

Solved Examples

JEE Main/Boards

Example 1: If z_1 and z_2 are $1 - i$, $-2 + 4i$ respectively.

Find $\text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$.

$$\begin{aligned}\text{Sol: } \frac{z_1 z_2}{\bar{z}_1} &= \frac{(1-i)(-2+4i)}{1+i} = \frac{-2+2i+4i+4}{1+i} \\ &= \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{2+6i-2i+6}{2} = 4+2i \\ \therefore \text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) &= 2.\end{aligned}$$

Example 2: Find the square root of $z = -7 - 24i$.

Sol: Consider $z_0 = x + iy$ be a square root then $z_0^2 = -7 - 24i$.

$$-7 - 24i = x^2 - y^2 + 2ixy$$

Equating real and imaginary parts we get

$$x^2 - y^2 = -7 \quad \dots (i)$$

$$\text{and } 2xy = -24 \quad \dots (ii)$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2$$

$$= (-7)^2 + (-24)^2 = 625$$

$$\therefore x^2 + y^2 = 25 \quad \dots (iii)$$

Solving (i) and (iii), we get,

$$(x, y) = (3, -4); (-3, 4) \text{ by (ii)}$$

$$\therefore z_0 = \pm(3 - 4i).$$

Example 3: If n is a positive integer and ω be an imaginary cube root of unity, prove that

$$1 + \omega^n + \omega^{2n} \begin{cases} 3, & \text{when } n \text{ is a multiple of } 3 \\ 0, & \text{when } n \text{ is not a multiple of } 3 \end{cases}$$

Sol: Case I: $n = 3m; m \in \mathbb{I}$

$$\therefore 1 + \omega^n + \omega^{2n} = 1 + \omega^{3m} + \omega^{6m}$$

$$= 1 + 1 + 1 \quad [\because \omega^3 = 1] = 3$$

Case II: $n = 3m + 1$ or $3m + 2; m \in \mathbb{I}$

(a) Let $n = 3m + 1$

$$\therefore \text{L.H.S} = 1 + \omega^{3m+1} + \omega^{6m+2} = 1 + \omega + \omega^2 = 0$$

(b) Let $n = 3m + 2$

$$1 + \omega^{3m+2} + \omega^{6m+4} = 1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0.$$

Example 4: Show that $\left| \frac{z-3}{z+3} \right| = 2$ represents a circle.

Sol: Consider $z = x + iy$ and then by taking modulus we will get the result.

$$\text{Let } z = x + iy$$

$$\therefore \left| \frac{z-3}{z+3} \right| = 2 \Rightarrow \left| \frac{x-3+iy}{x+3+iy} \right| = 2$$

$$\therefore |x-3+iy|^2 = 2^2 |x+3+iy|^2$$

$$\text{or } (x-3)^2 + y^2 = 4((x+3)^2 + y^2)$$

$$\Rightarrow 3x^2 + 3y^2 + 30x + 27 = 0$$

which represents a circle.

Example 5: If $|z_1| = |z_2| = \dots = |z_n| = 1$

prove that $|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$

Sol: $|z_j| = 1 \Rightarrow z_j \bar{z}_j = 1 \quad \forall j = 1, \dots, n$

$$(\because z \bar{z} = |z|^2)$$

L.H.S.

$$|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \dots + \frac{1}{\bar{z}_n} \right| =$$

$$\left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \dots + \frac{1}{z_n} \right|$$

$$= \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \dots + \frac{1}{z_n} \right| = \text{R.H.S.}$$

Example 6: If $|z_1 + z_2| = |z_1 - z_2|$, prove that

$$\arg z_1 - \arg z_2 = \text{odd multiple of } \frac{\pi}{2}.$$

Sol: As we know $|z| = \sqrt{z \bar{z}}$. Apply this formula and

consider $z = r(\cos\theta + i \sin\theta)$.

$$|z_1 + z_2|^2 = |z_1 - z_2|^2$$

$$\Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \text{ or}$$

$$z_1\bar{z}_1 + z_2\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_2 = z_1\bar{z}_1 + z_2\bar{z}_2 - z_2\bar{z}_1 - z_1\bar{z}_2$$

$$\text{or } 2(z_2\bar{z}_1 + z_1\bar{z}_2) = 0; \operatorname{Re}(z_1\bar{z}_2) = 0$$

$$\text{Let } z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2);$$

$$\text{then } z_1\bar{z}_2 = r_1r_2(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$\therefore \cos(\theta_1 - \theta_2) = 0 \text{ (as } \operatorname{Re}(z_1\bar{z}_2) = 0)$$

$$\theta_1 - \theta_2 = \text{odd multiple of } \frac{\pi}{2}.$$

Example 7: If $|z - 1| < 3$, prove that $|iz + 3 - 5i| < 8$.

Sol: Here we have to reduce $iz + 3 - 5i$ as the sum of two complex numbers containing $z - 1$. because we have to use

$$|z - 1| < 3.$$

$$|iz + 3 - 5i| = |iz - i + 3 - 4i|$$

$$= |3 - 4i + i(z - 1)| \leq |3 - 4i| + |i(z - 1)|$$

$$\text{(by triangle inequality)} < 5 + 1 \cdot 3 = 8$$

Example 8: If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, then show that

$$(a) a_0 - a_2 + a_4 + \dots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$$

$$(b) a_1 - a_3 + a_5 + \dots = 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$$

Sol: Simply put $x = i$ in the given expansion and then by using formula

$$z = r(\cos\theta + i \sin\theta) \text{ and } (\cos\theta + i \sin\theta)^n$$

$$= \cos n\theta + i \sin n\theta, \text{ we can solve this problem.}$$

Put $x = i$ in the given expansion

$$(1 + i)^n = a_0 + a_1i + a_2i^2 + \dots + a_ni^n.$$

$$\left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n$$

$$= (a_0 - a_2 + a_4 - \dots) + i(a_1 - a_3 + a_5 - \dots)$$

$$2^{n/2} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right)$$

$$= (a_0 - a_2 + a_4 + \dots) + i(a_1 - a_3 + a_5 + \dots)$$

Equating real and imaginary parts.

$$2^{\frac{n}{2}} \cos \frac{n\pi}{4} = a_0 - a_2 + a_4 + \dots$$

$$2^{\frac{n}{2}} \sin \frac{n\pi}{4} = a_1 - a_3 + a_5 + \dots$$

Example 9: Solve the equation $z^{n-1} = \bar{z}; n \in \mathbb{N}$

Sol: Apply modulus on both side.

$$|z|^{n-1} = |\bar{z}|; \quad |z|^{n-1} = |\bar{z}| = |z|$$

$$\therefore |z| = 0 \text{ or } |z| = 1 \quad \text{If } |z| = 0 \text{ then } z = 0,$$

$$\text{Let } |z| = 1; \text{ then, } z^n = z \bar{z} = 1$$

$$\therefore z = \cos \frac{2m\pi}{n} + i \sin \frac{2m\pi}{n}; m = 0, 1, \dots, n-1$$

Example 10: If $z = x + iy$ and $\omega = \frac{1-iz}{z-i}$ with $|\omega| = 1$, show that, z lies on the real axis.

Sol: Substitute value of ω in $|\omega| = 1$.

$$|\omega| = \left| \frac{1-iz}{z-i} \right| = 1 \Rightarrow |1-iz| = |z-i|$$

$$\text{or, } |1-ix+y| = |x+i(y-1)|$$

$$\text{or, } (1+y)^2 + x^2 = x^2 + (y-1)^2 \text{ or, } 4y = 0$$

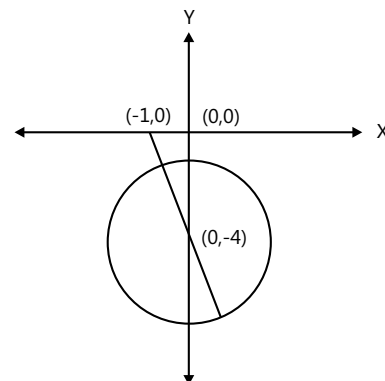
Hence z lies on the real axis.

Example 11: If a complex number z lies in the interior or on the boundary of a circle of radius as 3 and centre at $(0, -4)$ then greatest and least value of $|z + 1|$ are-

$$(A) 3 + \sqrt{17}, \sqrt{17} - 3 \quad (B) 6, 1$$

$$(C) \sqrt{17}, 1 \quad (D) 3, 1$$

Sol: Greatest and least value of $|z + 1|$ means maximum and minimum distance of circle from the point $(-1, 0)$. In circle greatest and least distance of it from any point is along the normal.



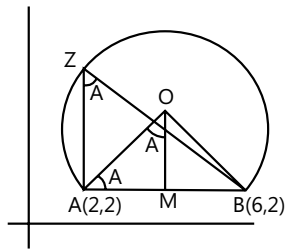
$$\therefore \text{Greatest distance} = 3 + \sqrt{1^2 + 4^2} = 3 + \sqrt{17}$$

$$\text{Least distance} = \sqrt{1^2 + 4^2} - 3 = \sqrt{17} - 3$$

Example 12: Find the equation of the circle for which

$$\arg \left(\frac{z-6-2i}{z-2-2i} \right) = \pi/4.$$

Sol: $\arg \left(\frac{z-6-2i}{z-2-2i} \right) = \pi/4$ represent a major arc of circle of which Line joining (6, 2) and (2, 2) is a chord that subtends an angle $\frac{\pi}{4}$ at circumference.



Clearly AB is parallel to real (x) axis, M is mid-point,

$$M \equiv (4, 2), OM = AM = 2$$

$\therefore O = (4, 4)$ and $OA^2 = OM^2 + AM^2 = 2\sqrt{2}$ Equation of required circle is

$$|z - 4 - 4i| = 2\sqrt{2}$$

Example 13: If $|z| \geq 3$, prove that the least value of

$$\left| z + \frac{1}{z} \right| \text{ is } \frac{8}{3}.$$

$$\text{Sol: Here } \left| z + \frac{1}{z} \right| \geq \left| z \right| - \frac{1}{|z|}.$$

$$\text{Now } |z| \geq 3$$

$$\therefore \frac{1}{|z|} \leq \frac{1}{3} \text{ or } -\frac{1}{|z|} \geq -\frac{1}{3} \quad \dots (i)$$

Adding the two like inequalities

$$\left| z \right| - \frac{1}{|z|} \geq 3 - \frac{1}{3} = \frac{8}{3} \quad \dots (ii)$$

$$\text{Hence from (i) and (ii), we get } \left| z + \frac{1}{z} \right| \geq \frac{8}{3}$$

$$\therefore \text{Least value is } \frac{8}{3}$$

Example 14: If z_1, z_2, z_3 are non-zero complex numbers such that $z_1 + z_2 + z_3 = 0$ and $z_1^{-1} + z_2^{-1} + z_3^{-1} = 0$ then prove that the given points are the vertices of an

equilateral triangle. Also show that $|z_1| = |z_2| = |z_3|$.

Sol: Use algebra to solve this problem.

Given $z_1 + z_2 + z_3 = 0$, and from 2nd relation $z_2 z_3 + z_3 z_1 + z_1 z_2 = 0$

$$\therefore z_2 z_3 = -z_1(z_2 + z_3) = -z_1(-z_1) = z_1^2$$

$$\therefore z_1^3 = z_1 z_2 z_3 = z_2^3 = z_3^3$$

$$\therefore |z_1|^3 = |z_2|^3 = |z_3|^3$$

Above shows that distance of origin from A, B, C is same.

Origin is circumcentre, but $z_1 + z_2 + z_3 = 0$

implies that centroid is also at the origin so that the triangle must be equilateral.

JEE Advanced/Boards

Example 1: For constant $c \geq 1$, find all complex numbers z satisfying the equation $z + c|z + 1| + i = 0$

Sol: Solve this by putting $z = x + iy$.

Let $z = x + iy$.

The equation $z + c|z + 1| + i = 0$ becomes

$$x + iy + c\sqrt{(x+1)^2 + y^2} + i = 0$$

$$\text{or } x + c\sqrt{(x+1)^2 + y^2} + i(y+1) = 0$$

Equating real and imaginary parts, we get

$$y + 1 = 0 \Rightarrow y = -1 \quad \dots (i)$$

$$\text{and } x + c\sqrt{(x+1)^2 + y^2} = 0 : x < 0 \quad \dots (ii)$$

Solving (i) and (ii), we get

$$x + c\sqrt{(x+1)^2 + 1} = 0 \text{ or } x^2 = c^2[(x+1)^2 + 1]$$

$$\text{or } (c^2 - 1)x^2 + 2c^2x + 2c^2 = 0$$

If $c = 1$, then $x = -1$. Let $c > 1$; then,

$$x = \frac{-2c^2 \pm \sqrt{4c^4 - 8c^2(c^2 - 1)}}{2(c^2 - 1)} = \frac{-c^2 \pm c\sqrt{2 - c^2}}{c^2 - 1}$$

As x is real and $c > 1$, we have: $1 < c \leq \sqrt{2}$

(Thus, for $c > \sqrt{2}$, there is no solution). Since both values of x satisfy (ii), both values are admissible.

Example 2: Find the sixth roots of $z = 64i$.

Sol: Here $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ and sixth root of z
i.e. $z_r = z^{1/6}$.

$$z = 64 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \quad \therefore z_r = z^{1/6}$$

$$= 2 \left[\cos \frac{2r\pi + \frac{\pi}{2}}{6} + i \sin \frac{2r\pi + \frac{\pi}{2}}{6} \right]$$

Where $r = 0, 1, 2, 3, 4, 5$

The roots $z_0, z_1, z_2, z_3, z_4, z_5$ are given by

$$z_0 = 2 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$z_1 = 2 \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

$$z_2 = 2 \left(\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} \right)$$

$$z_3 = 2 \left(\cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12} \right) = -2 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$z_4 = 2 \left(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right) = -2 \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

$$z_5 = 2 \left(\cos \frac{21\pi}{12} + i \sin \frac{21\pi}{12} \right) = -2 \left(\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} \right)$$

Example 3: Locate the region in the Argand plane for the complex number z satisfying

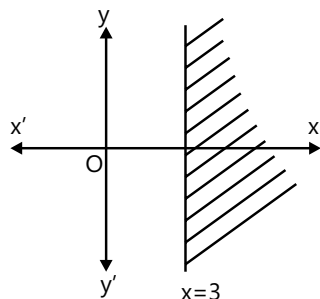
(a) $|z - 4| < |z - 2|$ (b) $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$

Sol: Consider $z = x + iy$ and solve by using properties of modulus and argument.

(a) Let $z = x + iy$

$$|x + iy - 4| < |x + iy - 2|$$

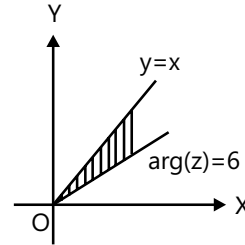
$$(x - 4)^2 + y^2 < (x - 2)^2 + y^2 \text{ or } -4x + 12 < 0$$



$\operatorname{Re}(z) > 3$ (see the Figure above)

(b) Let $z = x + iy$, then, $x > 0$ and $y > 0$

$$\arg z = \tan^{-1} \frac{y}{x} \quad \tan \frac{\pi}{6} \leq \frac{y}{x} \leq \tan \frac{\pi}{4}$$



$$\frac{1}{\sqrt{3}} \leq \frac{y}{x} \leq 1; \quad x \leq \sqrt{3}y \text{ and } y \leq x$$

Hence the given inequality represents the region bounded by the rays $y = x$ and $y = \frac{1}{\sqrt{3}}x$ except the origin.

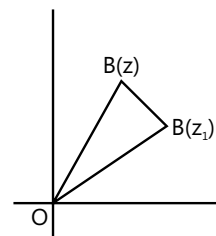
Example 4: If $z_1^2 + z_2^2 - 2z_1z_2 \cos \theta = 0$, show that the points z_1, z_2 and the origin, in the argand plane, are the vertices of an isosceles triangle.

Sol: By using formula of roots of quadratic equation we can solve it.

$$z_1^2 + z_2^2 - 2z_1z_2 \cos \theta = 0$$

$$\Rightarrow \left(\frac{z_1}{z_2} \right)^2 - 2 \left(\frac{z_1}{z_2} \right) \cos \theta + 1 = 0$$

$$\Rightarrow \left(\frac{z_1}{z_2} \right) = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2}$$



$$= \cos \theta \pm i \sin \theta$$

$$\Rightarrow \left| \frac{z_1}{z_2} \right| = |\cos \theta \pm i \sin \theta| = 1$$

$$\Rightarrow \left| \frac{z_1}{z_2} \right| = 1 \Rightarrow |z_1| = |z_2| \text{ or } OA = OB$$

Hence points $A(z)$, $B(z)$ and the origin are the vertices of an isosceles triangle.

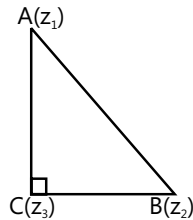
Example 5: Let three vertices A, B, C (taken in clock wise order) of an isosceles right angled triangle with right angle at C, be affixes of complex numbers z_1, z_2, z_3 respectively. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol: Here $\frac{z_2 - z_3}{z_1 - z_3} = e^{-i\pi/2}$. Therefore solve it using algebra method.

Given $CB = CA$ and angle $\angle C = \frac{\pi}{2}$

$$\frac{z_2 - z_3}{z_1 - z_3} = e^{-i\pi/2} \quad \text{or} \quad (z_3 - z_2)^2 = i^2(z_1 - z_3)^2$$

$$(z_3 - z_2)^2 = -(z_1 - z_3)^2$$



or

$$z_3^2 + z_2^2 - 2z_2z_3 + z_1^2 + z_3^2 - 2z_1z_3 = 0$$

Add and subtract $2z_1z_2$, we get

$$z_1^2 + z_2^2 - 2z_1z_2 + 2z_3^2 - 2z_2z_3 - 2z_1z_3 + 2z_1z_2 = 0, \text{ or}$$

$$(z_1 - z_2)^2 + 2[z_3(z_3 - z_2) - z_1(z_3 - z_2)] = 0 \text{ or}$$

$$(z_1 - z_2)^2 + 2(z_3 - z_1)(z_3 - z_2) = 0, \text{ or}$$

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$

Example 6: If A, B, C be the angles of triangle then

prove that $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is purely real.

Sol: Here $A+B+C = \pi$, therefore $e^{i\pi} = \cos \pi + i \sin \pi = -1$. And by using properties of matrices we can solve this problem.

$$e^{-i\pi} = -1 \quad \dots (i)$$

$$e^{i(B+C)} = e^{i(\pi-A)} = e^{i\pi} e^{-iA} = -e^{-iA}$$

$$e^{-i(B+C)} = -e^{iA} \quad \dots (ii)$$

Take e^{iA} , e^{iB} and e^{iC} common from R_1 , R_2 and R_3 respectively. $\Delta = e^{i(A+B+C)}$

$$\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(B+A)} \\ e^{-i(B+C)} & e^{-i(C+A)} & e^{iC} \end{vmatrix}$$

$$= -1 \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}, \text{ by (2)}$$

Take e^{iA} , e^{iB} and e^{iC} common from C_1 , C_2 and C_3 and again put $e^{i(A+B+C)} = e^{i\pi} = -1$.

$$\therefore \Delta = (-1)(-1) \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Now make two zeros and expand

$\Delta = -4$ which is purely real.

Example 7: Prove that $|a + b|^2 + |a - b|^2$

$= 2(|a|^2 + |b|^2)$. Interpret the result geometrically and

deduce that $|c + \sqrt{c^2 - d^2}| + |c - \sqrt{c^2 - d^2}| = |c + d| + |c - d|$; all numbers involved being complex

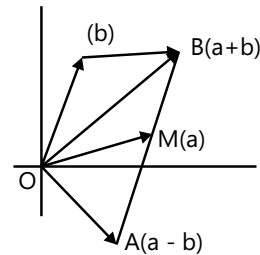
Sol: By using algebra of complex number and modulus property we can prove given expression. And then by using Appollonius theorem we can interpret the result geometrically.

$$S = |a + b|^2 + |a - b|^2$$

$$= |a|^2 + |b|^2 + 2\operatorname{Re}(a\bar{b}) + |a|^2 + |b|^2 - 2\operatorname{Re}(a\bar{b}) = 2(|a|^2 + |b|^2) \text{ (proved)}$$

$$\text{Now } |a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$$

This is nothing but Appollonius theorem. In $\triangle OAB$, M is midpoint of AB on applying Appollonius theorem we get



$$OA^2 + OB^2 = 2(AM^2 + OM^2)$$

$$\text{Now take } a = \sqrt{\frac{c+d}{2}} \text{ and } b = \sqrt{\frac{c-d}{2}}$$

Then using result

$$|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$$

$$\text{RHS} = 2 \left(\left| \sqrt{\frac{c+d}{2}} \right|^2 + \left| \sqrt{\frac{c-d}{2}} \right|^2 \right) = |c + d| + |c - d|$$

$$\text{L.H.S.} = \left(\left| \sqrt{\frac{c+d}{2}} + \sqrt{\frac{c-d}{2}} \right| \right)^2 + \left(\left| \sqrt{\frac{c+d}{2}} - \sqrt{\frac{c-d}{2}} \right| \right)^2$$

On simplifying we get

$$\text{L.H.S.} = \left| c + \sqrt{c^2 - d^2} \right| + \left| c - \sqrt{c^2 - d^2} \right|$$

Example 8: Show that the triangles whose vertices are z_1, z_2, z_3 and a, b, c are

$$\text{similar if } \begin{vmatrix} z_1 & a & 1 \\ z_2 & b & 1 \\ z_3 & c & 1 \end{vmatrix} = 0.$$

Sol: Consider triangle ABC and DEF are similar, therefore

$$\frac{AB}{DE} = \frac{BC}{EF} \text{ and } \angle ABC = \angle DEF.$$

Suppose z_1, z_2, z_3 are given by A, B, C respectively and a, b, c are given by D, E, F respectively. Since the triangle

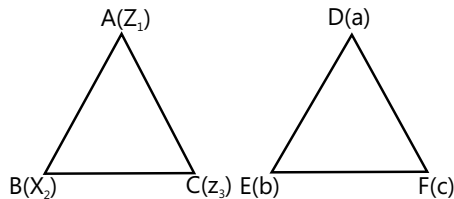
ABC and DEF are similar $\frac{AB}{DE} = \frac{BC}{EF}$ and

$$\angle ABC = \angle DEF = \alpha (\text{say})$$

$$\text{We have } \angle B = \arg \left(\frac{z_1 - z_2}{z_3 - z_2} \right) = \arg \left(\frac{a - b}{c - b} \right)$$

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC} (\cos \alpha + i \sin \alpha) \quad \dots (i)$$

$$\text{and } \frac{a - b}{c - b} = \frac{DE}{EF} (\cos \alpha + i \sin \alpha) \quad \dots (ii)$$



$$\text{Since } \frac{AB}{DE} = \frac{BC}{EF} \text{ we have } \frac{AB}{BC} = \frac{DE}{EF}$$

Thus, from (i) and (ii) we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{a - b}{c - b} \Rightarrow \begin{vmatrix} z_1 - z_2 & a - b \\ z_3 - z_2 & c - b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z_1 - z_2 & a - b & 0 \\ z_2 & b & 1 \\ z_3 - z_2 & c - b & 0 \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 + R_2$

$$\text{we get } \begin{vmatrix} z_1 & a & 1 \\ z_2 & b & 1 \\ z_3 & c & 1 \end{vmatrix} = 0$$

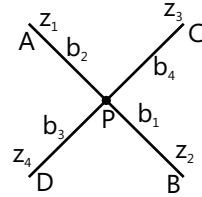
Example 9: If $b_1 + b_2 + b_3 + b_4 = 0$ where b_1 etc. are non-zero real numbers, sum of no two being zero, and $b_1 z_1 + b_2 z_2 + b_3 z_3 + b_4 z_4 = 0$ where no three of the points z_1, z_2, z_3, z_4 are collinear then prove that the four point concyclic if $b_1 b_2 |z_1 - z_2|^2 = b_3 b_4 |z_3 - z_4|^2$.

Sol: Here the four points A, B, C, D will be concyclic if $PA \cdot PB = PC \cdot CD$. therefore obtain PA, PB, PC and CD and simplify.

$$\text{Here } b_1 + b_2 = -(b_3 + b_4)$$

$$\text{Also } b_1 z_1 + b_2 z_2 = -(b_3 z_3 + b_4 z_4)$$

$$\text{Dividing these, } \frac{b_1 z_1 + b_2 z_2}{b_1 + b_2} = \frac{b_3 z_3 + b_4 z_4}{b_3 + b_4}$$



The left side gives the point that divides the line segment joining $A(z_1), B(z_2)$ in the ratio $b_2 : b_1$ and the right side gives the point that divides the line segment joining the points $C(z_3), D(z_4)$ in the ratio $b_4 : b_3$. So the line segments intersect at P which is

$$\text{Represented by } \frac{b_1 z_1 + b_2 z_2}{b_1 + b_2} \text{ as well as } \frac{b_3 z_3 + b_4 z_4}{b_3 + b_4}$$

$$\text{Now, } AB = |z_1 - z_2|$$

$$\therefore PA = \frac{b_2}{b_1 + b_2} (z_1 - z_2) \text{ \& } PB = \frac{b_1}{b_1 + b_2} (z_1 - z_2)$$

$$\text{Also, } CD = |z_3 - z_4|$$

$$\therefore PC = \frac{b_4}{b_3 + b_4} (z_3 - z_4) \text{ \& } PD = \frac{b_3}{b_3 + b_4} (z_3 - z_4)$$

The four points A, B, C, D will be concyclic if $PA \cdot PB = PC \cdot CD$

$$\text{i.e. } \frac{b_1 b_2}{(b_1 + b_2)^2} |z_1 - z_2|^2 = \frac{b_3 b_4}{(b_3 + b_4)^2} |z_3 - z_4|^2$$

$$\text{i.e. } b_1 b_2 |z_1 - z_2|^2 = b_3 b_4 |z_3 - z_4|^2$$

$$(\because b_1 + b_2 = -(b_3 + b_4))$$

Example 10: Show that all the roots of the equation $z^n \cos q_0 + z^{n-1} \cos q_1 + z^{n-2} \cos q_2 + \dots + z \cos q_{n-1} + \cos q_n = 2$ lie outside the circle $|z| = \frac{1}{2}$ where q_0, q_1 etc. are real.

Sol: By using triangle inequality.

$$\text{Here } |z^n \cos q_0 + z^{n-1} \cos q_1 + z^{n-2} \cos q_2 + \dots + z \cos q_{n-1} + \cos q_n| = 2 \quad \dots (i)$$

By triangle inequality.

$$\begin{aligned} 2 &= |z^n \cos q_0 + z^{n-1} \cos q_1 + z^{n-2} \cos q_2 + \dots + z \cos q_{n-1} + \cos q_n| \leq |z^n \cos q_0| + |z^{n-1} \cos q_1| + \\ &|z^{n-2} \cos q_2| + \dots + |z \cos q_{n-1}| + |\cos q_n| \\ &= |z_n| |\cos q_n| + |z^{n-1}| |\cos q_n| + \dots + |z| |\cos q_{n-1}| + |\cos q_n| \leq |z|^n + |z|^{n-1} + \dots + |z| + 1 \end{aligned}$$

$$(\because |\cos q_1| \leq 1 \text{ and } |z^{n+1}| = |z|^{n+1})$$

$$= \frac{1 - |z|^{n+1}}{1 - |z|} < \frac{1}{1 - |z|} \therefore 2 < \frac{1}{1 - |z|}$$

$$\text{So } 1 - |z| \text{ is positive and } 1 - |z| < \frac{1}{2}$$

$$\therefore |z| > 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \text{All } z \text{ satisfying (i) lie outside the circle } |z| = \frac{1}{2}$$

Example 11: If $z + \frac{1}{z} = 2 \cos \theta$, prove that $\left| \frac{z^{2n} - 1}{z^{2n} + 1} \right| = |\tan n\theta|$.

