6. COMPLEX NUMBER

1. INTRODUCTION

The number system can be briefly summarized as $N \subset W \subset I \subset Q \subset R \subset C$, where N, W, I, Q, R and C are the standard notations for the various subsets of the numbers belong to it.

N - Natural numbers = {1, 2, 3 …. n}

W - Whole numbers = {0, 1, 2, 3 …. n}

I - Integers = {….2, -1, 0, 1, 2 …..}

Q – Rational numbers = $\left\{\frac{1}{2}, \frac{3}{5} .....\right\}$

IR – Irrational numbers = $\left\{\sqrt{2}, \sqrt{3}, \pi\right\}$

C – Complex numbers

A complex number is generally represented by the letter "z". Every complex number z, can be written as, $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$.

x is called the real part of complex number, and

y is the imaginary part of complex number.

Note that the sign + does not indicate addition as normally understood, nor does the symbol "i" denote a number. These are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers (x, y) to form a single complex number.

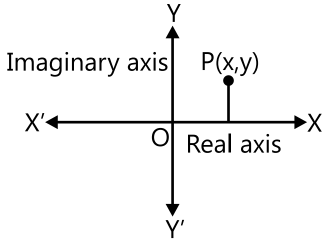


Figure 6.1: Representation of a complex number on a plane

Swiss-born mathematician Jean Robert Argand, after a systematic study on complex numbers, represented every complex number as a set of ordered pair (x, y) on a plane called complex plane.

All complex numbers lying on the real axis were called purely real and those lying on imaginary axis as purely imaginary.

Hence, the complex number $0 + 0i$ is purely real as well as purely imaginary but it is not imaginary.

Note

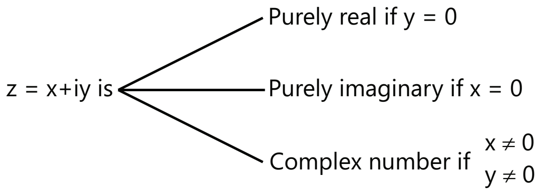


Figure 6.2: Classification of a complex number

(a) The symbol i combines itself with real number as per the rule of algebra together with

$i^2 = -1$; $i^3 = -i$; $i^4 = 1$; $i^{2014} = -1$; $i^{2015} = -i$ and so on.

In general, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $n \in I$ and $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

Hence, $1 + i^1 + i^2 + \ldots\ldots + i^{2014} + i^{2015} = 0$

(b) The imaginary part of every real number can be treated as zero. Hence, there is one-one mapping between the set of complex numbers and the set of points on the complex plane.

MASTERJEE CONCEPTS

Complex number as an ordered pair: A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a, b). For a complex number to be uniquely specified, we need two real numbers in a particular order.

Vaibhav Gupta (JEE 2009, AIR 54)

2. ALGEBRA OF COMPLEX NUMBERS

(a) Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$

(b) Subtraction: $(a+ib)-(c+id) = (a-c)+i(b-d)$

(c) Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

(d) Reciprocal: If at least one of a, b is non-zero, then the reciprocal of $a+ib$ is given by

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

(e) Quotient: If at least one of c, d is non-zero, then quotient of $a+ib$ and $c+id$ is given by

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

(f) Inequality in complex numbers is not discussed/defined. If a + ib > c + id is meaningful only if b = d = 0. However, equalities in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. (Geometrically, the position of complex number z_1 on complex plane)

(g) In real number system if $p^2 + q^2 = 0$ implies, $p = 0 = q$. But if z_1 and z_2 are complex numbers then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$. For e.g. $z_1 = i$ and $z_2 = 1$.

However if the product of two complex numbers is zero then at least one of them must be zero, same as in case of real numbers.

(h) In case x is real, then $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$ but in case of complex number z, $|z|$ means the distance of the point z from the origin.

MASTERJEE CONCEPTS

- The additive inverse of a complex number $z = a + ib$ is $-z$ (i.e. $-a - ib$).
- For every non-zero complex number z, the multiplicative inverse of z is $\frac{1}{z}$.
- $|z| \geq |\mathrm{Re}(z)| \geq \mathrm{Re}(z)$ and $|z| \geq |\mathrm{Im}(z)| \geq \mathrm{Im}(z)$.
- $\frac{z}{|z|}$ is always a uni-modular complex number if $z \neq 0$.

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 1: Find the square root of 5 + 12i. **(JEE MAIN)**

Sol: $z = 5 + 12i$

Let the square root of the given complex number be a + ib. Use algebra to simplify and get the value of a and b.

Let its square root $= a + ib \quad \Rightarrow 5 + 12i = a^2 - b^2 + 2abi$

$\Rightarrow a^2 - b^2 = 5$ ... (i)

$\Rightarrow 2ab = 12$ ... (ii)

$\Rightarrow (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 \quad \Rightarrow (a^2 + b^2)^2 = 25 + 144 = 169 \quad \Rightarrow a^2 + b^2 = 13$ ... (iii)

(i) + (iii) $\Rightarrow 2a^2 = 18 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

If $a = 3 \Rightarrow b = 2$ If $a = -3 \Rightarrow b = -2$

$\therefore$ Square root $= 3 + 2i, -3 - 2i$ $\therefore$ Combined form $\pm(3 + 2i)$

Illustration 2: If $z = (x, y) \in C$. Find z satisfying $z^2 \times (1 + i) = (-7 + 17i)$. **(JEE MAIN)**

Sol: Algebra of Complex Numbers.

$(x + iy)^2 (1 + i) = -7 + 17i$

$\Rightarrow (x^2 - y^2 + 2xyi)(1 + i) = -7 + 17i$; $x^2 - y^2 + i(x^2 - y^2) + 2xyi - 2xy = -7 + 17i$

$\Rightarrow (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy) = -7 + 17i \Rightarrow x = 3, y = 2 \quad \Rightarrow x = -3, y = -2$

$\Rightarrow z = -3 + i(-2) = -3 - 2i$

Illustration 3: If $x^2 + 2(1 + 2i)x - (11 + 2i) = 0$. Solve the equation. **(JEE ADVANCED)**

Sol: Use the quadratic formula to find the value of x.

$$\therefore x = \frac{-2(1 + 2i) \pm \sqrt{4 - 16 + 16i + 44 + 8i}}{2}$$

$\Rightarrow 2x = (-2)(1 + 2i) \pm \sqrt{32 + 24i}$

$\Rightarrow x = (-1)(1 + 2i) \pm \sqrt{8 + 6i} = -1 - 2i \pm (3 + i)$; $x = 2 - i, -4 - 3i$

Illustration 4: If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$. Find $f(3+2i)$. **(JEE ADVANCED)**

Sol: Let x = 3+2i, and square it to form a quadratic equation. Then try to represent f(x) in terms of this quadratic.

$x = 3 + 2i$

$\Rightarrow (x-3)^2 = -4 \qquad \Rightarrow x^2 - 6x + 13 = 0$

$x^4 - 4x^3 + 4x^2 + 8x + 44 = x^2\,(x^2 - 6x + 13) + 2x^3 - 9x^2 + 8x + 44$

$\Rightarrow f(x) = x^2\,(x^2 - 6x + 13) + 2(x^3 - 6x^2 + 13x) + 3(x^2 - 6x + 13) + 5 \qquad \Rightarrow f(x) = 5$

3. IMPORTANT TERMS ASSOCIATED WITH COMPLEX NUMBER

Three important terms associated with complex number are conjugate, modulus and argument.

(a) Conjugate: If z = x + iy then its complex conjugate is obtained by changing the sign of its imaginary part and denoted by $\bar{z}$ i.e. $\bar{z}$ = x – iy (see Fig 6.3).

The conjugate satisfies following basic properties

(i) $z + \bar{z}$ = 2Re (z)

(ii) $z - \bar{z}$ = 2i Im (z)

(iii) $z\bar{z} = x^2 + y^2$

(iv) If z lies in 1st quadrant then $\bar{z}$ lies in 4th quadrant and $-\bar{z}$ in the 2nd quadrant.

(v) If x + iy = f (a + ib) then x – iy = f (a – ib)

For e.g. If $(2 + 3i)^3$ = x + i y then $(2 - 3i)^3$ = x – iy

and, $\sin(\alpha + i\beta)$ = x + iy $\Rightarrow \sin(\alpha - i\beta)$ = x – iy

(vi) $z + \bar{z} = 0 \qquad \Rightarrow z$ is purely imaginary

(vii) $z - \bar{z} = 0 \qquad \Rightarrow z$ is purely real

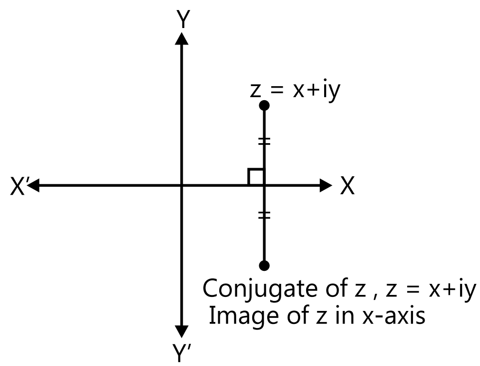


Figure 6.3: Conjugate of a complex number

(b) Modulus: If P denotes a complex number z = x + iy then, OP = | z | = $\sqrt{x^2 + y^2}$. Geometrically, it is the distance of a complex number from the origin.

Hence, note that $|z| \ge 0$, | i | = 1 i.e. $\left|\sqrt{-1}\right|$ = 1.

All complex number satisfying |z| = r lie on the circle having centre at origin and radius equal to 'r'.

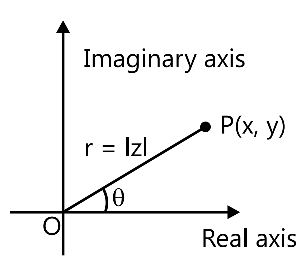


Figure 6.4: Modulus of a complex number

(c) Argument: If OP makes an angle θ (see Fig 6.4) with real axis in anticlockwise sense, then θ is called the argument of z. General values of argument of z are given by $2n\pi + \theta, n \in I$. Hence any two successive arguments differ by 2π.

Note: A complex number is completely defined by specifying both modulus and argument. However for the complex number 0 + 0i the argument is not defined and this is the only complex number which is completely defined by its modulus only.

(i) Amplitude (Principal value of argument): The unique value of θ such that $-\pi < \theta \le \pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

(ii) Least positive argument: The value of θ such that $0 < \theta \le 2\pi$ is called the least positive argument.

If $\phi = \tan^{-1}\left|\frac{y}{x}\right|$.

