

Solved Examples

JEE Main/Boards

Example 1: In any triangle PQR, prove that, $(b+c)\cos P + (c+a)\cos Q + (a+b)\cos R = a+b+c$.

Sol: Simply, by using projection rule, we can solve the above problem.

$$\begin{aligned} \text{given L.H.S. } & (b+c)\cos P + (c+a)\cos Q + (a+b)\cos R \\ & = b\cos P + c\cos P + c\cos Q + a\cos Q + a\cos R + b\cos R \\ & = (b\cos P + a\cos Q) + (c\cos P + a\cos R) + (c\cos Q + b\cos R) \\ & = c + b + a = \text{R.H.S. [By using projection Rule]} \end{aligned}$$

Example 2: In any $\triangle ABC$,

If $a = 2, b = \sqrt{3} + 1$ and $C = 60^\circ$, solve the triangle.

Sol: Here, by using $\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2}$,

we can obtain the value of $B - A$.

Two sides and the included angle is given.

$$\therefore \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2} \cot 30^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+3} \sqrt{3}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$$

$$\tan(60^\circ - 45^\circ) = \tan 15^\circ$$

$$\therefore \frac{B-A}{2} = 15^\circ \text{ or } B-A = 30^\circ \quad \dots(i)$$

We know, $A+B+C = 180^\circ$

$$\Rightarrow A+B = 120^\circ \quad \dots(ii)$$

Solving (i) and (ii), we get $B = 75^\circ$ & $A = 45^\circ$

To find side c , we use sine rule

$$\frac{a}{\sin A} = \frac{c}{\sin 60^\circ}; \text{ or } c = 2\sqrt{2} \frac{\sqrt{3}}{2} = \sqrt{6}$$

Thus $A = 45^\circ, B = 75^\circ$ and $c = \sqrt{6}$.

Example 3: If $A = 30^\circ, a = 100, c = 100\sqrt{2}$, solve the triangle

Sol: Here, simply by using sine rule, we can obtain the required values.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \sin C = \frac{1}{2} \therefore C = 135^\circ \text{ or } 45^\circ$$

$$C = 45^\circ \Rightarrow B = 105^\circ; b = \frac{c \sin B}{\sin C} = \frac{100}{\sqrt{2}} (\sqrt{3} + 1)$$

$$C = 135^\circ \Rightarrow B = 15^\circ; b = \frac{c \sin B}{\sin C} = \frac{100}{\sqrt{2}} (\sqrt{3} - 1)$$

Example 4: In a triangle ABC, if $a=3, b=4$ and $\sin A = \frac{3}{4}$, then find the value of $\angle B$.

Sol: By using sine rule, we can obtain $\angle B$.

$$\text{We have, } \frac{\sin A}{a} = \frac{\sin B}{b} \text{ or } \sin B = \frac{b}{a} \sin A$$

Since, $a=3, b=4, \sin A = 3/4$,

$$\text{We get, } \sin B = \frac{4}{3} \times \frac{3}{4} = 1$$

$$\therefore \angle B = 90^\circ$$

Example 5: Find the smallest angle of the triangle whose sides are $6 + \sqrt{12}, \sqrt{48}, \sqrt{24}$.

Sol: The smallest angles of a triangle are those angles whose opposite sides are small.

$$\text{Let } a = 6 + \sqrt{12}, b = \sqrt{48}, c = \sqrt{24}$$

Here, c is the smallest side.

$\angle C$ is the smallest angle of the triangle.

$$\begin{aligned} \text{Now } \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{(48 + 24\sqrt{3}) + 48 - 24}{4(3 + \sqrt{3}) \cdot 4\sqrt{3}} = \frac{\sqrt{3}}{2} \end{aligned}$$

$$\text{So, } \angle C = \pi / 6$$

Example 6: In a $\triangle ABC$, $\tan A \tan B \tan C = 9$. For such triangles, if $\tan^2 A + \tan^2 B + \tan^2 C = \lambda$ then find the value of λ .

Sol: Here, by solving

$$(\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2 > 0,$$

we can obtain the value of λ .

$$\Rightarrow 2 \left(\frac{\tan^2 A + \tan^2 B + \tan^2 C - \tan A \tan B}{-\tan B \tan C - \tan C \tan A} \right) > 0$$

$$\begin{aligned} &\Rightarrow 3(\tan^2 A + \tan^2 B + \tan^2 C) - (\tan A + \tan B + \tan C)^2 > 0 \\ &\Rightarrow 3\lambda - (\tan A \cdot \tan B \cdot \tan C)^2 > 0; \\ &\Rightarrow 3\lambda - 81 > 0 \therefore \lambda > 27 \end{aligned}$$

Example 7: In a triangle, a, b and A are given and c_1, c_2 are two values of the third side c. Find the sum of the areas of two triangles with side a, b, c_1 and a, b, c_2 .

Sol: Here, as we know $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, therefore, by solving this equation we can obtain c_1 and c_2 .

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ or } c^2 - (2b \cos A)c + (b^2 - a^2) = 0$$

Which is a quadratic in c, whose roots are c_1 and c_2 ;

$$\therefore c_1 + c_2 = 2b \cos A \text{ and } c_1 c_2 = b^2 - a^2$$

\therefore Sum of areas of two Δ 's with sides a, b, c_1 & a, b, c_2 ;

$$= \frac{1}{2}bc_1 \sin A + \frac{1}{2}bc_2 \sin A$$

$$= \frac{1}{2}b(c_1 + c_2) \sin A = \frac{1}{2}b \cdot 2b \cos A \cdot \sin A = \frac{1}{2}b^2 \sin 2A$$

Example 8: In any triangle ABC, if $\tan \theta = \frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}$

Prove that $c = (a-b) \sec \theta$

Sol: As given, $\tan \theta = \frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}$. Hence, by solving this

and using formula $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, we can solve the above problem.

$$\therefore \tan \theta = \frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}$$

$$\therefore (a-b)^2 \tan^2 \theta = 4ab \sin^2 \frac{C}{2}$$

$$\text{or } (a-b)^2 (\sec^2 \theta - 1) = 4ab \sin^2 \frac{C}{2}$$

$$\text{or } (a-b)^2 \sec^2 \theta = (a-b)^2 + 4ab \sin^2 \frac{C}{2}$$

$$\text{or } (a-b)^2 \sec^2 \theta = a^2 + b^2 - 2ab \left(1 - 2 \sin^2 \frac{C}{2} \right)$$

$$\text{or } (a-b)^2 \sec^2 \theta = a^2 + b^2 - 2ab \cos C$$

$$\left[\because \cos C = \frac{a^2 + b^2 - c^2}{2ab} \right]$$

$$\text{or } (a-b)^2 \sec^2 \theta = c^2; \therefore c = (a-b) \sec \theta$$

Example 9: In any Δ , prove that,

$$\frac{1}{a} \cos^2 \frac{A}{2} + \frac{1}{b} \cos^2 \frac{B}{2} + \frac{1}{c} \cos^2 \frac{C}{2} = \frac{s^2}{abc}$$

Sol: Here, simply by using trigonometric ratios of half angle formula we can prove the above example.

$$\text{Given L.H.S.} = \frac{1}{a} \cos^2 \frac{A}{2} + \frac{1}{b} \cos^2 \frac{B}{2} + \frac{1}{c} \cos^2 \frac{C}{2}$$

$$= \frac{1}{a} \cdot \left(\frac{s(s-a)}{bc} \right) + \frac{1}{b} \cdot \left(\frac{s(s-b)}{ac} \right) + \frac{1}{c} \cdot \left(\frac{s(s-c)}{ab} \right)$$

$$= \frac{s}{abc} (s-a + s-b + s-c)$$

$$= \frac{s}{abc} \{3s - (a+b+c)\} = \frac{s}{abc} (3s - 2s)$$

$$= \frac{s^2}{abc} = \text{R.H.S}$$

Hence proved.

Example 10: In any ΔABC , prove that

$$a \cos A + b \cos B + c \cos C = \frac{8\Delta^2}{abc}$$

Sol: By using sine rule, we can obtain values of a, b and c and then by substituting these values in L.H.S. we can prove this.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (let) [by sine rule]}$$

Then, $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$

Now, $a \cos A + b \cos B + c \cos C$

$$= k \sin A \cos A + k \sin B \cos B + k \sin C \cos C$$

$$= \frac{k}{2} [\sin 2A + \sin 2B + \sin 2C]$$

$$= \frac{k}{2} [4 \sin A \sin B \sin C] = 2k \sin A \sin B \sin C$$

$$= 2a \sin B \sin C = 2a \cdot \frac{2\Delta}{ac} \cdot \frac{2\Delta}{ab}$$

$$[\because \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B \therefore \sin B = \frac{2\Delta}{bc}, \sin C = \frac{2\Delta}{ab}]$$

$$= \frac{8\Delta^2}{abc} = \text{R.H.S}$$

Example 11: In a ΔABC , $\angle C = 90^\circ$, $a = 3$, $b = 4$ and D is a point on AB so that $\angle BCD = 30^\circ$. Find the length of CD.

Sol: Here, by using Pythagoras theorem and sine rule, we can obtain the length of CD.

In $\triangle ABC$, $\angle C = 90^\circ$

$$\therefore AB^2 = 3^2 + 4^2 = 25$$

$$AB = \sqrt{3^2 + 4^2} = 5$$

and

$$\therefore \sin A = \frac{3}{5}, \sin B = \frac{4}{5}$$

in $\triangle BCD$, by sine rule

$$\frac{BD}{\sin 30^\circ} = \frac{CD}{\sin B}$$

$$\therefore BD = \frac{\sin 30^\circ}{\sin B} \cdot CD = \frac{1}{2} \times \frac{5}{4} CD = \frac{5}{8} CD \quad \dots (i)$$

And in $\triangle ACD$, by sine rule

$$\frac{AD}{\sin 60^\circ} = \frac{CD}{\sin A}$$

$$\Rightarrow AD = \frac{\sin 60^\circ}{\sin A} CD = \frac{\sqrt{3}}{2} \times \frac{5}{3} CD = \frac{5\sqrt{3}}{6} CD \quad \dots (ii)$$

$$\text{But } BD + AD = AB \therefore \frac{5}{8} CD + \frac{5\sqrt{3}}{6} CD = 5$$

$$\text{Or } \frac{15 + 20\sqrt{3}}{24} CD = 5$$

$$\text{Or } CD = \frac{24}{3 + 4\sqrt{3}} = \frac{24(4\sqrt{3} - 3)}{48 - 9}$$

$$\text{Hence } CD = \frac{24}{39}(4\sqrt{3} - 3) = \frac{8}{13}(4\sqrt{3} - 3)$$

JEE Advanced/Boards

Example 1: Prove that, $a \cos \frac{B-C}{2} = (b+C) \sin \frac{A}{2}$.

Sol: By using sine rule i.e.

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A}, \text{ we can prove the above example.}$$

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A}, \text{ [using sine Rule]}$$

$$= \frac{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{2 \cos \frac{A}{2} \cos \frac{B-C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\cos \frac{B-C}{2}}{\sin \frac{A}{2}}$$

$$\therefore (b+c) \sin \frac{A}{2} = a \cos \frac{B-C}{2}$$

Example 2: If a_1, a_2, a_3 are the altitudes of the triangle PQR,

$$\text{prove that, } \frac{\cos P}{a_1} + \frac{\cos Q}{a_2} + \frac{\cos R}{a_3} = \frac{1}{R}$$

Sol: Here, simply by using sine rule i.e.

$(\cos P)a = 2R \sin P \cos P$ and so on, we can prove the above problem.

In $\triangle PQR$, consider, $PX = a_1, QY = a_2, RZ = a_3$.