Sol: By using formula of roots of quadratic equation, we can solve this problem.

$$\text{Here } z + \frac{1}{z} = 2 \cos \theta;$$

$$\therefore z^2 - 2 \cos \theta \cdot z + 1 = 0$$

$$\therefore z = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cos \theta \pm i \sin \theta$$

$$\text{Taking positive sign, } z = \cos \theta + i \sin \theta$$

$$\therefore \frac{1}{z} = (\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta$$

$$\therefore \frac{z^{2n} - 1}{z^{2n} + 1} = \frac{z^n - \frac{1}{z^n}}{z^n + \frac{1}{z^n}} = \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n}$$

$$= \frac{\cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta)}{\cos n\theta + i \sin n\theta + (\cos n\theta - i \sin n\theta)}$$

$$= \frac{2i \sin n\theta}{2 \cos n\theta} = i \tan n\theta. \text{ Taking negative sign,}$$

$$\text{similarly we get } \frac{z^{2n} - 1}{z^{2n} + 1} = \frac{-2i \sin n\theta}{2 \cos n\theta} = -i \tan n\theta$$

$$\therefore \left| \frac{z^{2n} - 1}{z^{2n} + 1} \right| = |\pm i \tan n\theta| = |\tan n\theta|,$$

$$\text{For } |\pm i| = 1.$$

Example 12: Find the complex number z which satisfies the condition $|z - 2 + 2i| = 1$ and has the least absolute value.

Sol: Here $z - 2 + 2i = \cos \theta + i \sin \theta$, therefore by obtaining modulus of z we can solve above problem.

$$|z - 2 + 2i| = 1$$

$$\Rightarrow z - 2 + 2i = \cos \theta + i \sin \theta$$

Where θ is some real number.

$$\Rightarrow z = (2 + \cos \theta) + (\sin \theta - 2)i$$

$$\Rightarrow |z| = [(2 + \cos \theta)^2 + (\sin \theta - 2)^2]^{1/2}$$

$$= [8 + \cos^2 \theta + \sin^2 \theta + 4(\cos \theta - \sin \theta)]^{1/2}$$

$$= \left[9 + 4\sqrt{2} \cos \left(\theta + \frac{\pi}{4} \right) \right]^{1/2}$$

$|z|$ will be least if $\cos(\theta + \pi/4)$ is least, that is, if $\cos(\theta + \pi/4) = -1$ or $\theta = \frac{3\pi}{4}$. Thus, least value of $|z|$ is

$$\left(9 - 4\sqrt{2} \right)^{1/2} \text{ for } z = \left(2 - \frac{1}{\sqrt{2}} \right) + i \left(\frac{1}{\sqrt{2}} - 2 \right)$$

Example 13: For every real number $c \geq 0$, find all the complex numbers z which satisfy the equation. $2|z| - 4cz + 1 + ic = 0$.

Sol: Substitute $z = x + iy$ and equate real and imaginary part to zero.

$$2\sqrt{x^2 + y^2} - 4c(x + iy) + 1 + ic = 0$$

$$\therefore -4cy + c = 0 \Rightarrow y = \frac{1}{4} \quad \dots (i)$$

$$2\sqrt{x^2 + \frac{1}{16}} - 4cx + 1 = 0 \text{ or}$$

$$4\left(x^2 + \frac{1}{16}\right) = (4cx - 1)^2$$

$$4x^2(4c^2 - 1) - 8cx + \frac{3}{4} = 0$$

$$\therefore x = \frac{8c \pm \sqrt{64c^2 - 12(4c^2 - 1)}}{8(4c^2 - 1)}$$

$$\text{or } x = \frac{4c \pm \sqrt{4c^2 + 3}}{4(4c^2 - 1)} \quad \dots (ii)$$

x is real as $c \geq 0$, $z = (x, y)$ as given by (i) and (ii), $c \geq 0$.

Example 14: Consider a square ABCD such that z_1, z_2, z_3 and z_4 represent its vertices A, B, C and D respectively. Express ' z_3 ' and ' z_4 ' in terms of z_1 & z_2 .

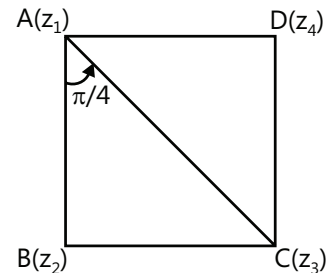
Sol: Consider the rotation of AB about A through an angle $\frac{\pi}{4}$.

$$\text{Therefore } \frac{z_3 - z_1}{z_2 - z_1} = \left| \frac{z_3 - z_1}{z_2 - z_1} \right| e^{i\frac{\pi}{4}}.$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \left| \frac{z_3 - z_1}{z_2 - z_1} \right| e^{i\frac{\pi}{4}} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$$

$$\text{Similarly } z_4 = z_1 + i(z_3 - z_1)$$



JEE Main/Boards

Exercise 1

Q.1 Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$.

Q.2 Express $\frac{1+2i+3i^2}{1-2i+3i^2}$ in the form $A + iB$.

Q.3 Find x and y if $(x + iy)(2 - 3i) = (4 + i)$

Q.4 Find x and y if $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$

Q.5 If $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$, where α and β are complex cube roots of unity, show that $xyz = a^3 + b^3$.

Q.6 $\frac{1+7i}{(2-i)^2}$ in the polar form.

Q.7 Find the square root of $-8 - 6i$.

Q.8 Find the value of smallest positive integer n , for which $\left(\frac{1+i}{1-i}\right)^n = 1$.

Q.9 Show that the complex number $z = x + iy$ which satisfies the equation $\left| \frac{z-5i}{z+5i} \right| = 1$ lies on the x -axis.

Q.10 If $z = 1 + i \tan \alpha$, where $\pi < \alpha < \frac{3\pi}{2}$, find the value of $|z| \cos \alpha$.

Q.11 If $1, \omega, \omega^2$ be the cube roots of unity, find the roots of the equation $(x-1)^3 + 8 = 0$.

Q.12 If $|z| < 4$, prove that $|iz + 3 - 4i| < 9$.

Q.13 $2 + i3$ is a vertex of square inscribed in circle $|z - 1| = 2$. Find other vertices.

Q.14 Find the centre and radius of the circle formed by the points represented by $z = x + iy$ satisfying the relation $\frac{|z - \alpha|}{|z - \beta|} = k (k \neq 1)$ where α & β are constant complex number's given by $\alpha = \alpha_1 + i\alpha_2$ & $\beta = \beta_1 + i\beta_2$

Q.15 Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^n a_r z^r = 1$ where $|a_r| < 2$.

Q.16 Let a complex number α , $\alpha \neq 1$, be a root of the equation $z^p + q - zp - zq + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \dots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \dots + \alpha^{q-1} = 0$, but not both together.

Q.17 Show that the area of the triangle on the Argand diagram formed by the complex numbers: z , iz and $z + iz$ is: $\frac{1}{2} |z|^2$.

Q.18 If $iz^3 + z^2 - z + i = 0$ then show that $|z| = 1$.

Q.19 Find the value of the expression

$$1(2 - \omega) (2 - \omega^2) + 2(3 - \omega) (3 - \omega^2) + \dots$$

+ $(n - 1) (n - \omega) (n - \omega^2)$ where ω is an imaginary cube root of unity.

Q.20 If $x = \frac{1}{2} (5 - \sqrt{3}i)$, then find the value of $x^4 - x^3 - 12x^2 + 23x + 12$.

Q.21 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that: $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Q.22 If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2 , prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

Q.23 Show that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \dots + \frac{H^2}{x-h} = x + \ell,$$

Where $A, B, C, \dots, a, b, c, \dots$ and ℓ are real, cannot have imaginary roots.

Q.24 Find the common roots of the equation

$$z^3 + 2z^2 + 2z + 1 = 0 \text{ and } z^{1985} + z^{100} + 1 = 0.$$

Q.25 If n is an odd integer greater than 3 but not a multiple of 3, prove that $[(x + y)^n - x^n - y^n]$ is divisible by $xy(x + y)(x^2 + xy + y^2)$.

Q.26 If α and β are any two complex numbers,

$$\begin{aligned} &\text{show that } \left| \alpha + \sqrt{\alpha^2 - \beta^2} \right| + \left| \alpha - \sqrt{\alpha^2 - \beta^2} \right| \\ &= |\alpha + \beta| + |\alpha - \beta| \end{aligned}$$

Q.27 Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{z - z_1}{z - z_2}$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

Q.28 If $|z| \leq 1, |w| \leq 1$, show that

$$|z - w|^2 \leq (|z| - |w|)^2 + (\arg z - \arg w)^2.$$

Q.29 Let A and B be two complex numbers such

that $\frac{A}{B} + \frac{B}{A} = 1$, prove that the origin and the points represented by A and B form the vertices of an equilateral triangle.

Q.30 Let z_1, z_2, z_3 be three complex numbers and a, b, c be real number not all zero, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$.

Show that z_1, z_2, z_3 are collinear.

Q.31 If $|z - 4 + 3i| \leq 2$, find the least and the greatest values of $|z|$ and hence find the limits between which $|z|$ lies.

Q.32 If $|z_1| < 1$ and $\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| < 1$, then show that $|z_2| < 1$.

Q.33 Find the locus of points z if $\log_{\sqrt{3}} \frac{|z|^2 - |z| + 1}{2 + |z|} < 2$.

Q.34 For complex numbers z and ω , prove that $|z|^2 \omega - |\omega|^2 z = z - \omega$ if and only if $z = \omega$ or $\bar{z}\omega = 1$.

Exercise 2

Single Correct Choice Type

Q.1 $|z + 4| \leq 3, Z \in \mathbb{C}$: then the greatest and least value of $|z + 1|$ are:

- (A) (7, 1) (B) (6, 1) (C) (6, 0) (D) None

Q.2 The maximum & minimum values of $|z + 1|$ when $|z + 3| \leq 3$ are

- (A) (5, 0) (B) (6, 0) (C) (7, 1) (D) (5, 1)

Q.3 The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is:

(A) $z = \frac{(3 + 2\sqrt{3})}{2} + \frac{\sqrt{3} + 2}{2}i$

(B) $z = 5 + 5i$

(C) $z = -1 - i$

(D) None of these

Q.4 If z_1, z_2, z_3, z_4 are the vertices of a square in that order, then which of the following do(es) not hold good?

- (A) $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary
 (B) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
 (C) $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary
 (D) None of these

Q.5 Let z_1, z_2 and z_3 be the complex numbers representing the vertices of a triangle ABC respectively and a, b, c are lengths of BC, CA, AB. If P is a point representing the complex number z_0 satisfying: $a(z_1 - z_0) + b(z_2 - z_0) + c(z_3 - z_0) = 0$, then w.r.t. the triangle ABC, the point P is its:

- (A) Centroid (B) Orthocentre
 (C) Circumcentre (D) Incentre

Q.6 Three complex numbers α, β & γ are represented in the Argand diagram by the three points A, B, C respectively. The complex number represented by D where A, B, C, D form a parallelogram with BD on a diagonal is:

- (A) $\alpha - \beta + \gamma$ (B) $-\alpha + \beta + \gamma$
 (C) $\alpha + \beta - \gamma$ (D) $\alpha - \beta - \gamma$

Q.7 If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is

- (A) $\frac{5}{3}$ (B) $\frac{8}{3}$ (C) $\frac{11}{3}$ (D) None of these

Q.8 Point z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 ($z_3 \neq z_1$) can be represented by:

- (A) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$
 (B) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 - z_2)$
 (C) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_2 - z_1)$
 (D) None of these

Q.9 If q_1, q_2, q_3 are the roots of the equation, $x^3 + 64 = 0$,

then the value of the determinant $\begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_1 \\ q_3 & q_1 & q_2 \end{vmatrix}$ is:

- (A) 1 (B) 4
 (C) 10 (D) none of these

Q.10 $z = (3 + 7i)(p + iq)$ where $p, q \in \mathbb{I} - \{0\}$ purely imaginary then minimum value of $|z|^2$ is

- (A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

Q.11 On the complex plane triangles OAP & OQR are similar and $\angle(OA) = 1$. If the points P and Q denotes the complex numbers z_1 & z_2 then the complex number 'z' denoted by the point R is given by:

- (A) $z_1 z_2$ (B) $\frac{z_1}{z_2}$ (C) $\frac{z_2}{z_1}$ (D) $\frac{z_1 + z_2}{z_2}$

Q.12 If A and B be two complex numbers satisfying $\frac{A}{B} + \frac{B}{A} = 1$. Then the two points represented by A and B and the origin form the vertices of

- (A) An equilateral triangle
 (B) An isosceles triangle which is not equilateral
 (C) An isosceles triangle which is not right angled
 (D) A right angled triangle

Q.13 The solutions of the equation in z , $|z|^2 - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$ are:

- (A) $2 + i, 1 - i$ (B) $1 + i, 1 - i$
 (C) $1 + 2i, -1 - i$ (D) $1 + i, 1 + i$

Q.14 If $z_1 = -3 + 5i$; $z_2 = -5 - 3i$ and z is a complex number lying on the line segment joining z_1 & z_2 then arg z can be:

- (A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

Q.15 The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are:

- (A) $\frac{1}{2}(7 \pm i\sqrt{3})$ (B) $\frac{1}{2}(3 \pm i\sqrt{7})$
 (C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

Q.16 Let z to be complex number having the argument θ , $0 < \theta < \frac{\pi}{2}$ and satisfying the equality $|z - 3i| = 3$. Then $\cot \theta - \frac{6}{z}$ is equal to:

- (A) 1 (B) -1 (C) i (D) $-i$

Q.17 The locus represented by the equation, $|z - 1| + |z + 1| = 2$ is:

- (A) An ellipse with foci $(1, 0)$; $(-1, 0)$
 (B) One of the family of circles passing through the points of intersection of the circles $|z + 1| = 1$
 (C) The radical axis of the circles $|z - 1| = 1$ and $|z + 1| = 1$
 (D) The portion of the real axis between the points $(1, 0)$ and $(-1, 0)$

Q.18 Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2} \cos\left(\frac{\pi}{4} + \theta\right)$ where $\theta = \text{amp } z$ if 'O' is the origin,

then the $\triangle OPQ$ is:

- (A) Isosceles but not right angled
 (B) Right angled but not isosceles
 (C) Right isosceles
 (D) Equilateral

Q.19 Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$.

If $z_1 + z_2 + z_3 = 3$ then z_1, z_2, z_3 must represent the vertices of:

- (A) An equilateral triangle
 (B) An isosceles triangles which is not equilateral
 (C) A right triangle
 (D) Nothing definite can be said

Q.20 If $p = a + b\omega + c\omega^2$; $q = b + c\omega + a\omega^2$; and $r = c + a\omega + b\omega^2$ where $a, b, c \neq 0$ and ω is the complex cube root of unity, then:

- (A) $p + q + r = a + b + c$
 (B) $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$
 (C) $p^2 + q^2 + r^2 = 2(pq + qr + rp)$
 (D) None of these

Previous Years' Questions

Q.1 The smallest positive integer n for which

$$\left(\frac{1+i}{1-i}\right)^n = 1, \text{ is} \quad (1980)$$

- (A) 8 (B) 16
 (C) 12 (D) None of these

Q.2 The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on

(1981)

- (A) The x -axis
 (B) The straight line $y = 5$
 (C) A circle passing through the origin
 (D) None of these

Q.3 If $z = x + iy$ and $w = (1 - iz) / (z - i)$, then $|w| = 1$ implies that, in the complex plane

(1983)

- (A) z lies on the imaginary axis
 (B) z lies on the real axis
 (C) z lies on the unit circle
 (D) None of these

Q.4 The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if

(1983)

- (A) $z_1 + z_4 = z_2 + z_3$ (B) $z_1 + z_3 = z_2 + z_4$
 (C) $z_1 + z_2 = z_3 + z_4$ (D) None of these

Q.5 If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(1987)

- (A) $-\pi$ (B) $-\frac{\pi}{2}$ (C) 0 (D) $\frac{\pi}{2}$

Q.6 The complex numbers $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other, for

(1988)

- (A) $x = n\pi$ (B) $x = 0$
 (C) $x = \left(n + \frac{1}{2}\right)\pi$ (D) No value of x

Q.7 If $\omega (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B are respectively

(1995)

- (A) 0, 1 (B) 1, 1 (C) 1, 0 (D) -1, 1

Q.8 Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$, then z equals **(1995)**

- (A) ω (B) $-\omega$ (C) $\bar{\omega}$ (D) $-\bar{\omega}$

Q.9 If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to **(1998)**

- (A) 128ω (B) -128ω (C) $128\omega^2$ (D) $-128\omega^2$

Q.10 The value of sum $\sum_{n=1}^{13} (i^n + i^{-n+1})$ where $i = \sqrt{-1}$ equals **(1998)**

- (A) i (B) $i - 1$ (C) $-i$ (D) 0

Q.11 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then **(1998)**

- (A) $x = 3, y = 1$ (B) $x = 1, y = 1$
(C) $x = 0, y = 3$ (D) $x = 0, y = 0$

Q.12 If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$, then $|z_1 + z_2 + z_3|$ is **(2000)**

- (A) Equal to 1 (B) Less than 1
(C) Greater than 3 (D) Equal to 3

Q.13 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then value of the determinant

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is **(2002)**

- (A) 3ω (B) $3\omega(\omega - 1)$ (C) $3\omega^2$ (D) $3\omega(1 - \omega)$

Q.14 If $\omega (\neq 1)$ be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is **(2004)**

- (A) 2 (B) 3 (C) 5 (D) 6

Q.15 A man walks a distance of 3 units from the origin towards the North-West ($N 45^\circ E$) direction. From there, he walks a distance of 4 units towards the North-West ($N 45^\circ W$) direction to reach a point P. Then, the position of P in the Argand plane is **(2007)**

- (A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$
(C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

Q.16 If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies **(2012)**

- (A) Either on the real axis or on a circle passing through the origin.
(B) On a circle with centre at the origin.
(C) Either on the real axis or on a circle not passing through the origin.
(D) On the imaginary axis.

Q.17 If z is a complex number of unit modulus and argument θ , then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals **(2013)**

- (A) $\frac{\pi}{2} - \theta$ (B) θ (C) $\pi - \theta$ (D) $-\theta$

Q.18 If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{z}\right|$ **(2014)**

- (A) Is equal to $\frac{5}{2}$
(B) Lies in the interval $(1, 2)$
(C) Is strictly greater than $\frac{5}{2}$
(D) Is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

Q.19 A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular.

Then the point z_1 lies on a: **(2015)**

- (A) Straight line parallel to y-axis
(B) Circle of radius 2
(C) Circle of radius $\sqrt{2}$
(D) Straight line parallel to x-axis

Q.20 A value of θ for which $\frac{2 + 3i \sin \theta}{1 - 2i \sin \theta}$ is purely imaginary, is: **(2016)**

- (A) $\frac{\pi}{6}$ (B) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$
(C) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (D) $\frac{\pi}{3}$

JEE Advanced/Boards

Exercise 1

Q.1 Prove that with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p, p', q, q' are all real.

(i) If the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$

(ii) If the equation has two equal roots then $p^2 - q'^2 = 4q$ and $pp' = 2q'$.

state whether these equal roots are real or complex.

Q.2 Let $z = 18 + 26i$ where $z_0 = x_0 + iy_0$ ($x_0, y_0 \in \mathbb{R}$) is the cube roots of z having least positive argument. Find the value of $x_0 y_0 (x_0 + y_0)$.

Q.3 Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the coordinate axes in the Argand's plane.

Further show that $|z| = \sqrt{\frac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z) + 1}}$

Q.4 Consider the diagonal matrix $A_n = \text{dia}(d_1, d_2, d_3, \dots, d_n)$ of order where

$D_i = a^{i-1}$, $1 \leq i \leq n$ and $\alpha = e^{\frac{i2\pi}{n}}$; $i = \sqrt{-1}$, is the n^{th} root of unity.

Let L : represent the value of $\text{Tr.}(A_7)^7$.

M : denotes the value of $\det(A_{2n+1}) + \det(A_{2n})$.

Find the value of $(L + M)$.

[Note: $T_r(A)$ denotes trace of square matrix A]

Q.5 Let $z_1, z_2 \in \mathbb{C}$ such that $z_1^2 + z_2^2 \in \mathbb{R}$. If $z_1(z_1^2 - 3z_2^2) = 10$ and $z_2(3z_1^2 - z_2^2) = 30$. Find the value of $(z_1^2 + z_2^2)$.

Q.6 If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \dots, z_7$, find the value of

(a) $\sum_{r=1}^7 \text{Re}(Z_r)$ and $\sum_{r=1}^7 \text{Im}(Z_r)$

Q.7 If z is one of the imaginary 7th roots of unity, then find the equation whose roots are $(z + z^4 + z^2)$ and $(z^6 + z^3 + z^5)$.

Q.8 If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

Q.9 Let z_1 & z_2 be any two arbitrary complex numbers then prove that:

$$|z_1| + |z_2| \geq \frac{1}{2}(|z_1| + |z_2|) \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right|$$

Q.10 Let z_i ($i = 1, 2, 3, 4$) represent the vertices of a square all of which lie on the sides of the triangle with vertices $(0, 0)$, $(2, 1)$ and $(3, 0)$. If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is m (where m and n are in their lowest form). Find the value of $(m + n)$.

Q.11 (i) Let C_r 's denotes the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in \mathbb{N}$. If the integers

$$a_n = C_0 + C_3 + C_6 + C_9 + \dots$$

$$b_n = C_1 + C_4 + C_7 + C_{10} + \dots$$

$$\text{and } c_n = C_2 + C_5 + C_8 + C_{11} + \dots$$

then prove that

$$(a) a_n^3 + b_n^3 + c_n^3 - 3a_n b_n c_n = 2^n.$$

$$(b) (a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2$$

(ii) Prove the identity:

$$(C_0 - C_2 + C_4 - C_6 + \dots)^2$$

$$+ (C_1 - C_3 + C_5 - C_7 + \dots)^2 = 2^n.$$

Q.12 Let z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that:

$$(i) 2z_2 = (1 + i)z_1 + (1 - i)z_3 \text{ \& (ii) } 2z_4 = (1 - i)z_1 + (1 + i)z_3$$

Q.13 A function f is defined on the complex number by $f(z) = (a + bi)z$, where ' a ' and ' b ' are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that

$b^2 = \frac{u}{v}$ where u and v are co-primes. Find the value of $(u + v)$.

Q.14 Prove that

$$(a) \cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \dots + {}^nC_n \cos (n+1)$$

$$x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos \left(\frac{n+2}{2} \right) x$$

$$(b) \sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \dots + {}^nC_n \sin(n+1)$$

$$x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin \left(\frac{n+2}{2} \right) x.$$

Q.15 Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1+i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$.

Q.16 Let $\omega_1, \omega_2, \omega_3, \dots, \omega_n$ be the complex numbers. A line L on the complex plane is called a mean line for the points $\omega_1, \omega_2, \omega_3, \dots, \omega_n$ if L contains the points (complex

numbers) $z_1, z_2, z_3, \dots, z_n$ such that $\sum_{k=1}^n (z_k - \omega_k) = 0$.

Now for the complex number $\omega_1 = 32 + 170i$, $\omega_2 = -7 + 64i$, $\omega_3 = -9 + 200i$, $\omega_4 = 1 + 27i$ and $\omega_5 = -14 + 43i$, there is a unique mean line with y -intercept 3. Find the slope of the line.

Q.17 A particle start to travel from a point P on the curve $C_1: |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P , the particle moves through an angle $\tan^{-1} \frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q . From Q , it comes down parallel to imaginary axis by 2 units and reaches at point R . Find the complex number corresponding to point R in the Argand plane.

Q.18 Evaluate: $\sum_{p=1}^{32} (3p-2) \left(\sum_{q=1}^{10} \left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right)^p$

Q.19 Let a, b, c be distinct complex numbers

$$\text{such that } \frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k.$$

Find the value of k .

Q.20 Let α, β be fixed complex numbers and z is a variable complex number such that

$$|z - \alpha|^2 + |z - \beta|^2 = k$$

Find out the limits for ' k ' such that the locus of z is a circle. Find also the centre and radius of the circle.

Q.21 C is the complex number. $f: C \rightarrow R$ is defined by $f(z) = |z^3 - z + 2|$. Find the maximum value of $f(z)$ if $|z| = 1$.

Q.22 Let a, b, c are distinct integers and ω, ω^2 are the imaginary cube roots of unity. If minimum value of $|a + bw + c\omega^2| + |a + b\omega^2 + c\omega|$ is $n^{\frac{1}{4}}$ where $n \in N$, then find the value of n .

Q.23 If the area of the polygon whose vertices are the solutions (in the complex plane) of the equation

$$x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

can be expressed in the simplest form as $\frac{a\sqrt{b}+c}{d}$, find the value of $(a + b + c + d)$.

Q.24 If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer.

Find N .

Q.25 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ ($a, b, c, d \in R$) has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of ' b '.

Q.26 Resolve $z^5 + 1$ into linear and quadratic factors with real coefficients. Deduce that:

$$4 \sin \frac{\pi}{10} \cos \frac{\pi}{5} = 1.$$

Q.27 If $x = 1 + i\sqrt{3}$, $y = 1 - i\sqrt{3}$ & $z = 2$,

prove that $x^p + y^p = z^p$ for every prime $p > 3$.

Q.28 Dividing $f(z)$ by $z - i$, we get the remainder i and dividing it by $z + i$ we get the remainder $1 + i$. Find the remainder upon the division of $f(z)$ by $z^2 + 1$.

Q.29 (a) Let $z = a + b$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

$$A = \{z | |z| \leq 2\} \text{ and}$$

$$B = \{z | (1-i)z + (1+i)\bar{z} \geq 4\}.$$

Find the area of the region $A \cap B$.

(b) For all real numbers x , let the mapping $f(x) = \frac{1}{x-i}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which $f(a), f(b), f(c)$ and $f(d)$ form a square on the complex plane. Find the area of the square.

Q.30

Column I	Column II
(A) Let ω be a non-real cube root of unity then the number of distinct elements in the set $\{(1 + \omega + \omega^2 + \dots + \omega^n)^m \mid m, n \in \mathbb{N}\}$	(p) 4
(B) Let $1, \omega, \omega^2$ be the cube root of unity. The least possible degree of a polynomial with real coefficients having roots $2\omega, (2+3\omega), (2+3\omega^2), (2-\omega-\omega^2)$, is	(q) 5
(C) $\alpha = 6+4i$ and $\beta = (2+4i)$ are two complex numbers on the complex plane. A complex number z . A complex number z satisfying $\arg\left(\frac{z-\alpha}{z-\beta}\right) = \frac{\pi}{6}$ moves on the major segment of a circle whose radius is	(r) 6
	(s) 7

Exercise 2**Single Correct Choice Type**

Q.1 The set of points on the complex plane such that $z^2 + z + 1$ is real and positive. (where $z = x + iy, x, y \in \mathbb{R}$) is:

- (A) Complete real axis only
 (B) Complete real axis or all points on the line $2x + 1 = 0$
 (C) Complete real axis or a line segment joining points

$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ & $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ excluding both.

(D) Complete real axis or set of points lying inside the rectangle formed by the lines.

$$2x + 1 = 0; 2x - 1 = 0; 2y - \sqrt{3} = 0 \text{ \& } 2y + \sqrt{3} = 0$$

Q.2 The complex numbers whose real and imaginary parts are integers and satisfy the relation $z\bar{z}^3 + z^3\bar{z} = 350$ form a rectangle on the Argand plane, the length of whose diagonal is-

- (A) 5 (B) 10 (C) 15 (D) 25

Q.3 Let z_1 and z_2 be non-zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$.

The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 .