MASTERJEE CONCEPTS

- If x > 0, y > 0 (i.e. z is in first quadrant), then arg z = $\theta = \tan^{-1}\left|\frac{y}{x}\right|$.
- If x < 0, y > 0 (i.e. z is in 2^{nd} quadrant, then arg z = $\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$
- If x < 0, y < 0 (i.e. z is in 3^{rd} quadrant), then arg z = $\theta = -\pi + \tan^{-1}\left|\frac{y}{x}\right|$
- If x > 0, y < 0 (i.e. z is in 4^{th} quadrant), then arg z = $\theta = -\tan^{-1}\left|\frac{y}{x}\right|$
- If y = 0 (i.e. z is on the X-axis), then arg (x + i0) = $\begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$
- If x = 0 (i.e. z is on the Y-axis), then arg (0 + iy) = $\begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ \frac{3\pi}{2}, & \text{if } y < 0 \end{cases}$

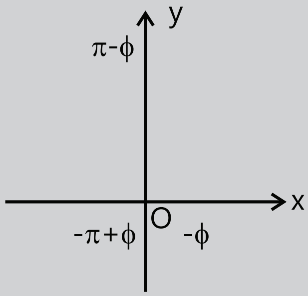


Figure 6.5

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 5: For what real values of x and y, are $-3 + ix^2y$ and $x^2 + y + 4i$ complex conjugate to each other? **(JEE MAIN)**

Sol: As $-3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugate of each other. Therefore $-3 + ix^2y = \overline{x^2 + y + 4i}$.

$-3 + ix^2y = x^2 + y - 4i$

Equating real and imaginary parts of the above question, we get

$-3 = x^2 + y \Rightarrow y = -3 - x^2$ ... (i)

and $x^2y = -4$ ... (ii)

Putting the value of y = $-3 - x^2$ from (i) in (ii), we get

$x^2(-3 - x^2) = -4 \quad \Rightarrow x^4 + 3x^2 - 4 = 0 \Rightarrow x^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = \frac{2}{2}, \frac{-8}{2} = 1, -4$

$\therefore\ x^2 = 1 \Rightarrow x = \pm 1$

Putting value of x = ± 1 in (i), we get $y = -3 - (1)^2 = -3 - 1 = -4$

Hence, x = ± 1 and y = – 4.

Illustration 6: Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$. **(JEE MAIN)**

Sol: As $|z| = \sqrt{x^2 + y^2}$, using algebra of complex number we will get the result.

Here, we have $\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} - \frac{(1-i)(1-i)}{(1+i)(1-i)}$

$= \frac{1 + i^2 + 2i}{1+1} - \frac{1 + i^2 - 2i}{1+1} = \frac{1 - 1 + 2i}{2} - \frac{1 - 1 - 2i}{2} = \frac{2i}{2} - \frac{(-2i)}{2} = i + i = 2i, \therefore \Rightarrow \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = |2i| = 2.$

Illustration 7: Find the locus of z if | z – 3 | = 3| z + 3|. **(JEE MAIN)**

Sol: Simply substituting z = x + iy and by using formula | z | = $\sqrt{x^2+y^2}$ we will get the result.

Let z = x + iy

| x + iy – 3 | = 3 | x + iy + 3 | | (x – 3) + iy | = 3 | (x + 3) + iy |

$\sqrt{(x-3)^2+y^2} = 3\sqrt{(x+3)^2+y^2}$; $(x-3)^2+y^2 = 9(x+3)^2+9y^2$.

Illustration 8: If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$. **(JEE ADVANCED)**

Sol: By using modulus and conjugate property, we can find out the value of $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

We have, $|\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1$

Now, $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta\bar{\beta}-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right| = \frac{|\beta-\alpha|}{|\beta||\overline{\beta-\alpha}|} = \frac{1}{|\beta|} = 1.$ $\{\text{as } |x+iy| = |\overline{x+iy}|\}$

Illustration 9: Find the number of non-zero integral solution of the equation $|1-i|^x = 2^x$. **(JEE ADVANCED)**

Sol: As | z | = $\sqrt{x^2+y^2}$, therefore by using this formula we can solve it.

We have, $|1-i|^x = 2^x$

$\Rightarrow \left[\sqrt{1^2+1^2}\right]^x = 2^x$ $\Rightarrow \left(\sqrt{2}\right)^x = 2^x$ $\Rightarrow 2^{\frac{x}{2}} = 2^x$ $\Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0$.

∴ The number of non zero integral solution is zero.

Illustration 10: If $\frac{a+ib}{c+id}$ = p + iq. Prove that $\frac{a^2+b^2}{c^2+d^2} = p^2+q^2$. **(JEE MAIN)**

Sol: Simply by obtaining modulus of both side of $\frac{a+ib}{c+id}$ = p + iq .

We have, $\frac{a+ib}{c+id}$ = p + iq

$\left|\frac{a+ib}{c+id}\right| = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \Rightarrow |p+iq| = \sqrt{p^2+q^2}$; $\left|\frac{a+ib}{c+id}\right| = |p+iq|$ $\Rightarrow \frac{a^2+b^2}{c^2+d^2} = p^2+q^2$.

Illustration 11: If $(x+iy)^{1/3}$ = a + ib. Prove that $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$. **(JEE ADVANCED)**

Sol: By using algebra of complex number. We have, $(x+iy)^{1/3}$ = a + ib

x + iy = $(a+ib)^3$ = $a^3+i^3b^3+3a^2ib+3a(ib)^2$ = $a^3-b^3i+3a^2bi-3ab^2$

x + iy = $(a^3-3ab^2)+(3a^2b-b^3)i$; x = a^3-3ab^2 = a (a^2-3b^2); y = $3a^2b-b^3$

$\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$.

4. REPRESENTATION OF COMPLEX NUMBER

4.1 Graphical Representation

Every complex number x + iy can be represented in a plane as a point P (x, y). X-coordinate of point P represents the real part of the complex number and y-coordinate represents the imaginary part of the complex number. Complex number x + 0i (real number) is represented by a point (x, 0) lying on the x-axis. Therefore, x-axis is called the real axis. Similarly, a complex number 0 + iy (imaginary number) is represented by a point on y-axis. Therefore, y-axis is called the imaginary axis.

The plane on which a complex number is represented is called complex number plane or simply complex plane or Argand plane (see Fig 6.6). The figure represented by the complex numbers as points in a plane is known as Argand Diagram.

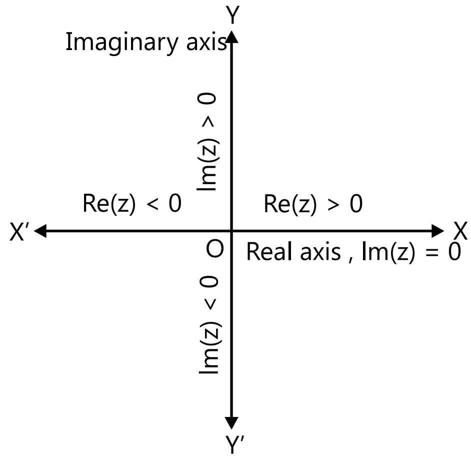


Figure 6.6: Graphical representation

4.2 Algebraic Form

If z = x + iy; then $|z| = \sqrt{x^2 + y^2}$; $\bar{z} = x - iy$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Generally this form is useful in solving equations and in problems involving locus.

4.3 Polar Form

Figure 6.7 shows the components of a complex number along the x and y-axes respectively. Then

$z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$ where $|z| = r$; amp $z = \theta$.

Aliter: $z = x + iy$

$$\Rightarrow z = \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right)$$

$$\Rightarrow z = |z|\left(\cos\theta + i\sin\theta\right) = r \text{ cis } \theta$$

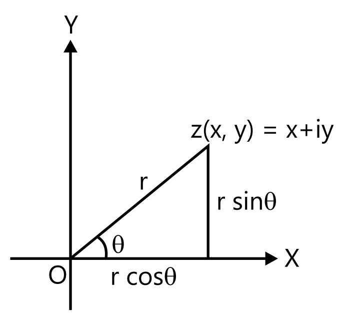


Figure 6.7: Polar form

Note: (a) $(\text{cis}\,\alpha)(\text{cis}\,\beta) = \text{cis}(\alpha + \beta)$

(b) $(\text{cis}\,\alpha)(\text{cis}(-\beta)) = \text{cis}(\alpha - \beta)$

(c) $\frac{1}{(\text{cis}\,\alpha)} = (\text{cis}\,\alpha)^{-1} = \text{cis}(-\alpha)$

MASTERJEE CONCEPTS

The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ & $y = r\sin\theta$ is known as the principal value of the argument.

The general value of argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of arg (z). While reducing a complex number to polar form, we always take the principal value.

The complex number z = $r(\cos\theta + i\sin\theta)$ can also be written as $r\,\text{cis}\,\theta$.

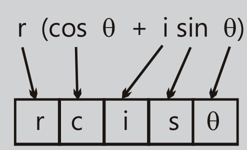


Figure 6.8

Nitish Jhawar (JEE 2009, AIR 7)

4.4 Exponential Form

Euler's formula, named after the famous mathematician Leonhard Euler, states that for any real number x, $e^{ix} = \cos x + i\sin x$.

Hence, for any complex number $z = r(\cos\theta + i\sin\theta)$, $z = re^{i\theta}$ is the exponential representation.

Note: (a) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Eulers identities.

(b) $\cos ix = \dfrac{e^{x} + e^{-x}}{2}$ = cos hx is always positive real $\forall\, x \in R$ and is > 1.

and, $\sin ix = i\,\dfrac{e^{x} - e^{-x}}{2}$ = i sin hx is always purely imaginary.

4.5 Vector Representation

The knowledge of vectors can also be used to represent a complex number z = x + iy. The vector $\overrightarrow{OP}$, joining the origin O of the complex plane to the point P (x, y), is the vector representation of the complex number z=x+iy, (see Fig 6.9). The length of the vector $\overrightarrow{OP}$, that is, $|\overrightarrow{OP}|$ is the modulus of z. The angle between the positive real axis and the vector $\overrightarrow{OP}$, more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as $\overrightarrow{OP}$ (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number z.

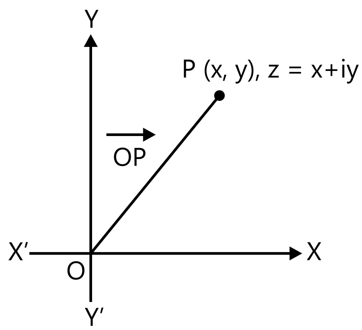


Figure 6.9 Vector representation

Illustration 12: Find locus represented by $\text{Re}\left(\dfrac{1}{x+iy}\right) < \dfrac{1}{2}$. **(JEE MAIN)**

Sol: Multiplying numerator and denominator by $x - iy$.

We have, $\text{Re}\left(\dfrac{1}{x+iy}\right) < \dfrac{1}{2}$ $\quad$ $\text{Re}\left(\dfrac{x-iy}{x^2+y^2}\right) < \dfrac{1}{2}$

$\Rightarrow \dfrac{x}{x^2+y^2} < \dfrac{1}{2}$ $\quad$ $\Rightarrow x^2 + y^2 - 2x > 0$

Locus is the exterior of the circle with centre (1, 0) and radius = 1.

Illustration 13: If $z = 1 + \cos\dfrac{6\pi}{5} + i\sin\dfrac{6\pi}{5}$. Find r and amp z. **(JEE MAIN)**

Sol: By using trigonometric formula we can reduce given equation in the form of $z = r\left(\cos\theta + i\sin\theta\right)$.