Area of triangle PQR

$$= \Delta = \frac{1}{2} a a_1 = \frac{1}{2} b a_2 = \frac{1}{2} c a_3$$

$$a_1 = \frac{2\Delta}{a}, a_2 = \frac{2\Delta}{b}, a_3 = \frac{2\Delta}{c}$$

$$\frac{\cos P}{a_1} + \frac{\cos Q}{a_2} + \frac{\cos R}{a_3} = \frac{(\cos P)a}{2\Delta} + \frac{(\cos Q)b}{2\Delta} + \frac{(\cos R)c}{2\Delta}$$

$$= \frac{1}{2\Delta} [2R \sin P \cos P + 2R \sin Q \cos Q + 2R \sin R \cos R]$$

[using sine rule]

$$= \frac{R}{2\Delta} [\sin 2P + \sin 2Q + \sin 2R]$$

$$= \frac{R}{2\Delta} [2 \sin(P+Q) \cos(P-Q) + 2 \sin R \cos R]$$

$$= \frac{2R \sin R}{2\Delta} [\cos(P-Q) - \cos(P+Q)]$$

[using $P+Q+R = \pi$]

$$= \frac{R \sin R}{\Delta} 2 \sin P \sin Q = \frac{2R}{\Delta} \sin P \sin Q \sin R$$

$$= \frac{2R}{\Delta} \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} \text{ [using sine rule]}$$

$$= \frac{abc}{4\Delta} \times \frac{1}{R^2} = \frac{R}{R^2} = \frac{1}{R} \text{ [by using } R = \frac{abc}{4\Delta}]$$

Example 3: If the sides of a triangle PQR are in A.P. and if its greatest angle exceeds the least angle by α ,

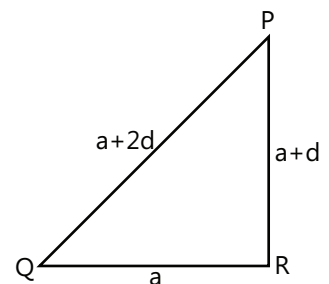
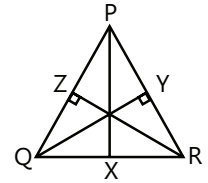
show that the sides are in the ratio $1-x : 1 : 1+x$

$$\text{where } x = \sqrt{\frac{1 - \cos \alpha}{7 - \cos \alpha}}$$

Sol: As the sides of a given triangle are in A.P., by considering the sides to be $a, a+d, a+2d$ and using sine rule, we can obtain the required result.

Consider the sides to be $a, a+d, a+2d : d > 0$

Let P be the least angle and R be the greatest angle.



Let $P = \phi$; then $R = \phi + \alpha$ and $Q = 180^\circ - (2\phi + \alpha)$

$$\frac{a}{\sin \phi} = \frac{a+2d}{\sin(\phi+\alpha)} = \frac{a+d}{\sin(\pi - (2\phi + \alpha))}$$

$$\Rightarrow \frac{a}{\sin \phi} = \frac{a+2d}{\sin(\phi+\alpha)} = \frac{a+d}{\sin(2\phi + \alpha)} = \frac{2(a+d)}{\sin \phi + \sin(\phi + \alpha)} \dots (i)$$

From the first and the second term,

$$\frac{a}{\sin \phi} = \frac{a+2d}{\sin(\phi+\alpha)} = \frac{a}{a+2d} = \frac{\sin \phi}{\sin(\phi+\alpha)}$$

By componendo and dividendo, we get,

$$\frac{a+a+2d}{a+2d-a} = \frac{\sin \phi + \sin(\phi + \alpha)}{-\sin \phi + \sin(\phi + \alpha)}$$

$$\Rightarrow \frac{2(a+d)}{2d} = \frac{2 \sin\left(\phi + \frac{\alpha}{2}\right) \cdot \cos \frac{\alpha}{2}}{2 \cos\left(\phi + \frac{\alpha}{2}\right) \cdot \sin \frac{\alpha}{2}} \Rightarrow \frac{a+d}{d} = \frac{\tan\left(\phi + \frac{\alpha}{2}\right)}{\tan \frac{\alpha}{2}}$$

$$\Rightarrow \frac{d}{a+d} = \frac{\tan \frac{\alpha}{2}}{\tan\left(\phi + \frac{\alpha}{2}\right)} \dots (ii)$$

From the third and the fourth term of equation (i) we get

$$\frac{a+d}{\sin(2\phi + \alpha)} = \frac{2(a+d)}{\sin \phi + \sin(\phi + \alpha)} \Rightarrow 2 = \frac{\sin \phi + \sin(\phi + \alpha)}{\sin(2\phi + \alpha)}$$

$$\Rightarrow 2 = \frac{\cos \frac{\alpha}{2}}{\cos\left(\phi + \frac{\alpha}{2}\right)} \Rightarrow \cos\left(\phi + \frac{\alpha}{2}\right) = \frac{\cos \frac{\alpha}{2}}{2}$$

$$\therefore \tan\left(\phi + \frac{\alpha}{2}\right) = \frac{\sqrt{4 - \cos^2 \frac{\alpha}{2}}}{\cos \frac{\alpha}{2}} \dots (iii)$$

From (ii) and (iii) we get,

$$\frac{d}{a+d} = \frac{\sin \frac{\alpha}{2}}{\sqrt{4 - \cos^2 \frac{\alpha}{2}}} = \frac{\sqrt{\sin^2 \frac{\alpha}{2}}}{\sqrt{4 - \cos^2 \frac{\alpha}{2}}}$$

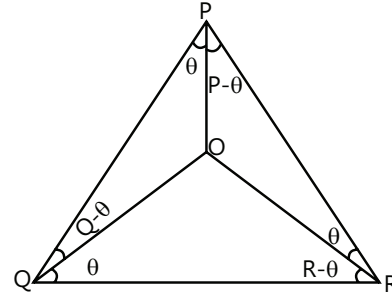
$$\Rightarrow \frac{d}{a+d} = \sqrt{\frac{1 - \cos \alpha}{4 - \frac{1 + \cos \alpha}{2}}} = \sqrt{\frac{1 - \cos \alpha}{7 - \cos \alpha}} = x$$

Required ratio is $a : a + d : a + 2d$

$$= 1 - \frac{d}{a+d} : 1 : 1 + \frac{d}{a+d} = 1 - x : 1 : 1 + x$$

Example 4: Let O be a point inside a triangle PQR such that $\angle OPQ = \angle OQR = \angle ORP = \theta$. Show that $\cot \theta = \cot P + \cot Q + \cot R$.

Sol: Simply, by applying sine rule in ΔPOQ and ΔQOR , we can prove the above problem.



$$\angle POQ = \pi - Q \text{ and } \angle QOR = \pi - R$$

Applying the sine rule in $\angle POQ$,

$$\text{we have } \frac{c}{\sin(\pi - Q)} = \frac{OQ}{\sin \theta} \Rightarrow OQ = \frac{c \sin \theta}{\sin Q} \dots (i)$$

Applying the sine rule in ΔQOR ,

$$\text{we have } \frac{OQ}{\sin(R - \theta)} = \frac{a}{\sin(\pi - R)} \Rightarrow OQ = \frac{a \sin(R - \theta)}{\sin R} \dots (ii)$$

$$\text{From (i) and (ii), we have } \frac{c \sin \theta}{\sin R} = \frac{a \sin(R - \theta)}{\sin R}$$

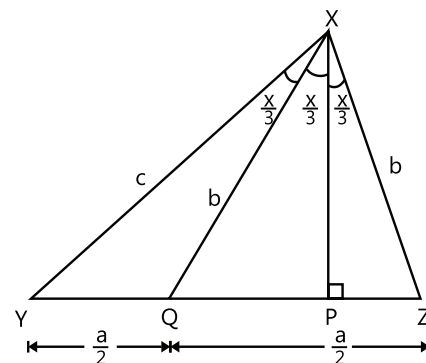
Using Sine Rule we have

$$\frac{2R \sin R \sin \theta}{\sin Q} = \frac{2R \sin P \sin(R - \theta)}{\sin R}$$

$$\frac{\sin R}{\sin P \sin Q} = \frac{\sin(R - \theta)}{\sin R \sin \theta}; \frac{\sin(P + Q)}{\sin P \sin Q} = \cot \theta - \cot R$$

$$\cot Q + \cot P = \cot \theta - \cot R \text{ or } \cot \theta = \cot P + \cot Q + \cot R.$$

Example 5: In a triangle XYZ , the median XQ and the perpendicular XP from the vertex X to the side QR divide angle X into three equal parts. Show that



$$\cos \frac{X}{3} \sin^2 \frac{X}{3} = \frac{3a^2}{32bc}$$

Sol: By using the cosine rule in ΔXYQ and ΔPQR and then subtracting them, we'll get the result.

As given, $\angle YXQ = \angle QXP = \angle PXZ = \frac{X}{3}$; $YQ = ZQ = \frac{a}{2}$

$QP = PZ = \frac{a}{4}$ [Since ΔXQP and ΔXPZ are congruent]

$XQ = XZ = b$ \therefore In ΔXYQ

$$\cos \frac{X}{3} = \frac{c^2 + b^2 - \frac{a^2}{4}}{2bc} = \frac{4c^2 + 4b^2 - a^2}{8bc} \quad \dots(i)$$

In ΔPQR , we have $\cos P = \frac{c^2 + b^2 - a^2}{2bc}$

$$4 \cos^3 \frac{X}{3} - 3 \cos \frac{X}{3} = \frac{c^2 + b^2 - a^2}{2bc} \quad \dots(ii)$$

Subtracting (ii) from (i) we get

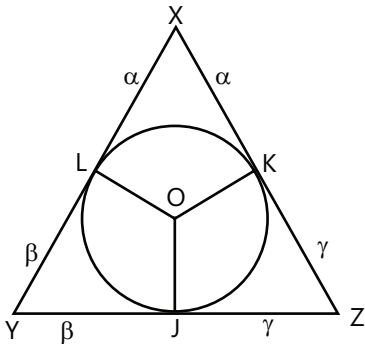
$$\begin{aligned} &\cos \frac{X}{3} - 4 \cos^3 \frac{X}{3} + 3 \cos \frac{X}{3} \\ &= \frac{4c^2 + 4b^2 - a^2}{8bc} - \frac{c^2 + b^2 - a^2}{2bc} \end{aligned}$$

$$4 \cos \frac{X}{3} \left(1 - \cos^2 \frac{X}{3} \right) = \frac{3a^2}{8bc};$$

$$\cos \frac{X}{3} \sin^2 \frac{X}{3} = \frac{3a^2}{32bc}.$$

Example 6: If α, β and γ are the distances of the vertices of a triangle XYZ from nearest points of contact of the incircle with sides of ΔXYZ ,

prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$.



Sol: Here, as we know, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $r = \frac{\Delta}{s}$.

Given $XL = XK = \alpha, YL = YJ = \beta, ZJ = ZK = \gamma$

$2s = XY + YZ + ZX = c + a + b = 2\alpha + 2\beta + 2\gamma$

Area of triangle $XYZ = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(\alpha + \beta + \gamma)\alpha\beta\gamma}$

$\therefore r = \frac{\Delta}{s} = \frac{\sqrt{(\alpha + \beta + \gamma)\alpha\beta\gamma}}{\alpha + \beta + \gamma}$

$\Rightarrow r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$.

Example 7: In any triangle ABC , if

$\cos \theta = \frac{a}{b+c}, \cos \phi = \frac{b}{a+c}, \cos \varphi = \frac{c}{a+b}$

where θ, ϕ, φ lie between 0 and π ,

prove that $\tan \frac{\theta}{2} \tan \frac{\phi}{2} \tan \frac{\varphi}{2} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

Sol: Here, by using formula

$\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$ and $\frac{a}{b+c} = \cos \theta$, we can

solve the above problem.

$\frac{a}{b+c} = \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$

By componendo and dividendo,

$\frac{2}{2 \tan^2 \frac{\theta}{2}} = \frac{a+b+c}{b+c-a}; \Rightarrow \tan^2 \frac{\theta}{2} = \frac{2s-2a}{2s} = \frac{s-a}{s}$

Similarly, $\tan^2 \frac{\phi}{2} = \frac{s-b}{s}$ and $\tan^2 \frac{\varphi}{2} = \frac{s-c}{s}$

$\therefore \tan^2 \frac{\theta}{2} \tan^2 \frac{\phi}{2} \tan^2 \frac{\varphi}{2} = \frac{(s-a)(s-b)(s-c)}{s^3} = \frac{\Delta^2}{s^4}$

$\therefore \tan \frac{\theta}{2} \tan \frac{\phi}{2} \tan \frac{\varphi}{2} = \frac{\Delta}{s^2} \quad \dots(i)$

Now $\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$

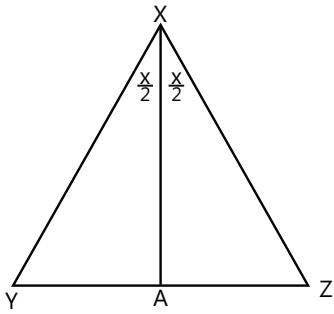
$= \sqrt{\frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{s(s-a)s(s-b)s(s-c)}}$

$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}} = \frac{\Delta}{s^2} \quad \dots(ii)$

From (i) and (ii)

$\tan \frac{\theta}{2} \tan \frac{\phi}{2} \tan \frac{\varphi}{2} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

Example 8: The bisector of angle X of triangle XYZ meets YZ at A. If $XA = \ell$ then, prove that



(i) $\ell = \frac{2bc}{b+c} \cos \frac{X}{2}$ (ii) $\frac{a}{b+c} = \sqrt{1 - \frac{\ell^2}{bc}}$

Sol: (i) Simply by using area of triangle formula, we can prove the above equation.