- (A) An isosceles right angled triangle
 (B) A right angled triangle which is not isosceles
 (C) An equilateral triangle
 (D) An isosceles triangle which is not right angled

Q.4 The set of points on the Argand diagram which satisfy both $|z| \leq 4$ & $\arg z = \frac{\pi}{3}$ is:

- (A) A circle and line (B) A radius of a circle
 (C) A sector of a circle (D) An infinite part line

Q.5 If z_1 & z_2 are two complex numbers & if $\arg \frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$ then the figure formed by the points represented by $0, z_1, z_2$ & $z_1 + z_2$ is:

- (A) A parallelogram but not a square rectangle or a rhombus
 (B) A rectangle but not a square
 (C) A rhombus but not a square
 (D) A square

Q.6 If z_1, z_2, z_3 are the vertices of the $\triangle ABC$ on the complex plane & are also the roots of the equation $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the $\triangle ABC$ to be equilateral triangle is:

- (A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$
 (C) $\alpha^2 = 2\beta$ (D) $\alpha = 3\beta^2$

Q.7 Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin then the orthocenter is represented by the complex number:

- (A) $z_1 + z_2 - z_3$ (B) $z_2 + z_3 - z_1$
 (C) $z_3 + z_1 - z_2$ (D) $z_1 + z_2 + z_3$

Q.8 Which of the following represents a point in an argand's plane, equidistant from the roots of equation $(z + 1)^4 = 16z^4$?

- (A) (0, 0) (B) $\left(-\frac{1}{3}, 0\right)$ (C) $\left(\frac{1}{3}, 0\right)$ (D) $\left(0, \frac{2}{\sqrt{5}}\right)$

Q.9 The equation of the radical axis of the two circles presented by the equations.

$|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is:

- (A) $3y + 1 = 0$ (B) $3y - 1 = 0$
(C) $2y - 1 = 0$ (D) None of these

Q.10 Number of real solution of the equation, $z^3 + iz - 1 = 0$ is

- (A) Zero (B) One (C) Two (D) Three

Q.11 A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are:

- (A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

Q.12 Let $z = 1 - \sin \alpha + i \cos \alpha$ where $\alpha \in \left(0, \frac{\pi}{2}\right)$, then the modulus and the principle value of the argument of z are respectively:

- (A) $\sqrt{2(1 - \sin \alpha)}, \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ (B) $\sqrt{2(1 - \sin \alpha)}, \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$
(C) $\sqrt{2(1 + \sin \alpha)}, \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ (D) $\sqrt{2(1 + \sin \alpha)}, \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

Q.13 z_1 and z_2 are complex numbers. Then

$$\text{Equ. } \left| \frac{z_1 + z_2}{2} + \sqrt{z_1 z_2} \right| + \left| \frac{z_1 + z_2}{2} - \sqrt{z_1 z_2} \right| =$$

- (A) $2 \left| \sqrt{z_1} + \sqrt{z_2} \right|$ (B) $2 \left| \sqrt{z_1} - \sqrt{z_2} \right|$
(C) $2 \left(\left| \sqrt{z_1} \right|^2 + \left| \sqrt{z_2} \right|^2 \right)$ (D) $\left(\left| \sqrt{z_1} \right|^2 + \left| \sqrt{z_2} \right|^2 \right)$

Q.14 If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot \theta = x + 1$, then $\frac{(x + \alpha)^n - (x + \beta)^n}{\alpha - \beta}$ is equal to:

- (A) $\frac{\sin n\theta}{\sin^n \theta}$ (B) $\frac{\cos n\theta}{\cos^n \theta}$ (C) $\frac{\sin n\theta}{\cos^n \theta}$ (D) $\frac{\cos n\theta}{\sin^n \theta}$

Q.15 If A_r ($r = 1, 2, 3, \dots, n$) are the vertices of a regular polygon inscribed in a circle of radius R , then

$$(A_1 A_2)^2 + (A_1 A_3)^2 + (A_1 A_4)^2 + \dots + (A_1 A_n)^2 =$$

- (A) $\frac{nR^2}{2}$ (B) $2nR^2$
(C) $4R^2 \cot \frac{\pi}{2n}$ (D) $(2n - 1) R^2$

Q.16 If the equation $z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero has a pure imaginary root then the expression

$$\frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3}$$
 has the value equal to

- (A) 0 (B) 1 (C) -2 (D) 2

Q.17 All roots of the equation $(1 + z)^6 + z^6 = 0$

- (A) Lie on a unit circle with centre at the origin
(B) Lie on a unit circle with centre at $(-1, 0)$
(C) Lie on the vertices of a regular polygon with centre at the origin
(D) Are collinear

Q.18 Number of roots of the equation $z^{10} - z^5 - 992 = 0$ with real part negative is:

- (A) 3 (B) 4 (C) 5 (D) 6

Q.19 z_1 and z_2 are two distinct points in argand plane.

If $a|z_1| = b|z_2|$, then the point $\frac{az_1}{bz_2} + \frac{bz_2}{az_1}$ is a point on the (a, b $\in \mathbb{R}$)

- (A) Line segment $[-2, 2]$ of the real axis
(B) Line segment $[-2, 2]$ of the imaginary axis
(C) Unit circle $|z| = 1$
(D) The line with $\arg z = \tan^{-1} 2$

Q.20 If ω is an imaginary cube root of unity, then the value of $(p + q)^3 + (p\omega + q\omega^2)^3 + (p\omega^2 + q\omega)^3$ is

- (A) $p^3 + q^3$
(B) $3(p^3 + q^3)$
(C) $3(p^3 + q^3) - pq(p + q)$
(D) $3(p^3 + q^3) + pq(p + q)$

Q.21 The solution set of the equation $(1 + i\sqrt{3})^x - 2^x = 0$

- (A) Form an A.P. (B) Form a G.P.
(C) Form an H.P. (D) Is a empty set

Multiple Correct Choice Type

Q.22 In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. If the sum of the squares of the roots is 8 then

- (A) $p = 3, q = -1$ (B) $p = 3, q = 1$
(C) $p = -3, q = -1$ (D) $p = -3, q = 1$

Q.23 Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

- (A) Zero (B) Real & positive
(C) Real and negative (D) Purely imaginary

Q.24 Given $a, b, x, y \in \mathbb{R}$ then which of the following statement(s) hold good?

- (A) $(a + ib)(x + iy)^{-1} = a - ib \Rightarrow x^2 + y^2 = 1$
(B) $(1 - ix)(1 + ix)^{-1} = a - ib \Rightarrow a^2 + b^2 = 1$
(C) $(a + ib)(a - ib)^{-1} = x - iy \Rightarrow |x + iy| = 1$
(D) $(y - ix)(a + ib)^{-1} = y + ix \Rightarrow |a - ib| = 1$

Q.25 If $z = x + iy = r(\cos \theta + i \sin \theta)$ then the values of $\sqrt{z}$ is equal to:

- (A) $\pm \frac{1}{\sqrt{2}}(\sqrt{r+x} + i\sqrt{r-x})$ for $y \geq 0$
(B) $\pm \frac{1}{\sqrt{2}}(\sqrt{r+x} - i\sqrt{r-x})$ for $y \geq 0$
(C) $\pm \frac{1}{\sqrt{2}}(\sqrt{r+x} + i\sqrt{r-x})$ for $y \leq 0$
(D) $\pm \frac{1}{\sqrt{2}}(\sqrt{r+x} - i\sqrt{r-x})$ for $y \leq 0$

Q.26 For two complex numbers z_1 and z_2

$$(az_1 + b\bar{z}_1)(cz_2 + d\bar{z}_2) = (cz_1 + d\bar{z}_1)(az_2 + b\bar{z}_2)$$

If $(a, b, c, d \in \mathbb{R})$:

- (A) $\frac{a}{b} = \frac{c}{d}$ (B) $\frac{a}{d} = \frac{b}{c}$
(C) $|z_1| = |z_2|$ (D) $\arg z_1 = \arg z_2$

Q.27 Let z_1, z_2 be two complex numbers represented by point on the circle $|z_1| = 1$ and $|z_2| = 2$ respectively, then:

- (A) $\text{Max } |2z_1 + z_2| = 4$ (B) $\text{Min } |z_1 - z_2| = 1$
(C) $\left| z_2 + \frac{1}{z_1} \right| \leq 3$ (D) None of these

Q.28 If α, β any two complex numbers such that

$$\left| \frac{\alpha - \beta}{1 - \bar{\alpha}\beta} \right| = 1, \text{ then}$$

- (A) $|\alpha| = 1$ (B) $|\beta| = 1$
(C) $\alpha = e^{i\theta}, \theta \in \mathbb{R}$ (D) $\beta = e^{i\theta}, \theta \in \mathbb{R}$

Q.29 On the argand plane, let $\alpha = -2 + 3z, \beta = -2 - 3z$ and $|z| = 1$. Then the correct statement is:

- (A) α moves on the circle, centre at $(-2, 0)$ and radius 3
(B) α and β describe the same locus
(C) α and β move on different circles
(D) $\alpha - \beta$ moves on a circle concentric with $|z| = 1$

Q.30 The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in \mathbb{I}$ is:

- (A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$
(C) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

Q.31 A complex number z satisfying the equation,

$$\log_{14}(13|z^2 - 4i|) + \log_{196} \frac{1}{(13 + |z^2 + 4i|)^2} = 0$$

- (A) Can be purely real
(B) Can be purely imaginary
(C) Must be imaginary
(D) Must be real or purely imaginary

Q.32 Let S be the set of real values of x satisfying the inequality

$$1 - \log_2 \frac{|x+1+2i|-2}{\sqrt{2}-1} \geq 0, \text{ then } S \text{ contains:}$$

- (A) $[-3, -1]$ (B) $(-1, 1]$
(C) $[-2, 2)$ (D) $[1, 2]$

Q.33 If $x = \cos \alpha$; $y = \cos \beta$; $z = \cos \gamma$; Where $\alpha, \beta, \gamma \in \mathbb{R}$, then

- (A) $\sum x = \Pi x \Rightarrow \cos(\alpha - \beta) = 1$
(B) $\Pi \frac{x-y}{z} = 8 \Pi \cos \frac{\alpha - \beta}{2}$
(C) $\Pi \frac{x+y}{z}$ is real
(D) $\sum (\operatorname{Re} x) = \cos(\sum \alpha)$, $\sum \operatorname{Im}(x) = \sin(\sum \alpha)$

Q.34 If $x_r = \cos\left(\frac{\pi}{2^r}\right)$ for $1 \leq r \leq n$; $r, n \in \mathbb{N}$ then:

- (A) $\lim_{n \rightarrow \infty} \operatorname{Re}\left(\prod_{r=1}^n x_r\right) = -1$ (B) $\lim_{n \rightarrow \infty} \operatorname{Re}\left(\prod_{r=1}^n x_r\right) = 0$
(C) $\lim_{n \rightarrow \infty} \operatorname{Im}\left(\prod_{r=1}^n x_r\right) = 1$ (D) $\lim_{n \rightarrow \infty} \operatorname{Im}\left(\prod_{r=1}^n x_r\right) = 0$

Q.35 If $1, z_1, z_2, z_3, \dots, z_{n-1}$ be the n^{th} roots of unity and ω be a non real complex cube root of unity then the product $\prod_{r=1}^{n-1} (\omega - z_r)$ can be equal to:

- (A) 0 (B) 1 (C) -1 (D) 2

Q.36 Identify the correct statement(s).

- (A) No non zero complex number z satisfies the equation, $\bar{z} = -4z$
(B) $\bar{z} = z$ implies that z is purely real
(C) $\bar{z} = -z$ implies that z is purely imaginary
(D) If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\operatorname{Im}(z_1 z_2) \neq 0$ then a, b, c must be real numbers

Q.37 If the complex numbers z_1, z_2, z_3 & z_1', z_2' and z_3' are representing the vertices of two triangles such that $z_3 = (1 - z_0) z_1 + z_0 z_2$ and $z_3' = (1 - z_0) z_1' + z_0 z_2'$ where z_0 is also a complex number then:

$$(A) \begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$$

- (B) The two triangles are congruent
(C) The two triangles are similar
(D) The two triangles have the same area.

Q.38 If ' z ' be any complex number in a plane ($|z| \neq 0$) then the complex number z for which the multiplication inverse is equal to the additive inverse is:

- (A) $0 + i$ (B) $0 - i$ (C) $1 - i$ (D) $1 + i$

Q.39 Given $z = a + bi = \frac{1 - ix}{1 + ix}$; $a, b, x \in \mathbb{R}$, then which of the following holds good?

- (A) $-\frac{\pi}{2} < \arg z \leq 0$ (B) $-\pi < \arg z \leq 0$
(C) $|z| = 1$ (D) $\arg z = \pi$; $|z| = 1$

Previous Years' Questions

Q.1 If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is **(2011)**

Q.2 Let $\omega = e^{i\frac{2\pi}{3}}$ and a, b, c, x, y, z be non-zero complex number such that $a + b + c = x$, $a + b\omega + c\omega^2 = y$,

$a + b\omega^2 + c\omega = z$. Then, the value of

$$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} \text{ is } \quad \textbf{(2011)}$$

Paragraph (for Q.3, 4, 5)

Read the following passage and answer the following questions.

Let A, B, C be three sets of complex number as defined below

$$A = \{z: |z| \geq 1\}$$

$$B = \{z: |z - 2 - i| = 3\}$$

$$C = \{z: \operatorname{Re}((1 - i)z) + \sqrt{2}\} \quad \textbf{(2008)}$$

Q.3 The number of elements in the set $A \cap B \cap C$ is

- (A) 0 (B) 1 (C) 2 (D) ∞

Q.4 Let z be any point in $A \cap B \cap C$. The $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

- (A) 25 and 29 (B) 30 and 34
(C) 35 and 39 (D) 40 and 44

Q.5 Let z be any point in $A \cap B \cap C$ and let w any point satisfying $|w - 2 - i| < 3$. Then $|z| - |w| + 3$ lies between

- (A) -6 and 3 (B) -3 and 6
(C) -6 and 6 (D) -3 and 9

Q.6 If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then (1982)

- (A) $\operatorname{Re}(z) = 0$
(B) $\operatorname{Im}(z) = 0$
(C) $\operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0$
(D) $\operatorname{Re}(z) > 0, \operatorname{Im}(z) < 0$

Q.7 The inequality $|z - 4| < |z - 2|$ represents the region given by (1982)

- (A) $\operatorname{Re}(z) \geq 0$ (B) $\operatorname{Re}(z) < 0$
(C) $\operatorname{Re}(z) > 0$ (D) None of these

Q.8 If a, b, c and u, v, w are the complex numbers representing the vertices of two triangles such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, then the two triangles (1985)

- (A) Have the same area (B) Are similar
(C) Are congruent (D) None of these

Q.9 The value of $\sum_{k=1}^6 \left(\sin \frac{2\pi k}{7} - i \cos \frac{2\pi k}{7} \right)$ is (1987)

- (A) -1 (B) 0 (C) -i (D) i

Q.10 Let z and w be two complex numbers such that $|z| \leq 1, |w| \leq 1$ and $|z + iw| = |z - iw| = 2$, then z equals (1995)

- (A) 1 or i (B) i or -i (C) 1 or -1 (D) i or -i

Q.11 For positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, Here $i = \sqrt{-1}$

is a real number, if and only if (1996)

- (A) $n_1 = n_2 + 1$ (B) $n_1 = n_2 - 1$
(C) $n_1 = n_2$ (D) $n_1 > 0, n_2 > 0$

Q.12 If $i = \sqrt{-1}$, then

$$4 + 5 \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^{334} + 3 \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^{365}$$

is equal to (1999)

- (A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

Q.13 If $\arg(z) < 0$, then $\arg(-z) - \arg(z)$ equal (2000)

- (A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

Q.14 If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfied by (1985)

- (A) $|w_1| = 1$ (B) $|w_2| = 1$
(C) $\operatorname{Re}(w_1 \bar{w}_2) = 0$ (D) None of these

Q.15 Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\arg(w)$ denotes the principal argument of a non-zero complex number w , then (2010)

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\arg(z - z_1) = \arg(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(D) $\arg(z - z_1) = \arg(z_2 - z_1)$

Q.16 Match the statements in column I with those in column II. (2010)

[Note: Here z takes values in the complex plane and $\operatorname{Im} z$ and $\operatorname{Re} z$ denote, respectively, the imaginary part and the real part of z .]

Column I	Column II
(A) The set of points z satisfying $ z - iz = z + iz $ is contained in or equal to	(p) An ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $ z + 4 + z - 4 = 10$ is contained in or equal to	(q) The set of points z satisfying $\operatorname{Im} z = 0$
(C) If $ w = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r) The set of points z satisfying $ \operatorname{Im} z \leq 1$

(D) If $ w = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) The set of points z satisfying $ \operatorname{Re} z \leq 2$
	The set of points z satisfying $ z \leq 3$

Q.17 Let ω be the complex number $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Then the number of distinct complex number z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$
 is equal to

Q.18 Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\operatorname{Arg}(w)$ denotes the principal argument of a nonzero complex number w , then

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z - z_2)$

(C) $\left| \frac{z - z_1}{z_2 - z_1} \right| = \left| \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$

(D) $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z_2 - z_1)$

Q.19 if z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 - 5i|$ is

Q.20 Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$

$$a + b\omega + c\omega^2 = y$$

$$a + b\omega^2 + c\omega = z$$

Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

Q.21 Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

$$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix} \text{ where each } a, b \text{ and } c \text{ is either } \omega \text{ or } \omega^2.$$

Then the number of distinct matrices in the set S is

- (A) 2 (B) 6 (C) 4 (D) 8

Q.22 Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a cannot take the value

- (A) -1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Q.23 Let complex numbers α and $\frac{1}{\alpha}$ lies on circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

- (A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

Q.24 Let $\omega = \frac{\sqrt{3} + i}{2}$ and $P = \{\omega^n : n = 1, 2, 3, \dots\}$. Further $H_1 = \left\{ z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2} \right\}$ and $H_2 = \left\{ z \in \mathbb{C} : \operatorname{Re} z < \frac{-1}{2} \right\}$, where $\mathbb{C}$ is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$

- (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Q.25 Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$

- (A) 57 (B) 55 (C) 58 (D) 56

Paragraph (for Q.26 and Q.27)

Let $S = S_1 \cap S_2 \cap S_3$, where

$$S_1 = \{z \in \mathbb{C} : |z| < 4\}, S_2 = \left\{ z \in \mathbb{C} : \operatorname{Im} \left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i} \right] > 0 \right\}$$

$$\text{and } S_3 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

Q.26 Area of $S =$

- (A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

Q.27 $\min_{z \in S} |1 - 3i - z| =$

- (A) $\frac{2 - \sqrt{3}}{2}$ (B) $\frac{2 + \sqrt{3}}{2}$ (C) $\frac{3 - \sqrt{3}}{2}$ (D) $\frac{3 + \sqrt{3}}{2}$

Q.28 Match the following: (2014)

Column I	Column II
(i) The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2 , satisfying $f(0) = 0$ and $\int_0^1 f(x) dx = 1$, is	(p) 8
(ii) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(q) 2
(iii) $\int_{-2}^2 \frac{3x^2}{(1+e^x)} dx$ equals	(r) 4
(iv) $\frac{\left(\int_{-1/2}^{1/2} \cos 2x \cdot \log \left(\frac{1+x}{1-x} \right) dx \right)}{\left(\int_0^{1/2} \cos 2x \cdot \log \left(\frac{1+x}{1-x} \right) dx \right)}$ equals	(s) 0

Codes:

	(i)	(ii)	(iii)	(iv)
(A)	r	q	s	p
(B)	q	r	s	p
(C)	r	q	p	s
(D)	q	r	p	s

Q.29 For any integer k , let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression.

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|} \text{ is} \quad (2015)$$

Q.30 Let $z = \frac{-1 + \sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$.

Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order

2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is (2016)

Q.31 Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{ z \in \mathbb{C} : z = \frac{1}{a + ibt}, t \in \mathbb{R}, t \neq 0 \right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on (2016)

(A) The circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) The circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) The x-axis for $a \neq 0, b = 0$

(D) The y-axis for $a = 0, b \neq 0$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.6	Q.9	Q.15
Q.18	Q.22	Q.24
Q.28	Q.31	Q.34

Exercise 2

Q.2	Q.8	Q.10
Q.13	Q.16	Q.18

Previous Years' Questions

Q.2	Q.4	Q.7
Q.10	Q.13	Q.15

JEE Advanced/Boards

Exercise 1

Q.7	Q.11	Q.13
Q.16	Q.18	Q.25
Q.29	Q.30	

Exercise 2

Q.2	Q.6	Q.9
Q.15	Q.19	Q.22
Q.25	Q.27	Q.31
Q.33	Q.36	Q.39

Previous Years' Questions

Q.2	Q.4	Q.8
Q.11	Q.14	Q.15

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $z = 0, i, \pm \frac{\sqrt{3}}{2} - \frac{i}{2}$

Q.2 $-i$

Q.3 $x = \frac{5}{13}, y = \frac{14}{13}$

Q.4 $x = 3, y = -1$

Q.6 $\sqrt{2} \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right]$

Q.7 $\pm (1 - 3i)$

Q.8 $n = 4$

Q.10 -1

Q.11 $-1, 1 - 2\omega, 1 - 2\omega^2$

Q.13 $z_1 = (1 - \sqrt{3}) + i; z_2 = -i\sqrt{3}; z_3 = (1 + \sqrt{3}) - i$

Q.14 Centre $\left[\frac{\alpha - K^2\beta}{1 - K^2} \right]$, radius $\frac{|\alpha + \beta - K|}{1 - K^2}$

Q.19 $\frac{(n-1)n}{4} [n^2 + 3n + 4]$

Q.20 5**Q.24** ω, ω^2 **Q.31** $3 \leq |z| \leq 7$ **Q.33** Interior of circle $x^2 + y^2 = 25$ **Exercise 2****Single Correct Choice Type****Q.1** C**Q.2** D**Q.3** B**Q.4** C**Q.5** D**Q.6** A**Q.7** B**Q.8** C**Q.9** D**Q.10** D**Q.11** A**Q.12** A**Q.13** D**Q.14** D**Q.15** B**Q.16** C**Q.17** D**Q.18** C**Q.19** A**Q.20** C**Previous Years' Questions****Q.1** D**Q.2** A**Q.3** B**Q.4** B**Q.5** C**Q.6** D**Q.7** B**Q.8** D**Q.9** D**Q.10** B**Q.11** D**Q.12** A**Q.13** B**Q.14** B**Q.15** D**Q.16** A**Q.17** B**Q.18** B**Q.19** B**Q.20** C**JEE Advanced/Boards****Exercise 1****Q.2** 12**Q.4** 7**Q.5** 10**Q.6** (a) $-\frac{7}{2}$, (b) zero**Q.7** $x^2 + x + 2 = 0$ **Q.8** 4**Q.10** 41**Q.13** 259**Q.15** 26**Q.16** 163**Q.17** $(3 + 7i)$ **Q.18** $48(1 - i)$ **Q.19** $-\omega$ or $-\omega^2$ **Q.20** $k > |\alpha - \beta|^2$ **Q.21** If (z) is maximum when $z = \omega$, when ω is the cube root unity ω and If $(z) = \sqrt{13}$ **Q.22** 144**Q.23** 8**Q.24** 198**Q.25** 51**Q.26** $(z + 1)(z^2 - 2z \cos 36^\circ + 1)(z^2 - 2z \cos 108^\circ + 1)$ **Q.28** $\frac{iz}{2} + \frac{1}{2} + i$ **Q.29** (a) $\pi - 2$; (b) $\frac{1}{2}$ **Q.30** $A \rightarrow s; B \rightarrow q; C \rightarrow p$ **Exercise 2****Single Correct Choice Type****Q.1** C**Q.2** B**Q.3** A**Q.4** B**Q.5** C**Q.6** A**Q.7** D**Q.8** C**Q.9** B**Q.10** A**Q.11** D**Q.12** A**Q.13** D**Q.14** A**Q.15** B**Q.16** B**Q.17** D**Q.18** A**Q.19** A**Q.20** B**Q.21** A

Multiple Correct Choice Type

Q.22 B, C	Q.23 A, D	Q.24 A, B, C, D	Q.25 A, D	Q.26 A, D	Q.27 A, B, C	Q.28 A, B, C, D
Q.29 A, B, D	Q.30 B, D	Q.31 A, B, D	Q.32 A, B	Q.33 D	Q.34 A, D	Q.35 A, B, D
Q.36 A, B, C	Q.37 A, C	Q.38 A, B	Q.39 A, B, C, D			

Previous Years' Questions

Q.1 5	Q.2 3	Q.3 B	Q.4 C	Q.5 D	Q.6 B	Q.7 D
Q.8 B	Q.9 D	Q.10 C	Q.11 D	Q.12 C	Q.13 A	Q.14 A, B, C
Q.15 A, C, D	Q.16 $A \rightarrow q, r; B \rightarrow p; C \rightarrow p, s, t; D \rightarrow q, r, s, t$	Q.17 3	Q.18 A, C, D	Q.19 5		
Q.20 3	Q.21 A	Q.22 D	Q.23 C	Q.24 C, D	Q.25 B, C, D	Q.26 B
Q.27 C	Q.28 C	Q.29 4	Q.30 1	Q.31 A, C, D		

Solutions**JEE Main/Boards****Exercise 1****Sol 1:** $\bar{z} = i(z^2) \Rightarrow$ Let $x = a + ib$

$$\Rightarrow a - ib = i(a^2 - b^2 + 2abi)$$

$$\Rightarrow a = -2ab \text{ and } -b = a^2 - b^2$$

$$a(1 + 2b) = 0 \text{ and } a^2 = b^2 - b$$

$$a = 0 \text{ or } b = -\frac{1}{2}$$

$$\text{if } a = 0 \Rightarrow b = 0, 1$$

$$\text{if } b = -\frac{1}{2}, a = \pm \frac{\sqrt{3}}{2}$$

$$\text{Complex numbers are } z = 0, i, \pm \frac{\sqrt{3}}{2} - \frac{i}{2}$$

$$\textbf{Sol 2: } \frac{1+3i^2+2i}{1+3i^2-2i} = \frac{-2+2i}{-2-2i} = \frac{1-i}{1+i}$$

$$= \left(\frac{1-i}{1+i} \right) \left(\frac{1-i}{1-i} \right) = \frac{1+i^2-2i}{1-i^2} = -i$$

$$\textbf{Sol 3: } (x + iy)(2-3i) = 4 + i$$

$$\Rightarrow (2x + 3y) + i(2y - 3x) = 4 + i$$

$$\Rightarrow 2x + 3y = 4 \text{ and } 2y - 3x = 1$$

$$\Rightarrow x = \frac{5}{13} \text{ and } y = \frac{14}{13}$$

$$\textbf{Sol 4: } \frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$$

$$= \frac{[x+i(x-2)][3-i] + [3+i][2y+i(1-3y)]}{10}$$

$$= \frac{[3x+x-2+i(3x-6-x)] + [6y+3y-1+i(2y+3-9y)]}{10}$$

$$= \frac{4x-2+i(2x-6)+(9y-1)+i(-7y+3)}{10}$$

$$= \frac{4x+9y-3+i(2x-7y-3)}{10} = i$$

$$2x - 7y - 3 = 10 \text{ and } 4x + 9y - 3 = 0$$

$$\Rightarrow x = 3 \text{ and } y = -1$$

Sol 5: $x = a + b$

$$y = \alpha a + b\beta$$

$$z = a\beta + b\alpha$$

 α and β are complex cube roots of unity

$$\Rightarrow \alpha\beta = 1, \alpha^2 = \beta, \beta^2 = \alpha$$

$$(\text{as } \alpha = \omega, \beta = \omega^2, \alpha^2 = \beta)$$

..... (i)

$$xyz = (a + b)(\alpha a + b\beta)(a\beta + b\alpha)$$

$$= (\alpha a^2 + \alpha ab + \beta ab + b^2\beta)(a\beta + b\alpha)$$

$$= \alpha a^3\beta + \alpha a^2b\beta + \alpha^2b\beta^2 + ab^2\beta^2$$

$$+ \alpha^2a^2b + \alpha^2ab^2 + ab^2a\beta + b^3\alpha\beta$$

$$= \alpha\beta(a^3 + b^3 + a^2b + ab^2) + \alpha^2(a^2b + b^2a)$$

$$+ \beta^2(a^2b + b^2a)$$

from eqⁿ. (i)

$$= a^3 + b^3 + a^2b + ab^2 + (a^2b + b^2a)(\alpha^2 + b^2)$$

$$= a^3 + b^3 + a^2b + ab^2 + (a^2b + b^2a)(-1)$$

$$= a^3 + b^3 + a^2b + ab^2 - a^2b - ab^2$$

$$= a^3 + b^3 \text{ hence proved.}$$

$$\text{Sol 6: } \frac{1+7i}{(2-i)^2} \times \frac{(2+i)^2}{(2+i)^2} = \frac{(1+7i)(2+i)^2}{25}$$

$$= \frac{(1+7i)(3+4i)}{25} = \frac{-25+25i}{25}$$

$$z = -1 + i$$

$$z(\theta) = |z|e^{i\theta}$$

$$|z| = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{1}{-1} = -1$$

$$\Rightarrow \theta = \tan^{-1}(-1) = \frac{3\pi}{4}$$

$$\Rightarrow z(\theta) = \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$\text{Sol 7: } \sqrt{-8-6i} = a + ib$$

$$-8-6i = a^2 - b^2 + i(2ab)$$

$$\Rightarrow a^2 - b^2 = -8$$

..... (i)

$$2ab = -6$$

$$\Rightarrow ab = -3$$

..... (ii)