$z = 2\cos^2\dfrac{3\pi}{5} + 2i\sin\dfrac{3\pi}{5}\cos\dfrac{3\pi}{5} = 2\cos\dfrac{3\pi}{5}\left[\cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5}\right]$

$= -2\cos\dfrac{2\pi}{5}\left[-\cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}\right] = 2\cos\dfrac{2\pi}{5}\left[\cos\dfrac{2\pi}{5} - i\sin\dfrac{2\pi}{5}\right]$ Hence, $|z| = 2\cos\dfrac{2\pi}{5}$; amp z = $-\dfrac{2\pi}{5}$

Illustration 14: Show that the locus of the point P(ω)denoting the complex number $z+\frac{1}{z}$ on the complex plane is a standard ellipse where | z | = a, where a ≠ 0, 1. **(JEE ADVANCED)**

Sol: Here consider w = x + iy and z = $\alpha + i\beta$ and then solve this by using algebra of complex number.

Let w = $z+\frac{1}{z}$ where z = $\alpha+i\beta$, $\alpha^2+\beta^2=a^2$ (as | z | = a)

$$x + iy = \alpha+i\beta+\frac{1}{\alpha+i\beta} = \alpha+i\beta+\frac{\alpha-i\beta}{\alpha^2+\beta^2} = \left(\alpha+\frac{\alpha}{a^2}\right)+i\left(\beta-\frac{\beta}{a^2}\right) \quad \therefore x=\alpha\left(1+\frac{1}{a^2}\right);\ y=\beta\left(1-\frac{1}{a^2}\right)$$

$$\therefore \frac{x^2}{\left(1+\frac{1}{a^2}\right)^2}+\frac{y^2}{\left(1-\frac{1}{a^2}\right)^2} = \alpha^2+\beta^2=a^2; \qquad \therefore \frac{x^2}{\left(a+\frac{1}{a}\right)^2}+\frac{y^2}{\left(a-\frac{1}{a}\right)^2} = 1.$$

5. IMPORTANT PROPERTIES OF CONJUGATE, MODULUS AND ARGUMENT

For z, z_1 and $z_2 \in C$,

(a) Properties of Conjugate:

(i) $z+\bar{z} = 2\mathrm{Re}\,(z)$

(ii) $z-\bar{z} = 2i\ \mathrm{Im}\,(z)$

(iii) $(\bar{\bar{z}}) = z$

(iv) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(v) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(vi) $\overline{z_1\ z_2} = \bar{z}_1\cdot\bar{z}_2$

(vii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2};\ z_2 \neq 0$

(b) Properties of Modulus:

(i) $|z| \geq 0;\ |z| \geq \mathrm{Re}(z);\ |z| \geq \mathrm{Im}(z);\ |z| = |\bar{z}| = |-z|$

(ii) $z\bar{z} = |z|^2$; if $|z|$ = 1, then z = $\frac{1}{\bar{z}}$

(iii) $|z_1 z_2| = |z_1|\cdot|z_2|$

(iv) $\left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$

(v) $|z^n| = |z|^n$

(vi) $|z_1+z_2|^2+|z_1-z_2|^2 = 2\left[|z_1|^2+|z_2|^2\right]$

(vii) $\big||z_1|-|z_2|\big| \leq |z_1+z_2| \leq |z_1|+|z_2|$ [Triangle Inequality]

(c) Properties of Amplitude:

(i) $\text{amp}(z_1 \cdot z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi,\ k \in I$

(ii) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi,\ k \in I$

(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where the value of k should be such that RHS lies in $(-\pi, \pi]$

Based on the above information, we have the following

- $|\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,|z|$
- $\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le |z_1| + |z_2|$. Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $\big||z_1| - |z_2|\big|$ is the least possible value of $|z_1 + z_2|$.
- If $\left|z + \frac{1}{z}\right| = a$, the greatest and least values of |z| are respectively $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$.
- $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_2 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$
- If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg z_1 = \arg z_2$
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$ i.e. z_1 and z_2 are parallel.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = 2n\pi$, where n is some integer.
- $|z_1 - z_2| = \big||z_1| - |z_2|\big| \Leftrightarrow \arg(z_1) - \arg(z_2) = 2n\pi$, where n is some integer.
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = (2n+1)\frac{\pi}{2}$, where n is some integer.
- If $|z_1| \le 1, |z_2| \le 1$, then $|z_1 + z_2|^2 \le \left(|z_1| - |z_2|\right)^2 + \left(\arg(z_1) - \arg(z_2)\right)^2$, and
  $|z_1 + z_2|^2 \ge \left(|z_1| + |z_2|\right)^2 - \left(\arg(z_1) - \arg(z_2)\right)^2$.

Illustration 15: If $z_1 = 3 + 5i$ and $z_2 = 2 - 3i$, then verify that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ **(JEE MAIN)**

Sol: Simply by using properties of conjugate.

$$\frac{z_1}{z_2} = \frac{3+5i}{2-3i} = \frac{(3+5i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)} = \frac{6 + 9i + 10i + 15i^2}{4 - 9i^2} = \frac{6 + 19i + 15(-1)}{4+9} = \frac{6 + 19i - 15}{13} = \frac{-9 + 19i}{13} = \frac{-9}{13} + \frac{19}{13}i$$

$$\text{L.H.S.} = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(-\frac{9}{13} + \frac{19}{13}i\right)} = -\frac{9}{13} - \frac{19}{13}i$$

$$\text{R.H.S.} = \frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{3+5i}}{\overline{2-3i}} = \frac{3-5i}{2+3i} = \frac{(3-5i)}{(2+3i)} \times \frac{(2-3i)}{(2-3i)}$$

$$= \frac{6 - 9i - 10i + 15i^2}{4 - 9i^2} = \frac{6 - 19i + 15(-1)}{4+9} = \frac{6 - 19i - 15}{13} = -\frac{9}{13} - \frac{19}{13}i \qquad \therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Illustration 16: If z be a non-zero complex number, then show that $\overline{(z^{-1})} = (\bar{z})^{-1}$. **(JEE MAIN)**

Sol: By considering z = a + ib and using properties of conjugate we can prove given equation.

Let z = a + ib Since, $z \neq 0$, we have $x^2 + y^2 > 0$

$$z^{-1} = \frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2} \Rightarrow \left(\overline{z^{-1}}\right) = \frac{a}{a^2+b^2} + \frac{ib}{a^2+b^2} \quad \text{... (i)}$$

$$\text{and } (\bar{z})^{-1} = \frac{1}{\bar{z}} = \frac{1}{\overline{a+ib}} = \frac{1}{a-ib} = \frac{1}{a-ib} \times \frac{a+ib}{a+ib} = \frac{a}{a^2+b^2} + i\frac{b}{a^2+b^2} \quad \text{... (ii)}$$

From (i) and (ii), we get $\overline{(z^{-1})} = (\bar{z})^{-1}$.

Illustration 17: If $\frac{(a+i)^2}{2a-i}$ = p + iq, then show that $p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}$. **(JEE MAIN)**

Sol: Multiply given equation to its conjugate.

We have, p + iq = $\frac{(a+i)^2}{2a-i}$ ... (i)

Taking conjugate of both sides, we get $\overline{p+iq} = \overline{\left(\frac{(a+i)^2}{(2a-i)}\right)}$

$$\Rightarrow p - iq = \frac{\overline{(a+i)^2}}{\overline{(2a-i)}} \quad \left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\right] \qquad \Rightarrow p - iq = \frac{(a-i)^2}{(2a+i)} \quad \text{... (ii)} \left[\text{using } \overline{(z^2)} = \overline{z \cdot z} = \bar{z} \cdot \bar{z} = (\bar{z})^2\right]$$

Multiplying (i) and (ii), we get (p + iq) (p – iq) = $\left(\frac{(a+i)^2}{2a-i}\right)\left(\frac{(a-i)^2}{2a+i}\right)$

$$\Rightarrow p^2 - i^2q^2 = \frac{(a^2-i^2)^2}{4a^2-i^2} \qquad \Rightarrow p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}.$$

Illustration 18: Let $z_1, z_2, z_3, \ldots\ldots z_n$ are the complex numbers such that $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$. If z = $\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then prove that

(i) z is a real number (ii) $0 < z \leq n^2$ **(JEE ADVANCED)**

Sol: Here $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$, therefore $z\bar{z} = 1 \Rightarrow z = \frac{1}{\bar{z}}$. Hence by substituting this to z = $\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$, we can solve above problem.

Now, z = $(z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right)$

$= (z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\bar{z}_1 + \bar{z}_2 + \ldots\ldots + \bar{z}_n\right) \qquad = (z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\overline{z_1 + z_2 + \ldots\ldots + z_n}\right)$

$= |z_1 + z_2 + z_3 + \ldots\ldots + z_n|^2$ which is real

$\leq \left(|z_1| + |z_2| + |z_3| + \ldots\ldots + |z_n|\right)^2 = n^2 \qquad \therefore \qquad 0 < z \leq n^2$.

Illustration 19: Let x_1, x_2 are the roots of the quadratic equation $x^2 + ax + b = 0$ where a, b are complex numbers and y_1, y_2 are the roots of the quadratic equation $y^2 + |a|y + |b| = 0$. If $|x_1| = |x_2| = 1$, then prove that $|y_1| = |y_2| = 1$.

(JEE ADVANCED)

Sol: Solve by using modulus properties of complex number.

Let $x^2 + ax + b = 0$ where x_1 and x_2 are complex numbers

$x_1 + x_2 = -a$ ... (i)

and $x_1 x_2 = b$ ... (ii)

From (ii) $|x_1||x_2| = |b| \Rightarrow |b| = 1$ Also $|-a| = |x_1 + x_2|$

$\therefore$ $|a| \le |x_1| + |x_2|$ or $|a| \le 2$

Now consider $y^2 + |a|y + |b| = 0$, $\langle \begin{matrix} y_1 \\ y_2 \end{matrix}$ where y_1 and y_2 are complex numbers

$$y_{1,2} = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2} = \frac{-|a| \pm \left(\sqrt{4 - |a|^2}\right)i}{2} \qquad \therefore \qquad |y_{1,2}| = \frac{\sqrt{|a|^2 + 4 - |a|^2}}{2} = 1$$

Hence, $|y_1| = |y_2| = 1$.

6. TRIANGLE ON COMPLEX PLANE

In a ΔABC, the vertices A, B and C are represented by the complex numbers z_1, z_2 and z_3 respectively, then

(a) Centroid: The centroid 'G' is given by $\frac{z_1 + z_2 + z_3}{3}$. Refer to Fig 6.10.

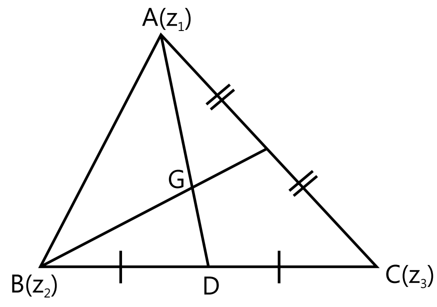


Figure 6.10: Centroid

(b) Incentre: The incentre 'I' is given by $\frac{az_1 + bz_2 + cz_3}{a + b + c}$. Refer to Fig 6.11.

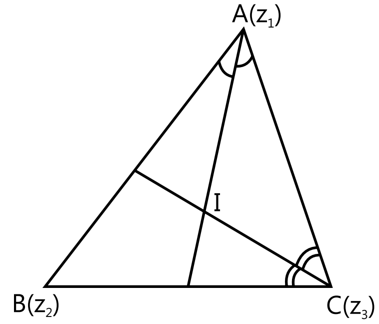


Figure 6.11: Incentre

(c) Orthocentre: The orhtocentre 'H' is given by $\dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\sum \tan A}$.

