 5i}{2 + 3i} = \frac{(3 - 5i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{6 - 9i - 10i + 15i^2}{4 - 9i^2} = \frac{6 - 19i + 15(-1)}{4 + 9} = \frac{6 - 19i - 15}{13} = \frac{-9 - 19i}{13} = \frac{-9}{13} - \frac{19}{13}i \]

\[
\therefore \frac{z_1}{z_2} = \frac{z_1}{z_2}
\]
Illustration 16: If $z$ be a non-zero complex number, then show that $\overline{(z^{-1})} = (\overline{z})^{-1}$.

(JEE MAIN)

Sol: By considering $z = a + ib$ and using properties of conjugate we can prove given equation.

Let $z = a + ib$ Since, $z \neq 0$, we have $x^2 + y^2 > 0$

\[
\frac{1}{z} = \frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} \Rightarrow \left(\frac{1}{z}\right) = \frac{a}{a^2 + b^2} + \frac{ib}{a^2 + b^2} \quad \cdots (i)
\]

and $\frac{1}{z^\ast} = \frac{1}{a + ib} = \frac{1}{a - ib} = \frac{a + ib}{a^2 + b^2} + \frac{b}{a^2 + b^2} \quad \cdots (ii)$

From (i) and (ii), we get $\overline{(z^{-1})} = (\overline{z})^{-1}$.

Illustration 17: If \( \frac{(a+i)^2}{2a-i} = p + iq \), then show that \( p^2 + q^2 = \frac{(a^2 + 1)^2}{4a^2 + 1} \).

(JEE MAIN)

Sol: Multiply given equation to its conjugate.

We have, \( p + iq = \frac{(a+i)^2}{2a-i} \) \quad \cdots (i)

Taking conjugate of both sides, we get \( p - iq = \frac{(a-i)^2}{2a+i} \) \quad \cdots (ii) using \((z^2) = z \cdot \overline{z} = \overline{z} \cdot z = (\overline{z})^2\)

Multiplying (i) and (ii), we get \( (p + iq)(p - iq) = \frac{(a+i)^2}{2a-i} \cdot \frac{(a-i)^2}{2a+i} \)

\[ p^2 - i^2q^2 = \frac{(a^2 - i^2)^2}{4a^2 - i^2} \Rightarrow p^2 + q^2 = \frac{(a^2 + 1)^2}{4a^2 + 1} \]

Illustration 18: Let \( z_1, z_2, z_3, \ldots, z_n \) are the complex numbers such that \( |z_1| = |z_2| = \ldots = |z_n| = 1 \). If \( z = \left( \sum_{k=1}^{n} z_k \right) \left( \sum_{k=1}^{n} \frac{1}{z_k} \right) \) then prove that

(i) \( z \) is a real number \hspace{1cm} (ii) \( 0 < z \leq n^2 \)

(JEE ADVANCED)

Sol: Here \( |z_1| = |z_2| = \ldots = |z_n| = 1 \), therefore \( z\overline{z} = 1 \) \( \Rightarrow z = \frac{1}{z} \). Hence by substituting this to \( z = \left( \sum_{k=1}^{n} z_k \right) \left( \sum_{k=1}^{n} \frac{1}{z_k} \right) \)

we can solve above problem.

Now, \( z = (z_1 + z_2 + z_3 + \ldots + z_n) \left( \frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n} \right) \)

\[ = (z_1 + z_2 + z_3 + \ldots + z_n) \left( \frac{z_1 + z_2 + \ldots + z_n}{z_1 + z_2 + \ldots + z_n} \right) = (z_1 + z_2 + z_3 + \ldots + z_n) \left( \frac{z_1 + z_2 + \ldots + z_n}{z_1 + z_2 + \ldots + z_n} \right) \]

\[ = |z_1 + z_2 + z_3 + \ldots + z_n|^2 \text{ which is real} \]

\[ \leq \left( |z_1| + |z_2| + |z_3| + \ldots + |z_n| \right)^2 = n^2 \quad \therefore \quad 0 < z \leq n^2. \]
Illustration 19: Let \( x_1, x_2 \) are the roots of the quadratic equation \( x^2 + ax + b = 0 \) where \( a, b \) are complex numbers and \( y_1, y_2 \) are the roots of the quadratic equation \( y^2 + |a|y + |b| = 0 \). If \( |x_1| = |x_2| = 1 \), then prove that \( |y_1| = |y_2| = 1 \).

(JEE ADVANCED)

Sol: Solve by using modulus properties of complex number.

Let \( x^2 + ax + b = 0 \) where \( x_1 \) and \( x_2 \) are complex numbers

\[
\begin{align*}
x_1 + x_2 &= -a \quad \cdots (i) \\
x_1 \cdot x_2 &= b \quad \cdots (ii)
\end{align*}
\]

From (ii) \( |x_1| \cdot |x_2| = |b| \Rightarrow |b| = 1 \) Also \( | -a | = |x_1 + x_2| \)

\[
\therefore \quad |a| \leq |x_1| + |x_2| \quad \text{or} \quad |a| \leq 2
\]

Now consider \( y^2 + |a|y + |b| = 0 \), where \( y_1 \) and \( y_2 \) are complex numbers

\[
y_{1,2} = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2} = \frac{-|a| \pm \left(\sqrt{4 - |a|^2}\right)i}{2} \quad \therefore \quad |y_{1,2}| = \frac{\sqrt{|a|^2 + 4 - |a|^2}}{2} = 1
\]

Hence, \( |y_1| = |y_2| = 1 \).

6. TRIANGLE ON COMPLEX PLANE

In a \( \Delta ABC \), the vertices \( A, B \) and \( C \) are represented by the complex numbers \( z_1, z_2 \) and \( z_3 \) respectively, then

(a) Centroid: The centroid ‘G’ is given by \( \frac{z_1 + z_2 + z_3}{3} \). Refer to Fig 6.10.

(b) Incentre: The incentre ‘I’ is given by \( \frac{az_1 + bz_2 + cz_3}{a + b + c} \). Refer to Fig 6.11.
(c) **Orthocentre:** The orthocentre ‘H’ is given by \( \sum \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A} \).

**Proof:** From section formula, we have \( z_D = \frac{p z_3 + q z_2}{a} \).

In \( \triangle ABD \) and \( \triangle ACD \), \( p = c \cos B \) and \( q = b \cos C \). Refer to Fig 6.12.

Therefore, \( z_D = \frac{b \cos C \ z_2 + c \cos B \ z_3}{a} \).

Now, \( AE = c \cos A; \ n = AH = AE \cosec C = c \cos A \cosec C \)
\( \Rightarrow n = 2R \cos A \quad [\text{Using Sine Rule}] \)
and \( m = c \cos B \cot C \quad \text{or, } m = 2R \cos B \cos C \quad [\text{Using Sine Rule}] \)

Hence, \( z_H = \frac{m z_1 + n z_D}{m + n} \).

\[
2R \cos B \cos C z_1 + 2R \cos A \left( \frac{b \cos C \ z_2 + c \cos B \ z_3}{a} \right)
\]

\[
= \frac{a \cos B \cos C z_1 + b \cos A \cos C z_2 + c \cos A \cos B z_3}{a(-\cos B + C) + \cos B \cos C}
\]

\[
= \frac{z_1 (\sin A \cos B \cos C) + z_2 (\sin B \cos C \cos A) + z_3 (\sin C \cos A \cos B)}{\sin A (\sin B \sin C)}
\]

\[
\therefore z_H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\sum \tan A} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\prod \tan A}
\]

[If \( A + B + C = \pi \), then \( \tan A + \tan B + \tan C = \tan A \tan B \tan C \)]

(d) **Circumcentre:**

Let \( R \) be the circumradius and the complex number \( z_0 \) represent the circumcentre of the triangle as shown in Fig 6.11.

\[
\therefore |z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|
\]

Consider, \( |z_1 - z_0|^2 = |z_2 - z_0|^2 \)

\((z_1 - z_0)(\overline{z}_1 - \overline{z}_0) = (z_2 - z_0)(\overline{z}_2 - \overline{z}_0)\)
6.14  Complex Number

\[ \overline{z}_1 (z_1 - z_0) - \overline{z}_2 (z_2 - z_0) = \overline{z}_0 \left[ (z_1 - z_0) - (z_2 - z_0) \right] \]

\[ \overline{z}_1 (z_1 - z_0) - \overline{z}_2 (z_2 - z_0) = \overline{z}_0 (z_1 - z_2) \]

Similarly 1\textsuperscript{st} and 3\textsuperscript{rd} gives

\[ \overline{z}_1 (z_1 - z_0) - \overline{z}_3 (z_3 - z_0) = \overline{z}_0 (z_1 - z_3) \]

On dividing (i) by (ii), \( z_0 \) gets eliminated and we obtain \( z_0 \).