Solving (i) and (ii), we get

$$a = -1 \text{ \& } b = +3$$

$$a = 1 \text{ \& } b = -3$$

So square root

$$= (-1 + 3i) \text{ and } (1 - 3i)$$

$$\text{Sol 8: } \left(\frac{1+i}{1-i}\right)^n = 1$$

$$\frac{(1+i)^{2n}}{[(1+i)(1-i)]^n} = 1$$

$$\frac{(1+i)^{2n}}{2^n} = 1 \Rightarrow \frac{[(1+i)^2]^n}{2^n} = 1 \Rightarrow \frac{[2i]^n}{2^n} = 1$$

$$\Rightarrow i^n = 1 \Rightarrow n = 4, 8, 12$$

Minimum value of n is 4

$$\text{Sol 9: } \left|\frac{z-5i}{z+5i}\right| = 1$$

$$z = x + iy$$

$$\Rightarrow |x + i(y-5)| = |x + i(y+5)|$$

$$\Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2$$

$$\Rightarrow y = 0$$

i.e. complex part of z is zero.

z is pure real i.e. it lies on x axis.

$$\text{Sol 10: } z = 1 + itan\alpha$$

$$|z| = \sqrt{1 + \tan^2 \alpha} = \sqrt{\sec^2 \alpha} = |\sec \alpha|$$

$$\text{For } \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$\sec \alpha < 0 \Rightarrow |\sec \alpha| = -\sec \alpha$$

$$|z| = -\sec \alpha$$

$$|z|\cos \alpha = -1$$

$$\text{Sol 11: } (x-1)^3 = -8$$

$$x-1 = (-8)^{1/3}$$

$$x-1 = (8^{1/3})(-1)^{1/3} \Rightarrow 1-x = (8)^{1/3}(1)^{1/3}$$

$$(-x+1) = 2, 2\omega, 2\omega^2$$

$$x = -1, 1-2\omega, 1-2\omega^2$$

Sol 12: $|z| < 4$

$$|3 + i(z - 4)| < 9 \text{ (To prove)}$$

We know that $|z_1 + z_2| \leq |z_1| + |z_2|$

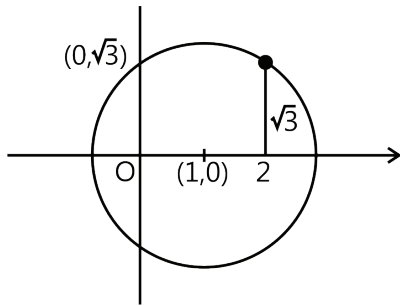
$$|iz + (3 - 4i)| \leq |z| + |3 - 4i|$$

$$|z| < 4$$

$$\Rightarrow |iz + (3 - 4i)| < 4 + 5$$

$$\Rightarrow |iz + (3 - 4i)| < 9 \text{ Hence proved}$$

Sol 13: $2 + i\sqrt{3}$ is vertex of square inscribed in $|z - 1| = 2$



Here one vertex is $A(2, \sqrt{3})$ and equation of circle is $(x-1)^2 + y^2 = 4$

Radius of the circle is 2. Hence side of the square will be $2\sqrt{2}$.

Points that lie on the circle and are at a distance $2\sqrt{2}$ from A are $B(1 - \sqrt{3}, 1)$ and $D(1 + \sqrt{3}, -1)$.

The point C will be the other end of diameter of A. Hence $C(0, -\sqrt{3})$.

Hence the four vertices are

$$2 + i\sqrt{3}, 1 - \sqrt{3} + i, -\sqrt{3}i \text{ and } 1 + \sqrt{3} - i$$

Sol 14: $\left| \frac{z - \alpha}{z - \beta} \right| = k$

$$\Rightarrow |z - \alpha|^2 = k^2 |z - \beta|^2$$

$$\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$$

$$\Rightarrow (x - \alpha_1)^2 + (y - \alpha_2)^2 = k^2 [(x - \beta_1)^2 + (y - \beta_2)^2]$$

$$\Rightarrow x^2 + \alpha_1^2 + y^2 + \alpha_2^2 - 2x\alpha_1 - 2y\alpha_2$$

$$= k^2 x^2 + k^2 y^2 + k^2 \beta_1^2 + k^2 \beta_2^2 - 2x\beta_1 k^2 - 2y\beta_2 k^2$$

$$\Rightarrow x^2(k^2 - 1) + y^2(k^2 - 1) + x(2\alpha_1 - 2\beta_1 k^2)$$

$$+ y(2\alpha_2 - 2\beta_2 k^2) + k^2 \beta_1^2 + k^2 \beta_2^2 - \alpha_1^2 - \alpha_2^2 = 0$$

$$\Rightarrow x^2 + y^2 + x \left[\frac{2\alpha_1 - 2\beta_1 k^2}{k^2 - 1} \right] + y \left[\frac{2\alpha_2 - 2\beta_2 k^2}{k^2 - 1} \right] + \left[\frac{k^2 \beta_1^2 + k^2 \beta_2^2 - \alpha_1^2 - \alpha_2^2}{k^2 - 1} \right] = 0$$

Eqⁿ. of circle

$$\text{with centre as } \left[\frac{\beta_1 k^2 - \alpha_1}{k^2 - 1}, \frac{\beta_2 k^2 - \alpha_2}{k^2 - 1} \right]$$

$$\text{or } \left[\frac{\alpha - k^2 \beta}{1 - k^2} \right]$$

$$\text{and radius} = \frac{(\alpha_1 + \beta_1 - k)^2 + (\alpha_2 + \beta_2)^2}{1 - k^2} = \frac{|\alpha + \beta - k|}{1 - k^2}$$

Sol 15: $|z| < \frac{1}{3}$ and $\sum_{r=1}^n a_r z^r = 1$... (i)

$$\Rightarrow |a_1 z^1 + a_2 z^2 + a_3 z^3 + \dots + a_n z^n| = 1$$

$$|z_1 + z_2 + z_3 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad \dots \text{ (ii)}$$

$$\Rightarrow 1 \leq |a_1 z| + |a_2 z^2| + |a_3 z^3| + \dots + |a_n z^n|$$

$$\Rightarrow |a_1| |z| + |a_2| |z|^2 + |a_3| |z|^3 + \dots + |a_n| |z|^n \geq 1$$

$$\Rightarrow |z| + |z|^2 + \dots + |z|^n \geq \frac{1}{2}$$

$$\Rightarrow \text{(Limiting case } n \rightarrow \infty)$$

$$\frac{|z|}{1 - |z|} \geq \frac{1}{2} \Rightarrow |z| \geq \frac{1}{3} \text{ --- (2)}$$

From (i) and (ii), we can say that there is no 'z' satisfying both conditions.

Sol 16: $z^{p+q} - z^p - z^q + 1 = 0$

$$z^p(z^q - 1)(z^q - 1) = 0$$

$$(z^p - 1)(z^q - 1) = 0$$

$$z^p = 1 \text{ or } z^q = 1$$

If α is roots, then $\alpha^0, \alpha, \alpha^2, \dots, \alpha^{n-1}$ also roots of $z^p = 1$ or $z^q = 1$

$$\text{Sum of roots} = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0$$

Sol 17: $z, iz, z + iz$

$$(x, y) \quad (-y, x) \quad (x - y, y + x)$$

$$\Delta = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix} = \frac{1}{2} | -x^2 - y^2 |$$

$$\Rightarrow |D| = \frac{1}{2} (x^2 + y^2) = \frac{1}{2} |z|^2$$

Sol 18: $iz^3 + i + z^2 - z = 0$

$$iz(z^2 + i) + i(z^2 + i) = 0$$

$$(iz + 1)(z^2 + i) = 0$$

Either $z = i$ or $z^2 = i$

If $z = i \Rightarrow |z| = |i| = 1$

If $z^2 = i \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

Hence, $|z| = 1$

Sol 19: $1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \dots (n - 1)(n - \omega)(n - \omega^2)$

$$T_n = n(n + 1 - \omega)(n + 1 - \omega^2)$$

$$= (n^2 + n - n\omega)(n + 1 - \omega^2)$$

$$= n^3 + n^2 - n^2\omega^2 + n^3 + n - n\omega^2 - n^2\omega - n\omega + n\omega^3$$

$$= n^3 + n^2(2 - \omega^2 - \omega) + n(1 - \omega - \omega^2 + 1)$$

$$= n^3 + 3n^2 + 3n$$

$$S_n = \sum T_n = \sum (n^3 + 3n^2 + 3n)$$

$$= \sum n^3 + 3\sum n^2 + 3\sum n$$

$$= \left[\frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + (2n+1) + 3 \right]$$

$$= \frac{n(n+1)}{4} [n^2 + n + 6 + 4n + 2]$$

$$S_n = \sum T_n = \frac{n(n+1)}{4} [n^2 + 5n + 8]$$

$$S_{n-1} = \frac{(n-1)n}{4} [n^2 + 1 - 2n + 5n - 5 + 8]$$

$$= \frac{(n-1)n}{4} [n^2 + 3n + 4]$$

Sol 20: $x = \frac{1}{2}(5 - i\sqrt{3}) = \frac{1}{2}(6 - 1 - i\sqrt{3}) = 3 + w$

$$x^4 - x^3 - 12x^2 + 23x + 12$$

$$= x^3(x - 3) + 2x^2(x - 3) - 6x(x - 3)$$

$$+ 5(x - 3) + 27$$

$$= (x - 3)[x^3 + 2x^2 - 6x + 5] + 27$$

$$= (x - 3)[(x - 3)(x^2 + 5x + 9) + 32] + 27$$

$$= (x - 3)^2[x^2 + 5x + 9] + 32(x - 3) + 27$$

$$= (x - 3)^2[x^2 - 3x + 8x - 24 + 33] + 32(x - 3) + 27$$

$$= (x - 3)^3(x + 8) + 33(x - 3)^2 + 32(x - 3) + 27$$

$$= \omega^3(x + 8) + 33(\omega^2) + 32\omega + 27$$

$$= \omega + 1 + 32\omega + 33\omega^2 + 27$$

$$= \omega + \omega^2 + 32\omega + 32\omega^2 + 38$$

$$= -1 - 32 + 38 = 5$$

Sol 21: $z_2 - z_1 = (z_3 - z_1)e^{-i\pi/3}$

$$z_3 - z_2 = (z_1 - z_2)e^{-i\pi/3}$$

$$\frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_1}{z_1 - z_2}$$

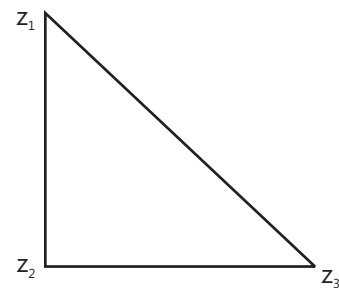
$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\left(\frac{z_1 + z_2 + z_3}{3} \right)^2 = z_0^2$$

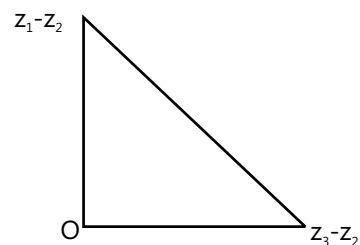
$$\Rightarrow \frac{z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)}{9} = z_0^2$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

Sol 22: z_1, z_2, z_3 of an isosceles Δ at z_2



Considering z_2 at origin



$$(z_1 - z_2) = (z_3 - z_2)e^{i\pi/2}$$

$$z_1 - z_2 = (z_3 - z_2)i$$

$$(z_1 - z_2)^2 = -(z_3 - z_1)^2$$

$$\Rightarrow z_1^2 + z_3^2 + 2z_1z_2 = 2z_1z_3 + 2z_1z_2$$

$$\Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

Sol 23: $\frac{A^2}{x-a} + \frac{B^2}{x-b} + \dots + \frac{H^2}{x-H} = x + \ell$ Let $x = p + iq$

$$(p + \ell) + iq = \sum \frac{A^2}{(p-a) + iq} = \sum \frac{A^2[(p-a) - iq]}{(p-a)^2 + q^2}$$

$$= \sum \frac{A^2(p-a)}{(p-a)^2 + q^2} - i \sum \frac{A^2q}{(p-a)^2 + q^2}$$

Equating the imaginary terms we get

$$q + \sum \frac{A^2q}{(p-a)^2 + q^2} = 0$$

$$\Rightarrow q \left(1 + \sum \frac{A^2}{(p-a)^2 + q^2} \right) = 0$$

$$\Rightarrow q = 0 \text{ or } \sum \frac{A^2}{(p-a)^2 + q^2} = -1$$

Here A, p, a and q are all real hence $q = 0$ is the only solution.

$\Rightarrow x$ cannot have imaginary roots.

Sol 24: $z^3 + 2z^2 + 2z + 1 = 0$

$$\Rightarrow z = -1, \omega, \omega^2$$

$$z^{1985} + z^{100} + 1 = 0$$

$$\Rightarrow z = \omega, \omega^2$$

Common roots are ω, ω^2

Sol 25: Let $f(x, y) = (x+y)^n - x^n - y^n$

$$xy(x+y)(x^2+xy+y^2) = (x-0)(y-0)(x+y)(x-0)(y-0)(x+y)$$

$$f(x=0)=0 \text{ and } f(y=0)=0$$

$$f(y=-x)=(x-x)^n - (x)^n - (-x)^n = 0$$

$$f(x=wy)=(wy+y)^n - w^n y^n - y^n ; \quad = y^n[w^{2n} + w^n + 1] = 0$$

$$\text{Similarly } f(x=w^2y)=0$$

$\therefore f(x)$ is divisible by $xy(x+y)(x^2+xy+y^2)$

Sol 26: $T = \left| \alpha + \sqrt{\alpha^2 - \beta^2} \right| + \left| \alpha - \sqrt{\alpha^2 - \beta^2} \right|$

$$T^2 = \left| \alpha + \sqrt{\alpha^2 - \beta^2} \right|^2 + \left| \alpha - \sqrt{\alpha^2 - \beta^2} \right|^2 +$$

$$2 \left| \alpha + \sqrt{\alpha^2 - \beta^2} \right| \left| \alpha - \sqrt{\alpha^2 - \beta^2} \right|$$

$$= \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right) \left(\overline{\alpha + \sqrt{\alpha^2 - \beta^2}} \right) + \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right)$$

$$\left(\overline{\alpha - \sqrt{\alpha^2 - \beta^2}} \right) + 2|\alpha^2 - \beta^2|$$

$$= 2 \left[|\alpha|^2 + \left| \sqrt{\alpha^2 - \beta^2} \right|^2 \right] + 2|\beta|^2$$

$$= 2|\alpha|^2 + 2|\beta|^2 + 2|\alpha^2 - \beta^2|$$

$$= 2|\alpha|^2 + 2|\beta|^2 + 2|\alpha + \beta||\alpha - \beta|$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\alpha} + \beta\bar{\beta} -$$

$$\alpha\bar{\beta} - \beta\bar{\alpha} + 2|\alpha + \beta||\alpha - \beta|$$

$$= |\alpha + \beta|^2 + |\alpha - \beta|^2 + 2|\alpha + \beta||\alpha - \beta|$$

$$= [|\alpha + \beta| + |\alpha - \beta|]^2$$

$$T = |\alpha + \beta| + |\alpha - \beta| \text{ Hence proved}$$

Sol 27: $z_1 = 10 + 6i; z_2 = 4 + 6i$

$$\frac{z - z_1}{z - z_2} = \frac{x - 10 + (y - 6)i}{x - 4 + (y - 6)i} \left(\frac{(x - 4) - i(y - 6)}{(x - 4) - i(y - 6)} \right)$$

$$= \frac{(x - 10)(x - 4) + (y - 6)^2 + i[(y - 6)(x - 4) + (x - 10)(6 - y)]}{(x - 4)^2 + (y - 6)^2}$$

$$\arg \frac{z - z_1}{z - z_2} = \frac{\pi}{4}$$

$$\Rightarrow (y - 6)(x - 4) + (x - 10)(6 - y)$$

$$= (x - 10)(x - 4) + (y - 6)^2$$

$$\Rightarrow (y - 6)(x - 4 - x + 10) = x^2 - 14x + 40 + (y - 6)^2$$

$$\Rightarrow 6(y - 6) = x^2 - 14x + 40 + y^2 + 36 - 12y$$

$$\Rightarrow x^2 + y^2 - 18y - 14x + 112 = 0$$

$$\Rightarrow (x - 7)^2 + (y - 9)^2 = 18$$

$$\Rightarrow |z - 7 - 9i| = 3\sqrt{2}$$

Sol 28: $|z - w|^2 = (z - w)(\bar{z} - \bar{w})$

$$= |z|^2 + |w|^2 - \bar{z}w - z\bar{w} + 2|z||w| - 2|z||w|$$

$$= (|z| - |w|)^2 - \bar{z}w - z\bar{w} + 2|z||w|$$

$$\text{Let } z = r_1 \cos \theta_1; w = r_2 \cos \theta_2$$

$$\begin{aligned}
&= (|z| - |w|)^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) + 2r_1r_2 \\
&= (|z| - |w|)^2 + 2r_1r_2 \left(2\sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) \right) \\
&= (|z| - |w|)^2 + 4r_1r_2 \left(\sin \frac{\theta_1 - \theta_2}{2} \right)^2
\end{aligned}$$

We know, $\sin \theta \leq \theta$ and $r_1, r_2 \leq 1$

$$\begin{aligned}
&\leq (|z| - |w|)^2 + 4 \times \left(\frac{\theta_1 - \theta_2}{2} \right)^2 \\
&\leq (|z| - |w|)^2 + (\theta_1 - \theta_2)^2 \\
&\leq (|z| - |w|)^2 + (\arg|z| - \arg|w|)^2
\end{aligned}$$

Sol 29: $\frac{A}{B} + \frac{B}{A} = 1$

$$A^2 + B^2 = AB$$

$$\frac{A}{B} = 1 - \frac{B}{A}$$

$$\frac{|A|}{|B|} = \frac{|A-B|}{|A|}$$

$$\frac{|A^2|}{|B|} = |A-B|$$

$$\frac{|B^2|}{|A|} = |B-A|$$

$$|A-B| = |B-A| \Rightarrow \left| \frac{A^2}{B} \right| = \left| \frac{B^2}{A} \right|$$

$$\Rightarrow |A^3| = |B^3|$$

$$\Rightarrow |A| = |B| = |A-B|$$

i.e. all sides are equal it forms an equilateral D

Sol 30: $a + b + c = 0 \Rightarrow b = -a - c$

$$az_1 + bz_2 + cz_3 = 0$$

$$az_1 + cz_3 - (a + c)z_2 = 0$$

$$z_2 = \frac{az_1 + cz_3}{a + c}$$

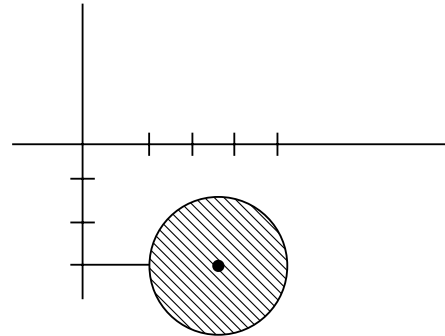
$$\Rightarrow a(z_1 - z_2) + c(z_3 - z_2) = 0$$

$$\Rightarrow (z_1 - z_2) = \frac{-c}{a} (z_3 - z_2)$$

$$z_1 - z_2 = k_1(z_3 - z_2)$$

$\Rightarrow z_1, z_2$ and z_3 are collinear (by vector)

Sol 31: $|z - 4 + 3i| \leq 2$



The shaded area show z

Min and max value = Distance of centre from origin $\pm$ radius

$$= 5 \pm 2 = 3, 7$$

$$\Rightarrow 3 \leq |z| \leq 7$$

Sol 32: $\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| < 1$

$$|z_1 - z_2|^2 < |1 - \bar{z}_1 z_2|^2$$

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2)$$

$$\Rightarrow |z_1|^2 + |z_2|^2 - z_1 \bar{z}_2 - \bar{z}_1 z_2$$

$$< -z_1 \bar{z}_2 - z_2 \bar{z}_1 + |z_1|^2 |z_2|^2 + 1$$

$$\Rightarrow |z_1|^2 - 1 < (|z_1|^2 - 1) |z_2|^2$$

$$|z_2|^2 < \frac{|z_1|^2 - 1}{|z_1|^2 - 1}$$

$$|z_2|^2 < 1 \Rightarrow |z_2| < 1$$

Sol 33: $\log_{\sqrt{3}} \left[\frac{|z|^2 - |z| + 1}{|z| + 2} \right] < 2$

$$\frac{|z|^2 - |z| + 1}{|z| + 2} < 3$$

$$\Rightarrow |z|^2 - |z| + 1 < 3|z| + 6$$

$$\Rightarrow |z|^2 - 4|z| - 5 < 0$$

$$\Rightarrow (|z| + 1)(|z| - 5) < 0$$

$$\Rightarrow |z| + 1 \geq 0$$

$$\text{So } (|z| - 5) < 0$$

$$\text{Interior of circle } x^2 + y^2 = 25$$

$$\text{Sol 34: } |z|^2 \omega - |\omega|^2 z = z - \omega$$

$$\Rightarrow z \bar{z} \omega - \omega \bar{\omega} z = z - \omega$$

$$\Rightarrow z \omega (\bar{z} - \bar{\omega}) = z - \omega$$

$$\Rightarrow z(\bar{z}\omega - 1) = \omega(z\bar{\omega} - 1)$$

$$\Rightarrow z(z\bar{\omega} - 1) = \bar{\omega}(\bar{z}\omega - 1)$$

$$\Rightarrow \text{Either } z\bar{\omega} = \bar{z}\omega = 1 \text{ or } \bar{z} = \bar{\omega}$$

$$\frac{x}{\omega} = \frac{\bar{\omega}}{z} \Rightarrow z\bar{z} = \omega\bar{\omega} \Rightarrow |z| = |\omega|$$

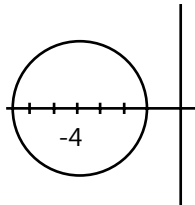
$$|z|^2(\omega - z) = (z - \omega)$$

$$(\omega - z)(|z|^2 + 1) = 0 \Rightarrow \omega = z$$

Exercise 2

Single Correct Choice Type

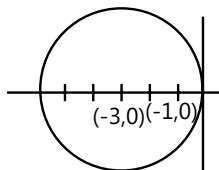
$$\text{Sol 1: (C) } |z + 4| \leq 3 \text{ Least and greatest value of } |z + 1|$$



i.e. distance of z from $(-1, 0)$

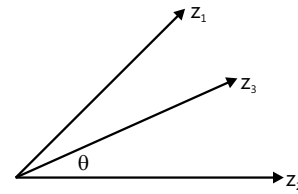
Least is 0; maximum is $2r = 6$

$$\text{Sol 2: (D) } |z + 3| \leq 3 \text{ Least \& greatest value of } |z + 1| \text{ ie its minimum and maximum distance from } (-1, 0) \text{ is 1 and 5.}$$



$$\text{Sol 3: (B) } z_1 = 3 + i\sqrt{3}$$

$$z_2 = 2\sqrt{3} + 6i$$



$$\hat{z}_3 = \hat{z}_2 e^{i\theta}; \hat{z}_1 = \hat{z}_3 e^{i\theta}$$

$$\hat{z}_3^2 = \hat{z}_1 \hat{z}_2 = \frac{24i}{24}$$

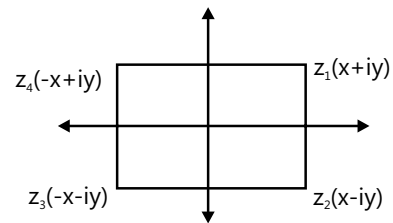
$$\dots (i) \quad \hat{z}_3 = \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

Unit vector along bisector is $\frac{1+i}{\sqrt{2}}$, complex number lying along this vector is $5(1+i)$

$$\text{Sol 4: (C) } z_1, z_2, z_3, z_4 \text{ vertices of square}$$

$$z_1 - z_2 = i2\text{Im}(z_1)$$

$$z_3 - z_2 = -2\text{Re}(z_1)$$



$$\frac{z_1 - z_2}{z_3 - z_2} \text{ is imaginary}$$

$$z_2 - z_4 = [-x + iy - (x - iy)]$$

$$= 2\text{Re}(z) - i2\text{Im}(z)$$

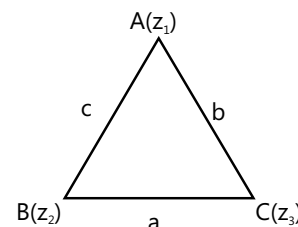
$$z_1 - z_3 = 2x + 2iy = 2\text{Re}(z) + i2\text{Im}(z)$$

$$\frac{z_1 - z_3}{z_2 - z_4} = \frac{2x + 2iy}{2x - 2iy} = \frac{x + iy}{x - iy} = \frac{x^2 - y^2 + 2ixy}{x^2 + y^2}$$

($x = y$ as it is aqueous)

$$= \frac{2ixy}{x^2 + y^2} \text{ (Purely Imaginary)}$$

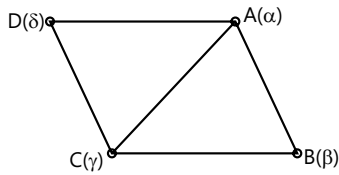
$$\text{Sol 5: (D) } \frac{az_1 + bz_2 + cz_3}{a + b + c} = z_0$$



z_0 is incentre.