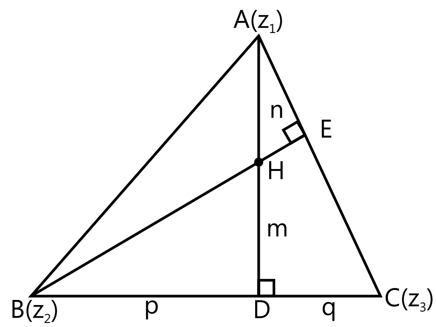


Figure 6.12: Orthocentre

Proof: From section formula, we have $z_D = \dfrac{p\, z_3 + q\, z_2}{a}$

In ΔABD and ΔACD, p = c cos B and q = b cos C. Refer to Fig 6.12.

Therefore, $z_D = \dfrac{b \cos C\, z_2 + c \cos B\, z_3}{a}$

Now, AE = c cos A; $n = AH = AE \operatorname{cosec} C$ = c cos A cosec C

$\Rightarrow n = 2R \cos A$ [Using Sine Rule]

and m = c cos B cot C or, m = 2R cos B cos C [Using Sine Rule]

Hence, $z_H = \dfrac{m z_1 + n Z_D}{m+n}$.

$$= \frac{2R \cos B \cos C z_1 + 2R \cos A \left(\dfrac{b \cos C\, z_2 + c \cos B\, z_3}{a} \right)}{2R\, (\cos A + \cos B \cos C)}$$

$$= \frac{a \cos B \cos C z_1 + b \cos A \cos C z_2 + c \cos A \cos B z_3}{a(-\cos(B+C) + \cos B \cos C)}$$

$$= \frac{z_1 (\sin A \cos B \cos C) + z_2 (\sin B \cos C \cos A) + z_3 (\sin C \cos A \cos B)}{\sin A\, (\sin B \sin C)}$$

$$\therefore\ Z_H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\sum \tan A} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\prod \tan A}$$

[If A + B + C = π, then tan A + tan B + tan C = tan A tan B tan C]

(d) Circumcentre:

Let R be the circumradius and the complex number z_0 represent the circumcentre of the triangle as shown in Fig 6.11.

$\therefore\ |z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$

Consider, $|z_1 - z_0|^2 = |z_2 - z_0|^2$

$(z_1 - z_0)(\overline{z}_1 - \overline{z}_0) = (z_2 - z_0)(\overline{z}_2 - \overline{z}_0)$

$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0\left[(z_1 - z_0) - (z_2 - z_0)\right]$

$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0(z_1 - z_2)$ ... (i)

Similarly 1st and 3rd gives

$\bar{z}_1(z_1 - z_0) - \bar{z}_3(z_3 - z_0) = \bar{z}_0(z_1 - z_3)$ ... (ii)

On dividing (i) by (ii), $\bar{z}_0$ gets eliminated and we obtain z_0.

Alternatively: From Fig 6.13, we have

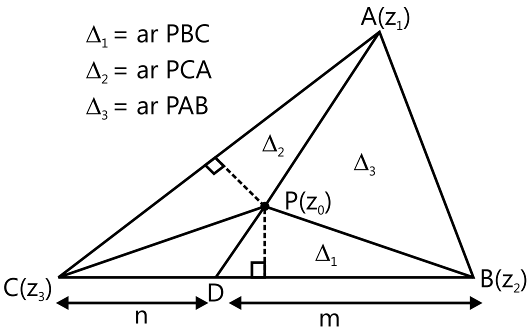


Figure 6.13: Circumcentre

$$\frac{BD}{DC} = \frac{m}{n} = \frac{Ar.\ \Delta ABD}{Ar.\ \Delta ADC} = \frac{Ar.\ \Delta PBD}{Ar.\ \Delta PDC}$$

$$\therefore \frac{m}{n} = \frac{Ar.\ \Delta ABD - Ar.\ \Delta PBD}{Ar.\ \Delta ADC - Ar.\ \Delta PDC} = \frac{\Delta_3}{\Delta_2}$$

$$\therefore \frac{m}{n} = \frac{\frac{R^2}{2}\sin 2C}{\frac{R^2}{2}\sin 2B} = \frac{\sin 2C}{\sin 2B}$$

Hence, $Z_D = \dfrac{\sin 2B\,(z_2) + \sin 2C\,(z_3)}{\sin 2B + \sin 2C}$

Now, $\dfrac{PA}{PD} = \dfrac{l}{k} = \dfrac{\Delta ABP}{\Delta PBD} = \dfrac{\Delta APC}{\Delta CPD} = \dfrac{\Delta ABP + \Delta APC}{\Delta PBD + \Delta CPD}$ $\therefore \dfrac{l}{k} = \dfrac{\Delta_3 + \Delta_2}{\Delta_1} = \dfrac{\sin 2C + \sin 2B}{\sin 2A}$

Hence, $z_0 = \dfrac{kz_1 + l z_D}{k + l} = \dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sum \sin 2A}$

MASTERJEE CONCEPTS

- The area of the triangle whose vertices are z, iz and z + iz is $\frac{1}{2}|z|^2$.
- The area of the triangle with vertices z, ωz and z + ωz is $\frac{\sqrt{3}}{4}|z|^2$.
- If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_0 be the circumcentre, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
- If $z_1, z_2, z_3, \ldots\ldots z_n$ be the vertices of a regular polygon of n sides and z_0 be its centroid, then $z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = nz_0^2$.
- If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral if $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ or $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$.
- If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2 then $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.
- If z_1, z_2, z_3 are the vertices of a right-angled isosceles triangle, then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.
- If z_1, z_2, z_3 be the affixes of the vertices A, B, C respectively of a triangle ABC, then its orthocentere is $\dfrac{a(\sec A)z_1 + b(\sec B)z_2 + c(\sec C)z_3}{a\sec A + b\sec B + c\sec C}$.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 20: If z_1, z_2, z_3 are the vertices of an isosceles triangle right angled at z_2 then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$ **(JEE MAIN)**

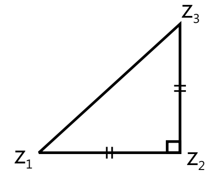


Figure 6.14

Sol: Here $(z_1 - z_2) = (z_3 - z_2)e^{\frac{i\pi}{2}}$. Hence by squaring both side we will get the result.

$\Rightarrow (z_1 - z_2)^2 = i^2(z_3 - z_2)^2$

$\Rightarrow z_3^2 + z_2^2 - 2z_3z_2 = -z_1^2 - z_2^2 + 2z_1z_2 \Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

Illustration 21: A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the points D, E, F in terms of z_1, z_2, z_3. **(JEE MAIN)**

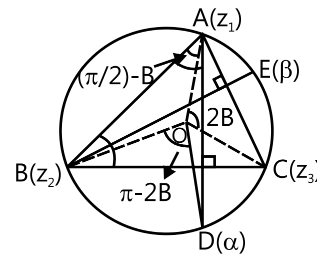


Figure 6.15

Sol: Here the $\angle BOD = \pi - 2B$, hence $\overrightarrow{OD} = \overrightarrow{OB}\, e^{i(\pi - 2B)}$.

From Fig 6.13, we have $\overrightarrow{OD} = \overrightarrow{OB}\, e^{i(\pi - 2B)}$;

$\alpha = z_2\, e^{i(\pi - 2B)} = -z_2\, e^{-i2B}$ ... (i)

also, $z_1 = z_3\, e^{i\,2B}$ ... (ii)

$\therefore\ \alpha z_1 = -z_2z_3 \qquad \Rightarrow \alpha = \dfrac{-z_2z_3}{z_1}$

Similarly, $\beta = \dfrac{-z_3z_1}{z_2}$ and $\gamma = \dfrac{-z_1z_2}{z_3}$.

Illustration 22: If z_r (r = 1, 2, ...,6) are the vertices of a regular hexagon then prove that $\sum_{r=1}^{6} z_r^2 = 6z_0^2$, where z_0 is the circumcentre of the regular hexagon. **(JEE MAIN)**

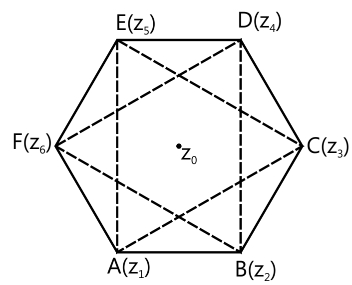


Figure 6.16

Sol: As we know If $z_1, z_2, z_3, \ldots\ldots. z_n$ be the vertices of a regular polygon of n sides and z_0 be its centroid, then $z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = nz_0^2$.

Here by the Fig 6.14,

$3z_0^2 = z_1^2 + z_3^2 + z_5^2$

and, $3z_0^2 = z_2^2 + z_4^2 + z_6^2 \qquad \Rightarrow 6z_0^2 = \sum_{r=1}^{6} z_r^2$.

Illustration 23: If z_1, z_2, z_3 are the vertices of an equilateral triangle then prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ and if z_0 is its circumcentre then $3z_0^2 = z_1^2 + z_2^2 + z_3^2$. **(JEE ADVANCED)**

Sol: By using triangle on complex plane we can prove

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ and by using $z_0 = \dfrac{z_1 + z_2 + z_3}{3}$ we can prove $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

To Prove, $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

As seen in the Fig 6.17,

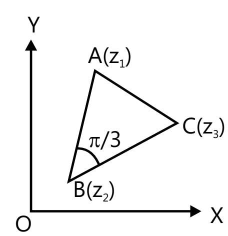


Figure 6.17

$\because \dfrac{z_1 - z_2}{z_2 - z_3} = \dfrac{(z_3 - z_2)\, e^{\frac{i\pi}{3}}}{(z_1 - z_3)\, e^{\frac{i\pi}{3}}} \Rightarrow (z_1 - z_2)(z_1 - z_3) = -(z_2 - z_3)^2$

$\Rightarrow z_1^2 - z_1z_3 - z_2z_1 + z_2z_3 + z_2^2 + z_3^2 - 2z_2z_3 = 0 \quad \therefore \sum z_1^2 = \sum z_1z_2$

Now if z_0 is the circumcentre of the Δ, then we need to prove $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

Since in an equilateral triangle, the circumcentre coincides with the centroid, we have $z_0 = \dfrac{z_1 + z_2 + z_3}{3}$

$\Rightarrow (z_1 + z_2 + z_3)^2 = (3z_0)^2$

$\Rightarrow \sum z_1^2 + 2\sum z_1 z_2 = 9z_0^2 \quad \therefore 3\sum z_1^2 = 9z_0^2$.

Illustration 24: Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle if and only if $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$. **(JEE ADVANCED)**

Sol: Consider ABC is the equilateral triangle with vertices z_1, z_2 and z_3 respectively.

Therefore $|z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2|$.

Let ABC be a triangle such that the vertices A, B and C are z_1, z_2 and z_3 respectively.