\section*{Alternatively:}

From Fig 6.13, we have

\[ \frac{BD}{DC} = \frac{m}{n} = \frac{\text{Ar. } \Delta ABD}{\text{Ar. } \Delta ADC} = \frac{\text{Ar. } \Delta PBD}{\text{Ar. } \Delta PDC} \]

\[ \therefore \frac{m}{n} = \frac{\text{Ar. } \Delta ABD - \text{Ar. } \Delta PBD}{\text{Ar. } \Delta ADC - \text{Ar. } \Delta PDC} = \frac{\Delta_3}{\Delta_2} \]

\[ \therefore \frac{m}{n} = \frac{\frac{R^2}{2} \sin 2C}{\frac{R^2}{2} \sin 2B} = \frac{\sin 2C}{\sin 2B} \]

Hence, \( z_0 = \frac{\sin 2B (z_2) + \sin 2C (z_3)}{\sin 2B + \sin 2C} \)

Now \( \frac{PA}{PD} = \frac{k}{l} = \frac{\Delta ABD}{\Delta PBD} = \frac{\Delta APC}{\Delta PCD} = \frac{\Delta ABP + \Delta APC}{\Delta PBD + \Delta CPD} \)

\[ \therefore \frac{k}{l} = \frac{\Delta_3 + \Delta_2}{\Delta_1} = \frac{\sin 2C + \sin 2B}{\sin 2A} \]

Hence, \( z_0 = \frac{kz_1 + lz_0}{k + l} = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sum \sin 2A} \)

\section*{MASTERJEE CONCEPTS}

- The area of the triangle whose vertices are \( z, iz \) and \( z + iz \) is \( \frac{1}{2} |z|^2 \).

- The area of the triangle with vertices \( z, \omega z \) and \( z + \omega z \) is \( \frac{\sqrt{3}}{4} |z|^2 \).

- If \( z_1, z_2, z_3 \) be the vertices of an equilateral triangle and \( z_0 \) be the circumcentre, then \( z_1^2 + z_2^2 + z_3^2 = 3z_0^2 \).

- If \( z_1, z_2, z_3, \ldots, z_n \) be the vertices of a regular polygon of \( n \) sides and \( z_0 \) be its centroid, then \( z_1^2 + z_2^2 + \ldots + z_n^2 = n z_0^2 \).

- If \( z_1, z_2, z_3 \) be the vertices of a triangle, then the triangle is equilateral if \( (z_1 - z_0)^2 + (z_2 - z_0)^2 + (z_3 - z_0)^2 = 0 \) or \( z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \) or \( \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0 \).

- If \( z_1, z_2, z_3 \) are the vertices of an isosceles triangle, right angled at \( z_1 \) then \( z_1^2 + 2z_2^2 + z_3^2 = 2z_2 (z_1 + z_3) \).

- If \( z_1, z_2, z_3 \) are the vertices of a right-angled isosceles triangle, then \( (z_1 - z_2)^2 = 2 (z_1 - z_3) (z_3 - z_2) \).

- If \( z_1, z_2, z_3 \) be the affixes of the vertices \( A, B, C \) respectively of a triangle \( ABC \), then its orthocentre is \( \frac{a (\sec A) z_1 + b (\sec B) z_2 + c (\sec C) z_3}{a \sec A + b \sec B + c \sec C} \).

Shivam Agarwal (JEE 2009, AIR 27)
Illustration 20: If \( z_1, z_2, z_3 \) are the vertices of an isosceles triangle right angled at \( z_2 \) then prove that \( z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3) \) (JEE MAIN)

Sol: Here \( (z_1 - z_2) = (z_3 - z_2) e^{i \frac{\pi}{2}} \). Hence by squaring both side we will get the result.
\[ \Rightarrow (z_1 - z_2)^2 = i(z_3 - z_2)^2 \]
\[ \Rightarrow z_1^2 + z_2^2 - 2z_2z_3 = -z_1^2 - z_2^2 + 2z_1z_2 \Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3). \]

Illustration 21: A, B, C are the points representing the complex numbers \( z_1, z_2, z_3 \) respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at \( D, E, F \) respectively. Find the complex numbers corresponding to the points \( D, E, F \) in terms of \( z_1, z_2, z_3 \). (JEE MAIN)

Sol: Here the \( \angle BOD = \pi - 2B \), hence \( \overrightarrow{OD} = \overrightarrow{OB} e^{i(\pi - 2B)} \).

From Fig 6.13, we have \( \overrightarrow{OD} = \overrightarrow{OB} e^{i(\pi - 2B)} \),
\[ \alpha = z_2 e^{i(\pi - 2B)} = -z_2 e^{-i2B} \]
also, \( z_1 = z_3 e^{i2B} \)
\[ \therefore \alpha z_1 = -z_2 z_3 \]
\[ \Rightarrow \alpha = \frac{-z_2 z_3}{z_1} \]
Similarly, \( \beta = \frac{-z_3 z_1}{z_2} \) and \( \gamma = \frac{-z_1 z_2}{z_3} \).

Illustration 22: If \( z_r \) \( (r = 1, 2, ...6) \) are the vertices of a regular hexagon then prove that \( \sum_{r=1}^{6} z_r^2 = 6z_0^2 \), where \( z_0 \) is the circumcentre of the regular hexagon. (JEE MAIN)

Sol: As we know if \( z_1, z_2, z_3, ......., z_n \) be the vertices of a regular polygon of \( n \) sides and \( z_0 \) be its centroid, then \( z_1^2 + z_2^2 + ....... + z_n^2 = nz_0^2 \).

Here by the Fig 6.14,
\[ 3z_0^2 = z_1^2 + z_2^2 + z_3^2 \]
and, \( 3z_0^2 = z_2^2 + z_3^2 + z_4^2 \) \[ = 6z_0^2 = \sum_{r=1}^{6} z_r^2 \].

Illustration 23: If \( z_1, z_2, z_3 \) are the vertices of an equilateral triangle then prove that \( z_1^2 + z_1^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \) and if \( z_0 \) is its circumcentre then \( 3z_0^2 = z_1^2 + z_2^2 + z_3^2 \). (JEE ADVANCED)

Sol: By using triangle on complex plane we can prove
\[ z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \] and by using \( z_0 = \frac{z_1 + z_2 + z_3}{3} \) we can prove \( 3z_0^2 = z_1^2 + z_2^2 + z_3^2 \).

To Prove, \( z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \)
As seen in the Fig 6.17,
\[ \therefore z_1 - z_2 = \frac{(z_3 - z_2) e^{i \frac{\pi}{3}}}{z_2 - z_3} \Rightarrow (z_1 - z_2) (z_1 - z_3) = -(z_2 - z_3)^2 \]
\[ \Rightarrow z_1^2 - z_1 z_3 - z_2 z_1 + z_2 z_3 + z_2^2 + z_3^2 - 2z_2 z_3 = 0 \]
\[ \therefore \sum z_1^2 = \sum z_1 z_2 \]
Now if \( z_0 \) is the circumcentre of the \( \Delta \), then we need to prove \( 3z_0^2 = z_1^2 + z_2^2 + z_3^2 \).