Sol 6: (A) $\delta - \gamma = \alpha - \beta$ [parallel vector]

$$\delta = \alpha + \gamma - \beta$$

**Sol 7: (B)** $|z| \geq 3$

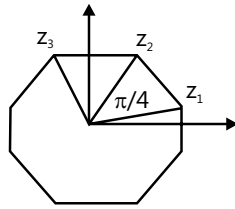
$$\left| z + \frac{1}{z} \right| \geq |z| - \frac{1}{|z|} = 3 - \frac{1}{3} = \frac{8}{3}$$

Sol 8: (C)

$$z_2 = z_1 e^{\pm i\pi/4} \quad z_3 = z_2 e^{\pm i\pi/4}$$

$$z_3 - z_2 = (z_2 - z_1) \frac{(1 \pm i)}{\sqrt{2}}$$

$$z_3 = z_2 + (z_2 - z_1) \frac{(1 \pm i)}{\sqrt{2}}$$

**Sol 9: (D)** $x^3 = (4)^3 (-1)^{1/3}$

$$x = -4, -4\omega, -4\omega^2$$

$$\begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_1 \\ q_3 & q_1 & q_2 \end{vmatrix} = \begin{vmatrix} -4 & -4\omega & -4\omega^2 \\ -4\omega & -4\omega^2 & -4 \\ -4\omega^2 & -4 & -4\omega \end{vmatrix}$$

$$= -64 \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = -64 \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$

Sol 10: (D) $z = (3 + 7i)(p + iq)$ is purely imaginary.

$$\Rightarrow 3p - 7q = 0$$

$$\Rightarrow p = \frac{7q}{3}$$

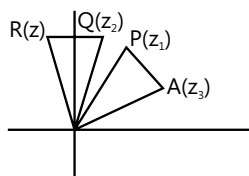
$$|z|^2 = |3 + 7i|^2 |p + iq|^2 = 58(\sqrt{p^2 + q^2})^2$$

for minimum $|z|$, $q = 3$, $p = 7$

$$|z|^2 = 58(49 + 9) = 3364$$

Sol 11: (A)

$$\frac{z_1}{z_3} = \frac{z}{z_2} \Rightarrow z = \frac{z_1 z_2}{z_3}$$



$$|z_3| = 1 \Rightarrow |z| = |z_1 z_2|$$

$$\Rightarrow z = z_1 z_2$$

Sol 12: (A) $\frac{A}{B} + \frac{B}{A} = 1$

$$\text{Let } y = \frac{A}{B}$$

$$y + \frac{1}{y} = 1$$

$$\Rightarrow y^2 - y + 1 = 0$$

$$\Rightarrow y = \frac{1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow \frac{A}{B} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow \left| \frac{A}{B} \right| = 1$$

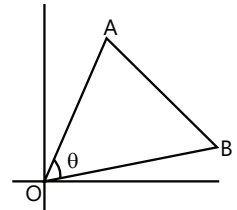
From Rotation Theorem

$$\frac{A}{B} = \left| \frac{A}{B} \right| e^{i\theta}$$

$$\Rightarrow e^{i\theta} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow \theta = 60^\circ$$

$$\text{and } |A| = |B| \Rightarrow \angle OAB = \angle OBA = 60^\circ$$

 $\Rightarrow$ AOB is equilateral triangle**Sol 13: (D)** $|z|^2 - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$

$$z\bar{z} - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$$

Let $z = a + ib$

$$a^2 + b^2 - 2a + i(2b) + 2 = 0$$

$$a^2 - 2a + 1 + b^2 - 2b + 1 = 0$$

$$(a - 1)^2 + (b - 1)^2 = 0$$

$$\Rightarrow a = b = 1$$

$$\Rightarrow z = 1 + i$$

Sol 14: (D) $z_1 = -3 + 5i$

$$z_2 = -5 - 3i$$

$$\text{Eq}^n \text{ of line } y = 4x + 17$$

$$\arg z = \tan^{-1} \left(\frac{4x + 17}{x} \right) = \tan^{-1} \left(4 + \frac{17}{x} \right)$$

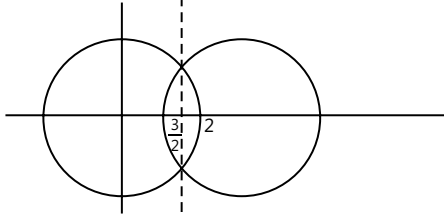
$$x \in [-3, -5] \Rightarrow \arg z \in \left[\tan^{-1} \frac{-5}{3}, \tan^{-1} \frac{-3}{5} \right]$$

Only option left is $\left(\frac{5\pi}{6} \right)$

Sol 15: (B) $|z-3| = 2 : (x-3)^2 + y^2 = 4$

$$|z| = 2 : x^2 + y^2 = 4$$

Points of intersection lie on the radical axis $S_1 - S_2 = 0$



Radical axis is $x = \frac{3}{2}$

$$x^2 + y^2 = 4$$

$$\Rightarrow \frac{9}{4} + y^2 = 4 \Rightarrow y^2 = 4 - \frac{9}{4}$$

$$\Rightarrow y^2 = \frac{7}{4} \Rightarrow y = \pm \frac{\sqrt{7}}{2}$$

Complex no. is $\frac{3}{2} \pm \frac{i\sqrt{7}}{2}$

Sol 16: (C) $|z-3i| = 3 \Rightarrow (x)^2 + (y-3)^2 = 9$

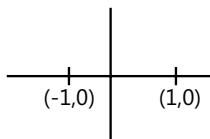
$$\Rightarrow \tan \theta = \frac{y}{x}$$

$$\cot \theta \cdot \frac{6}{z} = \frac{x}{y} - \frac{6(x-iy)}{(x+iy)(x-iy)} = \frac{x}{y} - \frac{6(x-iy)}{x^2+y^2}$$

$$x^2 + y^2 = 6y$$

$$\Rightarrow \frac{x}{y} - \frac{6(x-iy)}{6y} = i$$

Sol 17: (D) $|z-1| + |z+1| = 2$



The portion of real axis between $(-1, 0)$ & $(1, 0)$ as the distance between both the point is 2

Sol 18: (C) $Q = \sqrt{2|z|^2} \operatorname{cis}\left(\frac{\pi}{4} + \theta\right)$

$$= |z| \sqrt{2} \left[\cos\left(\frac{\pi}{4} + \theta\right) + i \sin\left(\frac{\pi}{4} + \theta\right) \right]$$

$$= |z|[(\cos \theta - \sin \theta) + i(\sin \theta + \cos \theta)]$$

$$= |z|[(\cos \theta + i \sin \theta) + \frac{|z|i}{i}(-\sin \theta + i \cos \theta)]$$

$$= |z| [\cos \theta + i \sin \theta] + \frac{|z|}{i} [-\cos \theta - i \sin \theta]$$

$$= P + iP$$

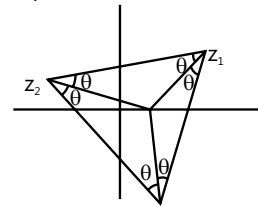
$$\Rightarrow \arg\left(\frac{Q-P}{P}\right) = \frac{\pi}{2}$$

This is right angle triangle with $|P| = |Q-P|$ and right angle at P.

Sol 19: (A) $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$

$$z_1 + z_2 + z_3 = 3$$

Centroid is at $z = 1$



Distance of vertical z_1, z_2 and z_3 from centroid is same, which mean centroid coincides with circumcenter.

Therefore, Δ is equilateral.

Sol 20: (C) $p = a + b\omega + c\omega^2$

$$q = b + c\omega + a\omega^2$$

$$r = c + a\omega + b\omega^2$$

$$p + q + r = a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2)$$

$$+ c(1 + \omega + \omega^2)$$

$$= (a + b + c)(1 + \omega + \omega^2) = 0$$

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2pq - 2qr - 2rp$$

$$= -2(pq + qr + rp)$$

$$pq + qr + rp = ab + ac\omega + a^2\omega^2 + b^2\omega^2 + b^2\omega + bc\omega^2 + ab\omega^3 + cb\omega^2 + c^2\omega^3 + ca\omega + \dots$$

$$= ab + 2ac\omega + ab + c^2 + (a^2 + 2bc)\omega^2 + b^2\omega$$

$$= 2ab + c^2 + 2ac\omega + (b^2 + 2bc)\omega^2 + a^2\omega + 2bc + a^2 + 2ba\omega + (b^2 + 2ac)\omega^2 + c^2\omega + 2ac + b^2 + 2bc\omega + (c^2 + a^2 + 2ab)\omega^2$$

$$= (2ab + 2bc + 2ac)(1 + \omega + \omega^2)$$

$$+ (c^2 + b^2 + a^2)(1 + \omega^2 + \omega)$$

$$p^2 + q^2 + r^2 = 0 = 2(pq + qr + rp)$$

Previous Years' Questions

Sol 1: (D) Since, $\left(\frac{1+i}{1-i}\right)^n = 1$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1$$

$$\Rightarrow \left(\frac{2i}{2}\right)^n = 1 \Rightarrow i^n = 1$$

The smallest positive integer n for which $i^n = 1$ is 4

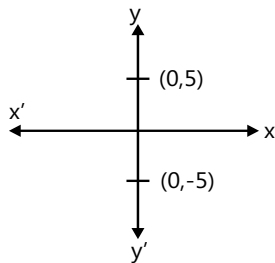
$$\therefore n = 4$$

Sol 2: (A) Given, $\left|\frac{z-5i}{z+5i}\right| = 1$

$$\Rightarrow |z-5i| = |z+5i|$$

$$(\text{if } |z-z_1| = |z-z_2|,$$

Then it is a perpendicular bisector of z_1



$\therefore$ Perpendicular bisector of $(0, 5)$ and $(0, -5)$ is x -axis.

Sol 3: (B) Since, $|w| = 1$

$$\Rightarrow \left|\frac{1-iz}{z-i}\right| = 1 \Rightarrow |z-i| = |1-iz|$$

$$\Rightarrow |z-i| = |z+i|$$

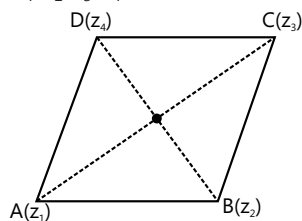
$$(\because |1-iz| = |-i||z+i| = |z+i|)$$

$\therefore$ It is a perpendicular bisector of $(0, 1)$ and $(0, -1)$

i.e., x -axis

Thus, z lies on real axis.

Sol 4: (B) Since, z_1, z_2, z_3, z_4 are the vertices of parallelogram.



$\therefore$ Mid-point of AC = mid-point of BD

$$\Rightarrow \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2} \Rightarrow z_1 + z_3 = z_2 + z_4$$

Sol 5: (C) Given, $|z_1 + z_2| = |z_1| + |z_2|$

On squaring both sides, we get

$$|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\arg z_1 - \arg z_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow 2|z_1||z_2|\cos(\arg z_1 - \arg z_2) = 2|z_1||z_2|$$

$$\Rightarrow \cos(\arg z_1 - \arg z_2) = 1$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = 0$$

Sol 6: (D) Since, $\overline{\sin x + i \cos 2x}$

$$= \cos x - i \sin 2x$$

$$\Rightarrow \sin x - i \cos 2x = \cos x - i \sin 2x$$

$$\Rightarrow \sin x = \cos x \text{ and } \cos 2x = \sin 2x < x$$

$$\Rightarrow \tan x = 1 \text{ and } \tan 2x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8} \text{ which is not possible at same time.}$$

Hence, no such value exists.

Sol 7: (B) $(1 + \omega)^7 = (1 + \omega)(1 + \omega)^6$

$$= (1 + \omega)(-\omega^2)^6$$

$$= 1 + \omega$$

$$\Rightarrow A + B\omega = 1 + 0 \Rightarrow A = 1, B = 1$$

Sol 8: (D) Since, $|z| = |\omega|$ and $\arg(z) = \pi - \arg(\omega)$

Let $\omega = re^{i\theta}$, then $\bar{\omega} = re^{-i\theta}$

$$\therefore z = re^{i(\pi-\theta)} = re^{i\pi} \cdot e^{-i\theta}$$

$$= -re^{-i\theta} = -\bar{\omega}$$

$\therefore z$ lies on the perpendicular bisector of the line joining $-i\omega$ and $-i\bar{\omega}$ is the mirror image of $-i\omega$ in the x -axis, the locus of z is the x -axis

$$\text{Let } z = x + iy \text{ and } y = 0$$

$$\text{Now, } |z| \leq 1 \Rightarrow x^2 + 0^2 \leq 1$$

$$\Rightarrow -1 \leq x \leq 1$$

..... (i)

$\therefore z$ may take values given in (i).

Sol 9: (D)

$$(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7 \quad (\because 1 + \omega + \omega^2 = 0)$$

$$= (-2\omega^2)^7 = (-2)^7 \omega^{14} = -128\omega^2$$

Sol 10: (B)

$$\begin{aligned}\sum_{n=1}^{13} (i^n + i^{n+1}) &= \sum_{n=1}^{13} i^n (1+i) = (1+i) \sum_{n=1}^{13} i^n \\ &= (1+i)(i + i^2 + i^3 + \dots + i^{13}) = (1+i) \left[\frac{i(1-i)}{1-i} \right] \\ &= (1+i)i = -1+i\end{aligned}$$

Alternate solution:

Since, sum of any four consecutive powers of iota is zero.

$$\begin{aligned}\therefore \sum_{n=1}^{13} (i^n + i^{n+1}) &= (i + i^2 + \dots + i^{13}) + (i^2 + i^3 + \dots + i^{14}) \\ &= i + i^2 = i - 1\end{aligned}$$

Sol 11: (D)

$$\begin{aligned}\text{Given, } \begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} &= x + iy \\ \Rightarrow -3i \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} &= x + iy \\ \Rightarrow x + iy &= 0 \quad (\because C_2 \text{ and } C_3 \text{ are identical}) \\ \Rightarrow x = 0, y &= 0\end{aligned}$$

Sol 12: (A)

$$\begin{aligned}\text{Given, } |z_1| &= |z_2| = |z_3| = 1 \\ \text{Now, } |z_1| &= 1 \\ \Rightarrow |z_1|^2 &= 1 \\ \Rightarrow \bar{z}_1 z_1 &= 1, \bar{z}_2 z_2 = 1\end{aligned}$$

$$\begin{aligned}\text{Again now, } \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| &= 1 \\ \Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| &= 1 \\ \Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| &= 1 \\ \Rightarrow |z_1 + z_2 + z_3| &= 1\end{aligned}$$

Sol 13: (B)

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

Applying $R_2 \rightarrow R_1 : R_3 \rightarrow R_3 - R_1$

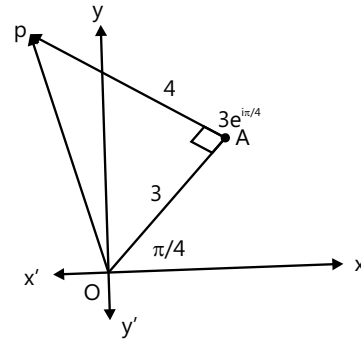
$$\begin{aligned}&= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-\omega^2 & \omega^2-1 \\ 0 & \omega^2-1 & \omega-1 \end{vmatrix} \\ &= (-2-\omega^2)(\omega-1) - (\omega^2-1)^2 \\ &= -2\omega + 2 - \omega^3 + \omega^2 - (\omega^4 - 2\omega^2 + 1) \\ &= 3\omega^2 - 3\omega = 3\omega(\omega-1) \quad (\because \omega^4 = \omega)\end{aligned}$$

Sol 14: (B)

$$\begin{aligned}\text{Given, } (1+\omega^2)^n &= (1+\omega^4)^n \\ \Rightarrow (-\omega)^n &= (-\omega^2)^n \quad (\because \omega^3 = 1 \text{ and } 1+\omega+\omega^2 = 0) \\ \Rightarrow \omega^n &= 1 \\ \Rightarrow n = 3 &\text{ is the least positive value of } n.\end{aligned}$$

Sol 15: (D) Let $OA=3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then

$$\begin{aligned}\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}} &= \frac{4}{3} e^{-i\pi/2} = -\frac{4i}{3} \\ \Rightarrow 3z - 9e^{i\pi/4} &= 12ie^{i\pi/4} \\ \Rightarrow z &= (3+4i)e^{i\pi/4}\end{aligned}$$



Sol 16: (A) Given: $\frac{z^2}{z-1}$ is real

Let $z = a + ib$, then

$$\begin{aligned}\frac{z^2}{z-1} &= \frac{(a+ib)^2}{a+ib-1} = \frac{a^2-b^2+2aib}{[(a-1)+ib]} \times \frac{[(a-1)-ib]}{[(a-1)-ib]} \\ \Rightarrow \frac{-b(a^2-b^2)+2ab(a-1)}{(a-1)^2+b^2} &= 0\end{aligned}$$

$$\Rightarrow -a^2b + b^3 + 2a^2b - 2ab = 0$$

$$\Rightarrow a^2b + b^3 - 2ab = 0$$

$$\Rightarrow b(a^2 + b^2 - 2a) = 0$$

$$\Rightarrow b = 0 \text{ or } a^2 + b^2 - 2a = 0$$

$\Rightarrow$ Either real axis or circle passing through origin.

Sol 17: (B) Let $\theta = \arg \left(\frac{1+z}{1+\bar{z}} \right)$

$$\Rightarrow \theta = \arg \left(\frac{1+z}{1+\frac{1}{z}} \right) \{ |z| = 1 \Rightarrow z\bar{z} = 1 \}$$

$$\Rightarrow \theta = \arg(z)$$

Sol 18: (B) Given: The expression $\left| z + \frac{1}{2} \right|$ and $|z| \geq 2$

Using triangle in equality

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \left| |z| - \left| \frac{1}{2} \right| \right|$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \left| z - \frac{1}{2} \right|$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{3}{2} \text{ lies in } (1, 2)$$

Sol 19: (B) $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2} = 1$

$$\Rightarrow |z_1 - 2z_2| = |2 - z_1\bar{z}_2|$$

$$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$$

$$\Rightarrow |z_1|^2 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4|z_2|^2$$

$$= 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + |z_1|^2 |z_2|^2$$

$$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_2|^2 |z_1|^2$$

$$\Rightarrow |z_1|^2 + (1 - |z_2|^2) - 4(1 - |z_2|^2) = 0$$

$$\Rightarrow (1 - |z_2|^2) (|z_1|^2 - 4) = 0$$

$$\Rightarrow |z_1|^2 = 4 \quad \{ |z_2| \neq 1 \}$$

$$\Rightarrow |z_1| = 2$$

Sol 20: (C) $\frac{2 + 3i \sin \theta}{1 - 2i \sin \theta}$

$$= \frac{(2 + 3i \sin \theta)(1 + 2i \sin \theta)}{1 + 4 \sin^2 \theta}$$

$$= \frac{2 + 4i \sin \theta + 3i \sin \theta - 6 \sin^2 \theta}{1 + 4 \sin^2 \theta}$$

$$\text{Given } \frac{2 - 6 \sin^2 \theta}{1 + 4 \sin^2 \theta} = 0$$

$$\Rightarrow \sin^2 \theta = \frac{1}{3}$$

$$\Rightarrow \sin \theta = \pm \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \sin^{-1} \left(\frac{1}{\sqrt{3}} \right), -\sin^{-1} \left(\frac{1}{\sqrt{3}} \right)$$

JEE Advanced/Boards

Exercise 1

Sol 1: $z^2 + (p + ip')z + (q + iq') = 0$

One real root

$$(z^2 + pz + q) + i(p'z + q') = 0$$

If z is real

$$p'z = -q' \Rightarrow z = \frac{-q'}{p'}$$

$$z^2 + pz + q = 0$$

$$\frac{q'^2}{p'^2} - \frac{pq'}{p'} + q = 0$$

$$q'^2 - pp'q' + qp'^2 = 0$$

If eqⁿ. has 2 equal roots

$$(p + ip')^2 = 4(q + iq')$$

$$p^2 - p'^2 = 4q \text{ and } p'^2 = 4q^2$$

The roots are $\frac{-(p + ip')}{2}$ i.e. roots (equal) are imaginary.

Sol 2: $z = 18 + 26i$

$z_0 = x_0 + iy_0$ is cube root of z

$$z_0 = (r\{\cos \theta + i \sin \theta\})^{1/3} = r^{1/3} e^{i\theta/3}$$

$$= r^{1/3} \left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)$$

$$r = (1000)^{1/2}$$

$$r^{1/3} = (10^{3/2})^{1/3} = 10^{1/2}$$

$$\cos \theta = \frac{18}{\sqrt{1000}} \quad \sin \theta = \frac{26}{\sqrt{1000}}$$

$$\cos \theta = 0.57, \sin \theta = 0.82$$

$$\cos \theta = 3 \cos \frac{\theta}{3} - 4 \cos^3 \left(\frac{\theta}{3} \right)$$

$$0.57 = 3t - 4t^3$$

$$\cos \left(\frac{\theta}{3} \right) = 0.2, 0.74, 0.95$$

$$\cos \frac{\theta}{3} = 0.95$$

$$\sin \frac{\theta}{3} = 0.3162$$

$$z = \sqrt{10} (0.95 + i 0.3162) = 3 + i$$

$$x_0 = 3, y_0 = 1$$

$$x_0 y_0 (x_0 + y_0) = 3.4 = 12$$

Sol 3: $z^3 + iz = 1$

$$z(z^2 + i) = 1$$

z can never be Purely real as $(z^2 + i)$ will be imaginary which multiplied with z will not be a real number, hence its locus will never cut x-coo axis Z can never be imaginary (pure) as $(z^2 + i)$ will be of form $(a + ib)$ which multiplied by z , cannot form 1 (real) therefore its locus can never cut y axis.

$$z(z^2 + i) = 1$$

$$(x + iy) [x^2 - y^2 + i(2xy + 1)] = 1$$

$$x^3 - xy^2 + i(2x^2y + x + x^2y - y^3) - 2xy^2 - y = 1$$

$$x^3 - xy^2 - 2xy^2 - y + i(3x^2y + x - y^3) = 1$$

$$x^3 - 3xy^2 - y = 1 \text{ \& } 3x^2y + x - y^3 = 0$$

$$x(x^2 - 3y^2) = 1 + y \text{ \& } y(y^2 - 3x^2) = x$$

$$|z|^2 = -\frac{1}{2} \left[\frac{1+y}{x} + \frac{x}{y} \right] \Rightarrow |z|^2 = -\frac{1}{2} \left[\frac{y + x^2 + y^2}{xy} \right]$$

$$|z|^2 = -\frac{1}{2xy} [y + |z|^2] \Rightarrow |z|^2 \left[\frac{2xy + 1}{2xy} \right] = \frac{-y}{2xy}$$

$$\Rightarrow |z| = \sqrt{\frac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z) + 1}}$$

Sol 4: $A_n = \text{dia}(d_1, d_2, \dots, d_n)$

$$d_i = a^{i-1}, \alpha = e^{i2\pi/n}$$

$$\Rightarrow d_n = e^{i\frac{2\pi}{n}(n-1)}$$

$$L = \text{Tr}(A_7)^7$$

$$M = \det A_{(2n+1)} + \det(A_{2n})$$

$$(A_7)^7 = e^{i\frac{2\pi 7(0)}{7}} + e^{i2\pi(1)} + \dots e^{i2\pi(6)}$$

$$= \cos 0 + i \sin 0 + \cos 2\pi + i \sin 2\pi \dots \cos 6\pi + i \sin 6\pi$$

$$= [1 + 1 + \dots 7 \text{ times} + i(0 + 0 \dots \dots)] = 7$$

$$\det A_{2n} = \prod e^{i\frac{2\pi}{2n}(K-1)} = e^{i\frac{\pi}{n} \left(2n \frac{(2n+1)}{2} - 2n \right)}$$

$$= e^{i\frac{\pi}{n} \left(\frac{4n^2 - 2n}{2} \right)} = e^{i\frac{\pi}{n} (2n^2 - n)} = e^{i\pi(2n-1)}$$

$$\det A_{2n+1} = e^{i2pn}$$

$$\det A_{2n} + \det A_{2n+1} = e^{i\pi(2n-1)} + e^{i2pn}$$

$$= \cos(2n-1)\pi + i \sin(2n-1)\pi + \cos 2n\pi + i \sin 2n\pi$$

$$M = 0 \Rightarrow L + M = 7$$

Sol 5: $z_1(z_1^2 - 3z_2^2) = 10$

$$z_2(3z_1^2 - z_2^2) = 30$$

$$z_1 = a + ib, z_2 = c + id$$

$$ab + cd = 0 \text{ as } z_1^2 + z_2^2 \text{ is real}$$

$$\Rightarrow z_1^3 - 3z_1z_2^2 + 3z_1^2z_2 - z_2^3 = 40$$

$$\Rightarrow z_1^2(z_1 + 3z_2) - z_2^2(z_2 + 3z_1) = 40$$

$$\Rightarrow (z_1 + iz_2)^3 = z_1^3 + iz_2^3 - 3z_1^2iz_2 - 3z_1z_2^2$$

$$= 10 - 30i$$

$$\Rightarrow (z_1 + iz_2)^3 = 10 + 30i$$

$$\Rightarrow ((z_1 + iz_2)^2)^3 = 1000 \Rightarrow z_1^2 + z_2^2 = 10$$

Sol 6: $(z + 1)^7 + z^7 = 0$ has roots $z_1, \dots, z_7$

$$\text{Re}(z_1) + \text{Re}(z_2) + \dots + \text{Re}(z_7) = ?$$

$$(z + 1)^7 = (-z)^7$$

$$\Rightarrow ((a+1) + ib)^7 = (-a - ib)^7$$

One of the solution is

$$\Rightarrow a + 1 = -a \Rightarrow a = \frac{-1}{2} \text{ and } b = 0$$

$$\Rightarrow z = \frac{-1}{2}$$

$$z_1 + z_2 + z_3 + \dots + z_7 = \frac{-7}{2} + i(0)$$

$$\text{Re}(z_1 + \dots + z_7) = \frac{-7}{2}$$

$$\text{Im}(z_1 + z_2 + \dots + z_7) = 0$$

Sol 7: $z = (1)^{1/7}$

$$\alpha + \beta = (z + z^2 + z^3 + z^4 + z^5 + z^6) = (-1) = -1$$

$$\begin{aligned}
 \alpha\beta &= (z + z^4 + z^2)(z^3 + z^5 + z^6) \\
 &= z^4 + z^6 + z^7 + z^7 + z^9 + z^{10} + z^5 + z^7 + z^8 \\
 &= z^4 + z^6 + 3 + z^5 + z + z^3 + z^2 \\
 &= 3 + z + z^4 + z^6 + z^5 + z^3 + z^2 = 3 - 1 = 2 \\
 \text{Eq}^n &\rightarrow x^2 + x + 2 = 0
 \end{aligned}$$

Sol 8: $z^5 - 32 = z^5 - 2^5$

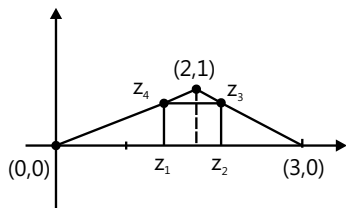
$$\begin{aligned}
 &= (z - 2)(z^2 - pz + 4)(z^2 - qz + 4) \\
 &= z^5 + z^4(-q - p - 2) + z^3(+pq + 8 + 2p + 2q) + z^2(-4p - 4q + (pq + 8) - 2) \\
 &\quad + z(16 + 8p + 8q) - 32 = 0 \\
 &\Rightarrow p + q + 2 = 0 \\
 &2p + 2q + 8 + pq = 0 \\
 &\Rightarrow pq + 4 = 0 \Rightarrow pq = -4 \\
 &p + q = -2 \Rightarrow p - \frac{4}{p} = -2 \\
 &\Rightarrow p^2 + 2p - 4 = 0 \\
 &\Rightarrow p^2 + 2p = 4
 \end{aligned}$$

Sol 9: $|z_1| + |z_2| \geq \frac{1}{2}(|z_1| + |z_2|) \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right|$

RHS

$$\begin{aligned}
 &\frac{1}{2}(|z_1| + |z_2|) \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right| \\
 &\left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right| \leq \left| \frac{z_1}{|z_1|} \right| + \left| \frac{z_2}{|z_2|} \right| = 2 \\
 &\frac{1}{2} \times 2(|z_1| + |z_2|) \geq \frac{1}{2}(|z_1| + |z_2|) \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right| \\
 &|z_1| + |z_2| \geq \frac{1}{2}(|z_1| + |z_2|) \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right|
 \end{aligned}$$