Further, let $\alpha = z_2 - z_3$, $\beta = z_3 - z_1$ and $\gamma = z_1 - z_2$. Then $\alpha + \beta + \gamma = 0$ ... (i)

As shown in Fig 6.16, let ΔABC be an equilateral triangle. Then, BC = CA = AB

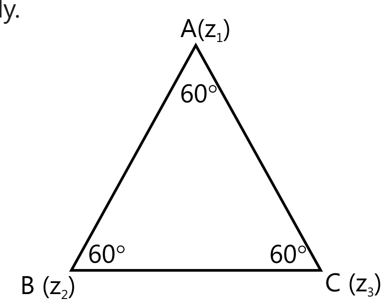


Figure 6.18

$\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \quad \Rightarrow |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |\alpha|^2 = |\beta|^2 = |\gamma|^2 = \lambda(\text{say})$

$\Rightarrow \alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = \lambda$

$\Rightarrow \bar{\alpha} = \dfrac{\lambda}{\alpha}, \bar{\beta} = \dfrac{\lambda}{\beta}, \bar{\gamma} = \dfrac{\lambda}{\gamma}$ ... (ii)

Now, $\alpha + \beta + \gamma = 0$ [from (i)]

$\Rightarrow \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0 \qquad \Rightarrow \dfrac{\lambda}{\alpha} + \dfrac{\lambda}{\beta} + \dfrac{\lambda}{\gamma} = 0$ [Using (ii)]

$\Rightarrow \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0 \Rightarrow \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$ which is the required condition.

Conversely, let ABC be a triangle such that

$\Rightarrow \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$ i.e. $\Rightarrow \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0$

Thus, we have to prove that the triangle is equilateral. We have, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0$

$\Rightarrow \dfrac{1}{\alpha} = -\left(\dfrac{1}{\beta} + \dfrac{1}{\gamma}\right) \quad \Rightarrow \dfrac{1}{\alpha} = -\left(\dfrac{\beta + \gamma}{\beta\gamma}\right) \quad \Rightarrow \dfrac{1}{\alpha} = \dfrac{\alpha}{\beta\gamma} \quad \Rightarrow \alpha^2 = \beta\gamma \quad \Rightarrow |\alpha|^2 = |\beta\gamma|$

$\Rightarrow |\alpha|^2 = |\beta||\gamma| \quad \Rightarrow |\alpha|^3 = |\alpha||\beta||\gamma|$

Similarly, $\Rightarrow |\beta|^3 = |\alpha||\beta||\gamma|$ and $|\gamma|^3 = |\alpha||\beta||\gamma|$

$\therefore \quad |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \quad \Rightarrow BC = CA = AB$

Hence, the given triangle is an equilateral triangle.

Illustration 25: Prove that the roots of the equation $\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}=0$ (where z_1, z_2, z_3 are pair wise distinct complex numbers) correspond to points on a complex plane, which lie inside a triangle with vertices z_1, z_2, z_3 excluding its boundaries. **(JEE ADVANCED)**

Sol: By using modulus and conjugate properties we can reduce given expression as $\frac{\bar{z}-\bar{z}_1}{|z-z_1|^2}+\frac{\bar{z}-\bar{z}_2}{|z-z_2|^2}+\frac{\bar{z}-\bar{z}_3}{|z-z_3|^2}$ = 0. Therefore by putting $|z-z_i|^2=\frac{1}{t_i}$, where i = 1, 2 and 3, we will get the result.

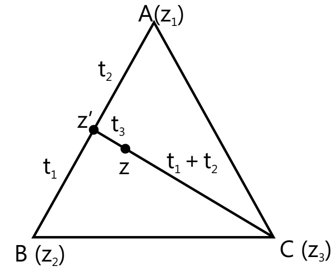


Figure 6.19

$t_1(\bar{z}-\bar{z}_1)+t_2(\bar{z}-\bar{z}_2)+t_3(\bar{z}-\bar{z}_3)=0$ where $|z-z_1|^2=\frac{1}{t_1}$ etc and $t_1, t_2, t_3 \in R^+$

$t_1(z-z_1)+t_2(z-z_2)+t_3(z-z_3)=0$

$(t_1+t_2+t_3)z=t_1z_1+t_2z_2+t_3z_3 \quad \Rightarrow z=\frac{t_1z_1+t_2z_2+t_3z_3}{t_1+t_2+t_3}$

$\Rightarrow z=\frac{t_1z_1+t_2z_2}{t_1+t_2}\cdot\frac{t_1+t_2}{t_1+t_2+t_3}+\frac{t_3z_3}{t_1+t_2+t_3}=\frac{t_1+t_2}{t_1+t_2+t_3}z'+\frac{t_3z_3}{t_1+t_2+t_3}$

$\Rightarrow z=\frac{(t_1+t_2)z'+t_3z_3}{t_1+t_2+t_3} \quad \Rightarrow$ z lies inside the $\Delta z_1z_2z_3$

If $t_1=t_2=t_3 \quad \Rightarrow$ z is the centroid of the triangle.

Also, it implies $|z-z_1|=|z-z_2|=|z-z_3| \quad \Rightarrow$ z is the circumcentre.

Illustration 26: Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB=\alpha\neq 0$ and OA = OB, where O is the origin, prove that $p^2=4q\cos^2\frac{\alpha}{2}$. **(JEE ADVANCED)**

Sol: Here $\overrightarrow{OB}=\overrightarrow{OA}e^{i\alpha}$. Therefore by using formula of sum and product of roots of quadratic equation we can prove this problem.

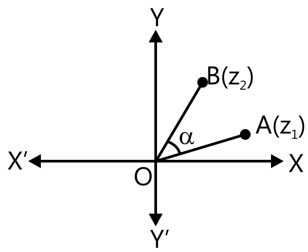


Figure 6.20

Since z_1 and z_2 are roots of the equation $z^2+pz+q=0$

$z_1+z_2=-p$ and $z_1z_2=q$ (1)

Since OA = OB. So $\overrightarrow{OB}$ is obtained by rotating $\overrightarrow{OA}$ in anticlockwise direction through angle α.

$\therefore \overrightarrow{OB}=\overrightarrow{OA}e^{i\alpha} \quad \Rightarrow z_2=z_1e^{i\alpha} \quad \Rightarrow \frac{z_2}{z_1}=e^{i\alpha}\Rightarrow\frac{z_2}{z_1}=\cos\alpha+i\sin\alpha$

$\Rightarrow \frac{z_2}{z_1}+1=1+\cos\alpha+i\sin\alpha\Rightarrow\frac{z_2+z_1}{z_1}=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)=2\cos\frac{\alpha}{2}e^{\frac{i\alpha}{2}}$

$\Rightarrow \frac{z_2+z_1}{z_1}=2\cos\frac{\alpha}{2}e^{\frac{i\alpha}{2}}\Rightarrow\left(\frac{z_2+z_1}{z_1}\right)^2=4\cos^2\frac{\alpha}{2}e^{i\alpha}$

$\Rightarrow \left(\frac{z_2+z_1}{z_1}\right)^2=4\cos^2\frac{\alpha}{2}\frac{z_2}{z_1} \quad \Rightarrow (z_2+z_1)^2=4z_1z_2\cos^2\frac{\alpha}{2}$

$\Rightarrow (-p)^2=4q\cos^2\frac{\alpha}{2} \quad \Rightarrow p^2=4q\cos^2\frac{\alpha}{2}$.

Illustration 27: On the Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with AC = BC and equal angles are θ. If z_4 is the incentre of the triangle then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec\theta)(z_4 - z_1)^2$ **(JEE ADVANCED)**

Sol: Here by using angle rotation formula we can solve this problem. From Fig 6.21, we have

$$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad \text{... (i) (clockwise)}$$

and $$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad \text{... (ii) (anticlockwise)}$$

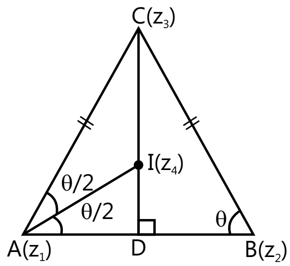


Figure 6.21

Multiplying (i) and (ii)

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{|(z_2 - z_1)||(z_3 - z_1)|}{|z_4 - z_1|^2} = \frac{AB|AC|}{(AI)^2} = \frac{2(AD)(AC)}{(AI)^2} = \frac{2(AD)^2}{(AI)^2}\cdot\frac{AC}{AD}$$

$$= 2\cos^2\frac{\theta}{2}\sec\theta = (1 + \cos\theta)\sec\theta.$$

7. REPRESENTATION OF DIFFERENT LOCI ON COMPLEX PLANE

(a) $|z - (1 + 2i)| = 3$ denotes a circle with centre (1, 2) and radius 3 (see Fig 6.22).

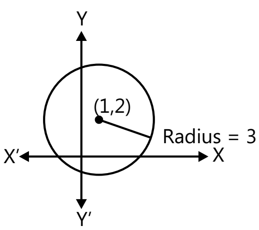


Figure 6.22: Circle on a complex plane

(b) $|z - 1| = |z - i|$ denotes the equation of the perpendicular bisector of join of (1, 0) and (0, 1) on the Argand plane (see Fig 6.24).

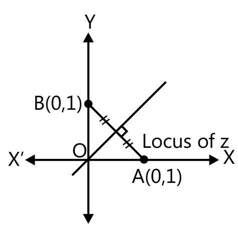


Figure 6.23: Perpendicular bisector complex plane

(c) $|z - 4i| + |z + 4i| = 10$ denotes an ellipse with foci at (0, 4) and (0, – 4); major axis 10; minor axis 6 with $e = \frac{4}{5}$ (see Fig 6.24).

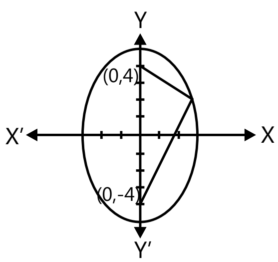


Figure 6.24: Ellipse on a complex plane

$e^2 = 1 - \frac{36}{100} = \frac{64}{100} \Rightarrow \quad e = \frac{4}{5} \left[\frac{x^2}{9} + \frac{y^2}{25} = 1\right]$

(d) | z – 1 | + | z + 1 | = 1 denotes no locus. (Triangle inequality).

(e) | z – 1 | < 1 denotes area inside a circle with centre (1, 0) and radius 1.

(f) $2 \le |z-1| < 5$ denotes the region between the concentric circles of radii 5 and 2. Centred at (1, 0) including the inner boundary (see Fig 6.25).

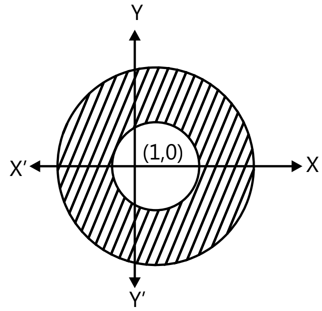


Figure 6.25: Circle disc on a complex plane

(g) $0 \le \arg z \le \frac{\pi}{4} (z \ne 0)$ where z is defined by positive real axis and the part of the line x = y in the first quadrant. It includes the boundary but not the origin. Refer to Fig 6.26.

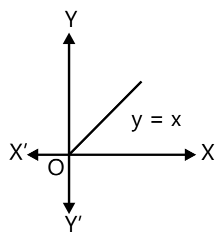


Figure 6.26

(h) $\text{Re}(z^2) > 0$ denotes the area between the lines x = y and x = – y which includes the x-axis.