Since in an equilateral triangle, the circumcentre coincides with the centroid, we have \( z_0 = \frac{z_1 + z_2 + z_3}{3} \)
\[\Rightarrow (z_1 + z_2 + z_3)^2 = (3z_0)^2\]
\[\Rightarrow \sum z_i^2 + 2\sum z_i z_j = 9z_0^2 \quad \therefore 3\sum z_i^2 = 9z_0^2.\]

**Illustration 24:** Prove that the triangle whose vertices are the points \( z_1, z_2, z_3 \) on the Argand plane is an equilateral triangle if and only if \( \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0 \). (JEE ADVANCED)

**Sol:** Consider \( \Delta ABC \) is the equilateral triangle with vertices \( z_1, z_2 \) and \( z_3 \) respectively.

Therefore \( |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \).

Let \( \Delta ABC \) be a triangle such that the vertices \( A, B \) and \( C \) are \( z_1, z_2 \) and \( z_3 \) respectively. Further, let \( \alpha = z_3 - z_1, \beta = z_2 - z_3 \) and \( \gamma = z_1 - z_2 \). Then \( \alpha + \beta + \gamma = 0 \) ... (i)

As shown in Fig 6.16, let \( \Delta ABC \) be an equilateral triangle. Then, \( BC = CA = AB \)
\[\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \quad \Rightarrow |\alpha| = |\beta| = |\gamma| \]
\[\Rightarrow |\alpha|^2 = |\beta|^2 = |\gamma|^2 = \lambda (say)\]
\[\Rightarrow \alpha \bar{\alpha} = \beta \bar{\beta} = \gamma \bar{\gamma} = \lambda.\]

Now, \( \alpha + \beta + \gamma = 0 \) [from (i)]
\[\Rightarrow \alpha + \beta + \gamma = 0 \quad \Rightarrow \frac{\lambda}{\alpha} + \frac{\lambda}{\beta} + \frac{\lambda}{\gamma} = 0 \quad \text{[Using (ii)]}\]
\[\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \Rightarrow \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0 \quad \text{which is the required condition.}\]

Conversely, let \( \Delta ABC \) be a triangle such that
\[\Rightarrow \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0 \quad \text{i.e.} \quad \Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0\]

Thus, we have to prove that the triangle is equilateral. We have, \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \)
\[\Rightarrow \frac{1}{\alpha} = -\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) \quad \Rightarrow \frac{\alpha}{\beta + \gamma} = \frac{1}{\alpha} \quad \Rightarrow \alpha^2 = \beta \gamma \quad \Rightarrow |\alpha|^2 = |\beta \gamma|\]
\[\Rightarrow |\alpha|^2 = |\beta||\gamma| \quad \Rightarrow |\alpha|^2 = |\beta||\gamma|\]

Similarly, \( \Rightarrow |\beta|^2 = |\alpha||\beta||\gamma| \quad \text{and} \quad |\gamma|^2 = |\alpha||\beta||\gamma|\]
\[\therefore |\alpha| = |\beta| = |\gamma|\]
\[\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \quad \Rightarrow BC = CA = AB\]

Hence, the given triangle is an equilateral triangle.
Illustration 25: Prove that the roots of the equation \( \frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3} = 0 \) (where \( z_1, z_2, z_3 \) are pair wise distinct complex numbers) correspond to points on a complex plane, which lie inside a triangle with vertices \( z_1, z_2, z_3 \) excluding its boundaries.  

\begin{align*}
\text{Sol:} & \quad \text{By using modulus and conjugate properties we can reduce given expression as } \\
& \quad \frac{z-z_1}{|z-z_1|^2} + \frac{z-z_2}{|z-z_2|^2} + \frac{z-z_3}{|z-z_3|^2} = 0. \text{ Therefore by putting } |z-z_1| = \frac{1}{t_1}, \text{ where } i = 1, 2, 3, \text{ we will get the result.} \\
& \quad t_1(z-z_1) + t_2(z-z_2) + t_3(z-z_3) = 0 \quad \text{where } |z-z_1| = \frac{1}{t_1} \text{ etc and } t_1, t_2, t_3 \in \mathbb{R}^+ \\
& \quad t_1(z-z_1) + t_2(z-z_2) + t_3(z-z_3) = 0 \\
& (t_1+t_2+t_3)z = t_1z_1 + t_2z_2 + t_3z_3 \Rightarrow z = \frac{t_1z_1 + t_2z_2 + t_3z_3}{t_1+t_2+t_3} \\
& \Rightarrow z = \frac{(t_1 + t_2)z' + t_3z_3}{t_1+t_2+t_3} \Rightarrow z \text{ lies inside the } \Delta z_1z_2z_3 \\
& \text{If } t_1 = t_2 = t_3 \Rightarrow z \text{ is the centroid of the triangle.} \\
& \text{Also, it implies } |z-z_1| = |z-z_2| = |z-z_3| \Rightarrow z \text{ is the circumcentre.}
\end{align*}

Illustration 26: Let \( z_1 \) and \( z_2 \) be roots of the equation \( z^2 + pz + q = 0 \), where the coefficients \( p \) and \( q \) may be complex numbers. Let \( A \) and \( B \) represent \( z_1 \) and \( z_2 \) in the complex plane. If \( \angle AOB = \alpha \neq 0 \) and \( OA = OB \), where \( O \) is the origin, prove that \( p^2 = 4q\cos^2\frac{\alpha}{2} \).  

\begin{align*}
\text{Sol:} & \quad \overline{OB} = \overline{OAe}^{i\alpha}. \text{ Therefore by using formula of sum and product of roots of quadratic equation we can prove this problem.} \\
& \text{Since } z_1 \text{ and } z_2 \text{ are roots of the equation } z^2 + pz + q = 0 \\
& z_1 + z_2 = -p \quad \text{and } z_1 z_2 = q \quad \text{(1)}
\end{align*}

\begin{align*}
& \text{Since } OA = OB \text{. So } \overline{OB} \text{ is obtained by rotating } \overline{OA} \text{ in anticlockwise direction through angle } \alpha. \\
& \therefore \overline{OB} = \overline{OAe}^{i\alpha} \Rightarrow z_2 = z_1 e^{i\alpha} \Rightarrow z_2 z_1 = e^{i\alpha} \Rightarrow \frac{z_2}{z_1} = \cos \alpha + i\sin \alpha \\
& \Rightarrow \frac{z_2}{z_1} + 1 = 1 + \cos \alpha + i\sin \alpha \Rightarrow \frac{z_2 + z_1}{z_1} = 2\cos \frac{\alpha}{2} \left( \cos \frac{\alpha}{2} + i\sin \frac{\alpha}{2} \right) = 2\cos \frac{\alpha}{2} e^{i\frac{\alpha}{2}} \\
& \Rightarrow \frac{z_2 + z_1}{z_1} = 2\cos \frac{\alpha}{2} e^{i\frac{\alpha}{2}} \Rightarrow \left( \frac{z_2 + z_1}{z_1} \right)^2 = 4\cos^2 \frac{\alpha}{2} e^{i\alpha} \\
& \Rightarrow \left( \frac{z_2 + z_1}{z_1} \right)^2 = 4\cos^2 \frac{\alpha}{2} \frac{z_2}{z_1} \Rightarrow \left( z_2 + z_1 \right)^2 = 4z_1z_2 \cos \frac{\alpha}{2} \\
& \Rightarrow (-p)^2 = 4q\cos^2 \frac{\alpha}{2} \Rightarrow p^2 = 4q\cos^2 \frac{\alpha}{2}.
\end{align*}
Illustration 27: On the Argand plane $z_1, z_2$ and $z_3$ are respectively the vertices of an isosceles triangle $ABC$ with $AC = BC$ and equal angles are $\theta$. If $z_4$ is the incentre of the triangle then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec \theta)(z_4 - z_1)^2$ \hspace{1cm} (JEE ADVANCED)