Sol 10:



$$z_1 = (a, 0), z_2 = (b, 0), z_3 = (b, c), z_4 = (a, c)$$

$$b - a = c^2$$

$$a = 2c; b = 3c; b + c = 3$$

$$4c = 3 \Rightarrow c = \frac{3}{4}, a = \frac{3}{2}, b = \frac{9}{4}$$

$$\Delta(0, z_3, z_4) = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ \frac{9}{4} & \frac{3}{4} & 1 \\ \frac{3}{2} & \frac{3}{4} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32} = \frac{m}{n}$$

$$\Rightarrow m + n = 32 + 9 = 41$$

Sol 11: $(1 + x)^n$

$$a_n = C_0 + C_3 + C_6 + \dots C_9$$

$$b_n = C_1 + C_4 + \dots$$

$$c_n = C_2 + C_5 + C_8 + \dots$$

$$a^3 + b^3 + c^3 - 3abc$$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots C_nx^n$$

$$2^n = C_0 + C_1 + C_2 + \dots C_n$$

$$(1+\omega)^n = C_0 + \omega(C_1 + C_4 + C_7 + \dots)$$

$$+ \omega^2(C_2 + C_5 + C_8 + \dots) + C_3 + C_6 + \dots$$

$$(1 + \omega^2)^n = C_0 + \omega^2(C_1 + C_4 + C_7 + \dots)$$

$$+ \omega(C_2 + C_5 + C_8 + \dots) + C_3 + C_6 + \dots$$

$$2^n + (1 + \omega)^n + (1 + \omega^2)^n = 3(C_0 + C_3 + C_6 + \dots)$$

$$2^n + (-\omega^2)^n + (-\omega)^n = 3(C_0 + C_3 + C_6 + \dots)$$

$$2^n\omega^2 + \omega(-\omega^2)^n + (-\omega)^n = 3\omega^2(C_1 + C_2 + C_7 + \dots)$$

$$a_n = \frac{2^n + (-\omega^2)^n + (-\omega)^n}{3}$$

$$b_n = \frac{2^n\omega^2 + \omega(-\omega^2)^n + (-\omega)^n}{3\omega^2}$$

$$c_n = \frac{\omega^2 2^n + (-\omega^2)^n + \omega(-\omega)^n}{3\omega^2}$$

Sol 12: $\frac{z_1 - z_2}{z_3 - z_2} = e^{i\pi/2} = i$

$$\frac{z_1 - iz_3}{1 - i} = z_2 \Rightarrow z_2 = \frac{z_1(1+i) + z_3(1-i)}{2}$$

$$\frac{z_1 - z_4}{z_3 - z_4} e^{-i\pi/2} = -i \Rightarrow z_1 - z_4 = -i(z_3 - z_4)$$

$$\Rightarrow z_4 = \frac{z_1(1-i) + z_3(1+i)}{2}$$

Sol 13: $f(z) = (a + ib)(z) = c + id$

$$z = (x + iy)$$

$$\text{Image} = c - id$$

$$(c - x)^2 + (y + d)^2 = c^2 + d^2$$

$$x^2 + y^2 - 2cx + 2dy = 0$$

$$a^2 + b^2 = 64$$

$$c = ax - by$$

$$d = bx - ay$$

$$\Rightarrow x^2 + y^2 - 2x(ax - by) + 2y(bx - ay) = 0$$

$$\Rightarrow x^2(1 - 2a) + y^2(1 - 2a) - 2bxy + 2bxy = 0$$

$$\Rightarrow (x^2 + y^2)(1 - 2a) = 0$$

$$\Rightarrow a = \frac{1}{2} \Rightarrow b^2 = 64 - \frac{1}{4} = \frac{255}{4} = \frac{u}{v}$$

$$\Rightarrow u + v = 259$$

Sol 14:

$$\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \dots + {}^nC_n \cos(n+1)x$$

$$= \cos x + {}^nC_1 \cos^2 x + {}^nC_2 \cos^3 x + \dots + {}^nC_n \cos^{n+1} x$$

$$= \cos x [1 + {}^nC_1 \cos x + {}^nC_2 \cos^2 x + \dots + {}^nC_n \cos^n x]$$

$$= \cos x [1 + \cos x]^n$$

$$= \cos x \left[2 \cos \frac{x}{2} \cos \frac{x}{2} \right]^n$$

$$= 2^n \cos^n \frac{x}{2} \cos \frac{n}{2} x$$

$$= 2^n \cos^n \frac{x}{2} \cos \left[\frac{n+2}{2} x \right]$$

$$= 2^n \cos^n \frac{x}{2} \left[\cos \frac{(n+2)x}{2} + i \sin \left[\frac{n+2}{2} x \right] \right]$$

By comparing, we get the desired results. Hence, proved

Sol 15: $f(x) = ax^3 + bx^2 + cx + d$ $f(i) = 0$

$$\Rightarrow -ai - b + ci + d = 0 + io \Rightarrow -a + c = 0 \text{ and } d - b = 0$$

$$\Rightarrow a = c \text{ and } b = d \quad \dots(i)$$

$$\text{And } f(1+i) = 5$$

$$a(1+i)^3 + b(1+i)^2 + c(1+i) + d = 5$$

$$\Rightarrow a(1+i^3 + 3i^2 + 3i) + b(1+i^2 + 2i) + c(1+i) + d = 5$$

$$\Rightarrow a(1-i-3+3i) + b(1-1+2i) + c(1+i) + d = 5$$

$$\Rightarrow a(-2+2i) + b(2i) + c(1+i) + d = 5$$

$$\Rightarrow -2a + c + d = 5 \text{ and } 2a + 2b + c = 0$$

From (i), we have

$$d - a = 5 \text{ and } 3a + 2d = 0$$

$$\Rightarrow a = c = -2 \text{ and } b = d = 3$$

$$\therefore a^2 + b^2 + c^2 + d^2 = (-2)^2 + 3^2 + (-2)^2 + 3^2$$

$$= 4 + 9 + 4 + 9$$

$$= 26$$

Sol 16: $\sum_{k=1}^n (z_k - w_k) = 0$

$$\Rightarrow z_1 + z_2 + z_3 + z_4 + z_5$$

$$= 32 + 170i - 7 + 64i - 9 + 200i + 1 + 27i - 14 + 43i$$

$$= 3 + 504i$$

If y-intercept is 3, then eq. of line

$$y = mx + 3$$

$$y_i = mx_i + 3 \text{ for } i=1,2,\dots,5$$

$$\text{Now, } z_1 + \dots + z_5 = 3 + 504i$$

$$(x_1 + x_2 + \dots + x_5) + i(y_1 + y_2 + \dots + y_5) = 3 + 504i +$$

$$i\{m(x_1 + x_2 + \dots + x_5) + 15\} = 3 + 504i$$

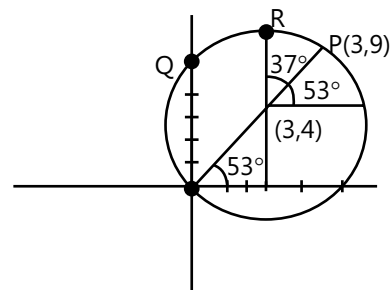
$$x_1 + x_2 + \dots + x_5 = 3$$

$$m(x_1 + x_2 + \dots + x_5) + 15 = 504$$

$$\Rightarrow 3m + 15 = 504$$

$$\Rightarrow m = 163$$

Sol 17:



$$|z - 3 - 4i| = 5 \tan^{-1} \frac{3}{4} (37^\circ) \text{ in clockwise direction}$$

It reaches Q (3, 9) above centre

Then it moves 2 unit downwards i.e. R (3, 7)

$$\begin{aligned}
 \text{Sol 18: } & \sum_{p=1}^{32} (3p+2) \left[\sum_{q=1}^{10} \left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right]^p \\
 &= \sum (3p+2) \left[-i \left(\sum_{q=0}^{10} e^{\frac{2q\pi i}{11}} - 1 \right) \right]^p \\
 &= \sum_{p=1}^{32} (3p+2) i^p = \sum_{p=1}^{32} (3p) i^p \left[2 \sum_1^{32} i^p = 0 \right] \\
 &= 3i(1-3+\dots-31) - 3(2-4+\dots-32) = 48(1-i)
 \end{aligned}$$

$$\text{Sol 19: } \frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$$

$$a = k - kb$$

$$b = k - kc$$

$$c = k - ka$$

$$a = k - k^2 + k^2(k - ka)$$

$$a = k - k^2 + k^3 - k^3a$$

$$a = \frac{k - k^2 + k^3}{1 + k^3} = b = c$$

$$\text{but } a \neq b \neq c$$

$$\text{i.e. } k^3 = -1 \Rightarrow k = -\omega, -\omega^2$$

$$\text{Sol 20: } |z - \alpha|^2 + |z - \beta|^2 = k$$

Locus of z is a circle

$$(x - \alpha_1)^2 + (x - \beta_1)^2 + (y - \alpha_2)^2 + (y - \beta_2)^2 = k$$

$$\Rightarrow 2x^2 + 2y^2 - 2x(\alpha_1 + \beta_1) - 2y(\alpha_2 + \beta_2)$$

$$+ \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - k = 0$$

$$\Rightarrow x^2 + y^2 - 2x \left(\frac{\alpha_1 + \beta_1}{2} \right) - 2y \left(\frac{\alpha_2 + \beta_2}{2} \right)$$

$$+ \left(\frac{\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - k}{2} \right) = 0$$

$$r > 0 \text{ i.e. } g^2 + f^2 - c > 0$$

$$\left(\frac{\alpha_1 + \beta_1}{2} \right)^2 + \left(\frac{\alpha_2 + \beta_2}{2} \right)^2 - 2 \left(\frac{\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - k}{4} \right) > 0$$

$$\Rightarrow -\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 + 2\alpha_1\beta_1 + 2\alpha_2\beta_2 + k > 0$$

$$\Rightarrow k > \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - 2\alpha_1\beta_1 - 2\alpha_2\beta_2$$

$$\Rightarrow k > (\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2 \Rightarrow k > |\alpha - \beta|^2$$

$$\text{Centre} = \left(\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_2 + \beta_2}{2} \right)$$

$$\text{Radius} = \sqrt{\frac{2k + 2\alpha_1\beta_1 + 2\alpha_2\beta_2 - \alpha_1^2 - \alpha_2^2 - \beta_1^2 - \beta_2^2}{4}}$$

$$\text{Sol 21: } f(z) = |z^3 - z + 2|z| = 1$$

$$(f(z))^2 = |z^3 - z + 2|^2 = (z^3 - z + 2)(\bar{z}^3 - \bar{z} + 2)$$

$$= 1 - z^2 + 2z^3 - \bar{z}^2 + z\bar{z} - 2z + 2\bar{z}^3 - 2\bar{z} + 4$$

$$= 6 - (z^2 - \bar{z}^2) - 2(z + \bar{z}) + 2(\bar{z}^3 + z^3)$$

$$= 6 - 2(a^2 - b^2) - 4(a + 2(z + \bar{z}))(z^2 + \bar{z}^2 - z\bar{z})$$

$$= 6 - 2(a^2 - b^2) - 4(a + 4a(2a^2 - 2b^2 - 1))$$

$$= 6 - 8a - 2(2a^2 - 1) + 8a(2a^2 - 1)$$

$$f(z) = 16a^3 - 4a^2 - 16a + 8$$

$$f'(z) = 48a^2 - 8a - 16 = 8(6a^2 - a - 2)$$

$$= 8(6a^2 - 4a + 3a - 2)$$

$$= 8(2a(3a - 2) + 1(3a - 1))$$

$$= 8(2a + 1)(3a - 2)$$

$$a = \frac{-1}{2}a = \frac{2}{3}$$

$$\text{For } a = \frac{2}{3}$$

$$f(z) = 16 \times \frac{8}{27} - 4 \times \frac{4}{9} - 16 \times \frac{2}{3} + 8$$

$$= \frac{128 - 48 - 288 + 202}{27} < 0$$

$$\text{For } a = \frac{-1}{2}; f(z) = -2 - 1 + 8 + 8 = 13$$

$$\text{Maximum value of } f(z) = \sqrt{13}$$

$$\text{Sol 22: } |a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega| \geq$$

$$|a + b\omega + c\omega^2 + a + b\omega^2 + c\omega|$$

$$= |2a - b - c| = |a - b + a - c|$$

Minimum value will occur when

$$a - b = k, a - c = -k$$

$$\text{i.e. } k = 1$$

$$a - b = 1 \text{ and } a - c = -1$$

with a & b being least possible value integer values

$$\Rightarrow a = 2, b = 1, c = 3$$

$$|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$$

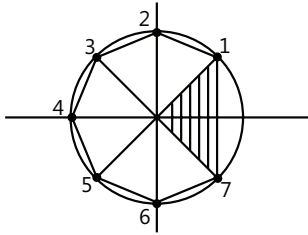
$$= 2|2 + \omega + 3\omega^2| = 2|1 + 2\omega^2|$$

$$= 2|\omega^2 - \omega| = 2|(-\sqrt{3})| = 2\sqrt{3}$$

$$= \sqrt{12} = n^{1/4} \Rightarrow n = 144$$

Sol 23: $x^7 + x^6 + \dots + 1 = 0$

$$\left(\frac{x^8 - 1}{x - 1} \right) = 0$$



Total area = unshaded area + shaded area

$$\text{Unshaded area} = 6 \times \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{6}{2} \times \frac{1}{\sqrt{2}} = \frac{6}{2\sqrt{2}}$$

$$\text{Shaded area} = 2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\text{Total area} = \frac{6}{2\sqrt{2}} + \frac{1}{2} = \frac{6 + \sqrt{2}}{2\sqrt{2}} = \frac{6\sqrt{2} + 2}{4}$$

$$= \frac{3\sqrt{2} + 1}{2} = \frac{a\sqrt{b} + c}{d}$$

$$a + b + c + d = 3 + 2 + 1 + 2 = 8$$

Sol 24: $N = (a + ib)^3 - 107i$

$$= a^3 - ib^3 + 3a^2ib - 3ab^2 - 107i$$

$$N = (-b^3 + 3a^2b - 107)i + a^3 - 3ab^2$$

$$\Rightarrow 3a^2b - b^3 = 107 \Rightarrow b(3a^2 - b^2) = 107$$

$$\Rightarrow b = 1 \text{ and } 3a^2 - b^2 = 107 \Rightarrow a^2 = 36$$

$$\Rightarrow N = a^3 - 3ab^2 = 216 - 18 = 198$$

Sol 25: $x^4 + ax^3 + bx^2 + cx + d$ has 4 non-real roots.

$$\alpha + \beta = 3 + 4i$$

$$\gamma\delta = 13 + i (\gamma = \bar{\alpha}, \delta = \bar{\beta})$$

$$\Rightarrow \bar{\alpha}\bar{\beta} = 13 + i, a\beta = 13 - i$$

$$\alpha + \beta = 3 + 4i$$

$$\Rightarrow a\beta + b\gamma + g\delta + d\alpha + a\gamma + b\delta = b$$

$$\Rightarrow a\beta + \beta\bar{\alpha} + \bar{\alpha}\bar{\beta} + \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} = b$$

$$\Rightarrow 13 - i + 13 + i + (\bar{\alpha} + \bar{\beta})(\alpha + \beta) = b$$

$$= 26 + (3 + 4i)(3 - 4i) = b$$

$$\Rightarrow b = 26 + 25 = 51$$

Sol 26: $z^5 + 1$

$$= (z^3 + 1)z^2 + (1 - z^2)$$

$$= (z + 1)(z^2 - z + 1)z^2 + (1 - z^2)$$

$$= (1 + z)[(z^2 - z + 1)z^2 + 1 - z]$$

$$= (1 + z)[z^4 - z^3 + z^2 - z + 1]$$

$$= (1 + z)[z^2 + az + 1][z^2 + bz + 1]$$

$$\Rightarrow b + a = -1$$

$$ba + 2 = 1$$

$$ab = -1$$

$$\Rightarrow a - \frac{1}{a} = -1 \Rightarrow a^2 + a - 1 = 0$$

$$\Rightarrow a = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow a = \frac{-1 + \sqrt{5}}{2} = -2\cos 108^\circ$$

$$\text{And } b = \frac{-1 - \sqrt{5}}{2} = -2\cos 36^\circ$$

$$\therefore \text{Factors are } (1 + z)(z^2 - 2\cos 36^\circ z + 1)(z^2 - 2\cos 108^\circ z + 1)$$

$$\text{Since } ab = -1$$

$$\Rightarrow 4\cos 36^\circ \cos 108^\circ = -1 \Rightarrow 4\cos \frac{\pi}{5} \cos \frac{\pi}{10} = 1$$

Sol 27: $x = 1 + i\sqrt{3}$

$$y = 1 - i\sqrt{3}$$

$$z = 2$$

$$x = -2\omega y = -2\omega^2 z = 2$$

$$x^p + y^p > 3 \text{ prime (P is odd)}$$

$$= -2^p \omega^p - 2^p \omega^{2p} = -2^p (\omega^p + \omega^{2p}) = 2^p = z^p$$

Sol 28: $f(z) = a(z - i) + i \Rightarrow f(i) = i$... (i)

$$f(z) = b(z + i) + (1 + i) \Rightarrow f(-i) = 1 + i$$
 ... (ii)

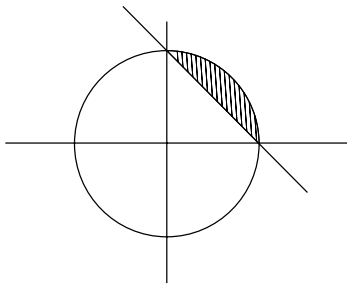
$$f(z) = c(z^2 + 1) + k_1 z + k_2$$
 ... (iii)

Substituting values from (i) and (ii) in (iii)

$$\Rightarrow i = k_1 i + k_2 \text{ and } i + 1 = -k_1 i + k_2$$

$$\Rightarrow k_2 = i + \frac{1}{2} \text{ and } k_1 = \frac{i}{2}$$

$$\therefore \text{Remainder} = \frac{iz}{2} + i + \frac{1}{2}$$

Sol 29: (a)

$$A = \{z \mid |z| \leq 2\}$$

$$B = \{z \mid (1-i)z + (1+i)\bar{z} \geq 4\}$$

$$\text{Let } z = a + ib$$

$$\Rightarrow (1-i)(a+ib) + (1+i)(a-ib) \geq 4$$

$$\Rightarrow a + b + a + b \geq 4 \Rightarrow a + b \geq 2$$

$$\text{area} = \frac{\pi r^2}{4} - \frac{1}{2}r^2 = \left(\frac{\pi}{4} - \frac{1}{2}\right)r^2 = (\pi - 2)$$

$$(b) f(x) = \frac{1}{x-i} = \frac{1}{x-i} \left(\frac{x+i}{x+i} \right)$$

$$= \frac{x}{x^2+1} + i \left(\frac{1}{x^2+1} \right)$$

$$x\text{-coordinate} = \frac{k}{k^2+1}$$

$$y\text{-coordinate} = \frac{1}{k^2+1}$$

$\therefore$ Locus of the function will be

$$x^2 + y^2 - y = 0$$

This is circle with diameter 1

$$\text{Hence the area of the square} = \frac{1}{2}$$

Sol 30: (a) $(1 + \omega + \dots + \omega^n)^m$, $m, n \in \mathbb{N}$

$$n \in 3k \Rightarrow 1$$

$$n \in 3k+1 \Rightarrow (-\omega^2)^m$$

$$n \in 3k+2 \Rightarrow 0$$

$$m=1 \Rightarrow (-\omega^2), m=2 \Rightarrow (\omega), m=3 \Rightarrow (-1)$$

$$m=4 \Rightarrow (\omega^2), m=5 \Rightarrow (-\omega), m=6 \Rightarrow (1)$$

No. of distinct elements are 7(S)

(b) Real coefficient,

$$\text{Root} \rightarrow 2\omega, 2+3\omega, 2+3\omega^2, 2-\omega-\omega^2$$

$$\text{Roots are } 2\omega, 2+3\omega, 2+3\omega^2, 3$$

Other root will be $2\bar{\omega}$

Total no. of roots are 5 (q)

$$(c) \alpha = 6 + 4i\beta = 2 + 4i$$

$$\text{amp} \left(\frac{z-\alpha}{z-\beta} \right) = \frac{\pi}{6}$$

$$\tan^{-1} \left[\frac{(x-6) + (y-4)i}{(x-2) + i(y-4)} \right] = \frac{\pi}{6}$$

$$\text{Re}(z) = (x-6)(x-2) + (y-4)^2$$

$$\text{Im}(z) = (x-2)(y-4) + (x-6)(y-4)$$

$$\Rightarrow \frac{(x-2)(y-4) + (x-6)(4-y)}{(x-6)(x-2) + (y-4)^2} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{-4x-2y+8+4x+6y-24}{x^2+y^2-8x-8y+12+16} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow x^2 + y^2 - 8x - 8y + 28 = 4\sqrt{3}y - 16\sqrt{3}$$

$$\Rightarrow x^2 + y^2 - 8x + y(-8-4\sqrt{3}) + 28 + 16\sqrt{3} = 0$$

$$\Rightarrow r = \sqrt{16 + (4+2\sqrt{3})^2 - 28 - 16\sqrt{3}}$$

$$= \sqrt{16 + 16 + 12 - 28 + 16\sqrt{3} - 16\sqrt{3}}$$

$$= \sqrt{16} = 4(p)$$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $z^2 + z + 1$ is real and +ve

$$x^2 - y^2 + x + 1 + i(2xy + y)$$

$$x^2 - y^2 + x + 1 > 0 \text{ and } 2xy + y = 0$$

$$\Rightarrow y(2x+1) = 0 \Rightarrow y = 0 \text{ or } x = -\frac{1}{2}$$

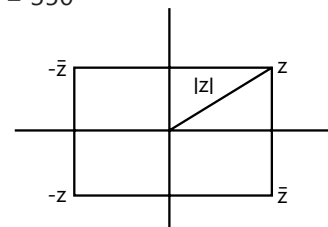
If $y = 0$ $x^2 + x + 1 > 0 \Rightarrow z$ represents real axis

$$\text{If } x = -\frac{1}{2}, \frac{3}{4} - y^2 > 0, \text{ Line segment joining}$$

$$\left[-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right] \text{ to } \left[-\frac{1}{2}, \frac{\sqrt{3}}{2} \right]$$

Sol 2: (B) $z\bar{z}^3 + z^3\bar{z} = 350$

$$|z|^2 (z^2 + \bar{z}^2) = 350$$



Length of diagonal $2|z|$

$$|z|^2 (x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy) = 350$$

$$(x^2 + y^2)(x^2 - y^2) = \frac{350}{2} = 175$$

As x and y are integers

$$x = \pm 4; y = \pm 3$$

$$\Rightarrow |z| = 5$$

Length of diagonal is 10

Sol 3: (A) $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$

$$(z_1 - z_2)^2 + z_2^2 = 0$$

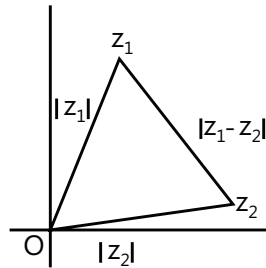
$$\Rightarrow (z_1 - z_2)^2 = -z_2^2 = |z_1 - z_2|^2 = |z_2|^2 \text{ and } (z_1 - z_2) = iz_2$$

and

$$\left(\frac{z_1}{z_2}\right)^2 - 2\frac{z_1}{z_2} + 2 = 0$$

$$\Rightarrow \frac{z_1}{z_2} = 1 + i$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{2}$$



Now,

$$\frac{z_1 - 0}{z_2 - 0} = \left|\frac{z_1}{z_2}\right| e^{i\angle z_1oz_2}$$

$$e^{i\angle z_1oz_2} = \frac{1+i}{\sqrt{2}} = e^{i\pi/4}$$

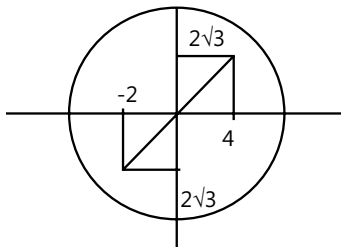
$$\Rightarrow \angle z_1oz_2 = 45^\circ = \angle oz_1z_2$$

Also,

$$\frac{z_2 - z_1}{z_2} = \left|\frac{z_2 - z_1}{z_2}\right| e^{i\angle oz_2z_1}$$

$$\Rightarrow e^{i\angle oz_2z_1} = i \Rightarrow \angle oz_2z_1 = 90^\circ$$

Sol 4: (B)



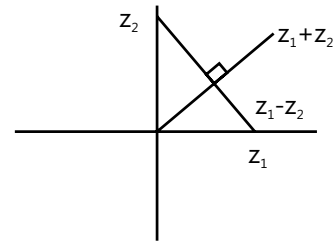
$$|z| \leq 4 \text{ and } \text{Arg}(z) = \frac{\pi}{3}$$

$$x^2 + y^2 \leq 16 \text{ and } y = \sqrt{3}x$$

$$\Rightarrow x \in [-2, 2] \text{ \& } y \in [-2\sqrt{3}, 2\sqrt{3}]$$

Set of points lie on radius of circle

Sol 5: (C)



$$\arg \frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$$

$$\tan^{-1} \left(\frac{z_1 + z_2}{z_1 - z_2} \right) = \frac{\pi}{2}$$

$$|z_1 + z_2| \neq |z_1 - z_2|$$

Diagonals are perpendicular but not equal ie figure represented is rhombus but not a square.