(ii) Here, by using cosine rule, we can prove it.

(i) Area of $\Delta XYZ = \text{Area of } \Delta XYA + \text{Area of } \Delta XAZ$

$$\frac{1}{2}bc \sin X = \frac{1}{2}c\ell \sin \frac{X}{2} + \frac{1}{2}b\ell \sin \frac{X}{2}$$

$$2bc \cos \frac{X}{2} = \ell(b+c); \Rightarrow \ell = \frac{2bc}{b+c} \cos \frac{X}{2}$$

(ii) $\frac{YA}{AZ} = \frac{XY}{XZ} = \frac{c}{b}$ or $\frac{YA}{c} = \frac{ZA}{b} = \frac{YA+ZA}{c+b} = \frac{a}{c+b}$

$$\Rightarrow \frac{YA}{c} = \frac{a}{c+b}$$

... (i)

In triangle XYA, $\cos \frac{X}{2} = \frac{c^2 + \ell^2 - YA^2}{2c\ell}$

$$\Rightarrow -2c\ell \cos \frac{X}{2} + c^2 + \ell^2 = YA^2$$

Substituting value of $\cos \frac{X}{2}$ from (i) we get

$$QA^2 = -2c\ell \frac{\ell(b+c)}{2bc} + c^2 + \ell^2 = \frac{c^2b - \ell^2c}{b}$$

Equation (A) gives $\frac{(QA)^2}{c^2} = \left(\frac{a}{c+b}\right)^2$

$$\text{or } \frac{c^2b - \ell^2c}{bc^2} = \left(\frac{a}{c+b}\right)^2 \text{ or } \sqrt{1 - \frac{\ell^2}{bc}} = \frac{a}{c+b}$$

Example 9: A cyclic quadrilateral ABCD of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB=1$ and

$\angle A$ is acute and the diagonal $BD = \sqrt{3}$, find the lengths of the other sides.

Sol: Here, area of cyclic quadrilateral = area of triangle ABC + area of triangle BCD. Therefore by using cosine rule in triangle ABD and BCD, we will be solving the above example.

Given $AB=1, BD=\sqrt{3}$

$OA=OB=OD=OC=1=R$ (O being center of the circle),

in triangle ABD,

$$\frac{BD}{\sin A} = 2R \Rightarrow \frac{\sqrt{3}}{2} = \sin A \text{ (}\because \text{ Given circle is circumcircle}$$

of ΔABD) $\Rightarrow A = \frac{\pi}{3}$; Hence $C = \frac{2\pi}{3}$

(\because ABCD is a cyclic quadrilateral)

Using cosine rule in triangle ABD,

$$\cos A = \frac{(AB)^2 + (AD)^2 - (BD)^2}{2AB \cdot AD}$$

$$\frac{1}{2} = \frac{1 + (AD)^2 - 3}{2AD} \text{ or } AD^2 - AD - 2 = 0$$

$$\text{or } (AD - 2)(AD + 1) = 0; \therefore AD = 2$$

Using cosine rule in triangle BCD, we have

$$\cos C = \frac{(BC)^2 + (CD)^2 - (BD)^2}{2(BC) \cdot (CD)} \Rightarrow -\frac{1}{2} = \frac{(BC)^2 + (CD)^2 - 3}{2(BC) \cdot (CD)}$$

$$\text{or } (BC)^2 + (CD)^2 + (BC)(CD) - 3 = 0 \quad \dots (i)$$

Area of cyclic quadrilateral = Area of triangle ABC + Area of triangle BCD

$$\frac{3\sqrt{3}}{4} = \frac{1}{2} \cdot 1 \cdot 2 \sin \frac{\pi}{3} + \frac{1}{2} \cdot BC \cdot CD \sin \frac{2\pi}{3}$$

$$3 = 2 + BC \cdot CD \text{ or } BC \cdot CD = 1 \quad \dots (ii)$$

Solving (i) and (ii), we get $BC=CD=1$

Hence length of sides of cyclic quadrilateral are

$$AD=2, BC=CD=1.$$

Example 10: The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ th of an equilateral triangle of the same perimeter. Prove that the sides are in the ratio 3:5:7.

Sol: Here, sides of triangle are in A.P. Hence, by considering the sides to be $a - d, a$ and $a + d$ and then by using area of triangle formula and the given conditions, we can prove the given ratios.

Let the sides be $a - d, a$ and $a + d$.

$2s = \text{sum of the sides} = 3a$

$$\therefore s = \frac{3a}{2}$$

Now, $\Delta_1 = \text{Area of the triangle whose sides are in A.P.}$

$$= \sqrt{\frac{3a}{2} \left[\frac{3a}{2} - a + d \right] \left[\frac{3a}{2} - a \right] \left[\frac{3a}{2} - a - d \right]}$$

$$= \frac{\sqrt{3a}}{4} \sqrt{(a+2d)(a-2d)} \quad \text{or} \quad \Delta_1 = \frac{\sqrt{3a}}{4} \sqrt{a^2 - 4d^2} \quad \dots (i)$$

Perimeter of equilateral triangle = Perimeter of the given triangle

$$\therefore 3 \times \text{One side of equilateral triangle} = 3a$$

$$\Rightarrow \text{Side of the equilateral triangle} = a$$

Now, $\Delta_2 = \text{Area of the equilateral triangle}$

$$= \frac{\sqrt{3}}{4} \times (\text{side})^2 = \frac{\sqrt{3}}{4} a^2 \quad \dots (ii)$$

$$\text{From question, } \frac{\Delta_1}{\Delta_2} = \frac{3}{5} \quad \text{or} \quad \frac{\sqrt{a^2 - 4d^2}}{a} = \frac{3}{5}$$

$$\text{or } 25a^2 - 100d^2 = 9a^2 \quad \text{or } 16a^2 = 100d^2$$

$$\Rightarrow \frac{a}{d} = \frac{5}{2}$$

Ratio of the sides = $a - d : a + d$

$$= \frac{a}{d} - 1 : \frac{a}{d} + 1 = \frac{5}{2} - 1 : \frac{5}{2} + 1$$

$$= \frac{3}{2} : \frac{7}{2} = 3 : 7$$

JEE Main/Boards

Exercise 1

Q.1. If $a=13$, $b=14$, $c=15$, find r and R .

Q.2. In an equilateral triangle, find the relation between the in radius and the circum radius.

Q.3. If in a triangle $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A - B)}{\sin(A + B)}$, prove that it is either a right angled or an isosceles triangle.

Q.4. If ΔABC is scalene and $\cos A + \cos B = 4 \sin^2 C / 2$ then prove that A, B, C are in A.P.

Q.5. Solve the triangle, if $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$.

Q.6. If $a=5$, $b=7$ and $\sin A = \frac{3}{4}$, solve the triangle, if possible.

Q.7. Two sides of the triangle are of length $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30° . How many such triangles are possible? Find the length of their third side and area.

Q.8. If in a triangle ABC , $\angle A = \frac{\pi}{3}$ and AD is a median then prove that $4AD^2 = b^2 + bc + c^2$.

Q.9. If $A=30^\circ$, $b=8$ and $a=6$, find c .

Q.10. The angles of a triangle are in the ratio 2:3:7. Find the ratio of its sides.

Q.11. In a triangle ABC , if $3a = b + c$, prove that:
 $\cot \frac{B}{2} \cot \frac{C}{2} = 2$.

Q.12. D is the mid point of BC in a triangle ABC . If AD is perpendicular to AC , prove that $\cos A \cos C = \frac{2(c^2 - a^2)}{3ac}$.

Q.13. In ΔABC , prove that:

$$\frac{\cos A}{b \cos C + c \cos B} + \frac{\cos B}{c \cos A + a \cos C} + \frac{\cos C}{a \cos B + b \cos A} = \frac{a^2 + b^2 + c^2}{2abc}$$

Q.14. Prove that

$$(b + c - a) \{ \cot(B/2) + \cot(C/2) \} = 2a \cot(A/2)$$

Q.15. If p_1, p_2, p_3 be the altitudes of a triangle ABC from the vertices A, B, C respectively and Δ be the area of

the triangle ABC , prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{2abc \cos^2 \frac{C}{2}}{\Delta(a+b+c)}$

Q.16. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Q.17. Prove that

$$a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0.$$

Q.18. With usual notations, if in a triangle,

$$ABC \quad \frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} \text{ then prove that}$$

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

Q.19. In a triangle ABC, prove that

$$\frac{a}{\cos B \cos C} + \frac{b}{\cos C \cos A} + \frac{c}{\cos A \cos B} = 2a \tan B \tan C \sec A$$

Q.20. Prove that the radius of the circle passing through the centre of the inscribed circle of the triangle ABC and through the end points of the base BC is $\frac{a}{2} \sec \frac{A}{2}$.

Q.21. In a triangle ABC, Prove that

$$(b+c-a) \tan \frac{A}{2} = (c+a-b) \tan \frac{B}{2} = (a+b-c) \tan \frac{C}{2}$$

Q.22. 3 circles of radius 3, 4, 5 touches externally. Find the distance from point of contact to intersection point of tangents.

Q.23. Perpendiculars are drawn from the vertices A, B, C of an acute angled triangle on the opposite sides, and produced to meet the circumcircle of the triangle. If these produced parts be α, β, γ respectively.

Show that $\frac{a}{\alpha} + \frac{a}{\beta} + \frac{a}{\gamma} = 2(\tan A + \tan B + \tan C)$.

Q.24. If in a triangle $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right angled.

Q.25. If Δ is the area of a triangle with side length a, b, c then show that $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$.

Q.26. In any triangle ABC, prove that

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

Q.27. In a triangle of base a, the ratio of the other two sides is $r (< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{a}{1-r^2}$.

Q.28. Let ABC be a triangle having O and I as its circumcentre and incentre respectively if R and r be the circumradius and the inradius respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled if and only if b is the A.M. of a and c.

Q.29. If α, β and γ are the altitudes of the ΔABC from the vertices A, B and C respectively then

show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{\Delta} (\cot A + \cot B + \cot C)$

Exercise 2

Single Correct Choice Type

Q.1. If A is the area and 2s the sum of the 3 sides of a triangle, then

(A) $A \leq \frac{s^2}{3\sqrt{3}}$ (B) $A = \frac{s^2}{2}$

(C) $A > \frac{s^2}{\sqrt{3}}$ (D) None

Q.2. In a triangle ABC, CH and CM are the lengths of the altitude and median to the base AB. If $a=10, b=26, c=32$ then length (HM) is

(A) 5 (B) 7 (C) 9 (D) None

Q.3. In a triangle ABC, CD is the bisector of the angle C. If $\cos \frac{C}{2}$ has the value $\frac{1}{3}$ and $l(CD) = 6$, then $\left(\frac{1}{a} + \frac{1}{b}\right)$ has the value equal to

(A) $\frac{1}{9}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) None

Q.4. With usual notations in a triangle ABC, $(II_1) \cdot (II_2) \cdot (II_3)$ has the value equal to

(A) $R^2 r$ (B) $2R^2 r$ (C) $4R^2 r$ (D) $16R^2 r$

Q.5. With usual notation in a ΔABC

$$\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \left(\frac{1}{r_2} + \frac{1}{r_3}\right) \left(\frac{1}{r_3} + \frac{1}{r_1}\right) = \frac{KR^3}{a^2 b^2 c^2}$$

Where K has the value equal to

(A) 1 (B) 16 (C) 64 (D) 128

Q.6. If the incircle of the $\triangle ABC$ touches its sides respectively at L, M and N and if x, y, z be the circumradii of the triangles MIN, NIL and LIM where I is the incentre then the product xyz is equal to

- (A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$

Q.7. The product of the distances of the incentre from the angular points of a $\triangle ABC$ is

- (A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(abc)R}{s}$ (D) $\frac{(abc)s}{R}$

Q.8. If x, y and z are the distances of incentre from the vertices of the triangle ABC respectively then $\frac{abc}{xyz}$ is equal to

- (A) $\prod \tan \frac{A}{2}$ (B) $\sum \cot \frac{A}{2}$
 (C) $\sum \tan \frac{A}{2}$ (D) $\sum \sin \frac{A}{2}$

Q.9. For each natural number k , let C_k denotes the circle with radius k centimeters and centre at the origin. On the circle C_k , a particle moves k centimeters in the counter-clockwise direction. After completing its motion on C_k , the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of the x -axis for the first time on the circle C_n then n equal to

- (A) 6 (B) 7 (C) 8 (D) 9

Q.10. If in a triangle ABC $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$

then the value of the angle is

- (A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Previous Years' Questions

Q.1. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D

divides BC internally in the ratio 1:3, then $\frac{\sin \angle BAD}{\sin \angle CAD}$

is equal to

- (A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{\frac{2}{3}}$ (1995)

Q.2. If in a triangle PQR, $\sin P, \sin Q, \sin R$ are in AP, then (1998)

- (A) The altitudes are in AP.
 (B) The altitudes are in HP.
 (C) The medians are in GP.
 (D) The medians are in AP.