Sol: Here by using angle rotation formula we can solve this problem. From Fig 6.21, we have

$$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad \text{... (i) (clockwise)}$$

and

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{-i\theta/2} \quad \text{... (ii) (anticlockwise)}$$

Multiplying (i) and (ii)

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{|(z_2 - z_1)||(z_3 - z_1)|}{|z_4 - z_1|^2} = \frac{AB|AC|}{(AI)^2} = \frac{2(AD)(AC)}{(AI)^2} = \frac{2(AD)^2}{(AI)^2} \frac{AC}{AD}$$

$$= 2\cos^2 \frac{\theta}{2} \sec \theta = (1 + \cos \theta) \sec \theta.$$

7. REPRESENTATION OF DIFFERENT LOCI ON COMPLEX PLANE

(a) $|z - (1 + 2i)| = 3$ denotes a circle with centre $(1, 2)$ and radius 3 (see Fig 6.22).

(b) $|z - 1| = |z - i|$ denotes the equation of the perpendicular bisector of join of $(1, 0)$ and $(0, 1)$ on the Argand plane (see Fig 6.24).

(c) $|z - 4i| + |z + 4i| = 10$ denotes an ellipse with foci at $(0, 4)$ and $(0, -4)$; major axis 10; minor axis 6 with $e = \frac{4}{5}$ (see Fig 6.24).
\[ e^2 = 1 - \frac{36}{100} = \frac{64}{100} \Rightarrow e = \frac{4}{5} \left[ \frac{x^2}{9} + \frac{y^2}{25} = 1 \right] \]

(d) \( |z - 1| + |z + 1| = 1 \) denotes no locus. (Triangle inequality).

(e) \( |z - 1| < 1 \) denotes area inside a circle with centre \((1, 0)\) and radius 1.

(f) \( 2 \leq |z - 1| < 5 \) denotes the region between the concentric circles of radii 5 and 2. Centred at \((1, 0)\) including the inner boundary (see Fig 6.25).

\[ \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 64 \end{bmatrix} \]

Figure 6.25: Circle disc on a complex plane

(g) \( 0 \leq \arg z \leq \frac{\pi}{4} \) \((z \neq 0)\) where \(z\) is defined by positive real axis and the part of the line \(x = y\) in the first quadrant. It includes the boundary but not the origin. Refer to Fig 6.26.

\[ \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \]

Figure 6.26

(h) \( \text{Re}(z^2) > 0 \) denotes the area between the lines \(x = y\) and \(x = -y\) which includes the \(x\)-axis.

Hint: \(x^2 - y^2 + 2xyi = 0 \Rightarrow x^2 - y^2 > 0 \Rightarrow (x - y)(x + y) > 0.\)

Illustration 28: Solve for \(z\), if \(z^2 + |z| = 0\) .

(JEE MAIN)

Sol: Consider \( z = x + iy \) and solve this using algebra of complex number.

Let \( z = x + iy \Rightarrow (x + iy)^2 + \sqrt{x^2 + y^2} = 0 \Rightarrow \left( x^2 - y^2 + \sqrt{x^2 + y^2} \right) + (2ixy) = 0 \)

\( \Rightarrow \) Either \( x = 0 \) or \( y = 0 \);

- \( x = 0 \Rightarrow -y^2 + |y| = 0 \Rightarrow y = 0, 1, -1 \Rightarrow z = 0, i, -i \)
- \( y = 0 \Rightarrow x^2 + |x| = 0 \Rightarrow x = 0 \Rightarrow z = 0 \)

Therefore, \( z = 0, z = i, z = -i \).

Illustration 29: If the complex number \(z\) is to satisfy \(|z| = 3, |z - (a(1 + i) - i)| \leq 3 \) and \(|z + 2a - (a + 1)i| > 3\) simultaneously for at least one \(z\) then find all \(a \in \mathbb{R}\).

(JEE ADVANCED)

Sol: Consider \( z = x + iy \) and solve these inequalities to get the result.
All \( z \) at a time lie on a circle \( |z| = 3 \) but inside and outside the circles \( |z - (a (1 + i) - i)| = 3 \) and \( |z + 2a - (a + 1) i| = 3 \), respectively.

Let \( z = x + iy \) then equation of circles are
\[
(x - a)^2 + (y - a + 1)^2 = 9 \tag{ii}
\]
and \( (x + 2a)^2 + (y - a - 1)^2 = 9 \tag{iii} \)

Circles (i) and (ii) should cut or touch then distance between their centres \( \leq \) sum of their radii.

\[
\Rightarrow (a - 0)^2 + (a - 1 - 0)^2 \leq 3 + 3 \Rightarrow a^2 + (a - 1)^2 \leq 36
\]
\[
\Rightarrow 2a^2 - 2a - 35 \leq 0 \Rightarrow a^2 - a - \frac{35}{2} \leq 0
\]
\[
\Rightarrow \left(a - \frac{1}{2}\right)^2 \leq \frac{71}{4} \Rightarrow \frac{1 - \sqrt{71}}{2} \leq a \leq \frac{1 + \sqrt{71}}{2} \tag{iv}
\]

Again circles (i) and (iii) should not cut or touch then distance between their centres > sum of the radii

\[
\Rightarrow \sqrt{(-2a - 0)^2 + (a + 1 - 0)^2} > 3 + 3 \Rightarrow \sqrt{5a^2 + 2a + 1} > 6 \Rightarrow 5a^2 + 2a + 1 > 36
\]
\[
\Rightarrow 5a^2 + 2a - 35 > 0 \Rightarrow a^2 + \frac{2a}{5} - 7 > 0
\]

Then
\[
\left(a - \frac{1 - 4\sqrt{11}}{5}\right)\left(a - \frac{1 + 4\sqrt{11}}{5}\right) > 0
\]
\[
\Rightarrow a \in \left(-\infty, \frac{1 - 4\sqrt{11}}{5}\right) \cup \left(\frac{1 - 4\sqrt{11}}{5}, \infty\right) \tag{v}
\]

The common values of \( a \) satisfying (iv) and (v) are
\[
a \in \left]\frac{1 - \sqrt{71}}{2}, \frac{1 - 4\sqrt{11}}{5}\right[ \cup \left(\frac{1 + 4\sqrt{11}}{5}, \frac{1 + \sqrt{71}}{2}\right]
\]

\section*{8. DEMOIVRE'S THEOREM}

\textbf{Statement:} \((\cos n\theta + i\sin n\theta)\) is the value or one of the values of \((\cos \theta + i\sin \theta)^n\), \(\forall n \in \mathbb{Q}\). Value if \(n\) is an integer.

One of the values if \(n\) is rational which is not integer, the theorem is very useful in determining the roots of any complex quantity.

\textbf{Note:} We use the theory of equations to find the continued product of the roots of a complex number.
8.1 Application

Cube root of unity

(a) The cube roots of unity are \(1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\)

[Note that \(1 - i\sqrt{3} = -2\) and \(1 + i\sqrt{3} = -2\omega^2\)]

(b) If \(\omega\) is one of the imaginary cube roots of unity then \(1 + \omega + \omega^2 = 0\).

In general \(1 + \omega^r + \omega^{2r} = 0\); where \(r = 1\), and not a multiple of 3.

(c) In polar form the cube roots of unity are: \(\cos 0 + i\sin 0; \cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}; \cos \frac{4\pi}{3} + i\sin \frac{4\pi}{3}\)

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

[Note that the 3 cube roots of i lies on the vertices of an isosceles triangle]

(e) The following factorization should be remembered.