Sol 6: (A) Condition for equilateral

$$\Delta: z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$z^3 - 3\alpha z^2 + 3\beta z + \gamma = 0$$

$$z_1, z_2, z_3$$

$$\Rightarrow z_1 + z_2 + z_3 = 3\alpha$$

$$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = 3\beta$$

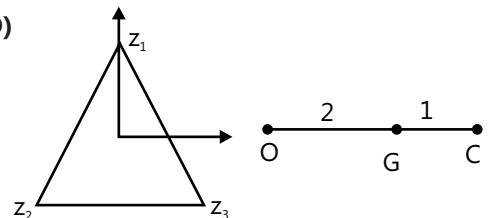
$$\Rightarrow z_1z_2z_3 = -\gamma$$

$$(z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(3\beta)$$

$$(3\alpha)^2 = 9\beta$$

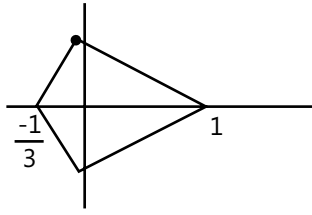
$$\alpha^2 = \beta$$

Sol 7: (D)



$$\frac{z_1 + z_2 + z_3}{3} = \text{Centroid, } z_c = 0$$

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2z_c + z_o}{3} \Rightarrow z_o = z_1 + z_2 + z_3$$

Sol 8: (C)

$$(z + 1)^4 = 16z^4$$

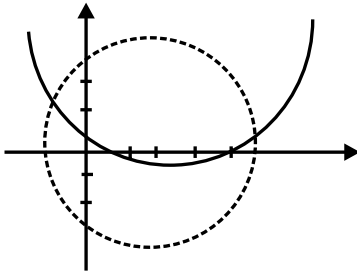
$$\Rightarrow z + 1 = 2z, 2iz, -2z, -2iz$$

$$\Rightarrow z = 1, \frac{1}{2i-1}, -\frac{1}{3}, \frac{1}{-1-2i} = -\frac{1}{3}, 1, \frac{-1-2i}{5}, \frac{-1+2i}{5}$$

$$\text{Point equidistant is } z = \frac{z_1 + z_2}{2}$$

$$\text{Where } z_1 = (1, 0), z_2 = \left(-\frac{1}{3}, 0\right)$$

$$z = \frac{1 - \frac{1}{3}}{2} = \frac{1}{3}$$

Sol 9: (B) $|z - 2| = 3$ 

$$|z - 2 - 3i| = 4$$

$$\Rightarrow S_1 = (x - 2)^2 + y^2 = 9$$

$$\Rightarrow S_2 = (x - 2)^2 + (y - 3)^2 = 16$$

Both circles are intersecting. So, radical axis will be

$$S_1 - S_2 = 0 \Rightarrow 9 - 6y = 7 \Rightarrow 3y - 1 = 0$$

Sol 10: (A) $z^3 + iz - 1 = 0$

Let $z = k$ (real)

$$\Rightarrow k^3 + ik - 1 = 0$$

$$k^3 + ik - 1 = 0 \text{ and } k = 0$$

If $k = 0$, then $0 - 1 = 0$

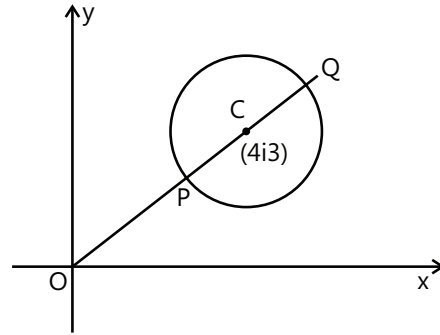
Not possible

Therefore $z \neq k$ (real)

Hence, no real solution

Sol 11: (D) $|z - 4 - 3i| = 2$

z lies on a circle shown in the figure $|z| = |z - 0|$ is nothing but distance from origin



$$\text{Minimum} = OP = OC - CP$$

$$= \sqrt{4^2 + 3^2} - 2 = 5 - 2 = 3$$

$$\text{Maximum} = OC + CQ = 5 + 2 = 7$$

Sol 12: (A) $z = 1 - \sin \alpha + i \cos \alpha$

$$|z| = \sqrt{1 + \sin^2 \alpha - 2 \sin \alpha + \cos^2 \alpha}$$

$$= \sqrt{2 - 2 \sin \alpha} = \sqrt{2(1 - \sin \alpha)}$$

$$\arg z = \tan^{-1} \frac{\cos \alpha}{1 - \sin \alpha} = \tan^{-1} \left(\frac{1}{\sec \alpha - \tan \alpha} \right)$$

$$= \tan^{-1} \frac{1 + \sin \alpha}{\cos \alpha} = \tan^{-1} \frac{\left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right)^2}{\left(\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2} \right)^2}$$

$$= \tan^{-1} \left(\frac{\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}} \right) = \tan^{-1} \left(\frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} \right)$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right] = \frac{\pi}{4} + \frac{\alpha}{2}$$

Sol 13: (D) $\left| \frac{z_1 + z_2}{2} + \sqrt{z_1 z_2} \right| + \left| \frac{z_1 + z_2}{2} - \sqrt{z_1 z_2} \right|$

$$= \frac{|z_1 + z_2 + 2\sqrt{z_1 z_2}| + |z_1 + z_2 - 2\sqrt{z_1 z_2}|}{2}$$

$$= \frac{(\sqrt{z_1} + \sqrt{z_2})^2 + (\sqrt{z_1} - \sqrt{z_2})^2}{2}$$

$$= \frac{2(z_1)^2 + 2(z_2)^2}{2} = |\sqrt{z_1}|^2 + |\sqrt{z_2}|^2$$

Sol 14: (A) $u^2 - 2u + 2 = 0$

$$\Rightarrow (u - 1)^2 = -1$$

$$\Rightarrow u - 1 = \pm i$$

$$\Rightarrow u = 1 + i, 1 - i = (\alpha, \beta)$$

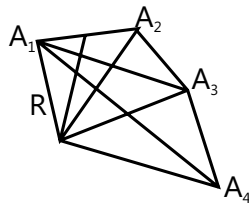
$$\cot \theta = x + 1$$

$$\frac{(x + \alpha)^n - (x + \beta)^n}{\alpha - \beta} = \frac{(\cot \theta + i)^n - (\cot \theta - i)^n}{2i}$$

$$= \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{2i \sin^n \theta}$$

$$= \frac{e^{i\theta n} - e^{-i\theta n}}{2i \sin^n \theta} = \frac{\sin n\theta}{\sin^n \theta}$$

Sol 15: (B) $A_1, \dots, A_n$ vertices of regular polygon in a circle of radius R



$$(A_1 A_2)^2 + (A_1 A_3)^2 + \dots + (A_1 A_n)^2$$

$$A_1 A_2 = 2R \sin \frac{\pi}{n}$$

$$A_1 A_3 = 2R \sin \frac{2\pi}{n}$$

$$A_1 A_4 = 2R \sin \frac{3\pi}{n}$$

$$\begin{aligned} & 4R^2 \left[\sin^2 \frac{\pi}{n} + \sin^2 \frac{2\pi}{n} + \sin^2 \frac{3\pi}{n} + \dots + \sin^2 \frac{(n-1)\pi}{n} \right] \\ &= -2R^2 \left[1 - n + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos(2n-1) \frac{\pi}{n} \right] \\ &= -2R^2 \left[1 - n + \frac{\sin \pi k}{\sin \left(\frac{\pi}{n} \right)} - \cos \frac{2n\pi}{n} \right] = 2nR^2 \end{aligned}$$

Sol 16: (B) $z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$

$$z = ib$$

$$b^4 - ib^3 a_1 - b^2 a_2 + ib a_3 + a_4 = 0$$

$$b^4 - b^2 a_2 + a_4 = 0 \Rightarrow b^4 a_2 + a_4 a_2 = b^2 a_2^2$$

$$\Rightarrow b a_3 - b^3 a_1 = 0 \Rightarrow a_3 = b^2 a_1$$

$$\frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} = \frac{b^2}{a_2} + \frac{a_4}{a_2 b^2} = \frac{b^4 a_2 + a_4 a_2}{a_2^2 b^2} = \frac{b^2 a_2^2}{b^2 a_2^2} = 1$$

Sol 17: (D) $(1+z)^6 + z^6 = 0$

$$\Rightarrow (1+z)^6 = -z^6$$

Now,

$$\Rightarrow \left(\frac{1+z}{z} \right)^6 = -1$$

(4, 3)

$$\Rightarrow 1 + \frac{1}{z} = (-1)^{1/6}$$

$$\Rightarrow \frac{1}{z} = \cos \frac{2K\pi + \pi}{6} - 1$$

$$\Rightarrow \frac{1}{z} = - \left\{ 2 \sin \frac{2K\pi + \pi}{12} \cos \left(\frac{2K\pi + \pi}{12} - \frac{\pi}{2} \right) \right\}$$

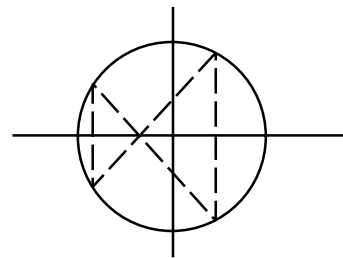
$$\Rightarrow z = - \frac{1}{2 \sin \frac{2K\pi + \pi}{12}} \left\{ \cos \left(\frac{\pi}{2} - \frac{2K\pi + \pi}{12} \right) \right\}$$

$$z = - \frac{1}{2} \left\{ 1 + i \cot \frac{2K\pi + \pi}{12} \right\}$$

$$\Rightarrow z \text{ lies on line } x = -\frac{1}{2}$$

$\Rightarrow$ All roots are collinear.

Sol 18: (A)



$$z^{10} - z^5 = 992$$

$$z^5(z^5 - 1) = 32(31)$$

$$z^5 = t$$

$$\Rightarrow t^2 - t = 992 \Rightarrow t^2 - 32t + t - 992 = 0$$

$$\Rightarrow t = 32, -31 \Rightarrow z^5 = 32, -31$$

$\Rightarrow z^5 = 32$ has 2 roots with -ve real Part

and $z^5 = -31$ has 3 roots with -ve real part

Sol 19: (A) $a|z_1| = b|z_2|$

$$\frac{a}{b} = \frac{|z_2|}{|z_1|} \quad T = \frac{az_1}{bz_2} + \frac{bz_2}{az_1} \quad \text{let } \frac{az_1}{bz_2} = z$$

$$T = z + \frac{1}{z} = z + \frac{\bar{z}}{|z|^2}$$

$$|z| = 1 \Rightarrow T = z + \bar{z} = 2\operatorname{Re}(z)$$

$$\operatorname{Re}(z) \in (-1, 1)$$

$$\Rightarrow T \in [-2, 2]$$

Sol 20: (B) $(p+q)^3 + (p\omega+q\omega^2)^3 + (p\omega^2+q\omega)^3$

$$= p^3+q^3+3p^2q+3pq^2+p^3+q^3+3pq^2\omega^5$$

$$+3p^2q\omega^4+p^3+q^3+3p^2q\omega^5+3pq^2\omega^4$$

$$= 3(p^3+q^3)$$

Sol 21: (A) $(1+i\sqrt{3})^x = 2^x \Rightarrow \left(\frac{1+i\sqrt{3}}{2}\right)^x = 1$

$$\Rightarrow (-\omega)^x = 1$$

$$\Rightarrow x = 6, 12, 18, \dots$$

It forms an AP

Multiple Correct Choice Type**Sol 22: (B, C)** $x^2+(p+iq)x+3i=0$

$$a^2+b^2=8$$

$$(\alpha+\beta)=-(p+iq)$$

$$a\beta=3i$$

$$\Rightarrow (\alpha+\beta)^2=8+6i=p^2-q^2+2ipq$$

$$p^2-q^2=8$$

$$2pq=6$$

$$\Rightarrow p=3, q=+1 \text{ or } p=-3, q=-1$$

Sol 23: (A, D) $|z_1| = |z_2| \Rightarrow a^2+b^2=c^2+d^2$

$$z_1 = a+ib, a>0$$

$$z_2 = c+id, d<0$$

$$\frac{z_1+z_2}{z_1-z_2} = \frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$$

(a+c) & (b+d) can be zero, so value can be zero

(a-c) & (b-d) can be simultaneously zero $\Rightarrow$ purely imaginary.**Sol 24: (A, B, C, D)** (a) $a, b, x, y \in \mathbb{R}$

$$\frac{a+ib}{x+iy} = a-ib$$

$$\Rightarrow x+iy = \frac{a^2-b^2+2abi}{a^2+b^2}$$

$$\Rightarrow x = \frac{a^2-b^2}{a^2+b^2}, y = \frac{2ab}{a^2+b^2}$$

$$\Rightarrow x^2+y^2=1 \text{ A is correct}$$

$$(b) \frac{1-ix}{1+ix} = a-ib$$

$$\Rightarrow \frac{1-x^2-2ix}{1+x^2} = a-ib$$

$$\Rightarrow a = \frac{1-x^2}{1+x^2}, b = \frac{2x}{1+x^2}$$

$$\Rightarrow a^2+b^2=1 \text{ B is correct}$$

$$(c) \frac{a+ib}{a-ib} = x-iy$$

$$\Rightarrow \frac{(a+ib)^2}{a^2+b^2} = x-iy$$

$$\Rightarrow x+iy = \frac{a^2-b^2+2iab}{a^2+b^2}$$

$$\Rightarrow x+iy = \frac{a^2-b^2-2iab}{a^2+b^2} \Rightarrow |x+iy|=1$$

$$(d) \frac{y-ix}{a+ib} = y+ix$$

$$\Rightarrow \frac{y^2-x^2-2ixy}{y^2+x^2} = a+ib$$

$$\Rightarrow a-ib = \frac{y^2-x^2+2ixy}{x^2+y^2} \Rightarrow |a-ib|=1$$

Sol 25: (A, D) $z = x+iy = r(\cos\theta + i\sin\theta)$

$$\sqrt{z} = \sqrt{x+iy} = \sqrt{r(\cos\theta + i\sin\theta)}$$

$$= \sqrt{r} e^{i\theta/2} = \sqrt{r} \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \right] \text{ for } y > 0$$

$$= \sqrt{r} \left[\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}} \right]$$

$$= \frac{\sqrt{r}}{\sqrt{2}} \left[\frac{\sqrt{r+x}+i\sqrt{r-x}}{\sqrt{r}} \right] = \frac{\sqrt{r+x}+i\sqrt{r-x}}{\sqrt{2}} [A]$$

Similarly

$$= \frac{1}{\sqrt{2}} \left[\sqrt{r+x} - i\sqrt{r-x} \right] \text{ for } y < 0$$

Sol 26: (A, D)

$$(az_1 + b\bar{z}_1)(cz_2 + d\bar{z}_2) = (cz_1 + d\bar{z}_1)(az_2 + b\bar{z}_2)$$

$$adz_1\bar{z}_2 + bc\bar{z}_1z_2 = bcz_1\bar{z}_2 + ad\bar{z}_1z_2$$

$$\Rightarrow ad = bc$$

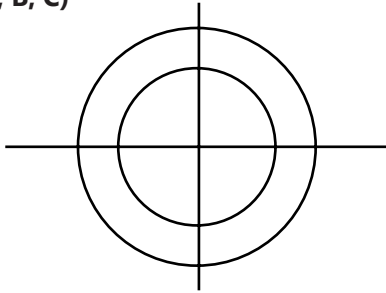
$$\frac{a}{b} = \frac{c}{d} \quad (\text{A is correct})$$

$$\Rightarrow \arg z_1 = \tan^{-1} \frac{b}{a}, \arg z_2 = \tan^{-1} \frac{d}{c}$$

$$\Rightarrow \arg z_1 = \arg z_2$$

$$\text{or } z_1\bar{z}_2 = \bar{z}_1z_2 \Rightarrow \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}$$

$$\Rightarrow \frac{a+ib}{a-ib} = \frac{c+id}{c-id} \Rightarrow -ad + bc = 0$$

Sol 27: (A, B, C)

$$|z_1| = 1 \text{ and } |z_2| = 2$$

$$\text{Max } |2z_1 + z_2|$$

$$|2z_1 + z_2| \leq 2|z_1| + |z_2| = 4 \quad (\text{A is correct})$$

$$\text{Min } |z_1 - z_2| \geq |z_1| - |z_2|$$

$$|z_1 - z_2| \geq 1 \quad (\text{B is correct})$$

$$\left| z_2 + \frac{1}{\bar{z}_1} \right| \leq |z_2| + \frac{1}{|z_1|} \leq 2 \quad (\text{C is correct})$$

$$\text{Sol 28: (A, B, C, D)} \quad \left| \frac{\alpha - \beta}{1 - \bar{\alpha}\beta} \right| = 1$$

$$|\alpha - \beta|^2 = |1 - \bar{\alpha}\beta|^2$$

$$\Rightarrow (\alpha - \beta)(\bar{\alpha} - \bar{\beta}) = (1 - \bar{\alpha}\beta)(1 - \alpha\bar{\beta})$$

$$|\alpha|^2 + |\beta|^2 - \alpha\bar{\beta} - \beta\bar{\alpha} = 1 - \alpha\bar{\beta} - \beta\bar{\alpha} + |\alpha|^2|\beta|^2$$

$$|\alpha|^2(\alpha - |\beta|^2) = (1 - |\beta|^2) \Rightarrow (|\alpha|^2 - 1)(1 - |\beta|^2) = 0$$

$$\Rightarrow |\alpha| = 1 \quad \text{A is correct}$$

$$\Rightarrow |\beta| = 1 \quad \text{B is correct}$$

$$\alpha = e^{i\theta} = \beta \quad [\text{C and D are correct}]$$

Sol 29: (A, B, D) $\alpha = 3z - 2$

$$\beta = -3z - 2$$

$$|z| = 1$$

$$|\alpha + 2| = 3 \rightarrow \text{A is correct}$$

$$|\beta + 2| = 3 \rightarrow \text{B is correct}$$

$$\alpha - \beta = 6z \quad \text{D is correct}$$

$$\Rightarrow \alpha - \beta \text{ and } z, \text{ both move on same circle}$$

Sol 30: (B, D) $(i^n + i^{-n})$ is

$$(A) \frac{2^{2n} + 2^{2n}}{2^n(1-i)^{2n}} = \frac{2 \cdot 2^n}{(1-i)^{2n}} = \frac{2^{n+1}}{2^{2n}}(1+i)^{2n} =$$

$$\frac{(2i)^n}{2^{n-1}} = 2i^n = i^n + i^n$$

$$(B) \frac{(1+i)^{2n} + (1-i)^{2n}}{2^n} = \frac{(2i)^n + (-2i)^n}{2^n} = i^n + (-i)^n$$

$$= i^n + i^{-n}$$

$$(C) \frac{-(2i)^n}{2^n} + \frac{2^n}{(-2i)^n} = -i^n + i^n = 0$$

$$(D) \frac{2^n}{(2i)^n} + \frac{2^n}{(-2i)^n} = i^{-n} + i^n$$

Sol 31: (A, B, D)

$$\log_{14} (13 + |z^2 - 4i|) + \log_{196} \left(\frac{1}{(13 + |z^2 + 4i|)^2} \right) = 0$$

$$\log_{14} \frac{13 + |z^2 - 4i|}{13 + |z^2 + 4i|} = 0$$

$$\Rightarrow 13 + |z^2 - 4i| = 13 + |z^2 + 4i|$$

$$\Rightarrow |z^2 - 4i| = |z^2 + 4i|$$

$$\Rightarrow |x^2 - y^2 + 2ixy - 4i| = |x^2 - y^2 + 2ixy + 4i|$$

$$\Rightarrow (x^2 - y^2)^2 + (2xy - 4)^2 = (x^2 - y^2)^2 + (2xy + 4)^2$$

$$\Rightarrow -16xy = 16xy \Rightarrow \text{either } x = 0 \text{ or } y = 0$$

If $y = 0$ can be purely real

$x = 0$ can be purely imaginary

Must be real or purely imaginary.

Sol 32: (A, B) $1 - \log_2 \left[\frac{|x+1+2i|-2}{\sqrt{2}-1} \right] \geq 0$

$$\Rightarrow \log_2 \frac{|x+1+2i|-2}{\sqrt{2}-1} \leq 1$$

$$\Rightarrow |x+1+2i|-2 \leq 2(\sqrt{2}-1)$$

$$\Rightarrow |x+1+2i| \leq 2\sqrt{2} \Rightarrow (x+1)^2 + 4 \leq 8$$

$$\Rightarrow (x+1)^2 \leq 4 \Rightarrow -2 \leq x+1 \leq 2$$

$$\Rightarrow -3 \leq x \leq 1 \Rightarrow x \in [-3, 1]$$

Sol 33: (D) $x = \cos \alpha$

$$y = \cos \beta$$

$$z = \cos \gamma$$

$$\Sigma x = \cos \alpha + \cos \beta + \cos \gamma$$

$$= \cos \alpha + \cos \beta + \cos \gamma + i(\sin \alpha + \sin \beta + \sin \gamma)$$

$$\Pi x = \cos \alpha \cos \beta \cos \gamma = e^{i\alpha} e^{i\beta} e^{i\gamma} = e^{i(\alpha + \beta + \gamma)}$$

Sol 34: (A, D) $x_r = \cos \left(\frac{\pi}{2^r} \right)$

$$\lim_{n \rightarrow \infty} \prod_{r=1}^n x_r = e^{i \left(\frac{\pi}{2} + \frac{\pi}{2^{r+1}} + \dots \right)} = e^{i \left(\frac{\pi}{2} + \frac{\pi}{4} + \dots \right)}$$

$$= e^{i \frac{\pi}{2 \left(1 - \frac{1}{2} \right)}} = e^{i\pi}$$

$$= \cos \pi + i \sin \pi = -1 + 0i$$

Real part is -1

Imaginary part is 0

Sol 35: (A, B, D) $\prod_{r=1}^{n-1} (\omega - z_r)$

$$(\omega - z_1)(\omega - z_2)(\omega - z_3) \dots (\omega - z_{n-1}) = \frac{\omega^n - 1}{\omega - 1}$$

If n is a multiple of 3, value can be zero as z_1 can be ω or ω^2

If $n = 3k + 1$ value is 1

If $n = 3k + 2$ value is $1 + \omega$

Sol 36: (A, B, C) $\bar{z} = -4z$

$$x - iy = -4x - 4iy$$

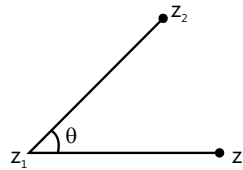
$$5x + i(3y) = 0$$

$$x = y = 0 \text{ (A is true)}$$

$\bar{z} = z$ implies z is purely real (B is correct)

$\bar{z} = -z$ implies z is purely imaginary (C is correct)

Sol 37: (A, C)



$$z_3 = (1 - z_0)z_1 + z_0z_2$$

$$z'_3 = (1 - z_0)z'_1 + z_0z'_2$$

$$z_0 = \frac{z_3 - z_1}{z_2 - z_1} = \frac{z'_3 - z'_1}{z'_2 - z'_1} \Rightarrow \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & z_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix} = 0$$

angle is same

The two triangles are similar

Sol 38: (A, B) Z multiplicative inverse is same as additive inverse

$$\frac{1}{a+ib} = -a-ib$$

$$\frac{a-ib}{a^2+b^2} = -a-ib$$

$$\Rightarrow a = -a^3 - ab^2 \Rightarrow a(1 + a^2 + b^2) = 0$$

$$\Rightarrow a = 0 \text{ and}$$

$$b = a^2b + b^3 \Rightarrow b(1 - a^2 - b^2) = 0$$

$$b \neq 0 \text{ so } a^2 + b^2 = 1 \Rightarrow b = \pm 1$$

$$z = 0 \pm i$$

Sol 39: (A, B, C, D) $Z = a + bi = \frac{(1-ix)(1-ix)}{(1+ix)(1-ix)}$

$$a + bi = \frac{1 - x^2 - 2ix}{1 + x^2}$$

$$\Rightarrow |z| = 1$$

$$\text{Arg } z = \tan^{-1} \left(\frac{2x}{x^2 - 1} \right)$$

$$\text{Arg } (z) \equiv (-\pi, \pi]$$

Previous Years' Questions

Sol 1: Given, $|z - 3 - 2i| \leq 2$

To find minimum of $|2z - 6 + 5i|$

$$\text{Or } 2 \left| z - 3 + \frac{5}{2}i \right|,$$

By using triangle inequality

$$\text{i.e., } ||z_1| - |z_2|| \leq |z_1 + \bar{z}_2|$$

$$\therefore \left| z - 3 + \frac{5}{2}i \right| = \left| z - 3 - 2i + 2i + \frac{5}{2}i \right| = \left| (z - 3 - 2i) + \frac{9}{2}i \right|$$

$$\geq \left| |z - 3 - 2i| - \frac{9}{2} \right|$$

$$\geq \left| 2 - \frac{9}{2} \right| \geq \frac{5}{2}$$

$$\Rightarrow \left| z - 3 + \frac{5}{2}i \right| \geq \frac{5}{2} \text{ or } |2z - 6 + 5i| \geq 5$$

Sol 2: $\omega = e^{i\frac{2\pi}{3}}$

$$\text{Then, } \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$$

Note: Here, $\omega = e^{i2\pi/3}$, then only integer solution exists.

Sol 3: (B)

$$A = \{z : \text{Im}(z) \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \text{Re}[(1-i)z + \sqrt{2}]\}$$

$$\text{Taking } |z - 2 - i| = 3$$

Let $z = x + iy$

$$|x + iy - 2 - i| = 3$$

$$\Rightarrow |(x-2) + i(y-1)|^2 = (3)^2$$

$$\Rightarrow (x-2)^2 + (y-1)^2 = 9$$

And

$$\text{Re}[(1-i)z] = \sqrt{2}$$

$$\text{Re}[(1-i)(x+iy)] = \sqrt{2}$$

$$\Rightarrow x + y = \sqrt{2}$$

....(ii)

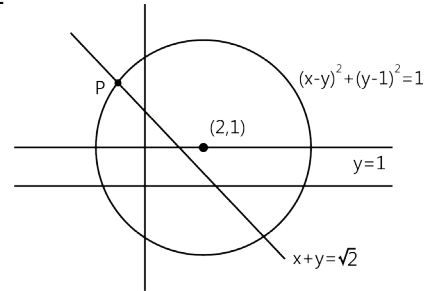
.... (i) And

$$\text{Im}(z) \geq 1$$

$$\Rightarrow \text{Im}(x + iy) \geq 1$$

$$\Rightarrow y \geq 1$$

....(iii)



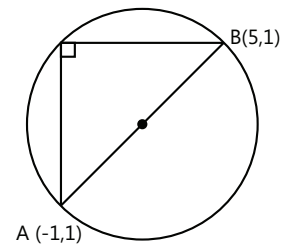
From (i), (ii) and (iii) $A \cap B \cap C$ has only one point P shown in the figure.

Sol 4: (C) $|z + 1 - i|^2 + |z - 5 - i|^2$

$$= |z - (-1 + i)|^2 + |z - (5 + i)|^2$$

The point A (-1, 1) and dB (5, 1) are end points of one of the diameter

In right angle $\triangle APB$



$$AP^2 + BP^2 = (AB)^2$$

$$\Rightarrow |z + 1 - i|^2 + |z - 5 - i|^2$$

$$= (6)^2 = 36$$

Sol 5: (D) $|w - 2 - i| < 3$

From triangle in equality

$$|z - w| > ||z| - |w||$$

$$\Rightarrow -|z - w| < |z| - |w| < |z - w|$$

....(i)

Given $|w - 2 - i| < 3$, which means that w lies inside the given circle. Since z is presented by point P and W has to be the other end of diameter

$|z - w|$ = length of diameter

From (i), we get

$$-6 < |z| - |w| < 6$$

$$\Rightarrow -6 + 3 < |z| - |w| + 3 < 6 + 3$$

$$\Rightarrow -3 < |z| - |w| + 3 < 9$$

Sol 6: (B) Given, $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$$\left(\because \omega = \frac{-1+i\sqrt{3}}{2} \text{ and } \omega^2 = \frac{-1-i\sqrt{3}}{2}\right)$$

$$\text{Now, } \frac{\sqrt{3}+i}{2} = -i \left(\frac{-1+i\sqrt{3}}{2}\right) = -i\omega$$

$$\text{And } \frac{\sqrt{3}-i}{2} = i \left(\frac{-1-i\sqrt{3}}{2}\right) = i\omega^2$$

$$\therefore z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$$

$$= i(\omega - \omega^2) = i(i\sqrt{3}) = -\sqrt{3}$$

$$\Rightarrow \operatorname{Re}(z) < 0 \text{ and } \operatorname{Im}(z) = 0$$

Alternate solution:

We know $z + \bar{z} = 2\operatorname{Re}(z)$

$$\text{If } z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5, \text{ then } z \text{ is purely real,}$$

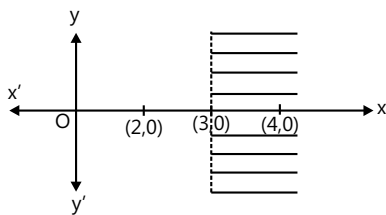
$$\text{i.e. } \operatorname{Im}(z) = 0$$

Sol 7: (D) Given, $|z-4| < |z-2|$

Since, $|z-z_1| > |z-z_2|$ represents the region on right side of perpendicular bisector of z_1 and z_2

$$\therefore |z-2| > |z-4|$$

$$\Rightarrow \operatorname{Re}(z) > 3 \text{ and } \operatorname{Im}(z) \in \mathbb{R}$$



Sol 8: (B) Since a, b, c and u, v, w are the vertices of two triangles.

$$\text{Also, } c = (1-r)a + rb \text{ and } w = (1-r)u + rv \quad \dots (i)$$

$$\text{Consider } \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix}$$

$$\text{Applying } R_3 \rightarrow R_3 - \{(1-r)R_1 + rR_2\}$$

$$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c - (1-r)a - rb & w - (1-r)u - rv & 1 - (1-r) - r \end{vmatrix}$$

$$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \text{ (from eq. (i))}$$

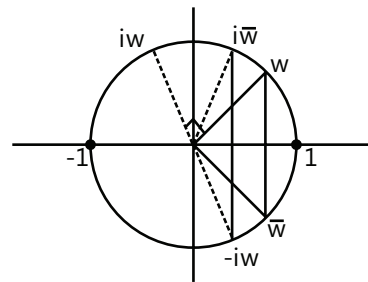
Hence, two triangles are similar.

Sol 9: (D) $\sum_{k=1}^6 \left(\sin \frac{2k\pi}{7} - i \cos \frac{2k\pi}{7} \right)$

$$\sum_{k=1}^6 i \left(\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7} \right) = -i \left\{ \sum_{k=1}^6 e^{\frac{i2k\pi}{7}} \right\}$$

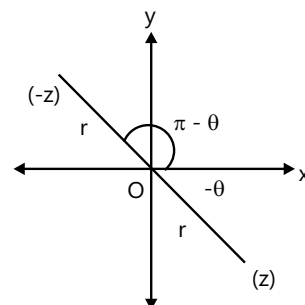
$$\left\{ \sum_{k=0}^6 e^{\frac{i2k\pi}{7}} = 0 \right\} = -i \left\{ \sum_{k=0}^6 e^{\frac{i2k\pi}{7}} - 1 \right\} = -i[0-1] = i$$

Sol 10: (C) Given, $|z+iw| = |z-i\bar{w}| \quad 2=2$
 $\Rightarrow |z-(-iw)| = |z-(i\bar{w})| = 2$
 $\Rightarrow |z-(-iw)| = |z-(-i\bar{w})|$



$\therefore z$ may take values given in (c).