Hint: $(x^2 - y^2) + 2xyi = 0 \quad \Rightarrow x^2 - y^2 > 0 \quad\quad \Rightarrow (x - y)(x + y) > 0.$

Illustration 28: Solve for z, if $z^2 + |z| = 0$. **(JEE MAIN)**

Sol: Consider z = $x + iy$ and solve this using algebra of complex number.

Let z = $x + iy \quad \Rightarrow (x+iy)^2 + \sqrt{x^2+y^2} = 0 \quad\quad \Rightarrow \left(x^2 - y^2 + \sqrt{x^2+y^2}\right) + (2ixy) = 0$

$\Rightarrow$ Either $x = 0$ or $y = 0$; $\quad x = 0 \Rightarrow -y^2 + |y| = 0 \quad\quad \Rightarrow y = 0, 1, -1 \quad \therefore z = 0, i, -i$

and, $y = 0 \Rightarrow x^2 + |x| = 0 \quad\quad \Rightarrow x = 0 \quad\quad \therefore z = 0$

Therefore, z = 0, z = i, z = – i.

Illustration 29: If the complex number z is to satisfy |z| = 3, $|z - \{a(1+i) - i\}| \le 3$ and | z + 2a – (a + 1) i | > 3 simultaneously for at least one z then find all $a \in R$. **(JEE ADVANCED)**

Sol: Consider z = x + iy and solve these inequalities to get the result.

All z at a time lie on a circle | z | = 3 but inside and outside the circles | z – {a (1 + i) – i} | = 3 and | z + 2a – (a + 1) i | = 3, respectively.

Let z = x + iy then equation of circles are $x^2 + y^2 = 9$ ... (i)

$(x - a)^2 + (y - a + 1)^2 = 9$ ... (ii)

and $(x + 2a)^2 + (y - a - 1)^2 = 9$ ... (iii)

Circles (i) and (ii) should cut or touch then distance between their centres $\le$ sum of their radii.

$\Rightarrow \sqrt{(a-0)^2 + (a-1-0)^2} \le 3+3 \Rightarrow a^2 + (a-1)^2 \le 36$

$\Rightarrow 2a^2 - 2a - 35 \le 0 \Rightarrow a^2 - a - \frac{35}{2} \le 0$

$\Rightarrow \left(a - \frac{1}{2}\right)^2 \le \frac{71}{4} \quad \therefore \frac{1-\sqrt{71}}{2} \le a \le \frac{1+\sqrt{71}}{2}$ ... (iv)

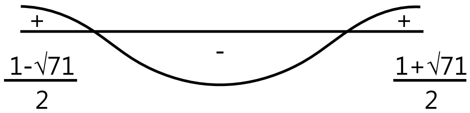


Figure 6.27

Again circles (i) and (iii) should not cut or touch then distance between their centres > sum of the radii

$\Rightarrow \sqrt{(-2a-0)^2 + (a+1-0)^2} > 3+3 \Rightarrow \sqrt{5a^2 + 2a + 1} > 6 \Rightarrow 5a^2 + 2a + 1 > 36$

$\Rightarrow 5a^2 + 2a - 35 > 0 \qquad \Rightarrow a^2 + \frac{2a}{5} - 7 > 0$

Then $\left(a - \frac{-1-4\sqrt{11}}{5}\right)\left(a - \frac{-1+4\sqrt{11}}{5}\right) > 0$

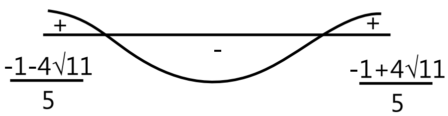


Figure 6.28

$\therefore \qquad a \in \left(-\infty, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \infty\right)$ ... (v)

The common values of a satisfying (iv) and v are

$a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$

8. DEMOIVRE'S THEOREM

Statement: $(\cos n\theta + i\sin n\theta)$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$, $\forall\, n \in Q$. Value if n is an integer. One of the values if n is rational which is not integer, the theorem is very useful in determining the roots of any complex quantity.

Note: We use the theory of equations to find the continued product of the roots of a complex number.

MASTERJEE CONCEPTS

The theorem is not directly applicable to $(\sin\theta + i\cos\theta)^n$, rather

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

8.1 Application

Cube root of unity

(a) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$

[Note that $1 - i\sqrt{3} = -2$ and $1 + i\sqrt{3} = -2\omega^2$]

(b) If ω is one of the imaginary cube roots of unity then $1+\omega+\omega^2=0$.

In general $1 + \omega^r + \omega^{2r} = 0$; where r = 1, and not a multiple of 3.

(c) In polar form the cube roots of unity are: $\cos 0 + i\sin 0$; $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$; $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

[Note that the 3 cube roots of i lies on the vertices of an isosceles triangle]

(e) The following factorization should be remembered.

For a, b, c $\in R$ and ω being the cube root of unity,

(i) $a^3-b^3=(a-b)(a-\omega b)(a-\omega^2 b)$

(ii) $x^2+x+1=(x-\omega)(x-\omega^2)$

(iii) $a^3+b^3=(a+b)(a+\omega b)(a+\omega^2 b)$

(iv) $a^3+b^3+c^3-3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$

n^{th} roots of unity: If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the n, n^{th} roots of unity then

(i) They are in G.P. with common ratio $e^{i\left(\frac{2\pi}{n}\right)} = \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$

(ii) $1^p+\alpha_1^p+\alpha_2^p+\ldots+\alpha_{n-1}^p=0$ if p is not an integral multiple of n

$1^p+(\alpha_1)^p+(\alpha_2)^p+\ldots+(\alpha_{n-1})^p=n$ if p is an integral multiple of n.

(iii) $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1})=n$.

Steps to determine nth roots of a complex number

(i) Represent the complex number whose roots are to be determined in polar form.

(ii) Add 2mπ to the argument.

(iii) Apply De Moivre's TheoremT

(iv) Put m = 0, 1, 2, 3, ... (n – 1) to get all the n^{th} roots.

Explanation: Let z = $1^{\frac{1}{n}} = (\cos 0 + i\sin 0)^{\frac{1}{n}} = (\cos 2m\pi + i\sin 2m\pi)^{\frac{1}{n}} = \left(\cos\frac{2m\pi}{n}+i\sin\frac{2m\pi}{n}\right)$

Put m = 0, 1, 2, 3, ... (n – 1), we get

$1, \underbrace{\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}}_{\alpha}, \cos\frac{4\pi}{n}+i\sin\frac{4\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)\pi}{n}$ (n, n^{th} roots in G.P.)

Now, $S = 1^p + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots\ldots + \alpha^{(n-1)p} = \frac{1-(\alpha^p)^n}{1-\alpha^p} = \frac{1-(\alpha^n)^p}{1-\alpha^p}$

$$= \frac{1-(\alpha^n)^p}{1-\alpha^p} = \begin{cases} \frac{0}{\text{non zero}} = 0, & \text{if p is not an integral multiple of n} \\ \frac{0}{0} = \text{indeterminant}, & \text{if p is an integral multiple of n} \end{cases}$$

Again, if x is one of the n^{th} root of unity then $x^n - 1 = (x-1)\ (x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1})$

$1 + x + x^2 + \ldots\ldots + x^{n-1} = \frac{x^n - 1}{x-1} \equiv (x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1})$

Put x = 1, to get $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = n$

Similarly put x = – 1, is to get other result.

MASTERJEE CONCEPTS

Square roots of z = a + ib are $\pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right]$ for b > 0.

If $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the n, n^{th} roots of unity then

$(1+\alpha_1)(1+\alpha_2)\ldots\ldots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

$1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \ldots\ldots \cdot \alpha_{n-1}$ = 1 or – 1 according as n is odd or even.

$$(\omega-\alpha_1)(\omega-\alpha_2)\ldots\ldots(\omega-\alpha_{n-1}) = \begin{cases} 0, & \text{if } n = 3k \\ 1, & \text{if } n = 3k+1 \\ 1+\omega, & \text{if } n = 3k+2 \end{cases}.$$

Ravi Vooda (JEE 2009, AIR 71)

Illustration 30: If $x = a+b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$, then prove that $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ **(JEE MAIN)**

Sol: Here $x+y+z=0$. Take cube on both side.

$x+y+z=0 \quad \Rightarrow x^3+y^3+z^3 = 3xyz \therefore$ LHS = 3xyz

$= 3(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega) = 3(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega.\omega^3) = 3\omega^3(a+b)(a+b\omega)(a+b\omega^2) = 3(a^3+b^3)$

Illustration 31: The value of expression $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$.

(JEE ADVANCED)

Sol: The given expression represent as $x^3 - 1 = (x-1)\ (x-\omega)(x-\omega^2)$. Therefore by putting x = 2, 3, 4 … n, we will get the result.

$x^3 - 1 = (x-1)\ (x-\omega)(x-\omega^2)$

Put x = 2 $\ 2^3 - 1 = 1 \cdot (2-\omega)(2-\omega)^2$ Put x = 3 $\ 3^3 - 1 = 2 \cdot (3-\omega)(3-\omega^2)$:

Put x = n $\ n^3 - 1 = (n-1)(n-\omega)(n-\omega^2)$

$\therefore \text{LHS} = (2^3 + 3^3 + \ldots\ldots + n^3) - (n-1) \quad = (1^3 + 2^3 + 3^3 + \ldots\ldots + n^3) - n = \left(\frac{n(n+1)}{2}\right)^2 - n$

9. SUMMATION OF SERIES USING COMPLEX NUMBER

(a) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \dfrac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cos\left(\frac{n+1}{2}\right)\theta$

(b) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta = \dfrac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin\left(\frac{n+1}{2}\right)\theta$

Note: If $\theta = \frac{2\pi}{n}$, then the sum of the above series vanishes.

9.1 Complex Number and Binomial Coefficients

Try the following questions using the binomial expansion of $(1+x)^n$ and substituting the value of x according to the binomial coefficients in the respective question.

Find the value of the following

(i) $C_0 + C_4 + C_8 + \ldots\ldots$ (ii) $C_1 + C_5 + C_9 + \ldots\ldots$

(iii) $C_2 + C_6 + C_{10} + \ldots\ldots$ (iv) $C_3 + C_7 + C_{11} + \ldots\ldots$

(v) $C_0 + C_3 + C_6 + C_9 + \ldots\ldots$

Hint (v): In the expansion of $(1+x)^n$, put $x = 1, \omega, \text{and}\, \omega^2$ and add the three equations.

Illustration 32: If $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ are n^{th} roots of unity, then the value of $(5-\omega)(5-\omega^2)\ldots\ldots(5-\omega^{n-1})$ is equal to **(JEE MAIN)**

Sol: Here consider x = $(1)^{\frac{1}{n}}$, therefore $x^n - 1 = 0$ (has n roots i.e. 1, $\omega, \omega^2, \ldots\ldots, \omega^{n-1}$).

$\Rightarrow x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots\ldots(x-\omega^{n-1})$ $\Rightarrow \dfrac{x^n-1}{x-1} = (x-\omega)(x-\omega^2)\ldots\ldots(x-\omega^{n-1})$

$\Rightarrow$ Putting x = 5 in both sides, we get $\therefore (5-\omega)(5-\omega^2)\ldots\ldots(5-\omega^{n-1}) = \dfrac{5^n-1}{4}$.