Q.3. In a triangle PQR, $\angle R = \frac{\pi}{2}$, if $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$

are the roots of the equation $ax^2 + bx + c = 0$ ($a \neq 0$), then (1999)

- (A) $a + b = c$ (B) $b + c = a$
 (C) $a + c = b$ (D) $b = c$

Q.4. In a triangle ABC, $2a \sin \frac{1}{2}(A - B + C) =$ (2000)

- (A) $a^2 + b^2 - c^2$ (B) $c^2 + a^2 - b^2$
 (C) $b^2 - c^2 - a^2$ (D) $c^2 - a^2 - b^2$

Q.5. In a triangle ABC, let $\angle C = \pi/2$, if r is the inradius and R is the circumradius of the triangle, then $2(r + R)$ is equal to (2000)

- (A) $a + b$ (B) $b + c$
 (C) $c + a$ (D) $a + b + c$

Q.6. The number of integral points (integral point means both the coordinates should be integers) exactly in the interior of the triangle with vertices $(0, 0)$, $(0, 21)$ and $(21, 0)$ is (2003)

- (A) 133 (B) 190 (C) 233 (D) 105

Q.7. The sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the angles of the triangle are in the ratio (2004)

- (A) 1:3:5 (B) 2:3:2 (C) 3:2:1 (D) 1:2:3

Q.8. There exists a triangle ABC satisfying the conditions (1986)

- (A) $b \sin A = a$, $A < \frac{\pi}{2}$ (B) $b \sin A > a$, $A > \frac{\pi}{2}$
 (C) $b \sin A > a$, $A < \frac{\pi}{2}$ (D) $b \sin A < a$, $A < \frac{\pi}{2}$, $b > a$

Q.9. A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is
(1987)

Q.10. The sides of a triangle inscribed in a given circle subtend angles α , β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$ and $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to
(1987)

Q.11. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm, then the area of the triangle is.....
(1988)

Q.12. If in a triangle ABC $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$ then the value of the angle A is degree.
(1993)

Q.13. If p_1, p_2, p_3 are the perpendiculars from the vertices of a triangle to the opposite sides, prove that $p_1 p_2 p_3 = \frac{a^2 b^2 c^2}{8R^3}$
(1978)

Q.14. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ the area of the triangle, then prove that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$
(1978)

Q.15. If in a triangle ABC, $a = 1 + \sqrt{3}$ cm, $b = 2$ cm and $\angle C = 60^\circ$, then find the other two angles and the third side.
(1978)

Q.16. The ex-radii r_1, r_2, r_3 of ΔABC are in HP, show that its sides a, b, c are in AP.
(1983)

Q.17. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.
(1991)

Q.18. Prove that a triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$.
(1998)

Q.19. I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that $I_n = \frac{O_n}{2} \left(1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2} \right)$
(2003)

Q.20. Circle with radii 3, 4 and 5 touch each other externally, if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the point of contact.
(2005)

Q.21 Consider a ΔABC and let a, b and c denote the lengths of the sides opposite to vertices A, B, C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangles is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to.....
(2010)

Q.22 In a ΔPQR , if $3 \sin P + 4 \cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, then the angle R is equal to
(2012)
(A) $\frac{6\pi}{6}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

Q.23 ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ABD = \theta$, $BC = p$ and $CD = q$ then AB is equal to :
(2013)

- (A) $\frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$ (B) $\frac{p^2 + q^2 \cos \theta}{p \cos \theta + q \sin \theta}$
(C) $\frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$ (D) $\frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)^2}$

Q.24 If the angles of elevation of the top of lower from three collinear points A, B and C, on a line leading to the foot of the lower, are 30° , 45° and 60° respectively, then the ratio, AB : BC, is
(2015)

- (A) $\sqrt{3} : 1$ (B) $\sqrt{3} : \sqrt{2}$
(C) $1 : \sqrt{3}$ (D) $2 : 3$

JEE Advanced/Boards

Exercise 1

Q.1 Given a triangle ABC with $AB=2$ and $AC=1$. Internal Bisector of $\angle BAC$ intersects BC at D. If $AD=BD$ and Δ is the area of triangle ABC, then find the value of $12\Delta^2$.

Q.2 In a triangle ABC, let angles A, B, C are in G.P. with common ratio 2. If circumradius of triangle ABC is 2, then find the value of $(b^{-1} + c^{-1} - a^{-1})$

Q.3 In a triangle ABC, BD is a median.

$$\text{If } l(\text{BD}) = \frac{\sqrt{3}}{4} \cdot l(\text{AB}) \text{ and } \angle \text{DBC} = \frac{\pi}{2}.$$

Determine the $\angle \text{ABC}$.

Q.4 In an isosceles ΔABC , if the altitudes intersect on the inscribed circle then find the secant of the vertical angle 'A'.

Q.5 ABCD is a rhombus. The circumradii of ΔABD and ΔACD are 12.5 and 25 respectively. Find the area of rhombus.

Q.6 In a triangle ABC if $a^2 + b^2 = 101c^2$ then find the value of $\frac{\cot C}{\cot A + \cot B}$.

Q.7 The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is 60° . If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Q.8 If in a ΔABC , $a = 6$, $b = 3$ and $\cos(A-B) = 4/5$ then find its area.

Q.9 The triangle ABC (with side lengths a,b,c as usual) satisfies $\log a^2 = \log b^2 + \log c^2 - \log(2bc \cos A)$. What can you say about this triangle?

Q.10 The sides of a triangle are consecutive integers n, n+1 and n+2 and the largest angle is twice the smallest angle. Find n.

Q.11 The triangle ABC is a right angled triangle, right angle at A. The ratio of the radius of the circle circumscribed to the radius of the circle escribed to the radius of the circle escribed to the hypotenuse is,

$\sqrt{2} : (\sqrt{3} + \sqrt{2})$. Find the acute angles B & C. Also find the ratio of the two sides of the triangle other than the hypotenuse.

Q.12 If a, b, c are the sides of triangle ABC satisfying $\log\left(1 + \frac{c}{a}\right) + \log a - \log b - \log 2$.

Also $a(1-x^2) + 2bx + c(1+x^2) = 0$ has two equal roots. Find the value of $\sin A + \sin B + \sin C$.

Q.13 Given a triangle ABC with sides $a = 7$, $b = 8$ and $c =$

5. If the value of the expression $(\sum \sin A) / \left(\sum \cot \frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\frac{p}{q}$ is in its lowest form find the value of $(p + q)$.

Q.14 If $r_1 = r + r_2 + r_3$, then prove that the triangle is a right angled triangle.

Q.15 If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Q.16 In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC, then prove

$$\text{that, } \frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

Q.17 In a right angled triangle ABC, $\angle C = 90^\circ$ and sides AC, AB are roots of the equation $2 + y^2 = 3y$. If the internal angle bisector of angle A intersects BC at D such that $BD : CD = x^2 + 1 : 2x$, then find the sum of all possible values of $\tan \frac{A}{2}$.

Q.18 Given a right triangle with $\angle A = 90^\circ$. Let M be the mid-point of BC. If the inradii of the triangle ABM and ACM are r_1 and r_2 then find the range of $\frac{r_1}{r_2}$.

Q.19 If the length of the perpendiculars from the vertices of a triangle A,B,C on the opposite sides are

p_1, p_2, p_3 , then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

- (A) $\frac{abc}{2R^2}$ (B) $\frac{R^3}{abc}$ (C) $\frac{4\Delta}{R^2}$ (D) $\frac{\Delta}{4R^2}$

Q.20 Tangents parallel to the three sides of ΔABC are drawn to its incircle. If x, y, z be the lengths of the parts of the tangents within the triangle (with respect to the sides a, b, c) then find the value of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$.

Q.7 In a ΔABC , a semicircle is inscribed, whose diameter lies on the side C . Then the radius of the semicircle is

- (A) $\frac{2\Delta}{a+b}$ (B) $\frac{2\Delta}{a+b-c}$ (C) $\frac{2\Delta}{s}$ (D) $\frac{c}{2}$

Where Δ is the area of the triangle ABC .

Exercise 2

Single Correct Choice Type

Q.1 If the median of a triangle ABC through A is perpendicular to AB then $\frac{\tan A}{\tan B}$ has the value equal to

- (A) $\frac{1}{2}$ (B) 2 (C) -2 (D) $-\frac{1}{2}$

Q.2 In a ΔABC , the value of $\frac{a \cos A + b \cos B + c \cos C}{a+b+c}$ is equal to

- (A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

Q.3 With usual notation in a ΔABC , if $R = k \frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{r_1 r_2 + r_2 r_3 + r_3 r_1}$ where k has the value equal to

- (A) 1 (B) 2 (C) $1/4$ (D) 4

Q.4 In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

- (A) $\frac{\pi}{3}$ & $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ & $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ & $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ & $\frac{3\pi}{10}$

Q.5 Let f, g, h be the lengths of the perpendiculars from the circumcentre of the ΔABC on the sides a, b and c respectively.

If $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh}$ then the value of λ is

- (A) $1/4$ (B) $1/2$ (C) 1 (D) 2

Q.6 If 'O' is the circumcentre of the ΔABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangle OBC, OCA and OAB respectively then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$ has the value equal to

Q.8 If in a ΔABC , $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ then the triangle is

- (A) Right angled (B) Isosceles
(C) Equilateral (D) Obtuse

Q.9 If $\cos A + \cos B + 2\cos C = 2$ then the sides of the ΔABC are in

- (A) A.P. (B) G.P. (C) H.P. (D) None

Q.10 The product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle is equal to:

- (A) Δ (B) 2Δ (C) 3Δ (D) 4Δ

[where Δ is the area of the triangle ABC]

Q.11 If in a triangle $\sin A : \sin C = \sin(A-B) : \sin(B-C)$ then $a^2 : b^2 : c^2$

- (A) Are in A.P. (B) Are in G.P.
(C) Are in H.P. (D) None of these

Q.12 The sines of two angles of a triangle are equal to $\frac{5}{13}$ & $\frac{99}{101}$. The cosine of the third angle is

- (A) $245/1313$ (B) $255/1313$
(C) $735/1313$ (D) $765/1313$

Multiple correct choice type

Q.13 If the side of a right angled triangle are $\{\cos 2\alpha + \cos 2\beta + 2\cos(\alpha + \beta)\}$ and $\{\sin 2\alpha + \sin 2\beta + 2\sin(\alpha + \beta)\}$, then the length of the hypotenuse is

- (A) $2[1 + \cos(\alpha - \beta)]$ (B) $2[1 + \cos(\alpha + \beta)]$
(C) $4\cos^2 \frac{\alpha - \beta}{2}$ (D) $4\sin^2 \frac{\alpha + \beta}{2}$

Match the Columns

Q.14 Let P be an interior point of $\triangle ABC$. Match the correct entries for the ratios of the Area of $\triangle PBC$: Area of $\triangle PCA$: Area of $\triangle PAB$ depending on the position of the point P w.r.t. $\triangle ABC$.

	Column I		Column II
(A)	If P is centroid (G)	(p)	$\tan A:\tan B:\tan C$
(B)	If P is incentre (I)	(q)	$\sin 2A:\sin 2B:\sin 2C$
(C)	If P is orthocenter (H)	(r)	$\sin A:\sin B:\sin C$
(D)	If P is circumcentre (S)	(s)	1:1:1
		(t)	$\cos A:\cos B:\cos C$

Q.15 In a $\triangle ABC$, $BC=2$, $CA=1+\sqrt{3}$ and $\angle C=60^\circ$. Feet of the perpendicular from A, B and C on the opposite sides BC, CA and AB are D, E and F respectively and are concurrent at P. Now match the entries of Column I with respective entries of Column II.