Alternate solution



$$|z + iw| \leq |z| + |iw| = |z| + |w| \leq 1 + 1 = 2$$

$$\therefore |z + iw| \leq 2$$

$$\Rightarrow |z + iw| = 2 \text{ holds when}$$

$$\arg z - \arg iw = 0$$

$$\text{Similarly, when } |z - i\bar{w}| = 2$$

$$\text{Then } \frac{z}{w} \text{ is purely imaginary}$$

Now, given relation

$$|z + iw| = |z - i\bar{w}| = 2$$

Put $w = i$, we get

$$|z + i^2| = |z + i^2| = 2$$

$$\Rightarrow |z - 1| = 2$$

$$\Rightarrow z = -1 \quad (\because |z| \leq 1)$$

Put $w = -i$, we get

$$|z - i^2| = |z - i^2| = 2$$

$$\Rightarrow |z + 1| = 2$$

$$\Rightarrow z = 1 \quad (\because |z| \leq 1)$$

$\therefore z = 1$ or -1 is the one correct option given.

Sol 11: (D)

$$\begin{aligned} & (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2} \\ &= [{}^{n_1}C_0 + {}^{n_1}C_1i + {}^{n_1}C_2i^2 + {}^{n_1}C_3i^3 + \dots] \\ & \quad + [{}^{n_1}C_0 - {}^{n_1}C_1i + {}^{n_1}C_2i^2 - {}^{n_1}C_3i^3 + \dots] \\ & \quad + [{}^{n_2}C_0 + {}^{n_2}C_1i + {}^{n_2}C_2i^2 + {}^{n_2}C_3i^3 + \dots] \\ & \quad + [{}^{n_2}C_0 - {}^{n_2}C_1i + {}^{n_2}C_2i^2 - {}^{n_2}C_3i^3 + \dots] \\ &= 2[{}^{n_1}C_0 + {}^{n_1}C_2i^2 + {}^{n_1}C_4i^4 + \dots] \\ & \quad + 2[{}^{n_2}C_0 + {}^{n_2}C_2i^2 + {}^{n_2}C_4i^4 + \dots] \\ &= 2[{}^{n_1}C_0 - {}^{n_1}C_2 + {}^{n_1}C_4 - \dots] \\ & \quad + 2[{}^{n_2}C_0 - {}^{n_2}C_2 + {}^{n_2}C_4 - \dots] \end{aligned}$$

This is a real number irrespective of the values of n_1 and n_2

Alternate solution

$$\{(1+i)^{n_1} + (1-i)^{n_1}\} + \{(1+i)^{n_2} + (1-i)^{n_2}\}$$

$$\Rightarrow \text{a real number for all } n_1 \text{ and } n_2 \in \mathbb{R}.$$

$$[\because z + \bar{z} = 2\operatorname{Re}(z) \Rightarrow (1+i)^{n_1} + (1-i)^{n_1}$$

is real number for all $n \in \mathbb{R}$]

Sol 12: (C) If in a complex number $a + ib$, the ratio $a : b$ is $1 : \sqrt{3}$, then always convert the complex number in the form of ω .

$$\text{Since, } \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\therefore 4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$$= 4 + 5\omega^{334} + 3\omega^{365}$$

$$= 4 + 5 \cdot (\omega^3)^{111} \cdot \omega + 3 \cdot (\omega^3)^{121} \cdot \omega^2$$

$$= 4 + 5\omega + 3\omega^2 \quad (\because \omega^3 = 1)$$

$$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$$

$$= 1 + 2\omega + 3(1 + \omega + \omega^2) = 1 + 2\omega + 3 \times 0$$

$$= 1 + (-1 + \sqrt{3}i) = \sqrt{3}i.$$

Sol 13: (A) Since, $\arg(z) < 0$

$$\Rightarrow \arg(z) = -\theta$$

$$\Rightarrow z = r \cos(-\theta) + i \sin(-\theta) = r(\cos \theta - i \sin \theta)$$

$$\text{and } -z = -r[\cos \theta - i \sin \theta]$$

$$= r[\cos(\pi - \theta) + i \sin(\pi - \theta)]$$

$$\therefore \arg(-z) = \pi - \theta$$

Thus, $\arg(-z) - \arg(z)$

$$= \pi - \theta - (-\theta) = \pi$$

Alternate solution:

$$\text{Reason: } \arg(-z) - \arg z = \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$$

$$\text{And also } \arg z - \arg(-z) = \arg\left(\frac{z}{-z}\right) = \arg(-1) = \pi$$

Sol 14: (A, B, C) Since, $z_1 = a + ib$ and $z_2 = c + id$

$$\Rightarrow |z_1|^2 = a^2 + b^2 = 1 \text{ and } |z_2|^2 = c^2 + d^2 = 1 \quad \dots(i)$$

$$(\because |z_1| = |z_2| = 1)$$

$$\text{Also, } \operatorname{Re}(z_1 \bar{z}_2) = 0 \Rightarrow ac + bd = 0$$

$$\Rightarrow \frac{a}{b} = -\frac{d}{c} = \lambda \quad (\text{say}) \dots(ii)$$

$$\text{From Eqs. (i) and (ii), } b^2 \lambda^2 + b^2 = c^2 + \lambda^2 c^2$$

$$\Rightarrow b^2 = c^2 \text{ and } a^2 = d^2$$

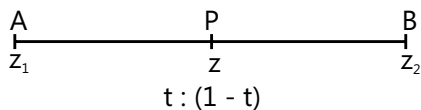
Also, given $w_1 = a + ic$ and $w_2 = b + id$

$$\text{Now, } |w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$$

$$|w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$$

$$\begin{aligned}\text{and } \operatorname{Re}(w_1 \bar{w}_2) &= ab + cd = (b\lambda)b + c(-\lambda c) \\ &= \lambda(b^2 - c^2) = 0\end{aligned}$$

Sol 15: (A, C, D) Given, $z = \frac{(1-t)z_1 + tz_2}{(1-t) + t}$



Clearly, z divides z_1 and z_2 in the ratio of t :

$$(1-t), 0 < t < 1$$

$$\Rightarrow AP + BP = AB$$

$$\text{ie, } |z - z_1| + |z - z_2| = |z_1 - z_2|$$

$\Rightarrow$ Option (a) is true.

$$\text{And } \arg(z - z_1) = \arg(z_2 - z) = \arg(z_2 - z_1)$$

$\Rightarrow$ (b) is false and (d) is true.

$$\text{Also, } \arg(z - z_1) = \arg(z_2 - z_1)$$

$$\Rightarrow \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$$

$\therefore \frac{z - z_1}{z_2 - z_1}$ is purely real.

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \text{ or } \left| \frac{z - z_1}{z_2 - z_1} \right| = \left| \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$$

Sol 16: (A) $|z - i| \cdot |z| = |z + i| \cdot |z|$

$$\Rightarrow |z - i| \cdot |z|^2 = |z + i| \cdot |z|^2$$

$$\Rightarrow (z - i|z|)(\bar{z} + i|z|) = (z + i|z|)(\bar{z} - i|z|)$$

$$\Rightarrow z\bar{z} + iz|z| - i|z|\bar{z} + |z|^2$$

$$= z\bar{z} - i|z| + i|z|\bar{z} + |z|^2$$

$$\Rightarrow 2iz|z| = 2i|z|\bar{z}$$

$$\Rightarrow 2i|z|(z - \bar{z}) = 0$$

$$\Rightarrow |z| = 0 \text{ or } z - \bar{z} = 0$$

$$\Rightarrow \operatorname{Im}(z) = 0$$

$$\text{Also } |\operatorname{Im}(z)| \leq 1$$

$$A \rightarrow q, r$$

$$(B) |z + 4| + |z - 4| = 0$$

Its an equation of ellipse having

$$aq = 4 \text{ and } 2a = 10$$

$$\Rightarrow e = \frac{4}{5}$$

$$B \rightarrow p$$

$$(C) z = w - \frac{1}{w}$$

$$\Rightarrow z = w - \frac{1}{w\bar{w}} \times \bar{w}$$

$$= w - \frac{\bar{w}}{|w|^2}$$

$$= w - \frac{\bar{w}}{4}$$

Let $w = p + iq$, then

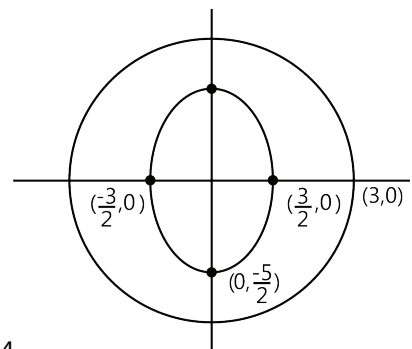
$$z = \frac{3p}{4} + i\frac{5q}{4}$$

Now, let $z = x + iy$

$$\Rightarrow x = \frac{3p}{4} \Rightarrow p = \frac{4x}{3}$$

$$\Rightarrow y = \frac{5q}{4} \Rightarrow q = \frac{4y}{5}$$

$$|w|^2 = p^2 + q^2 = 4$$



$$\Rightarrow \frac{16x^2}{9} + \frac{16y^2}{25} = 4$$

$$\Rightarrow \frac{x^2}{9/4} + \frac{y^2}{25/4} = 1, \text{ its an ellipse}$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

$$= 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow e = \frac{4}{5}$$

From figure, we can conclude that

$$|\operatorname{Re}(z)| \leq 2$$

$$\text{And } |z| \leq 3$$

$$C \rightarrow p, s, t$$

$$(D) \quad z = w + \frac{1}{w} = w + \frac{\bar{w}}{w \bar{w}} = w + \frac{\bar{w}}{|w|^2} = w + \bar{w}$$

$$\text{Let } w = a + ib$$

$$z = a + ib + a - ib = 2a$$

$$\text{Now, } |z| = 2|a| \text{ and } \operatorname{Im}(z) = 0, \operatorname{Im}(z) \leq 1$$

$$\Rightarrow \operatorname{Re}(z) \leq 2 \text{ and } |z| \leq 3$$

$$D \rightarrow q, r, s \text{ and } t$$

Sol 17: (B)

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix} = 0$$

$$\left\{ \omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right\}$$

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

Expanding the determinant, we get

$$z \left[(z + \omega^2)(z + \omega) - 1 - \omega(z + \omega - 1) + \omega^2(1 - z - \omega^2) \right] = 0$$

$$\Rightarrow z \left[z^2 + z\omega + z\omega^2 + \omega^2 - 1 - z\omega - \omega^2 + \omega + \omega^2 - \omega^2 z - \omega^4 \right] = 0$$

$$\Rightarrow z \left[z^2 \right] = 0$$

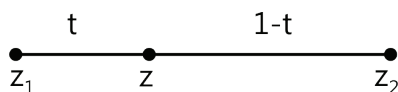
$$\Rightarrow z = 0$$

Sol 18: (A, C, D) Given $z = (1-t)z_1 + tz_2$

This equation represents line segment between z_1 and z_2

From fig.

$$|z - z_1| + |z - z_2| = |z_1 - z_2|$$



$$\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z_2 - z_1)$$

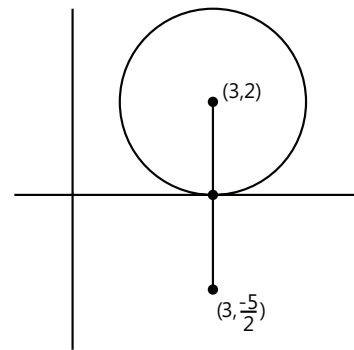
$$\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$$

$$\Rightarrow \left| \frac{z - z_1}{z_2 - z_1} \cdot \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$$

Sol 19: $|2z - 6 - 5i|$

$$= 2 \left| z - 3 + \frac{5}{2}i \right|$$

Here $\left| z - \left(3 - \frac{5}{2}i \right) \right|$ is nothing but the distance between any point on the circle $|z - 3 - 2i| \leq 2$ and point $\left(3, \frac{-5}{2} \right)$



$$\therefore \text{Minimum value} = \frac{5}{2}$$

$$|2z - 6 + 5i|_{\min} = 2 \times \frac{5}{2} = 5$$

Sol 20: Given: $a + b + c = x$

$$a + b\omega + c\omega^2 = y$$

$$a + b\omega^2 + c\omega = z$$

$$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{x\bar{x} + y\bar{y} + z\bar{z}}{|a|^2 + |b|^2 + |c|^2}$$

$$\begin{aligned} &= \frac{(a+b+c)(\bar{a} + \bar{b} + \bar{c}) + (a+b\omega+c\omega^2)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2) + (a+b\omega^2+c\omega)(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega})}{|a|^2 + |b|^2 + |c|^2} \end{aligned}$$

$$\begin{aligned} &\left\{ \begin{array}{l} \omega = e^{i\pi/3} \\ \bar{\omega} = e^{-i\pi/3} \\ \omega^2 = e^{2i\pi/3} = -\bar{\omega} \end{array} \right. \\ &= \frac{(a+b+c)(\bar{a} + \bar{b} + \bar{c}) + (a+b\omega-\bar{\omega}c)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2) + (a-b\bar{\omega}+c\omega)(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega})}{|a|^2 + |b|^2 + |c|^2} \end{aligned}$$

$$= \frac{3(|a|^2 + |b|^2 + |c|^2)}{|a|^2 + |b|^2 + |c|^2} = 3$$

Sol 21: (A) Given $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

Determinant (D) = $\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix}$

$$|D| = 1(1 - c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2)$$

$$\Rightarrow |D| = 1 - (a + c)\omega + a\omega^2$$

$$|D| \neq 0, \text{ only when } a = \omega \text{ and } c = \omega$$

$$\therefore (a, b, c) \equiv (\omega, \omega, \omega) \text{ or } (\omega, \omega^2, \omega)$$

$\therefore$ Two non-singular matrices are possible.

Sol 22: (D) $a = z^2 + z + 1 = 0$

$$\Rightarrow z^2 - z + 1 - a = 0$$

Discriminant < 0 {Since imaginary part of z is not zero}

$$1 - 4(1 - a) < 0$$

$$\Rightarrow 1 - 4 + 4a < 0$$

$$\Rightarrow 4a < 3$$

$$\Rightarrow a < \frac{3}{4}$$

Sol 23: (C) $(x - x_0)^2 + (y - y_0)^2 = r$

$$\Rightarrow |z - z_0| = r$$

α lies on it, then

$$|\alpha - z_0| = r$$

$$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$$

$$= |\alpha|^2 - \bar{z}_0 \alpha - z_0 \bar{\alpha} + |z_0|^2 = r^2$$

....(i)

Similarly,

$$(x - x_0)^2 + (y - y_0)^2 = 2r$$

$$\Rightarrow |z - z_0| = 2r$$

$\frac{1}{\alpha}$ lies on it, then

$$\left| \frac{1}{\alpha} - z_0 \right| = 2r$$

$$\Rightarrow \left(\frac{1}{\alpha} - z_0 \right) \left(\frac{1}{\alpha} - \bar{z}_0 \right) = 4r^2$$

$$\Rightarrow \left(\frac{1}{\alpha} - \frac{\bar{z}_0}{\alpha} - \frac{z_0}{\alpha} + |z_0|^2 \right) = 4r^2$$

$$\left\{ \alpha \bar{\alpha} = |\alpha|^2 \Rightarrow \bar{\alpha} = \frac{|\alpha|^2}{\alpha} \right\}$$

$$\Rightarrow \frac{1}{|\alpha|^2} - \frac{\bar{z}_0}{|\alpha|^2} - \frac{z_0}{|\alpha|^2} + |z_0|^2 = 4r^2$$

$$\Rightarrow 1 - z_0 \bar{\alpha} - \bar{z}_0 \alpha + |\alpha|^2 |z_0|^2 = 4r^2 |z|^2 \quad \dots(ii)$$

Subtracting (ii) from (i), we get

$$1 - |\alpha|^2 + |\alpha|^2 |z_0|^2 - |z_0|^2 = r^2 (4|\alpha|^2 - 1)$$

$$1 - |\alpha|^2 + \frac{r^2 + 2}{2} (|\alpha|^2 - 1) = r^2 (4|\alpha|^2 - 1)$$

$$\Rightarrow (|\alpha|^2 - 1) \left[\frac{r^2 + 2}{2} - 1 \right] = r^2 (4|\alpha|^2 - 1)$$

$$\Rightarrow (|\alpha|^2 - 1) \left[\frac{r^2}{2} \right] = r^2 (4|\alpha|^2 - 1)$$

$$\Rightarrow |\alpha|^2 - 1 = 8|\alpha|^2 - 2$$

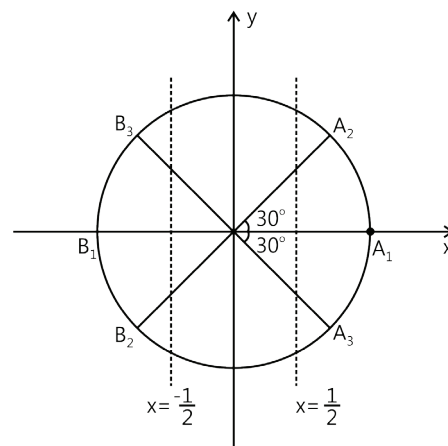
$$\Rightarrow 7|\alpha|^2 = 1$$

$$\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$$

Sol 24: (C, D) $P = \{w^n : n = 1, 2, 3, \dots\}$

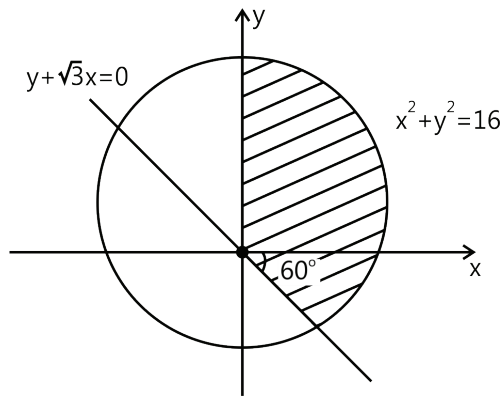
$$w = \frac{\sqrt{3}}{2} + \frac{1}{2} \Rightarrow |w| = 1$$

$\Rightarrow$ All the complex number belong to set P lie on circle of unit radius, centre at origin.



All the complex number belong to PnH_1 lie right of the line $x = \frac{1}{2}$ on circle. Possible positions are shown in the figure as A_1, A_2, A_3 .

Similarly, all the complex number belong to PnH_2 lie left of the line $x = \frac{-1}{2}$ on circle. Possible positions are shown in the figure as B_1, B_2, B_3 . z_1 z_2 = angle between A_2 and $B_3 = \frac{2\pi}{3}$ or angle between A_1 and $B_3 = \frac{5\pi}{6}$



Sol 25: (B, C, D) $P = [p_{ij}]_{n \times n}$ $p_{ij} = \omega^{i+j}$

$$P = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 & \dots & \omega^{n+1} \\ \omega^3 & \omega^4 & \omega^5 & \dots & \omega^{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^{2n+1} & \omega^{n+2} & \omega^{n+3} & \dots & \omega^{n+n} \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} \omega^4 + \omega^6 + \omega^8 + \dots & \omega^5 + \omega^7 + \omega^9 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{n+3} + \omega^{n+5} + \dots & \dots & \dots & \dots \end{bmatrix}$$

Lets take element $P_{11} = \omega^4 + \omega^6 + \omega^8 + \dots n$ terms

$$= \frac{\omega^4 (\omega^{2n} - 1)}{\omega - 1}$$

If n is multiple of 3, then this element will vanish. Which is the case for every element.

$\therefore n$ can not be multiple of 3, for $P^2 \neq 0$

$\therefore$ Possible values of n are 55, 58, 56

Sol 26: (B) $S_1 = \{z \in \mathbb{C} : |z| < 4\}$

z lies inside of a circle given by $|z| = 4$

$$S_2 = \left\{ z \in \mathbb{C} : \operatorname{Im} \left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i} \right] > 0 \right\}$$

$$\operatorname{Im} \left[\frac{(z-1+\sqrt{3}i)(1+\sqrt{3}i)}{4} \right] > 0$$

Let $z = x + iy$

$$\Rightarrow \operatorname{Im} \left[\frac{(x+iy)(1+\sqrt{3}i) - (1-\sqrt{3}i)(1+\sqrt{3}i)}{4} \right] > 0$$

$$\Rightarrow \operatorname{Im} \left[\frac{(x+\sqrt{3}y) + i(\sqrt{3}x+y) - 4}{4} \right] > 0$$

$$\Rightarrow \frac{\sqrt{3}x+y}{4} > 0 \Rightarrow \sqrt{3}x+y > 0$$

$$S_3 \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$$

$\Rightarrow z$ lies in either first or fourth quadrant.

Now, points of intersection of circle $x^2 + y^2 = 16$ and

$$\sqrt{3}x + y = 0 \text{ is } x^2 + 3x^2 = 16$$

$$\Rightarrow 4x^2 = 16 \Rightarrow x = \pm 2$$

$$\Rightarrow y = \pm 2\sqrt{3}$$

$$\text{Area} = \frac{1}{2} (4)^2 \times \frac{5\pi}{6}$$

$$= \frac{1}{2} \times 16 \times \frac{5\pi}{6} = \frac{20\pi}{3} \text{ .sq. unit}^2$$

Sol 27: (C) $|1-3i-z|$

$$= |z-1+3i| = |z-(1-3i)|$$

$\operatorname{Min}(|z-(1-3i)|) = \text{distance of point } (1, -3) \text{ from line}$

$$\sqrt{3}x + y = 0$$

$$= \left| \frac{\sqrt{3} \times 1 - 3}{\sqrt{3} + 1} \right| = \left| \frac{\sqrt{3} - 3}{2} \right|$$

$$= \frac{3-\sqrt{3}}{2}$$

Sol 28: (C) $z_k = \cos \frac{2\pi k}{10} + i \sin \frac{2\pi k}{10}, k = 1, 2, \dots, 9$

$$(p) z_k = e^{i \frac{2\pi k}{10}}$$

$$\Rightarrow z_k \cdot z_j = e^{i \frac{2\pi}{10}(k+j)} = 1$$

$$\Rightarrow \cos \frac{2\pi}{10}(k+j) + i \sin \frac{2\pi}{10}(k+j) = 1$$

If $k + j = 10m$ (multiple of 10), then above equation is True.

$$(q) z_1 \cdot z = z_k$$

$$\Rightarrow z = \frac{z_k}{z_1} = \frac{e^{i \frac{2\pi k}{10}}}{e^{i \frac{2\pi}{10}}} = e^{i \frac{(k-1)2\pi}{10}}$$

Clearly, this equation has many solutions

$$\Rightarrow Q \text{ is False}$$

$$(r) \text{ Consider, } z^{10} = 1$$

$$z^{10} - 1 = (z - 1)(z - z_1)(z - z_2)(z - z_3) \dots (z - z_9)$$

Where z_k represents the roots of equation $z^{10} = 10$

$$= 1 + z + z^2 + \dots + z^9$$

$$\Rightarrow \frac{z^{10} - 1}{z - 1} = (z - z_1)(z - z_2) \dots (z - z_9)$$

$$= 1 + z + z^2 + z^3 + \dots + z^9$$

$$\frac{z^{10} - 1}{z - 1} = (z - z_1)(z - z_2) \dots (z - z_9)$$

$$= 1 + 1 + 1 \dots 10$$

$$= 10$$

(s) We know that sum of roots of $z^{10} - 1 = 0$ is zero

$$\Rightarrow 1 + \sum_{k=1}^9 \cos \frac{2k\pi}{10} = 0 \Rightarrow \sum_{k=1}^9 \cos \frac{2k\pi}{10} = -1$$

$$\Rightarrow 1 - \sum_{k=1}^9 \cos \frac{2k\pi}{10} = 2$$

$$\text{Sol 29: Given } \alpha_k = \cos \frac{k\pi}{7} + i \sin \frac{k\pi}{7} = e^{i \frac{k\pi}{7}}$$

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|} = \frac{\sum_{k=1}^{12} \left| e^{i \frac{(k+1)\pi}{7}} - e^{i \frac{k\pi}{7}} \right|}{\sum_{k=1}^3 \left| e^{i \frac{(4k-1)\pi}{7}} - e^{i \frac{(4k-2)\pi}{7}} \right|}$$

$$= \frac{\sum_{k=1}^{12} \left| e^{i \frac{k\pi}{7}} \right| \left| e^{i \frac{\pi}{7}} - 1 \right|}{\sum_{k=1}^3 \left| e^{i \frac{(4k-2)\pi}{7}} \right| \left| e^{i \frac{\pi}{7}} - 1 \right|} = \frac{\sum_{k=1}^{12} \left| e^{i \frac{\pi}{7}} - 1 \right|}{\sum_{k=1}^3 \left| e^{i \frac{\pi}{7}} - 1 \right|}$$

$$= \frac{\sum_{k=1}^{12} 2 \sin \frac{\pi}{14}}{\sum_{k=1}^3 2 \sin \frac{\pi}{14}} = \frac{12 \times 2 \sin \frac{\pi}{14}}{3 \times 2 \sin \frac{\pi}{14}} = 4$$

$$\text{Sol 30: } P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-z)^{2r} + (z^{4s}) & (-z)^r z^{2s} + z^r z^{2s} \\ (-z)^r z^{2s} + z^r z^{2s} & z^{2r} + z^{4s} \end{bmatrix}$$

$$= \begin{bmatrix} z^{2r} + z^{4s} & z^{2s} [(-z)^r + z^r] \\ z^{2s} [(-z)^r + z^r] & z^{2r} + z^{4s} \end{bmatrix}$$

$$\text{Given that } P^2 - I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow z^{2r} + z^{4s} = -1$$

$$\text{And } z^{2s} [(-z)^r + z^r] = 0$$

$$\text{Now, we have } z = \frac{-1 + i\sqrt{3}}{2} = \omega$$

$$\Rightarrow \omega^{2r} + \omega^{4s} = -1 \text{ and } \omega^{2s} [(-\omega)^r + \omega^r] = 0$$

Only $(r, s) \equiv (1, 1)$ satisfies both the equation.

Only one pair exists.

$$\text{Sol 31: (A, C, D) } S = \left\{ z \in \mathbb{C} : z = \frac{1}{a + ibt}, t \in \mathbb{R}, t \neq 0 \right\}$$

$$z = \frac{1}{a + ibt} \times \frac{a - ibt}{a - ibt}$$

$$\Rightarrow z = \frac{a - ibt}{a^2 + b^2 t^2} = x + iy \text{ (Let)}$$

$$\Rightarrow x = \frac{a}{a^2 + b^2 t^2} \text{ and } y = \frac{bt}{a^2 + b^2 t^2}$$

$$\Rightarrow \frac{x}{y} = \frac{a}{bt} \Rightarrow t = \frac{ay}{bx}$$

Substituting 't' in $x = \frac{a}{a^2 + b^2 t^2}$

$$x = \frac{a}{a^2 + b^2 \frac{a^2 y^2}{b^2 x^2}}$$

$$\Rightarrow x = \frac{ax^2}{a^2 x^2 + a^2 y^2} \Rightarrow a^2 x^2 + a^2 y^2 - ax = 0$$

$$\Rightarrow x^2 + y^2 - \frac{x}{a} = 0$$

$$\left(x - \frac{1}{2a}\right)^2 + y^2 = \left(\frac{1}{2a}\right)^2$$

$$\text{Centre} \equiv \left(\frac{1}{2a}, 0\right)$$

$$\text{Radius} = \frac{1}{2a}, \text{ when } a > 0, b \neq 0$$

$$\text{If } b = 0, a \neq 0$$

$$y = 0 \Rightarrow x\text{-axis}$$

$$\text{If } a = 0, b \neq 0$$

$$x = 0 \Rightarrow y\text{-axis}$$