For \(a, b, c \in \mathbb{R}\) and \(\omega\) being the cube root of unity,

(i) \(a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)\)

(ii) \(x^2 + x + 1 = (x - \omega)(x - \omega^2)\)

(iii) \(a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)\)

(iv) \(a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)\)

\(n^{th}\) roots of unity: If \(1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}\) are the \(n^{th}\) roots of unity then

(i) They are in G.P. with common ratio \(e^{\frac{2\pi}{n}} = \cos \frac{2\pi}{n} + i\sin \frac{2\pi}{n}\)

(ii) \(1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0\) if \(p\) is not an integral multiple of \(n\)

\(1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p = n\) if \(p\) is an integral multiple of \(n\).

(iii) \(1^{n} - \alpha_1^n (1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n\).

Steps to determine \(n^{th}\) roots of a complex number

(i) Represent the complex number whose roots are to be determined in polar form.

(ii) Add \(2m\pi\) to the argument.

(iii) Apply De Moivre's Theorem

(iv) Put \(m = 0, 1, 2, 3, \ldots, (n - 1)\) to get all the \(n^{th}\) roots.

Explanation: Let \(z = \left(\cos 0 + i\sin 0\right)^\frac{1}{n} = \left(\cos 2m\pi + i\sin 2m\pi\right)^\frac{1}{n} = \left(\cos \frac{2m\pi}{n} + i\sin \frac{2m\pi}{n}\right)\)

Put \(m = 0, 1, 2, 3, \ldots, (n - 1)\), we get

\(1, \cos \frac{2\pi}{n} + i\sin \frac{2\pi}{n}, \cos \frac{4\pi}{n} + i\sin \frac{4\pi}{n}, \ldots, \cos \frac{2(n-1)\pi}{n} + i\sin \frac{2(n-1)\pi}{n}\) (\(n, n^{th}\) roots in G.P.)
Now, \( S = 1^p + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots + \alpha^{(n-1)p} = \frac{1-(\alpha^p)^n}{1-\alpha^p} = \frac{1-(\alpha^n)^p}{1-\alpha^p} \)

\[
\begin{align*}
\frac{1-(\alpha^n)^p}{1-\alpha^p} &= \begin{cases} 
0 & \text{non zero} \\
0 & \text{indeterminant, }
\end{cases} \quad \text{if } p \text{ is not an integral multiple of } n \\
&= \begin{cases} 
1 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd} \\
0, & \text{if } n = 3k + 1
\end{cases}
\end{align*}
\]

Again, if \( x \) is one of the \( n \)th root of unity then \( x^n - 1 = (x-1)(x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_{n-1}) \)

\[
1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1} = (x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_{n-1})
\]

Put \( x = 1 \), to get \((1-\alpha_1)(1-\alpha_2) \ldots (1-\alpha_{n-1}) = n \)

Similarly put \( x = -1 \), to get other result.

**MASTERJEE CONCEPTS**

Square roots of \( z = a + ib \) are \( \pm \left[ \sqrt{\frac{|z| + a}{2}} + \sqrt{\frac{|z| - a}{2}} \right] \) for \( b > 0 \).

If \( 1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \) are the \( n \)th roots of unity then

\[
(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = 0 \quad \text{if } n \text{ is even and 1 if } n \text{ is odd.}
\]

\[
1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \ldots \cdot \alpha_{n-1} = 1 \text{ or } -1 \text{ according as } n \text{ is odd or even.}
\]

\[
(\omega - \alpha_1)(\omega - \alpha_2) \ldots (\omega - \alpha_{n-1}) = \begin{cases} 
0, & \text{if } n = 3k \\
1, & \text{if } n = 3k + 1, \\
1 + \omega, & \text{if } n = 3k + 2
\end{cases}
\]

Ravi Vooda (JEE 2009, AIR 71)

**Illustration 30:** If \( x = a + b, \quad y = a\omega + b\omega^2 \) and \( z = a\omega^2 + b\omega \), then prove that \( x^3 + y^3 + z^3 = 3(a^3 + b^3) \) \hspace{1cm} (JEE MAIN)

**Sol:** Here \( x + y + z = 0 \). Take cube on both side.

\[
x + y + z = 0 \implies x^3 + y^3 + z^3 = 3xyz \implies \text{LHS} = 3xyz = 3(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) = 3(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) = 3\omega^3(a + b)(a + b\omega)(a + b\omega^2) = 3(a^3 + b^3)
\]

**Illustration 31:** The value of expression \( 1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2) \) \hspace{1cm} (JEE ADVANCED)

**Sol:** The given expression represent as \( x^3 - 1 = (x-1)(x-\omega)(x-\omega^2) \). Therefore by putting \( x = 2, 3, 4 \ldots n \), we will get the result.

\[
x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)
\]

Put \( x = 2 \quad 2^3 - 1 = 1 \cdot (2-\omega)(2-\omega^2) \quad \text{Put } x = 3 \quad 3^3 - 1 = 2 \cdot (3-\omega)(3-\omega^2) \):

Put \( x = n \quad n^3 - 1 = (n-1)(n-\omega)(n-\omega^2) \)

\[
\therefore \text{LHS} = (2^3 + 3^3 + \ldots + n^3) - (n-1) = (1^3 + 2^3 + 3^3 + \ldots + n^3) - n = \left( \frac{n(n+1)}{2} \right)^2 - n
\]
9. SUMMATION OF SERIES USING COMPLEX NUMBER

(a) \[ \cos \theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin \left(\frac{n\theta}{2}\right)}{\sin \left(\frac{\theta}{2}\right)} \cos \left(\frac{n+1}{2}\right)\theta \]

(b) \[ \sin \theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin \left(\frac{n\theta}{2}\right)}{\sin \left(\frac{\theta}{2}\right)} \sin \left(\frac{n+1}{2}\right)\theta \]

Note: If \( \theta = \frac{2\pi}{n} \), then the sum of the above series vanishes.

9.1 Complex Number and Binomial Coefficients

Try the following questions using the binomial expansion of \((1 + x)^n\) and substituting the value of \(x\) according to the binomial coefficients in the respective question.

Find the value of the following

(i) \( C_0 + C_4 + C_8 + \ldots \ldots \)  
(ii) \( C_1 + C_5 + C_9 + \ldots \ldots \)

(iii) \( C_2 + C_6 + C_{10} + \ldots \ldots \)  
(iv) \( C_3 + C_7 + C_{11} + \ldots \ldots \)

(v) \( C_0 + C_3 + C_6 + C_9 + \ldots \ldots \)

Hint (v): In the expansion of \((1 + x)^n\), put \(x = 1, \omega, \text{and} \omega^2\) and add the three equations.

Illustration 32: If \(1, \omega, \omega^2, \ldots, \omega^{n-1}\) are \(n^{th}\) roots of unity, then the value of \((5 - \omega)(5 - \omega^2) \ldots (5 - \omega^{n-1})\) is equal to \(JEE \text{ MAIN}\)

Sol: Here consider \(x = \frac{1}{(1)^n}\), therefore \(x^n - 1 = 0\) (has \(n\) roots i.e. \(1, \omega, \omega^2, \ldots, \omega^{n-1}\)).

\[ x^n - 1 = (x-1)(x-\omega)(x-\omega^2) \ldots (x-\omega^{n-1}) \Rightarrow \frac{x^n - 1}{x-1} = (x-\omega)(x-\omega^2) \ldots (x-\omega^{n-1}) \]

\[ \Rightarrow \text{ Putting } x = 5 \text{ in both sides, we get } \Rightarrow (5 - \omega)(5 - \omega^2) \ldots (5 - \omega^{n-1}) = \frac{5^n - 1}{4}. \]

10. APPLICATION IN GEOMETRY

10.1 Distance Formula

Distance between \(A(z_1)\) and \(B(z_2)\) is given by \(AB = |z_2 - z_1|\). Refer Fig 6.29.
10.2 Section Formula

The point $P(z)$ which divides the join of $A(z_1)$ and $B(z_2)$ in the ratio $m: n$ is given by $z = \frac{mz_2 + nz_1}{m + n}$. Refer Fig 6.30.