	Column I		Column II
(A)	Radius of the circle circumscribing the $\triangle DEF$, is	(p)	$\frac{\sqrt{6}-\sqrt{2}}{4}$
(B)	Area of the $\triangle DEF$, is	(q)	$\frac{1}{\sqrt{2}}$
(C)	Radius of the circle inscribed in the $\triangle DEF$, is	(r)	$\frac{\sqrt{3}}{4}$
		(s)	$\frac{\sqrt{6}+\sqrt{2}}{4}$

Previous Years' Questions

Q.1 Which of the following pieces of data does not uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)? **(2002)**

- (A) $a, \sin A, \sin B$ (B) a, b, c
 (C) $a, \sin B, R$ (D) $a, \sin A, R$

Q.2 If the angles of a triangle are in the ratio 4:1:1, then the ratio of the longest side to the perimeter is **(2003)**

- (a) $\sqrt{3}:(2+\sqrt{3})$ (b) 1:3
 (c) $1:2+\sqrt{3}$ (d) 2:3

Q.3 Orthocentre of triangle with vertices (0,0), (3,4) and (4,0) is **(2003)**

- (A) $\left(3, \frac{5}{4}\right)$ (B) (3,12)
 (C) $\left(3, \frac{3}{4}\right)$ (D) (3,9)

Q.4 In a $\triangle ABC$, among the following which one is true? **(2005)**

- (A) $(b+c)\cos\frac{A}{2} = a\sin\left(\frac{B+C}{2}\right)$
 (B) $(b+c)\cos\left(\frac{B+C}{2}\right) = a\sin\frac{A}{2}$
 (C) $(b-c)\cos\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
 (D) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$

Q.5 In radius of a circle which is inscribed in a isosceles triangle one of whose angle is $2\pi/3$, is $\sqrt{3}$, then area of triangle is **(2006)**

- (A) $4\sqrt{3}$ (B) $12-7\sqrt{3}$
 (C) $12+7\sqrt{3}$ (D) None of these

Q.6 If the angles A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the lengths of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is **(2010)**

- (A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

Paragraph (Q.7 to Q.9):

Read the following Paragraph and answer the questions.

Consider the circle $x^2+y^2=9$ and the parabola $y^2=8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. **(2007)**

Q.7 The ratio of the areas of the triangle PQS and PQR is

- (A) $1:\sqrt{2}$ (B) 1:2 (C) 1:4 (D) 1:8

Q.8 The radius of the circumcircle of the triangle PRS is

- (A) 5 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

Q.9 The radius of the incircle of the triangle PQR is

- (A) $\frac{4}{8}$ (B) 3 (C) $\frac{8}{3}$ (D) 2

Q.10 Internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and side AB at F. If a, b, c represent sides of ΔABC , then **(2006)**

- (A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$
 (C) $EF = \frac{4bc}{b+c} \sin \frac{A}{2}$ (D) The ΔAEF is isosceles

Q.11 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then **(2008)**

- (A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$
 (C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ (D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Q.12 Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) **(2010)**

- (A) $-(2 + \sqrt{3})$ (B) $1 + \sqrt{3}$
 (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

Q.13 In a triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23^\circ$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B = \dots\dots$ **(1994)**

Q.14 In a triangle ABC, $a:b:c = 4:5:6$. The ration of the radius of the circumcircle to that of the incircle is $\dots\dots\dots$ **(1996)**

Q.15 ABC is a triangle, D is the middle point of BC. If AD is perpendicular to AC, then prove that

$$\cos A \cos C = \frac{2(c^2 - a^2)}{3ac} \quad \text{(1980)}$$

Q.16 For a triangle ABC it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral. **(1984)**

Q.17 With usual notation, if in a triangle ABC

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}, \text{ then}$$

prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$. **(1984)**

Q.18 In a triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11 - 6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45° . Find the length of the side BC. **(1985)**

Q.19 If in a triangle ABC $\cos A \cos B + \sin A \sin B \sin C = 1$, show that $a : b : c = 1 : 1 : \sqrt{2}$. **(1986)**

Q.20 In a triangle of base a, the ratio of the other two sides is $r (< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. **(1991)**

Q.21 Let A_1, A_2, \dots, A_n be the vertices of an n-sided regular polygon such that $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$. Find the value of n. **(1994).**

Q.22 Consider the following statements concerning a triangle ABC

- (i) The sides a, b, c and area of Δ are rational
- (ii) $a, \tan \frac{B}{2}, \tan \frac{C}{2}$ are rational
- (iii) $a, \sin A, \sin B, \sin C$ are rational.

Prove that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) **(1994)**

Q.23 Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. Find all positive values of P such that A, B, C are the angles of triangle. **(1997)**

Q.24 Let ABC be a triangle having O and I as its circumcentre and incentre, respectively. If R and r are the circumradius and the inradius, respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled triangle if and only if b is the arithmetic mean of a and c. **(1999)**

Q.25 Let ABC be a triangle with incentre I and radius r . Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1 , r_2 and r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF and CEID respectively,

prove that:
$$\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$$
 (2000)

Q.26 If Δ is the area of a triangle with side lengths a , b , c then show that $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$.

Also show that the equality occurs in the above inequality if and only if $a=b=c$. (2001)

Q.27 Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is (2007)

- (A) 3 (B) 2 (C) 3/2 (D) 1

Paragraph (Q.28 to 30)

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the

equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Further, it is given that the origin and the centre of C are on the same side of the line PQ. (2008)

Q.28 The equation of circle C is

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

(B) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Q.29 Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$ (B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(C) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Q.30 Equation of the sides QR, RP are

(A) $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

(B) $y = \frac{1}{\sqrt{3}}x, y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$

(D) $y = \sqrt{3}x, y = 0$

Q.31 Let $z = \cos \theta + i \sin \theta$. Then the value of

$\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$ at $\theta = 2^\circ$ is (2009)

(A) $\frac{1}{\sin 2^\circ}$ (B) $\frac{1}{3 \sin 2^\circ}$

(C) $\frac{1}{2 \sin 2^\circ}$ (D) $\frac{1}{4 \sin 2^\circ}$

Q.32 In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$. If a , b and c

denote the length of the sides of the triangle opposite to the angles A, B and C, respectively, then (2009)

(A) $b + c = 4a$

(B) $b + c = 2a$

(C) Locus of point A is an ellipse

(D) Locus of point A is a pair of straight lines

Q.33 For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^6 \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

is (are) (2009)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Q.34 The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is (2009)

Q.35 Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30^\circ$. The absolute value of the difference between the areas

of these triangles is **(2009)**

Q.36 Let ABC be a rectangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) **(2010)**

- (A) $-(2 + \sqrt{3})$ (B) $1 + \sqrt{3}$
 (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

Q.37 Let $\theta, \phi \in [0, 2\pi]$ be such that $2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$.

Then ϕ cannot satisfy **(2012)**

- (A) $0 < \phi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$
 (C) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \phi < 2\pi$

Q.38 Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are lengths of the sides of the triangle opposite to the angle at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **(2012)**

- (A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Q.39 In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the length of PN, QL and RM are consecutive even integer. Then possible length(s) of the side(s) of the triangle is (are) **(2013)**

- (A) 16 (B) 18 (C) 24 (D) 22

Q.40 In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circumradius of the triangle is **(2014)**

- (A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

Q.41 Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangles ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P? **(2015)**

- (A) $(4, 2\sqrt{2})$ (B) $(9, 3\sqrt{2})$
 (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1, \sqrt{2})$

Q.42 The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3 , respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3 , respectively. If Q_2 and Q_3 lie on the y-axis, then **(2016)**

- (A) $Q_2Q_3 = 12$
 (B) $R_2R_3 = 4\sqrt{6}$
 (C) Area of the triangle OR_2R_3 is $6\sqrt{2}$
 (D) Area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Q.43 In a triangle XYZ, let x, y, z be the length of sides opposite to the angle X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of triangle XYZ is $\frac{8\pi}{3}$, then **(2016)**

- (A) Area of the triangle XYZ is $6\sqrt{6}$
 (B) The radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
 (C) $\sin\frac{X}{2} \sin\frac{Y}{2} \sin\frac{Z}{2} = \frac{4}{35}$
 (D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Q.44 The orthocenter of the triangle F_1MN is **(2016)**

- (A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$
 (C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.10 Q.13 Q.15 Q.22
Q.24 Q.29

Exercise 2

Q.3 Q.4 Q.6 Q.9

Previous Years' Questions

Q.5 Q.12 Q.14 Q.19
Q.22

JEE Advanced/Boards

Exercise 1

Q.8 Q.13 Q.17 Q.20
Q.27

Exercise 2

Q.3 Q.6 Q.13 Q.14

Previous Years' Questions

Q.4 Q.6 Q.11 Q.12
Q.13 Q.22 Q.25

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $r = 4, R = \frac{65}{8}$

Q.2 $2r = R$

Q.5 $A = \frac{\pi}{4}, B = \frac{\pi}{3}, C = \frac{5\pi}{12}$

Q.6 No triangle can be formed

Q.7 sides: $2\sqrt{3} \pm \sqrt{2}$; Area: $2\sqrt{3} - \sqrt{2}, 2\sqrt{3} \pm \sqrt{2}$

Q.9 $c = 4\sqrt{3} \pm 2\sqrt{5}$

Q.10 $\sqrt{2} : 2 : \sqrt{3} + 1$

Q.16 4, 5 and 6

Q.22. $\sqrt{5}$

Exercise 2

Single Correct Choice Type

Q.1 A

Q.2 C

Q.3 A

Q.4 D

Q.5 C

Q.6 C

Q.7 B

Q.8 B

Q.9 B

Q.10 D

Previous Years' Questions

Q.1 A

Q.2 B

Q.3 A

Q.4 B

Q.5 A

Q.6 B

Q.7 D

Q.8. (A, D)

Q.9. $\operatorname{cosec} 20^\circ$

Q.10. $-\frac{\sqrt{3}}{2}$

Q.11. $\frac{1+\sqrt{3}}{2}$ sq. unit

Q.12. 90°

Q.15 $c = \sqrt{6}, \angle B = 45^\circ$ and $\angle A = 75^\circ$

Q.17 4, 5, 6 unit

Q.20 $\sqrt{5}$

Q.21 3

Q.22 B

Q.23 A

Q.24 A

JEE Advanced/Boards

Exercise 1

Q.1 9

Q.2 $\frac{1}{4\sin A}$

Q.3 120°

Q.4 $1/(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$

Q.5 400

Q.6 50

Q.6 3 cms & 2 cms

Q.7 two triangles: $(2\sqrt{3} - \sqrt{2})$ & $(2\sqrt{3} + \sqrt{2})$; $(2\sqrt{3} - \sqrt{2})$ & $(2\sqrt{3} + \sqrt{2})$ sq. units

Q.9 Isosceles

Q.10 4, 5, 6

Q.11 $B = \frac{5\pi}{12}$; $C = \frac{\pi}{12}$; $\frac{b}{c} = 2 + \sqrt{3}$

Q.12 $\frac{12}{5}$

Q.13 10

Q.14 90°

Q.15 right angled triangle

Q.16 $\frac{2}{r}$

Q.17 8

Q.18 bc $\left[\frac{\sin C \sin B}{(1 + \sin C)(1 + \sin B)} \right]$

Q.19 $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Q.20 1

Exercise 2

Single Correct Choice Type

Q.1 C

Q.2 A

Q.3 C

Q.4 B

Q.5 A

Q.6 C

Q.7 A

Q.8 C

Q.9 A

Q.10 B

Q.11 A

Q.12 B

Multiple correct choice type

Q.13 A, C

Match the Columns

Q.14 $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow p$; $D \rightarrow q$

Q.15 $A \rightarrow q$; $B \rightarrow r$; $C \rightarrow p$

Previous Years' Questions

Q.1 D

Q.2 A

Q.3 C

Q.4 D

Q.5 C

Q.6 D

Q.7 C

Q.8 B

Q.9 D

Q.10 A, B, C, D

Q.11 B, D

Q.12 B

Q.13. 113°

Q.14. $\frac{16}{7}$

Q.15 $\frac{2(c^2 - a^2)}{3ac}$

Q.16 Equilateral

Q.17 7 : 19 : 25

Q.18. 2

Q.19 1 : 1 : $\sqrt{2}$

Q.20 $p \leq \frac{ar}{1 - r^2}$

Q.21. 7

Q.22 (iii) \Rightarrow (i).