10. APPLICATION IN GEOMETRY

10.1 Distance Formula

Distance between $A(z_1)$ and $B(z_2)$ is given by AB = $|z_2 - z_1|$. Refer Fig 6.29.

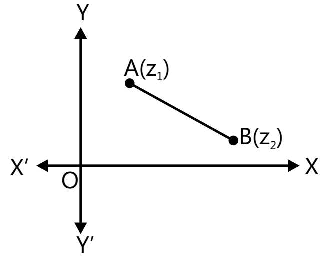


Figure 6.29

10.2 Section Formula

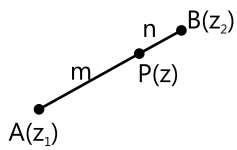


Figure 6.30

The point P(z) which divides the join of A(z_1) and B(z_2) in the ratio m: n is given by $z = \frac{mz_2 + nz_1}{m+n}$. Refer Fig 6.30.

10.3 Midpoint Formula

Mid-point M(z) of the segment AB is given by $z = \frac{1}{2}(z_1 + z_2)$.

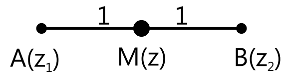


Figure 6.31 Mid point formula

10.4 Condition For Four Non-Collinear Points

Condition(s) for four non-collinear $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ to represent vertices of a

(a) Parallelogram: The diagonals AC and BD must bisect each other

$\Leftrightarrow \quad \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$

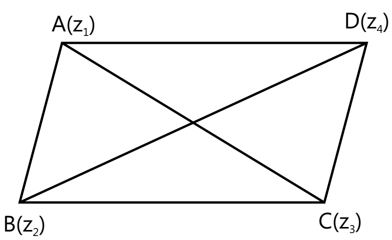


Figure 6.32

(b) Rhombus:

(i) The diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$, and

(ii) A pair of two adjacent sides are equal, for instance AD = AB

$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$

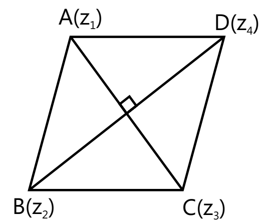


Figure 6.33

(c) Square:

(i) The diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$

(ii) A pair of adjacent sides are equal; for instance, AD = AB

$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$

(iii) The two diagonals are equal, that is AC = BD

$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|$

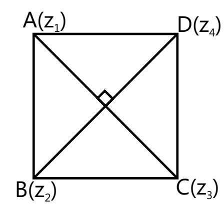


Figure 6.34

(d) Rectangle:

(i) The diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$

(ii) The diagonals AC and BD are equal

$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|$

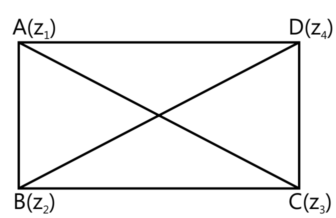


Figure 6.35

10.5 Triangle

In a triangle ABC, let the vertices A, B and C be represented by the complex numbers z_1, z_2, and z_3 respectively. Then

(a) Centroid: The centroid (G), is the point of intersection of medians of ΔABC. It is given by the formula

$$z = \frac{1}{3}(z_1 + z_2 + z_3)$$

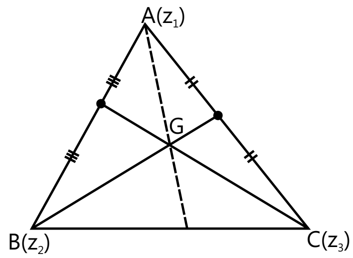


Figure 6.36 (a)

(b) Incentre: The incentre (I) of ΔABC is the point of intersection of internal angular bisectors of angles of ΔABC. It is given by the formula

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c},$$

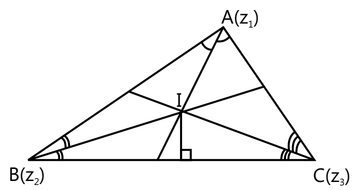


Figure 6.36 (b)

(c) Circumcentre: The circumcentre (S) of ΔABC is the point of intersection of perpendicular bisectors of sides of ΔABC. It is given by the formula

$$z = \frac{|z_1|^2 (z_2 - z_3) + |z_2|^2 (z_3 - z_1) + |z_3|^2 (z_1 - z_2)}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)} = \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}}$$

Also, $z = \dfrac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$

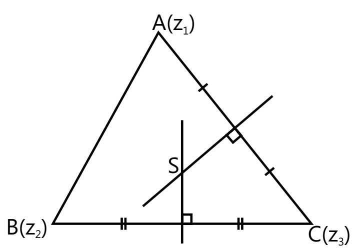


Figure 6.36 (c)

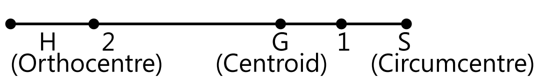


Figure 6.37

(d) Euler's Line: The orthocenter H, the centroid G and the circumcentre S of a triangle which is not equilateral lies on a straight line. In case of an equilateral triangle these points coincide.

G divides the join of H and S in the ratio 2 : 1 (see Fig 6.37).

Thus, $z_G = \frac{1}{3}(z_H + 2z_S)$

10.6 Area of a Triangle

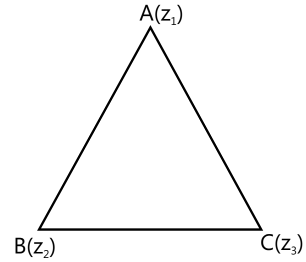


Figure 6.38

Area of ΔABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left| \frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right| = \left| \frac{1}{2}\text{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1) \right|$$

10.7 Conditions for Triangle to be Equilateral

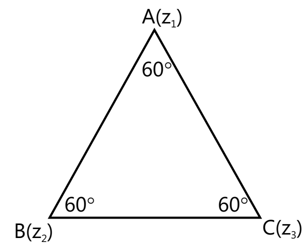


Figure 6.39

The triangle ABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is equilateral

iff $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$

$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2 \quad \Leftrightarrow z_1 \bar{z}_2 = z_2 \bar{z}_3 = z_3 \bar{z}_1 \Leftrightarrow z_1^2 = z_2 z_3$ and $z_2^2 = z_1 z_3$

$\Leftrightarrow \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0 \quad \Leftrightarrow \dfrac{z_2 - z_1}{z_3 - z_2} = \dfrac{z_3 - z_2}{z_1 - z_2}$

$\Leftrightarrow \dfrac{1}{z - z_1} + \dfrac{1}{z - z_2} + \dfrac{1}{z - z_3} = 0$ where $z = \dfrac{1}{3}(z_1 + z_2 + z_3)$.

10.8 Equation of a Straight line

(a) Non-parametric form: An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\text{Arg}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1 \bar{z}_2 - z_2 \bar{z}_1 = 0$

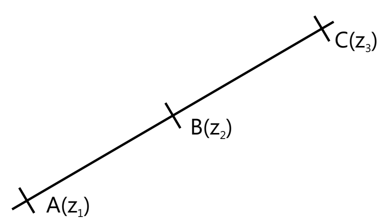


Figure 6.40

(b) Parametric form: An equation of the line segment between the points $A(z_1)$ and $B(z_2)$ is $z = tz_1 + (1 - t)z_2,\ t(0,1)$ where t is a real parameter.

(c) General equation of a straight line: The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$ where, a is non-zero complex number and b is a real number.

10.9 Complex Slope of a Line

If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be $\mu = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

Two lines with complex slopes μ_1 and μ_2 are

(i) Parallel, if $\mu_1 = \mu_2$ (ii) Perpendicular, if $\mu_1 + \mu_2 = 0$

The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\left(\dfrac{-a}{\bar{a}}\right)$.

10.10 Length of Perpendicular from a Point to a Line

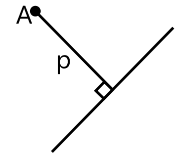


Figure 6.41

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$.

Where $a \in C - \{0\}$, and $b \in R$ is given by $p = \dfrac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$

10.11 Equation of Circle

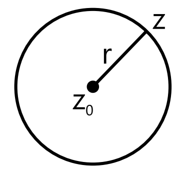


Figure 6.42

(a) An equation of the circle with centre z_0 and radius r is $|z - z_0| = r$ or

$z = z_0 + re^{i\theta}, 0 \le \theta < 2\pi$ (parametric form) or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

(b) General equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ ... (i)

Where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$. Centre of (i) is – a and its radius is $\sqrt{a\bar{a} - b}$

(c) Diameter form of a circle: An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is $(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$

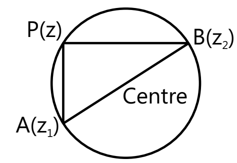


Figure 6.43

(d) An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$

is $(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) + i\,k\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$ where k is a real parameter.

(e) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$ and P(z) be any point on the circle through A, B and C.

Then either $\angle ACB = \angle APB$ [when angles are in the same segment]

or, $\angle ACB + \angle APB = \pi$ [when angles are in the opposite segment] (see Fig 6.44).

$$\Rightarrow \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0 \text{ or, } \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$$

$$\Rightarrow \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = 0$$

$$\text{or, } \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = \pi$$

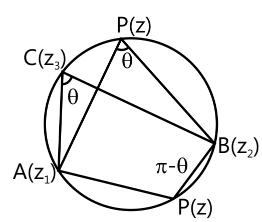


Figure 6.44

In any case, we get $\dfrac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

(f) Condition for four points to be concyclic.

Four points z_1, z_2, z_3 and z_4 will lie on the same circle if and only if $\dfrac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} = \frac{(\bar{z}_4 - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z}_4 - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

MASTERJEE CONCEPTS

Three points z_1, z_2 and z_3 are collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$.

If three points $A(z_1), B(z_2)$ and $C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$\Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 33: If the imaginary part of $\frac{2z+1}{iz+1}$ is – 4, then the locus of the point representing z in the complex plane is

(a) A straight line (b) A parabola (c) A circle (d) An ellipse **(JEE MAIN)**

Sol: Put z = x + iy and then equate its imaginary part to – 4.

Let z = x + iy, then $\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+2iy}{(1-y)+ix} = \frac{[(2x+1)+2iy][(1-y)-ix]}{(1-y)^2+x^2}$

As $\text{Im}\left(\frac{2z+1}{iz+1}\right) = -4$, we get $\frac{2y(1-y)-x(2x+1)}{x^2+(1-y)^2} = -4$

$\Rightarrow 2x^2+2y^2+x-2y = 4x^2+4(y^2-2y+1)$ $\Rightarrow 2x^2+2y^2-x-6y+4=0$. It represents a circle.

Illustration 34: The roots of $z^5 = (z-1)^5$ are represented in the argand plane by the points that are

(a) Collinear (b) Concyclic

(c) Vertices of a parallelogram (d) None of these **(JEE MAIN)**

Sol: Apply modulus on both the side of given expression.

Let z be a complex number satisfying $z^5 = (z-1)^5$.

$\Rightarrow |z^5| = |(z-1)^5|$ $\Rightarrow |z|^5 = |z-1|^5$ $\Rightarrow |z| = |z-1|$

Thus, z lies on the perpendicular bisector of the segment joining the origin and (1 + i0) i.e. z lies on Re(z) = $\frac{1}{2}$.