10.3 Midpoint Formula

Mid-point $M(z)$ of the segment $AB$ is given by $z = \frac{1}{2} (z_1 + z_2)$.

10.4 Condition For Four Non-Collinear Points

Condition(s) for four non-collinear $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ to represent vertices of a

(a) Parallelogram: The diagonals $AC$ and $BD$ must bisect each other

\[ \iff \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4) \]

\[ \iff z_1 + z_3 = z_2 + z_4 \]

(b) Rhombus:

(i) The diagonals $AC$ and $BD$ bisect each other

\[ \iff z_1 + z_3 = z_2 + z_4 , \text{ and} \]

(ii) A pair of two adjacent sides are equal, for instance $AD = AB$

\[ \iff |z_4 - z_1| = |z_2 - z_1| \]

(c) Square:

(i) The diagonals $AC$ and $BD$ bisect each other

\[ \iff z_1 + z_3 = z_2 + z_4 \]

(ii) A pair of adjacent sides are equal; for instance, $AD = AB$

\[ \iff |z_4 - z_1| = |z_2 - z_1| \]

(iii) The two diagonals are equal, that is $AC = BD$

\[ \iff |z_3 - z_1| = |z_4 - z_2| \]

(d) Rectangle:

(i) The diagonals $AC$ and $BD$ bisect each other

\[ \iff z_1 + z_3 = z_2 + z_4 \]

(ii) The diagonals $AC$ and $BD$ are equal

\[ \iff |z_3 - z_1| = |z_4 - z_2| \]
10.5 Triangle

In a triangle ABC, let the vertices A, B and C be represented by the complex numbers $z_1, z_2$, and $z_3$ respectively. Then

(a) **Centroid**: The centroid (G), is the point of intersection of medians of $\triangle ABC$. It is given by the formula

$$z = \frac{1}{3} (z_1 + z_2 + z_3)$$

![Figure 6.36 (a)]

(b) **Incentre**: The incentre (I) of $\triangle ABC$ is the point of intersection of internal angular bisectors of angles of $\triangle ABC$. It is given by the formula

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c},$$

![Figure 6.36 (b)]

(c) **Circumcentre**: The circumcentre (S) of $\triangle ABC$ is the point of intersection of perpendicular bisectors of sides of $\triangle ABC$. It is given by the formula

$$z = \frac{|z_1|^2 (z_2 - z_3) + |z_2|^2 (z_3 - z_1) + |z_3|^2 (z_1 - z_2)}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)} = \begin{vmatrix} \frac{|z_1|^2}{\bar{z}_1} & z_1 & 1 \\ \frac{|z_2|^2}{\bar{z}_2} & z_2 & 1 \\ \frac{|z_3|^2}{\bar{z}_3} & z_3 & 1 \end{vmatrix}$$

Also, $z = \frac{z_1 (\sin 2A) + z_2 (\sin 2B) + z_3 (\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$

![Figure 6.36 (c)]

(d) **Euler’s Line**: The orthocenter H, the centroid G and the circumcentre S of a triangle which is not equilateral lies on a straight line. In case of an equilateral triangle these points coincide.

G divides the join of H and S in the ratio 2 : 1 (see Fig 6.37).

Thus, $z_G = \frac{1}{3}(z_H + 2z_S)$

![Figure 6.37]
10.6 Area of a Triangle

Area of $\Delta ABC$ with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left| \frac{1}{4!} \begin{vmatrix} 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \\ 1 & z_3 & \bar{z}_3 \end{vmatrix} \right| = \left| \frac{1}{2} \text{Im}(z_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1) \right|$$

10.7 Conditions for Triangle to be Equilateral

The triangle $ABC$ with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral iff

$$\frac{1}{z_2-z_3} + \frac{1}{z_3-z_1} + \frac{1}{z_1-z_2} = 0$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2 \quad \iff \quad z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1 \quad \iff \quad z_1^2 = z_2z_3 \quad \text{and} \quad z_2^2 = z_1z_3$$

$$\Rightarrow \frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3} = 0 \quad \Rightarrow \quad \frac{z_2-z_1}{z_3-z_1} = \frac{z_3-z_2}{z_1-z_2}$$

$$\Rightarrow \frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3} = 0 \quad \text{where} \quad z = \frac{1}{3}(z_1 + z_2 + z_3).$$

10.8 Equation of a Straight line

(a) **Non-parametric form:** An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\text{Arg} \left( \frac{z - z_1}{z_2 - z_1} \right) = 0 \quad \Rightarrow \quad \left| \begin{vmatrix} 1 & z & \bar{z} \\ 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \end{vmatrix} \right| = 0$$

or $$\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

or $$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

(b) **Parametric form:** An equation of the line segment between the points $A(z_1)$ and $B(z_2)$ is $z = tz_1 + (1-t)z_2$, $t(0,1)$ where $t$ is a real parameter.

(c) **General equation of a straight line:** The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$ where, $a$ is non-zero complex number and $b$ is a real number.

10.9 Complex Slope of a Line

If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of $AB$ is defined to be $\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

Two lines with complex slopes $\mu_1$ and $\mu_2$ are

(i) Parallel, if $\mu_1 = \mu_2$ (ii) Perpendicular, if $\mu_1 + \mu_2 = 0$

The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\left( \frac{-a}{\bar{a}} \right)$. 
10.10 Length of Perpendicular from a Point to a Line

Length of perpendicular of point $A(\omega)$ from the line $\overline{a}z + a\overline{z} + b = 0$.

Where $a \in C \setminus \{0\}$, and $b \in R$ is given by $p = \frac{|\overline{a}\omega + a\overline{\omega} + b|}{|a|}$

10.11 Equation of Circle

(a) An equation of the circle with centre $z_0$ and radius $r$ is $|z - z_0| = r$ or $z = z_0 + re^{i\theta}, 0 \leq \theta < 2\pi$ (parametric form) or $zz - z_0z - \overline{z}_0z + z_0\overline{z}_0 - r^2 = 0$

(b) General equation of a circle is $zz + a\overline{z} + \overline{a}z + b = 0 \quad \cdots (i)$

Where $a$ is a complex number and $b$ is a real number such that $a\overline{a} - b \geq 0$. Centre of (i) is $-a$ and its radius is $\sqrt{a\overline{a} - b}$

(c) Diameter form of a circle: An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is $(z - z_1)(\overline{z} - \overline{z}_2) + (\overline{z} - \overline{z}_1)(z - z_2) = 0$

(d) An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$

is $(z - z_1)(\overline{z} - \overline{z}_2) + (\overline{z} - \overline{z}_1)(z - z_2) + ik\begin{vmatrix} z & z_1 & 1 \\ z_2 & \overline{z}_2 & 1 \end{vmatrix} = 0$ where $k$ is a real parameter.

(e) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1), B(z_2)$ and $C(z_2)$ and $P(z)$ be any point on the circle through $A, B$ and $C$.