Q.23. $p \in (-\infty, 0) \cup (3 + 2\sqrt{2}, \infty)$

Q.24 $2b = a + c$

Q.25 $\frac{r_1 r_2 r_3}{(r - r_1)(r - r_2)(r - r_3)}$

Q.26 Equilateral

Q.27 B

Q.28 D

Q.29 A

Q.30 D

Q.31 D

Q.32 B, C

Q.33 C, D

Q.34 8

Q.35 4

Q.36 A, B

Q.37 A, C, D

Q.38 C

Q.39 B, D

Q.40 B

Q.41 A, D

Q.42 A, B, C

Q.43 A, C, D

Q.44 A

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $r = \frac{\Delta}{S}$; $R = \frac{abc}{4\Delta}$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = 84$$

$$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4 \times 84} = \frac{65}{8}$$

$$r = \frac{84}{21} = 4$$

Sol 2: $r = \frac{\Delta}{S} = \frac{\frac{\sqrt{3}}{4}a^2}{3a} \times a^2 = \frac{a}{2\sqrt{3}}$

$$R = \frac{abc}{4\Delta} = \frac{a^3}{4\sqrt{3}a^2} = \frac{a}{\sqrt{3}}$$

$$2r = R$$

Sol 3: $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin A \cos B - \cos A \sin B}{\sin A \cos B + \cos A \sin B}$

$$\frac{a^2}{b^2} = \frac{\sin A \cos B}{\cos A \sin B} = \frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A}$$

$$\Rightarrow \frac{a}{b} = \frac{\cos B}{\cos A} \Rightarrow a \cos A = b \cos B$$

$$\Rightarrow \text{either } \cos A = 0 \text{ or } \cos B \text{ or } a = b$$

Sol 4: ΔABC

$$\cos A + \cos B = 4 \sin^2 \frac{C}{2}$$

$$2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2}$$

$$2 \sin \frac{C}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2}$$

$$\cos \frac{A-B}{2} = 2 \sin \frac{C}{2} \Rightarrow \frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}} = 2$$

$$\Rightarrow \frac{2 \cos \frac{C}{2} \cos \frac{A-B}{2}}{2 \cos \frac{C}{2} \sin \frac{C}{2}} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\sin C} = 2$$

$$\Rightarrow \frac{\sin A + \sin B}{\sin C} = 2 \Rightarrow \frac{a+b}{c} = 2$$

Sol 5: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Using cosine formula, we get

$$A = \frac{\pi}{4}, B = \frac{\pi}{3}, C = \frac{5\pi}{12}$$

Sol 6: $a = 5, b = 7, \sin A = \frac{4}{3}$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{28}{15} > 1$$

no triangle can be formed.

Sol 7: $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\frac{\sqrt{6}}{\sin 30^\circ} = \frac{4}{\sin B}$$

$$\sin B = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

$$\sin B = \sqrt{\frac{2}{3}} \cos B = \pm \frac{1}{\sqrt{3}}$$

$$\sin A = \frac{1}{2} \cos A = \frac{\sqrt{3}}{2}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \cos(180 - (A + B))$$

$$= -\cos A \cos B + \sin A \sin B$$

$$\Rightarrow \frac{22 - c^2}{8\sqrt{6}} = \frac{\sqrt{2} \pm \sqrt{3}}{2\sqrt{3}}; \text{ we get } c = 2\sqrt{3} \pm \sqrt{2}$$

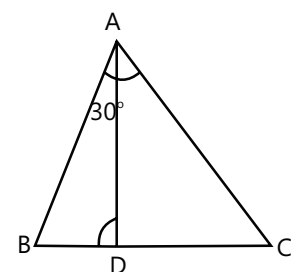
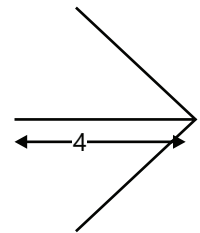
$$\text{Area} = \frac{1}{2} ab \sin C = 2\sqrt{3} - \sqrt{2} \text{ or } 2\sqrt{3} + 2$$

Sol 8: $\angle A = \frac{\pi}{3}$

Length of median

$$AD^2 = AC^2 + CD^2 - 2AC \times CD \times \cos C$$

$$AD^2 = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$$



$$4AD^2 = 2b^2 + 2c^2 - a^2$$

$$4AD^2 = (b^2 + c^2 + bc) + (b^2 + c^2 - a^2 - ac)$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}$$

$$b^2 + c^2 - a^2 - bc = 0$$

Sol 9: $\frac{6}{\sin 30^\circ} = \frac{8}{\sin B} = \frac{c}{\sin C}$

$$\sin B = \frac{2}{3} \Rightarrow \cos B = \frac{\pm\sqrt{5}}{3}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$\Rightarrow \sin C = \sin(180 - 30 - B) = \sin(150 - B)$$

$$= \sin 150 \cos B - \cos 150 \sin B$$

$$= \pm \frac{\sqrt{5}}{6} + \frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{\pm\sqrt{5} + 2\sqrt{3}}{6}$$

$$C = \frac{6}{\sin 30} \cdot \sin C = 4\sqrt{3} \pm 2\sqrt{5}$$

Sol 10: $\angle A = 2x; \angle B = 3x; \angle C = 7x$

$$\angle A + \angle B + \angle C = 180 \Rightarrow 12x = 180 \Rightarrow x = 15^\circ$$

$$\Rightarrow \angle A = 30^\circ; \angle B = 45^\circ; \angle C = 105^\circ$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

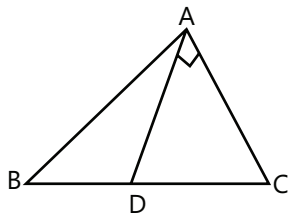
$$a : b : c = \sin A : \sin B : \sin C$$

$$= \frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{2} : 2 : \sqrt{3} + 1$$

Sol 11: $a = \frac{b+c}{3}, S = 2a$

$$\cot \frac{B}{2} \cot \frac{C}{2} = \frac{S}{S-a} = \frac{2a}{2a-a} = 2$$

Sol 12: $AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}; AC = b$



$$CD = \frac{a}{2} = \sqrt{b^2 + \frac{2b^2 + 2c^2 - a^2}{4}}$$

$$\frac{a^4}{4} = \frac{6b^2 + 2c^2 - a^2}{4}$$

$$a^2 - c^2 = 3b^2$$

$$\cos A \cos C = \frac{(b^2 + c^2 - a^2)(a^2 + b^2 - c^2)}{4b^2ac}$$

$$= \frac{(-2b^2)(4b^2)}{4b^2ac} = \frac{-2b^2}{ac} = \frac{2(-3b^2)}{3ac} = \frac{2(c^2 - a^2)}{3ac}$$

Sol 13: From projection formula

$$a \cos B + b \cos A = C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{So L.H.S.} = \frac{b^2 + c^2 - a^2}{abc} + \frac{c^2 + a^2 - b^2}{abc} + \frac{a^2 + b^2 - c^2}{abc}$$

$$= \frac{(a^2 + b^2 + c^2)}{abc}$$

Sol 14: $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$$\cot \frac{B}{2} + \cot \frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\frac{(s-b)\sqrt{s} + (s-c)\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

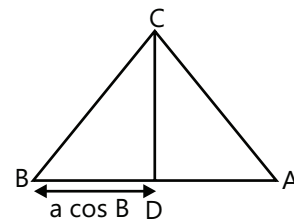
$$\text{L.H.S.} = (b+c-a) \frac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$= \frac{2a\sqrt{s}(s-a)}{\sqrt{(s-a)(s-b)(s-c)}} = 2a \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= 2a \cot \frac{A}{2} = \text{R.H.S.}$$

Sol 15: Length of CD = (a sin B)

$$\sin C = \sin(180 - (A + B)) = \sin A \cos B$$



$$\text{L.H.S.} = \frac{1}{a \sin B} + \frac{1}{c \sin B} - \frac{1}{c \sin A}$$

$$\begin{aligned}
 &= \frac{2R}{bc} + \frac{2R}{ca} - \frac{2R}{ab} = 2R \frac{(a+b-c)}{abc} \\
 &= 2R \frac{2(s-c)}{abc} = \frac{4R(s-c)}{abc} = \frac{s-c}{\Delta} \\
 &= \frac{ab \cos^2 \frac{C}{2}}{4\Delta S} = \frac{2ab \cos^2 \frac{C}{2}}{(a+b+c)\Delta}
 \end{aligned}$$

$$\text{Sol 16: } \frac{a}{\sin A} = \frac{a+1}{\sin(180-A-2A)} = \frac{a+2}{\sin 2A}$$

$$A < 180 - A - 2A < 2A$$

$$A < 180 - 3A < 2A$$

$$4A < 180 < 5A$$

$$A < 45^\circ; A > \frac{180}{5} \Rightarrow A > 36^\circ$$

$$a = \frac{a+1}{3-4\sin^2 A} = \frac{a+2}{2\cos A}$$

$$a = \frac{a+1}{4\cos^2 A - 1} = \frac{a+2}{2\cos A}$$

$$a = \frac{a+1}{\left(\frac{a+2}{a}\right)^2 - 1} = \frac{(a+1)a^2}{4+4a}$$

$$\Rightarrow 4 + 4a = (a+1)a$$

$$\Rightarrow a = 4$$

$$\text{Sol 17: L.H.S.} = a \sin(B-C) + b \sin(C-A) + c \sin(A-B)$$

$$= a \sin B \cos C - a \cos B \sin C + b \sin C \cos A - b \cos C \sin A + c \sin A \cos B - c \cos A \sin B$$

$$= \cos C (a \sin B - b \sin A) + \cos B (C \sin A - a \sin C) + \cos C (a \sin B - b \sin A)$$

$$= 0 = \text{R.H.S. (as } a \sin B = b \cos A)$$

Hence Proved.

$$\text{Sol 18: } \frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$$

$$\left. \begin{aligned}
 b+c &= 11x \\
 c+a &= 12x \\
 a+b &= 13x
 \end{aligned} \right\} \begin{aligned}
 a &= 7x \\
 b &= 6x \\
 c &= 5x
 \end{aligned}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36 + 25 - 49}{2(6)(5)} = \frac{1}{5}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49 + 25 - 36}{2(7)(5)} = \frac{38}{70} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 36 - 25}{2(6)(7)} = \frac{5}{7}$$

$$5 \cos A = \frac{35 \cos B}{19} = \frac{75 \cos B}{75}$$

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

$$\text{Sol 19: } \frac{a \cos A + b \cos B + c \cos C}{\cos A \cos B \cos C}$$

$$= 2a \tan B \tan C \sec A$$

$$\Rightarrow a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$$

$$\Rightarrow a \cos A + b \cos B + c \cos C = 2(2R \sin A) \sin B \sin C$$

$$\Rightarrow a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$$

$$\text{L.H.S.} = a \cos A + b \cos B + c \cos C$$

$$= (2R \sin A) \cos A + 2R \sin B \cos B + 2R \sin C \cos C$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

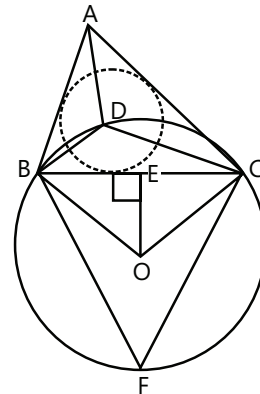
from property of a triangle

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

(we can prove this property using basic trigonometric formulas)

$$= 4R \sin A \sin B \sin C$$

Sol 20:



$$BE = EC = a/2$$

$$\angle BOE = \angle EOC$$

$$\angle DBE = \angle B/2$$

$$\angle DCE = \angle C/2$$

$$\angle BDC = 180^\circ - \left(\frac{B+C}{2}\right) = 180^\circ - \left(\frac{180-A}{2}\right) = 90^\circ + \frac{A}{2}$$

$$\angle BFC = 180^\circ - \left(90 + \frac{A}{2}\right) = 90^\circ - \frac{A}{2}$$

$$\angle BOC = 2\left(90 - \frac{A}{2}\right) = 180^\circ - A$$

$$\angle BOE = 90^\circ - \frac{A}{2}$$

$$\angle EBO = 90^\circ - \left(90 - \frac{A}{2}\right) = \frac{A}{2}$$

$$\Rightarrow BE = \frac{a}{2} \Rightarrow BO = \frac{a}{2} \sec \frac{A}{2}$$

Sol 21: $(b + c - a) \tan \frac{A}{2} = (c + a - b) \tan \frac{B}{2}$
 $= (a + b - c) \tan \frac{C}{2}$

$$b + c - a = 2(s - a)$$

$$c + a - b = 2(s - b)$$

$$a + b - c = 2(s - c)$$

$$r = (s - a) \tan \left(\frac{A}{2}\right)$$

Hence proved.

Sol 22: $C_1C_3 = 3 + 5 = 8$

$$C_3C_2 = 5 + 4 = 9$$

$$C_1C_2 = 3 + 4 = 7$$

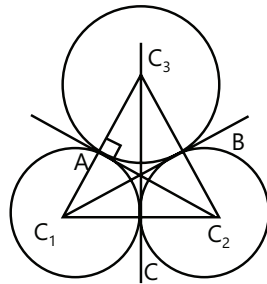
$$OA = OB = OC = k(\text{let's say})$$

since C_3C is angle bisector of angle $\angle C_1C_3C_2$.