Illustration 35: Let z_1 and z_2 be two non-zero complex numbers such that $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$, then the origin and points represented by z_1 and z_2

(a) Lie on straight line (b) Form a right triangle

(c) Form an equilateral triangle (d) None of these **(JEE ADVANCED)**

Sol: Here consider z = $\frac{z_1}{z_2}$ and z_1 and z_2 are represented by A and B respectively and O be the origin.

Let z = $\frac{z_1}{z_2}$, then z + $\frac{1}{z}$ = 1 $\Rightarrow z^2 - z + 1 = 0$

$\Rightarrow z = \frac{1 \pm \sqrt{3}i}{2}$ $\Rightarrow \frac{z_1}{z_2} = \frac{1 \pm \sqrt{3}i}{2}$

If z_1 and z_2 are represented by A and B respectively and O be the origin, then

$$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \left|\frac{1 \pm \sqrt{3}i}{2}\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1 \quad \Rightarrow OA = OB$$

Also, $$\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left|1-\frac{z_1}{z_2}\right| = \left|1-\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right| = \left|\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$\Rightarrow AB = OB$ Thus, OA = OB = AB $\therefore$ ΔOAB is an equilateral triangle.

Illustration 36: If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, then the value of $(z_1 - z_2)^2 + (z_2 - z_3)^2$ is

(a) -1 (b) 0 (c) $(z_1 - z_3)^2$ (d) None of these **(JEE ADVANCED)**

Sol: Here use distance and argument formula of complex number to solve this problem.

As ABC is an isosceles right angled triangle with right angle at B,

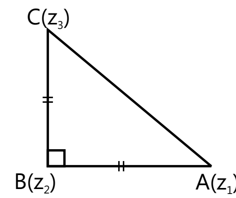


Figure 6.45

BA = BC and $\angle ABC = 90° \Rightarrow |z_1 - z_2| = |z_3 - z_2|$ and $\arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$

$$\Rightarrow \frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = i$$

$\Rightarrow (z_3 - z_2)^2 = -(z_1 - z_2)^2 \qquad \Rightarrow (z_1 - z_2)^2 + (z_2 - z_3)^2 = 0$.

11. CONCEPTS OF ROTATION OF COMPLEX NUMBER

Let z be a non-zero complex number. We can write z in the polar form as follows:

$z = r\,(\cos\theta + i\sin\theta) = re^{i\theta}$ where r = | z | and arg (z) = θ (see Fig 6.46).

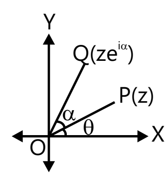


Figure 6.46

Consider a complex number $ze^{i\alpha}$.

$ze^{i\alpha} = (re^{i\theta})e^{i\alpha} = re^{i(\theta+\alpha)}$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$.

Geometrically, $ze^{i\alpha}$ can be obtained by rotating the line segment joining

O and P(z) through an angle α in the anticlockwise direction.

Corollary: If $A(z_1)$ and $B(z_2)$ are two complex number such that

$\angle AOB = \theta$, then $z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$ (see Fig 6.47).

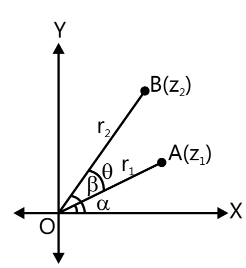


Figure 6.47

Let $z_1 = r_1e^{i\alpha}$ and $z_2 = r_2e^{i\beta}$ where $|z_1| = r_1, |z_2| = r_2$.

Then $\frac{z_2}{z_1} = \frac{r_2e^{i\beta}}{r_1e^{i\alpha}} = \frac{r_2}{r_1}e^{i(\beta-\alpha)}$

Thus, $\frac{z_2}{z_1} = \frac{r_2}{r_1}e^{i\theta}$ $(\because \beta - \alpha = \theta)$ $\Rightarrow z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$

MASTERJEE CONCEPTS

Multiplication of a complex number,z with i.

Let z = r $(\cos\theta + i\sin\theta)$ and i = $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, then iz = $r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$.

Hence, iz can be obtained by rotating the vector z by right angle in the positive sense. And so on, to multiply a vector by – 1 is to turn it through two right angles.

Thus, multiplying a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

Anvit Tawar (JEE 2009, AIR 9)

Illustration 37: Suppose A(z_1), B(z_2) and C(z_3) are the vertices of an equilateral triangle inscribed in the circle | z | = 2. If $z_1 = 1 + \sqrt{3}i$, then z_2 and z_3 are respectively.

(a) $-2, 1-\sqrt{3}i$ (b) $-1+\sqrt{3}i, -2$

(c) $-2, -1+\sqrt{3}i$ (d) $-2, 2+\sqrt{3}i$ **(JEE ADVANCED)**

Sol: As we know x + iy = $re^{i\theta}$. Hence by using this formula we can obtain z_2 and z_3.

$z_1 = 1+\sqrt{3}i = 2e^{\frac{i\pi}{3}}$

Since, $\angle AOC = \frac{2\pi}{3}$ and $\angle BOC = \frac{2\pi}{3}$, $z_2 = z_1 e^{\frac{2\pi i}{3}}$ and $z_3 = z_2 e^{\frac{2\pi i}{3}}$

$\Rightarrow z_3 = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$ and $z_3 = 2e^{\frac{5\pi i}{3}}$

$= 2\left[\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right]$

$= 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 1-\sqrt{3}i.$

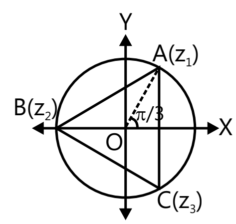


Figure 6.48

PROBLEM-SOLVING TACTICS

(a) On a complex plane, a complex number represents a point.

(b) In case of division and modulus of a complex number, the conjugates are very useful.

(c) For questions related to locus and for equations, use the algebraic form of the complex number.

(d) Polar form of a complex number is particularly useful in multiplication and division of complex numbers. It directly gives the modulus and the argument of the complex number.

(e) Translate unfamiliar statements by changing z into x+iy.

(f) Multiplying by $\cos\theta$ corresponds to rotation by angle θ about O in the positive sense.

(g) To put the complex number $\frac{a+ib}{c+id}$ in the form A + iB we should multiply the numerator and the denominator by the conjugate of the denominator.

(h) Care should be taken while calculating the argument of a complex number. If z = a + ib, then arg(z) is not always equal to $\tan^{-1}\left(\frac{b}{a}\right)$. To find the argument of a complex number, first determine the quadrant in which it lies, and then proceed to find the angle it makes with the positive x-axis.

For example, if z = – 1 – i, the formula $\tan^{-1}\left(\frac{b}{a}\right)$ gives the argument as $\frac{\pi}{4}$, while the actual argument is $\frac{-3\pi}{4}$.

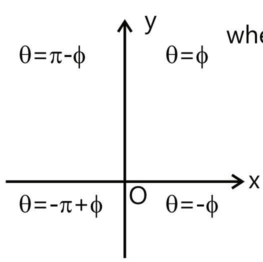
FORMULAE SHEET

(a) Complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.

(b) If z = x + iy then its conjugate $\bar{z} = x - iy$.

(c) Modulus of z, i.e. $|z| = \sqrt{x^2 + y^2}$

(d) Argument of z, i.e. $\theta = \begin{cases} \tan^{-1}\left|\frac{y}{x}\right| & x > 0, \quad y > 0 \\ \pi - \tan^{-1}\left|\frac{y}{x}\right| & x < 0, \quad y > 0 \\ -\pi + \tan^{-1}\left|\frac{y}{x}\right| & x < 0, \quad y < 0 \\ -\tan^{-1}\left|\frac{y}{x}\right| & x > 0, \quad y < 0 \end{cases}$



where, $\phi = \tan^{-1}\left|\frac{y}{x}\right|$

(e) If y=0, then argument of z, i.e. $\theta = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$

(f) If x=0, then argument of z, i.e. $\theta = \begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ \frac{3\pi}{2}, & \text{if } y < 0 \end{cases}$

(g) In polar form $x = r\cos\theta$ and $y = r\sin\theta$, therefore $z = r(\cos\theta + i\sin\theta)$

(h) In exponential form complex number $z = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.

(i) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

(j) Important properties of conjugate

(i) $z + \bar{z} = 2\text{Re}(z)$ and $z - \bar{z} = 2\,\text{Im}(z)$

(ii) $z = \bar{z} \Leftrightarrow z$ is purely real

(iii) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(iv) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vi) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(k) Important properties of modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \le \text{Re}(z) \le |z|$

(iv) $-|z| \le \text{Im}(z) \le |z|$

(v) $z\bar{z} = |z|^2$

If z_1, z_2 are two complex numbers, then

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$

(iv) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1 \bar{z}_2)$

(l) Important properties of argument

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

In fact $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

$$\text{where, } k = \begin{cases} 0, & \text{if } -\pi < \arg(z_1) + \arg(z_2) \le \pi \\ 1, & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \le -\pi \\ -1, & \text{if } \pi < \arg(z_1) + \arg(z_2) \le 2\pi \end{cases}$$

(iii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(v) $|z_1+z_2|=|z_1-z_2| \quad \Leftrightarrow \arg(z_1)-\arg(z_2)=\frac{\pi}{2}$

(vi) $|z_1+z_2|=|z_1|+|z_2| \quad \Leftrightarrow \arg(z_1)=\arg(z_2)$

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then

(vii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$

(viii) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

(m) Triangle on complex plane

(i) Centroid (G), $z_G=\frac{z_1+z_2+z_3}{3}$

(ii) Incentre (I), $z_I=\frac{az_1+bz_2+cz_3}{a+b+c}$

(iii) Orthocentre (H), z_H $\frac{z_1\tan A+z_2\tan B+z_3\tan C}{\sum\tan A}$

(iv) Circumcentre (S), z_S $\frac{z_1(\sin 2A)+z_2(\sin 2B)+z_3(\sin 2C)}{\sin 2A+\sin 2B+\sin 2C}$

(n) $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

(o) $\sqrt{z}=\sqrt{x+iy}=\pm\left[\sqrt{\frac{|z|+x}{2}}+i\sqrt{\frac{|z|-x}{2}}\right]$ for $y>0$

(p) Distance between $A(z_1)$ and $B(z_2)$ is given by $|z_2-z_1|$

(q) Section formula: The point P (z) which divides the join of the segment AB in the ratio m : n is given by $z=\frac{mz_2+nz_1}{m+n}$.

(r) Midpoint formula: $z=\frac{1}{2}(z_1+z_2)$.

(s) Equation of a straight line

(i) Non-parametric form: $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+z_1\bar{z}_2-z_2\bar{z}_1=0$

(ii) Parametric form: $z=tz_1+(1-t)z_2$

(iii) General equation of straight line: $\bar{a}z+a\bar{z}+b=0$

(t) Complex slope of a line, $\mu=\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$. Two lines with complex slopes μ_1 and μ_2 are

(i) Parallel, if $\mu_1=\mu_2$

(ii) Perpendicular, if $\mu_1+\mu_2=0$

(u) Equation of a circle: $|z-z_0|=r$