Then either $\angle ACB = \angle APB$ [when angles are in the same segment]

or $\angle ACB + \angle APB = \pi$ [when angles are in the opposite segment] (see Fig. 6.44).

\[
\Rightarrow \arg \left( \frac{z_3 - z_2}{z_3 - z_1} \right) - \arg \left( \frac{z - z_2}{z - z_1} \right) = 0 \quad \text{or} \quad \arg \left( \frac{z_3 - z_2}{z_3 - z_1} \right) + \arg \left( \frac{z - z_1}{z - z_2} \right) = \pi
\]

\[
\Rightarrow \arg \left[ \frac{z_3 - z_2}{z_3 - z_1} \cdot \frac{z - z_1}{z - z_2} \right] = 0
\]

or $\arg \left[ \frac{z_3 - z_2}{z_3 - z_1} \cdot \frac{z - z_1}{z - z_2} \right] = \pi$

In any case, we get $\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

\[
\Leftrightarrow \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}
\]

(f) Condition for four points to be concyclic.

Four points $z_1, z_2, z_3$ and $z_4$ will lie on the same circle if and only if $\frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

\[
\Leftrightarrow \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}
\]
Three points \( z_1, z_2 \) and \( z_3 \) are collinear if 
\[
\begin{vmatrix}
  z_1 & 1 \\
  z_2 & 1 \\
  z_3 & 1 \\
\end{vmatrix} = 0.
\]

If three points \( A(z_1), B(z_2) \) and \( C(z_3) \) are collinear then slope of \( AB = \) slope of \( BC = \) slope of \( AC \)
\[
\Rightarrow \frac{z_1 - z_2}{z_1 - z_2} = \frac{z_2 - z_3}{z_2 - z_3} = \frac{z_1 - z_3}{z_1 - z_3}
\]

**Illustration 33:** If the imaginary part of \( \frac{2z+1}{iz+1} \) is \(-4\), then the locus of the point representing \( z \) in the complex plane is
(a) A straight line  (b) A parabola  (c) A circle  (d) An ellipse (JEE MAIN)

**Sol:** Put \( z = x + iy \) and then equate its imaginary part to \(-4\).

Let \( z = x + iy \), then
\[
\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+2iy}{(1-y)+ix} = \frac{[(2x+1)+2iy][(1-y)-ix]}{(1-y)^2+x^2}
\]

As \( \text{Im} \left( \frac{2z+1}{iz+1} \right) = -4 \), we get
\[
\Rightarrow 2y(1-y) - x(2x+1) = -4
\]
\[
\Rightarrow 2x^2 + 2y^2 + x - 2y = 4x^2 + 4(y^2 - 2y + 1) \Rightarrow 2x^2 + 2y^2 - x - 6y + 4 = 0
\]
It represents a circle.

**Illustration 34:** The roots of \( z^5 = (z-1)^5 \) are represented in the argand plane by the points that are
(a) Collinear  (b) Concyclic  (c) Vertices of a parallelogram  (d) None of these (JEE MAIN)

**Sol:** Apply modulus on both the side of given expression.

Let \( z \) be a complex number satisfying \( z^5 = (z-1)^5 \).
\[
\Rightarrow |z^5| = |(z-1)^5| \Rightarrow |z|^5 = |z-1|^5 \Rightarrow |z| = |z-1|
\]
Thus, \( z \) lies on the perpendicular bisector of the segment joining the origin and \((1 + i0)\) i.e. \( z \) lies on \( \text{Re}(z) = \frac{1}{2} \).

**Illustration 35:** Let \( z_1 \) and \( z_2 \) be two non-zero complex numbers such that \( \frac{z_1 + z_2}{z_1} = 1 \), then the origin and points represented by \( z_1 \) and \( z_2 \)
(a) Lie on straight line  (b) Form a right triangle  (c) Form an equilateral triangle  (d) None of these (JEE ADVANCED)

**Sol:** Here consider \( z = \frac{z_1}{z_2} \) and \( z_1 \) and \( z_2 \) are represented by \( A \) and \( B \) respectively and \( O \) be the origin.

Let \( z = \frac{z_1}{z_2} \), then
\[
z + \frac{1}{z} = 1 \Rightarrow z^2 - z + 1 = 0
\]
\[
\Rightarrow z = \frac{1 \pm \sqrt{3}i}{2} \Rightarrow z_1 = \frac{1 \pm \sqrt{3}i}{2}
\]
If $z_1$ and $z_2$ are represented by $A$ and $B$ respectively and $O$ be the origin, then

$$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \frac{1\pm\sqrt{3}i}{2} = \frac{1 + \frac{3}{4}}{1 - \frac{3}{4}} = 1 \implies OA = OB$$

Also, \[
\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left| 1 - \frac{z_1}{z_2} \right| = \left| 1 - \left( \frac{1 \pm \frac{3}{2}}{2} \right) \right| = \left| \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right| = \frac{1 + \frac{3}{4}}{4} = 1
\]

$\implies$ $AB = OB$ \hspace{0.5cm} Thus, $OA = OB = AB \therefore \triangle OAB$ is an equilateral triangle.

**Illustration 36:** If $z_1, z_2, z_3$ are the vertices of an isosceles triangle, right angled at the vertex $z_2$, then the value of $(z_1 - z_2)^2 + (z_2 - z_3)^2$ is

(a) -1 \hspace{0.5cm} (b) 0 \hspace{0.5cm} (c) $(z_1 - z_3)^2$ \hspace{0.5cm} (d) None of these \hspace{0.5cm} (JEE ADVANCED)

**Sol:** Here use distance and argument formula of complex number to solve this problem.

As $ABC$ is an isosceles right angled triangle with right angle at $B$,

$$BA = BC \text{ and } \angle ABC = 90^\circ \implies |z_1 - z_2| = |z_3 - z_2| \text{ and } \arg \left( \frac{z_3 - z_2}{z_1 - z_2} \right) = \frac{\pi}{2}
$$

$\implies$ \[
\frac{z_3 - z_2}{z_1 - z_2} = \left| \frac{z_3 - z_2}{z_1 - z_2} \right| \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) = i
\]

$\implies (z_3 - z_2)^2 = -(z_1 - z_2)^2 \quad \implies (z_1 - z_2)^2 + (z_2 - z_3)^2 = 0.$

**11. CONCEPTS OF ROTATION OF COMPLEX NUMBER**

Let $z$ be a non-zero complex number. We can write $z$ in the polar form as follows:

$$z = r (\cos \theta + i \sin \theta) = re^{i\theta} \text{ where } r = |z| \text{ and } \arg(z) = \theta \text{ (see Fig 6.46).}$$

Consider a complex number $ze^{i\alpha}$.

$$ze^{i\alpha} = (re^{i\theta})e^{i\alpha} = re^{i(\theta + \alpha)}$$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is $r$ and argument is $\theta + \alpha$.

Geometrically, $ze^{i\alpha}$ can be obtained by rotating the line segment joining $O$ and $P(z)$ through an angle $\alpha$ in the anticlockwise direction.

**Corollary:** If $A(z_1)$ and $B(z_2)$ are two complex number such that

$$\angle AOB = \theta, \text{ then } z_2 = \frac{z_2}{z_1} z_1 e^{i\theta} \text{ (see Fig 6.47).}$$

Let $z_1 = r_1 e^{i\alpha}$ and $z_2 = r_2 e^{i\beta}$ where $|z_1| = r_1, |z_2| = r_2$.

Then $\frac{z_2}{z_1} = \frac{r_2 e^{i\beta}}{r_1 e^{i\alpha}} = \frac{r_2}{r_1} e^{i(\beta - \alpha)}$

Thus, \[
\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i\beta} \quad (\because \beta - \alpha = \theta) \quad \implies z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}
\]
Illustration 37: Suppose $A(z_1)$, $B(z_2)$ and $C(z_3)$ are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If \( z_1 = 1 + \sqrt{3}i \), then \( z_2 \) and \( z_3 \) are respectively.