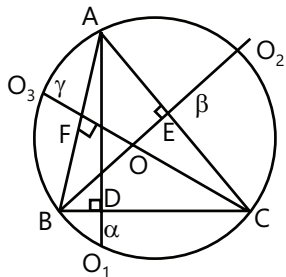
So OA is radius of incircle of triangle $\Delta C_1C_2C_3$.

$$OA = r = \frac{\Delta}{S} = \frac{\sqrt{12(12-7)(12-8)(12-9)}}{12}$$

$$= \sqrt{5}$$



Sol 23: $DO_1 = a, EO_2 = b, FO_3 = g$



$$\angle BOA = 2C$$

$$\angle A$$

Sol 24: $8R^2 = a^2 + b^2 + c^2$

$$a^2 = 4R^2 \sin^2 A$$

$$\text{So, } 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$2 = \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} + \frac{1 - \cos 2C}{2}$$

$$4 = 3 - \cos 2A - \cos 2B - \cos 2C$$

$$\cos 2A + \cos 2B + \cos 2C = 1$$

$$2\cos(A + B) \cos(A - B) + 2\cos^2 C - 1 = 1$$

$$2\cos C [\cos(A + B) + \cos(A - B)] = 0$$

$$\cos C [\cos(A + B) + \cos(A - B)] = 0$$

$$\cos A \cos B \cos C = 0$$

So one angle should be 90° .

Sol 25: $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$

$$\text{R.H.S.} = \frac{1}{4} \sqrt{(a+b+c)abc} = \sqrt{S \frac{(abc)}{8}}$$

$$= \sqrt{\frac{S}{2} \left(\frac{abc}{4\Delta}\right) \Delta} = \sqrt{\left(\frac{S}{\Delta}\right) \left(\frac{abc}{4\Delta}\right) \frac{\Delta^2}{2}} = \Delta \sqrt{\frac{R}{2r}}$$

$$= \Delta \sqrt{\frac{1}{8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}}$$

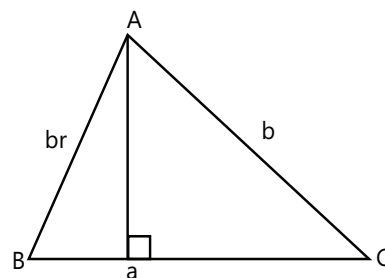
Sol 26: $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\text{L.H.S.} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s - (a+b+c)}{s} = 1$$

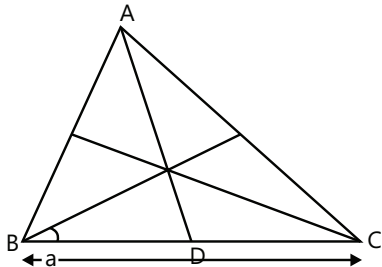
Sol 27: $b \cos B \leq \frac{a}{1-r^2}$



$$b \left(\frac{a^2 + b^2 r^2 - b^2}{2abr} \right) \leq \frac{a}{1-r^2}$$

$$\frac{a^2 + b^2(r^2 - 1)}{2a^2 r} \leq \frac{1}{1-r^2}$$

Sol 28:

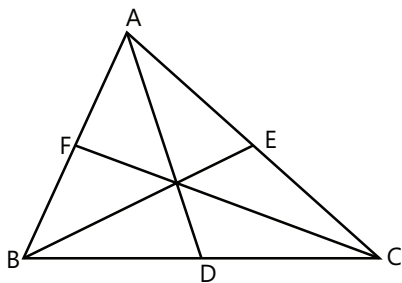


let's say B lie on center and BC lies on x-axis B(0, 0), C(a, 0)

A(c cosB, csinB)

$$BD = \left(\frac{c}{b+c} \right) a$$

Sol 29: $AD = \alpha = C \sin B = b \sin C = (2R) \sin B \sin C$



$$\text{R.H.S.} = \sum \frac{1}{\Delta} (\cot A) = \sum \frac{1}{\Delta^2} \Delta \cot A$$

$$= \frac{1}{\Delta^2} \sum \frac{1}{2} bc \sin A \frac{\cos A}{\sin A} = \frac{1}{\Delta^2} \sum \frac{1}{2} bc \cos A$$

$$= \frac{1}{2\Delta^2} \sum \frac{b^2 + c^2 - a^2}{2} = \frac{1}{4\Delta^2} \sum a^2$$

$$= \sum \left(\frac{a}{2\Delta} \right)^2 = \sum \frac{1}{\alpha^2}$$

Exercise 2

Sol 1: (A) Area = $\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} ab \sin C$

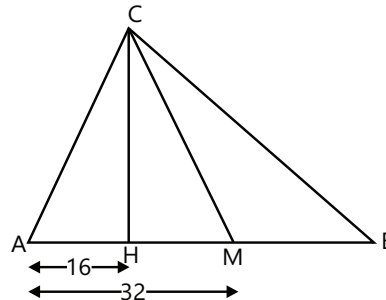
$$\frac{(s-a) + (s-b) + (s-c)}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$$

$$\left(\frac{s}{3} \right)^{3/2} \geq [(s-a)(s-b)(s-c)]^{1/2}$$

$$\frac{\sqrt{5}(s)^{3/2}}{3\sqrt{3}} \geq [s(s-a)(s-b)(s-c)]^{1/2}$$

$$A \leq \frac{s^2}{3\sqrt{3}}$$

Sol 2: (C) $CH = b \sin C = 2R \sin B \sin C$



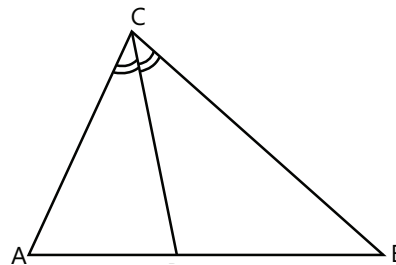
AM = 16, AC = b = 26, CB = 10 = a

$$AH = b \cos A = b \left(\frac{b^2 + c^2 - a^2}{2bc} \right)$$

$$= \frac{(26)^2 + (32)^2 - (10)^2}{2(32)} = 25$$

$$HM = 25 - 16 = 9$$

Sol 3: (A) $AB = C$; $AD = \left(\frac{b}{b+a} \right) C$



length of angle bisector = 6

$$= \frac{2ab \cos \frac{C}{2}}{a+b} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{9}$$

Sol 4: (D) $\Pi_1 = 4R \sin \frac{A}{2}$

$$(\Pi_1)(\Pi_2)(\Pi_3) = 64R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 16R^2 \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = 16 R^2 r$$

Sol 5: (C) $r_1 = \frac{\Delta}{s-a}$

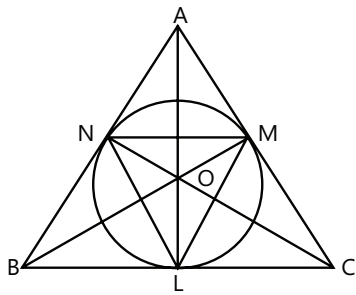
$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{s-a+s-b}{\Delta} = \frac{c}{\Delta}$$

$$\text{L.H.S.} = \frac{abc}{\Delta^3} = \frac{a^3b^3c^3}{64\Delta^3} \cdot \frac{64}{a^2b^2c^2} = \frac{64R^2}{a^2b^2c^2}$$

$K = 64$

Sol 6: (C) $ON = r = OL = OM$

$\angle NOL = 180 - B$



$$\angle ONL = \frac{180 - (180 - B)}{2} = \frac{B}{2}$$

$$R \text{ of } \triangle NOL = \frac{r}{2\sin\angle ONL}$$

$$xyz = \frac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

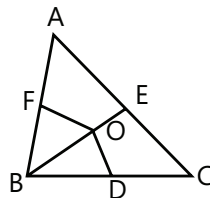
$$xyz = \frac{r^2R}{2}$$

Sol 7: (B) $\angle ABO = \angle B/2$

$OF = r$; $OB = \frac{r}{\sin\frac{B}{2}}$

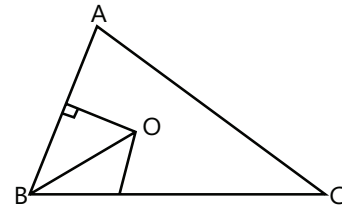
$$(OB)(OC)(OA) = \frac{4r^3R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{4r^3R}{r} = 4r^2R$$



Sol 8: (B) $OB = x = \frac{r}{\sin\frac{B}{2}}$

$$r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$



$$R = \frac{a}{2\sin A}$$

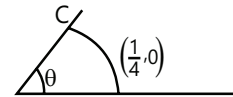
$$xyz = abc \left(\frac{\sin^2\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A \sin B \sin C} \right)$$

$$xyz = abc \left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \right)$$

$$\frac{abc}{xyz} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \Sigma\cot\frac{A}{2}$$

Sol 9: (B) arc length = k

radius of circle = k



$k(\theta) = k \Rightarrow \theta = 1$ radius

Particle moves to next circle after completing arc length = k.

To cross positive x-axis

$n(\theta) > 2\pi$

$n(\theta) > 6.28$

$\theta = 1$ radian

$n > 6.28$

$n = 7$

Sol 10: (D) L.H.S. = $\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c}$

by cosine rule

$$= \frac{2b^2 + 2c^2 - 2a^2 + 2a^2 + 2c^2 - 2b^2 + 2a^2 + 2b^2 - 2c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{abc} = \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} = \frac{a}{bc} + \frac{b}{ca}$$

then we can conclude that there should not $\frac{c}{ab}$ term in L.H.S.. It is possible only if $\cos A = 0$

$$A = \frac{\pi}{2}$$

Previous Years' Questions

Sol 1: (A) In $\triangle ABD$, applying sine rule, we get

$$\frac{AD}{\sin \pi/3} = \frac{x}{\sin \alpha}$$

$$\Rightarrow AD = \frac{\sqrt{3}}{2} x \sin \alpha \quad \dots (i)$$

And in $\triangle ACD$, applying sine rule, we get

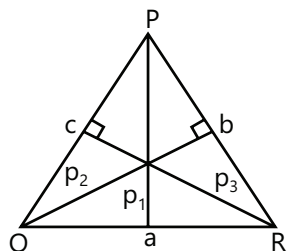
$$\frac{AD}{\sin \pi/4} = \frac{x}{\sin \beta}$$

$$\Rightarrow AD = \frac{3}{\sqrt{2}} x \sin \beta \quad \dots (ii)$$

From Eqs. (i) and (ii)

$$\frac{\sqrt{3}x}{2 \sin \alpha} = \frac{3x}{\sqrt{2} \sin \beta} \Rightarrow \frac{\sin \alpha}{\sin \beta} = \frac{1}{\sqrt{6}}$$

Sol 2: (B) By the law of sine rule



$$\frac{a}{\sin P} = \frac{b}{\sin Q} = \frac{c}{\sin R} = k \text{ (say)}$$

Also, $\frac{1}{2} a p_1 = \Delta \Rightarrow \frac{2\Delta}{a} = p_1 \Rightarrow p_1 = \frac{2\Delta}{k \sin P}$

Similarly, $p_2 = \frac{2\Delta}{k \sin Q}$ and $p_3 = \frac{2\Delta}{k \sin R}$

Since, $\sin P, \sin Q, \sin R$ are in AP, we get that p_1, p_2, p_3 are in HP.

Sol 3: (A) It is given that $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the quadratic equation

$$ax^2 + bx + c = 0 \text{ and } \angle R = \frac{\pi}{2}$$

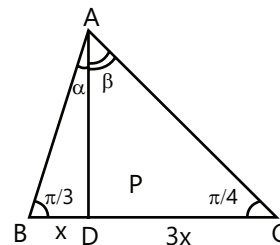
$$\therefore \tan(P/2) + \tan(Q/2) = -b/a$$

$$\text{and } \tan(P/2) \tan(Q/2) = c/a$$

Since, $P + Q + R = 180^\circ$

$$\Rightarrow P + Q = 90^\circ \Rightarrow \frac{P+Q}{2} = 45^\circ$$

$$\Rightarrow \tan\left(\frac{P+Q}{2}\right) = \tan 45^\circ$$



$$\Rightarrow \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)} = 1$$

$$\Rightarrow \frac{-b/a}{1 - c/a} = 1 \Rightarrow \frac{-b/a}{\frac{a-c}{a}} = 1 \Rightarrow \frac{-b}{a-c} = 1$$

$$\Rightarrow -b = a - c \Rightarrow a + b = c$$

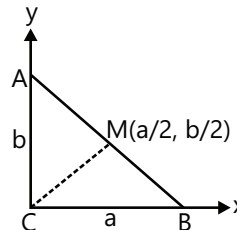
Sol 4: (B) We know that, $A + B + C = 180^\circ$

$$\Rightarrow A + C - B = 180 - 2B.$$

Now, $2ac \sin\left[\frac{1}{2}(A - B + C)\right] = 2ac \sin(90^\circ - B)$

$$= 2ac \cos B = \frac{2ac \cdot (a^2 + c^2 - b^2)}{2ac} = a^2 + c^2 - b^2$$

Sol 5: (A)



Here, $R^2 = MC^2 = \frac{1}{4}(a^2 + b^2)$ (by distance from origin)