(a) $-2, 1 - \sqrt{3}i$  
(b) $-1 + \sqrt{3}i, -2$  
(c) $-2, -1 + \sqrt{3}i$  
(d) $-2, 2 + \sqrt{3}i$  

(JEE ADVANCED)

Sol: As we know \( x + iy = re^{i\theta} \). Hence by using this formula we can obtain \( z_2 \) and \( z_3 \).

\[
z_1 = 1 + \sqrt{3}i = 2e^{i\pi/3}
\]

Since, \( \angle AOC = \frac{2\pi}{3} \) and \( \angle BOC = \frac{2\pi}{3} \), \( z_2 = z_1 e^{\frac{2\pi i}{3}} \) and \( z_3 = z_2 e^{\frac{5\pi i}{3}} \)

\[
\Rightarrow z_3 = 2e^{\pi i} = 2(\cos \pi + i\sin \pi) = -2 \quad \text{and} \quad z_3 = 2e^{\pi i/3}
\]

\[
= 2\left[ \cos \left(2\pi - \frac{\pi}{3}\right) + i\sin \left(2\pi - \frac{\pi}{3}\right) \right]
\]

\[
= 2\left[ \cos \frac{\pi}{3} - i\sin \frac{\pi}{3} \right] = 2\left[ \frac{1}{2} - \frac{\sqrt{3}}{2}i \right] = 1 - \sqrt{3}i.
\]

PROBLEM-SOLVING TACTICS

(a) On a complex plane, a complex number represents a point.
(b) In case of division and modulus of a complex number, the conjugates are very useful.
(c) For questions related to locus and for equations, use the algebraic form of the complex number.
(d) Polar form of a complex number is particularly useful in multiplication and division of complex numbers. It directly gives the modulus and the argument of the complex number.
(e) Translate unfamiliar statements by changing \( z \) into \( x+iy \).
(f) Multiplying by \( \cos \theta \) corresponds to rotation by angle \( \theta \) about \( O \) in the positive sense.
(g) To put the complex number \( \frac{a + ib}{c + id} \) in the form \( A + iB \) we should multiply the numerator and the denominator by the conjugate of the denominator.

(h) Care should be taken while calculating the argument of a complex number. If \( z = a + ib \), then \( \arg(z) \) is not always equal to \( \tan^{-1}\left(\frac{b}{a}\right) \). To find the argument of a complex number, first determine the quadrant in which it lies, and then proceed to find the angle it makes with the positive x-axis.

For example, if \( z = -1 - i \), the formula \( \tan^{-1}\left(\frac{b}{a}\right) \) gives the argument as \( \frac{\pi}{4} \), while the actual argument is \( -\frac{3\pi}{4} \).

**FORMULAE SHEET**

(a) Complex number \( z = x + iy \), where \( x, y \in \mathbb{R} \) and \( i = \sqrt{-1} \).

(b) If \( z = x + iy \) then its conjugate \( \overline{z} = x - iy \).

(c) Modulus of \( z \), i.e. \( |z| = \sqrt{x^2 + y^2} \)

(d) Argument of \( z \), i.e. \( \theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0, \ y > 0 \\ \pi - \tan^{-1}\left(\frac{y}{x}\right) & x < 0, \ y > 0 \\ -\pi + \tan^{-1}\left(\frac{y}{x}\right) & x < 0, \ y < 0 \\ -\tan^{-1}\left(\frac{y}{x}\right) & x > 0, \ y < 0 \end{cases} \)

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]

\[ x \]

\[ y \]

\[ \theta = \pi - \phi \]

\[ \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ \theta = \pi + \phi \]

\[ \theta = -\pi + \phi \]

\[ \theta = 0 \]

\[ \theta = -\phi \]

\[ \theta = \pi \]

\[ \theta = -\pi \]
(i) \[ \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

(j) Important properties of conjugate

(i) \( z + \bar{z} = 2 \text{Re}(z) \) and \( z - \bar{z} = 2 \text{Im}(z) \)
(ii) \( z = \bar{z} \iff z \) is purely real
(iii) \( z + \bar{z} = 0 \iff z \) is purely imaginary
(iv) \( z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2 \)
(v) \( \bar{z}_1 + z_2 = \bar{z}_1 + z_2 \)
(vi) \( \bar{z}_1 - z_2 = \bar{z}_1 - z_2 \)
(vii) \( z_1\bar{z}_2 = \bar{z}_1 z_2 \)
(viii) \( \frac{z_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2} \) if \( z_2 \neq 0 \)

(k) Important properties of modulus

If \( z \) is a complex number, then

(i) \( |z| = 0 \iff z = 0 \)
(ii) \( |z| = |\bar{z}| = |-z| = |-\bar{z}| \)
(iii) \( -|z| \leq \text{Re}(z) \leq |z| \)
(iv) \( -|z| \leq \text{Im}(z) \leq |z| \)
(v) \( z\bar{z} = |z|^2 \)

If \( z_1, z_2 \) are two complex numbers, then

(i) \( |z_1 z_2| = |z_1||z_2| \)
(ii) \( \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \), if \( z_2 \neq 0 \)
(iii) \( |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2) \)
(iv) \( |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1 \bar{z}_2) \)

(l) Important properties of argument

(i) \( \arg(z) = -\arg(z) \)
(ii) \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \)

In fact \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \)

where \( k = \begin{cases} 
0, & \text{if } -\pi < \arg(z_1) + \arg(z_2) \leq \pi \\
1, & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \leq -\pi \\
-1, & \text{if } \pi < \arg(z_1) + \arg(z_2) \leq 2\pi 
\end{cases} \)

(iii) \( \arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2) \)
(iv) \( \arg\left( \frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2) \)
(v) \[ |z_1 + z_2| = |z_1 - z_2| \quad \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \]

(vi) \[ |z_1 + z_2| = |z_1| + |z_2| \quad \Leftrightarrow \arg(z_1) = \arg(z_2) \]

If \( z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \), then

(vii) \[ |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2 |z_1| |z_2| \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2 r_1 r_2 \cos(\theta_1 - \theta_2) \]

(viii) \[ |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 |z_1| |z_2| \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2) \]

(m) Triangle on complex plane

(i) Centroid (G), \( z_G = \frac{z_1 + z_2 + z_3}{3} \)

(ii) Incentre (I), \( z_I = \frac{az_1 + bz_2 + cz_3}{a + b + c} \)

(iii) Orthocentre (H), \( z_H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\sum \tan A} \)

(iv) Circumcentre (S), \( z_S = \frac{z_1 (\sin 2A) + z_2 (\sin 2B) + z_3 (\sin 2C)}{\sin 2A + \sin 2B + \sin 2C} \)

(n) \( (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \)

(o) \( \sqrt{z} = \sqrt{x + iy} = \pm \left[ \sqrt{\frac{|z| + x}{2}} + i \sqrt{\frac{|z| - x}{2}} \right] \) for \( y > 0 \)

(p) Distance between \( A(z_1) \) and \( B(z_2) \) is given by \( |z_2 - z_1| \)

(q) Section formula: The point \( P(z) \) which divides the join of the segment AB in the ratio \( m:n \)

is given by \( z = \frac{mz_2 + nz_1}{m + n} \).

(r) Midpoint formula: \( z = \frac{1}{2} (z_1 + z_2) \).

(s) Equation of a straight line

(i) Non-parametric form: \( z (\overline{z}_1 - \overline{z}_2) - \overline{z} (z_1 - z_2) + z_1 \overline{z}_2 - \overline{z}_1 z_2 = 0 \)

(ii) Parametric form: \( z = tz_1 + (1-t)z_2 \)

(iii) General equation of straight line: \( az + a\overline{z} + b = 0 \)

(t) Complex slope of a line, \( \mu = \frac{z_1 - z_2}{\overline{z}_1 - \overline{z}_2} \). Two lines with complex slopes \( \mu_1 \) and \( \mu_2 \) are

(i) Parallel, if \( \mu_1 = \mu_2 \)

(ii) Perpendicular, if \( \mu_1 + \mu_2 = 0 \)

(u) Equation of a circle: \( |z - z_0| = r \)