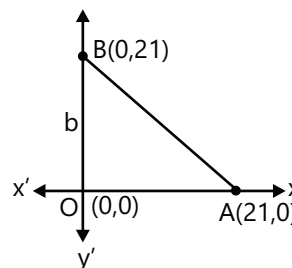
$$\frac{1}{4}c^2 \text{ (Pythagoras theorem)}$$

$$\Rightarrow R = \frac{c}{2}$$

Next, $r = (s - c) \tan(C/2) = (s - c) \tan \pi/4 = s - c$

$$\therefore 2(r+R) = 2r + 2R = 2s - 2c + c = a + b + c - c = a + b$$

Sol 6: (B)



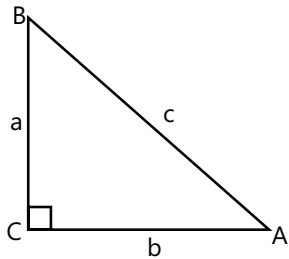
The given vertices of triangle are (0, 0) (0, 21) and (21, 0). To find number of integral points inside the $\triangle AOB$, then

$$x > 0, y > 0 \text{ and } x + y < 21$$

\therefore Number of points exactly in the interior of the triangle

$$= \frac{20 \times 20 - 20}{2} = 190$$

Sol 7: (D)



$$\text{Let } a : b : c = 1 : \sqrt{3} : 2 \Rightarrow c^2 = a^2 + b^2$$

\therefore Triangle is right angled at C

$$\text{or } \angle C = 90^\circ \text{ and } \frac{a}{b} = \frac{1}{\sqrt{3}}$$

$$\text{In } \triangle BAC, \tan A = \frac{a}{b} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow A = 30^\circ \text{ and } B = 60^\circ \quad (A + B = 90^\circ)$$

\therefore Ratio of angles, $A : B : C = 30^\circ : 60^\circ : 90^\circ$

$$\Rightarrow A : B : C = 1 : 2 : 3$$

Sol 8: (A, D) The sine formula is

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a \sin B = b \sin A$$

$$\text{(a) } b \sin A = a \Rightarrow a \sin B = a \Rightarrow B = \frac{\pi}{2}$$

Since, $\angle A < \frac{\pi}{2}$ therefore, the triangle is possible

$$\text{(b) and (c) : } b \sin A > a \Rightarrow a \sin B > a \Rightarrow \sin B > 1$$

$\therefore \triangle ABC$ is not possible

$$\text{(d): } b \sin A < a \Rightarrow a \sin B < a$$

$$\Rightarrow \sin B < 1 \Rightarrow \angle B \text{ exists}$$

$$\text{Now, } b > a \Rightarrow B > A \text{ since } A < \frac{\pi}{2}$$

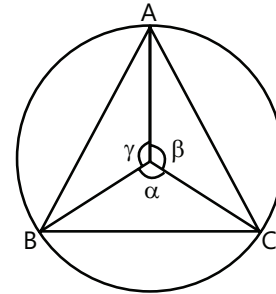
\therefore The triangle is possible.

$$\text{Sol 9: Here, central angle} = \frac{360^\circ}{9} = 40^\circ$$

$$\text{In } \triangle ACM, \frac{1}{r} = \sin 20^\circ \Rightarrow r = \operatorname{cosec} 20^\circ$$

$$\therefore \text{Radius of circle} = \operatorname{cosec} 20^\circ$$

Sol 10:



Since, sides of a triangle subtends α, β, γ at the center.

$$\therefore \alpha + \beta + \gamma = 2\pi \quad \dots (i)$$

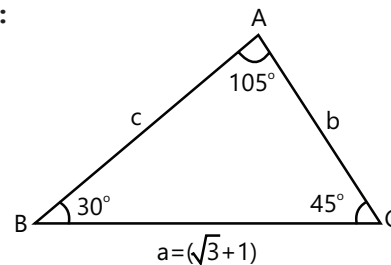
\therefore Arithmetic mean

$$= \frac{\cos\left(\frac{\pi}{2} + \alpha\right) + \cos\left(\frac{\pi}{2} + \beta\right) + \cos\left(\frac{\pi}{2} + \gamma\right)}{3}$$

The value is minimum when the three angles are equal.

$$\text{The minimum value of the AM is } -\frac{\sqrt{3}}{2}.$$

Sol 11:



$$\text{By sine rule, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{\sqrt{3}+1}{\sin(105^\circ)} = \frac{b}{\sin 30^\circ} \Rightarrow b = \frac{(\sqrt{3}+1)\sin 30^\circ}{\sin 105^\circ}$$

\therefore Area of triangle

$$= \frac{1}{2} ab \sin 45^\circ = \frac{1}{2} (\sqrt{3}+1) \frac{(\sqrt{3}+1)\sin 30^\circ \sin 45^\circ}{\sin 105^\circ}$$

$$= \frac{1}{2} \cdot \frac{(\sqrt{3}+1)^2}{(\sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ)} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}} \frac{(3+1+2\sqrt{3})}{\left(\frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}\right)} = \frac{(4+2\sqrt{3})}{4\sqrt{2}(1+\sqrt{3})} \cdot 2\sqrt{2}$$

$$= \frac{(1+\sqrt{3})^2}{2(1+\sqrt{3})} = \frac{1+\sqrt{3}}{2} \text{ sq unit}$$

$$\text{Sol 12: Given, } \frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca} \quad \dots (i)$$

$$\text{We know that, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

On putting these values in Eq. (i), we get

$$\begin{aligned} & \frac{2(b^2 + c^2 - a^2)}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{2(a^2 + b^2 - c^2)}{2abc} \\ &= \frac{a}{bc} + \frac{b}{ca} \\ &\Rightarrow \frac{2(b^2 + c^2 - a^2) + c^2 + a^2 - b^2 + 2(a^2 + b^2 - c^2)}{2abc} \end{aligned}$$

$$= \frac{a^2 + b^2}{abc} \Rightarrow 3b^2 + c^2 + a^2 = 2a^2 + 2b^2$$

$$\Rightarrow b^2 + c^2 = a^2$$

Hence, the angle A is 90°

$$\text{Sol 13: } p_1 p_2 p_3 = \frac{8\Delta^3}{abc}$$

$$\text{Since, } \Delta = \frac{abc}{4R}$$

$$\therefore p_1 p_2 p_3 = \frac{8}{abc} \cdot \frac{(abc)^3}{64R^3} = \frac{(abc)^2}{8R^3}$$

$$\text{Sol 14: Since, } \Delta = \frac{1}{2} a p_1 \Rightarrow \frac{1}{p_1} = \frac{a}{2\Delta}$$

$$\text{Similarly, } \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$$

$$\therefore \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2\Delta} (a + b + c)$$

$$= \frac{2(s - c)}{2\Delta} = \frac{s - c}{\Delta} = \frac{s(s - c)}{ab} \cdot \frac{ab}{s\Delta}$$

$$= \frac{ab}{\left(\frac{a+b+c}{2}\right)\Delta} \cdot \cos^2 \frac{C}{2} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$$

Sol 15: Given that,

$$a = 1 + \sqrt{3}, b = 2 \text{ and } \angle C = 60^\circ$$

$$\text{We have, } c^2 = a^2 + b^2 - 2ab \cos C$$

$$\Rightarrow c^2 = (1 + \sqrt{3})^2 + 4 - 2(1 + \sqrt{3}) \cdot 2 \cos 60^\circ$$

$$\Rightarrow c^2 = 1 + 2\sqrt{3} + 3 + 4 - 2 - 2\sqrt{3}$$

$$\Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}$$

Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{1 + \sqrt{3}}{\sin A} = \frac{2}{\sin B} = \frac{\sqrt{6}}{\sin 60^\circ}$$

$$\therefore \sin B = \frac{2 \sin 60^\circ}{\sqrt{6}} = \frac{2 \times \frac{\sqrt{3}}{2}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

$$\therefore \angle B = 45^\circ$$

$$\Rightarrow \angle A = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$$

Sol 16: Since, r_1, r_2, r_3 are ex-radii of ΔABC are in HP.

$$\therefore \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3} \text{ are in AP}$$

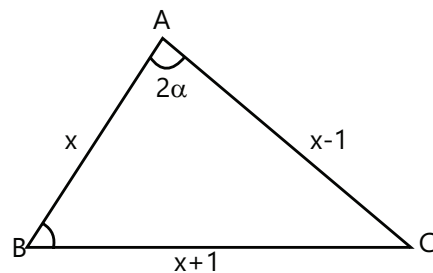
$$\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in AP}$$

$$\Rightarrow s-a, s-b, s-c \text{ are in AP}$$

$$\Rightarrow -a, -b, -c \text{ are in AP}$$

$$\Rightarrow a, b, c \text{ are in AP.}$$

Sol 17: Let ABC be the triangle such that the lengths of its sides CA, AB and BC are $x-1, x$ and $x+1$ respectively where $x \in \mathbb{N}$ and $x > 1$. Let $\angle B = \alpha$ be the smallest angle and $\angle A = 2\alpha$ be the largest angle.



Then, by sine rule, we have

$$\frac{\sin \alpha}{x-1} = \frac{\sin 2\alpha}{x+1}$$

$$\Rightarrow \frac{\sin 2\alpha}{\sin \alpha} = \frac{x+1}{x-1} \Rightarrow 2 \cos \alpha = \frac{x+1}{x-1}$$

$$\therefore \cos \alpha = \frac{x+1}{2(x-1)} \quad \dots (i)$$

$$\text{Also, } \cos \alpha = \frac{x^2 + (x+1)^2 - (x-1)^2}{2x(x+1)}, \text{ using cosine law}$$

$$\Rightarrow \cos \alpha = \frac{x+4}{2(x-1)} \quad \dots \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x+1}{2(x-1)} = \frac{x+4}{2(x+1)}$$

$$\Rightarrow (x+1)^2 = (x+4)(x-1)$$

$$\Rightarrow x^2 + 2x + 1 = x^2 + 3x - 4 \Rightarrow x = 5$$

Hence, the lengths of the sides of the triangle are 4, 5 and 6 unit.

Sol 18: If the triangle is equilateral, then

$$A = B = C = 60^\circ$$

$$\Rightarrow \tan A + \tan B + \tan C = 3 \tan 60^\circ = 3\sqrt{3}$$

Conversely assume that

$$\tan A + \tan B + \tan C = 3\sqrt{3}$$

But in triangle ABC, $A + B = 180^\circ - C$

Taking tan on both sides, we get

$$\tan(A+B) = \tan(180^\circ - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C + \tan A \tan B \tan C = 3\sqrt{3}$$

\Rightarrow none of the $\tan A, \tan B, \tan C$ can be negative

$\Rightarrow \Delta ABC$ cannot be obtuse angle triangle

Also, $AM \geq GM$

$$\frac{1}{3} [\tan A + \tan B + \tan C] \geq [\tan A \tan B \tan C]^{1/3}$$

$$\Rightarrow \frac{1}{3} (3\sqrt{3}) \geq (3\sqrt{3})^{1/3} \Rightarrow \sqrt{3} \geq \sqrt{3}$$

So, the equality can hold if and only if

$\tan A = \tan B = \tan C$ or $A = B = C$ or when the triangle is equilateral.

Sol 19: We know, $I_n = \frac{n}{2} r^2 \sin \frac{2\pi}{n}$ (I_n is area of regular polygon)

$$\Rightarrow \frac{2I_n}{n} = \sin \frac{2\pi}{n} \quad (r=1) \quad \dots \text{(i)}$$

and $O_n = nr^2 \tan \frac{\pi}{2}$ (O_n is area of circumscribing polygon)

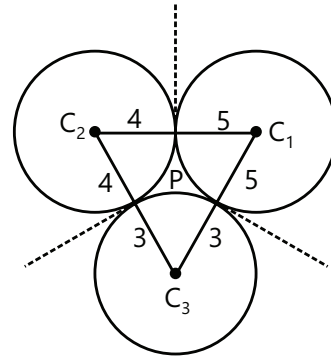
$$\frac{O_n}{n} = \tan \frac{\pi}{n} \quad \dots \text{(ii)}$$

$$\therefore \frac{2I_n}{O_n} = \frac{\sin \frac{2\pi}{n}}{\tan \frac{\pi}{n}} \therefore \frac{I_n}{O_n} = \cos^2 \frac{\pi}{n} = \frac{1 + \cos \frac{2\pi}{n}}{2}$$

$$\frac{I_n}{O_n} = \frac{1 + \sqrt{1 - (2I_n/n)^2}}{2} \quad [\text{from Eq. (i)}]$$

$$I_n = \frac{O_n}{2} \left(1 + \sqrt{1 - (2I_n/n)^2} \right)$$

Sol 20: Since, the circles with radii 3, 4 and 5 touch each other externally and P is the point of intersection of tangents



$\Rightarrow P$ is incentre of $\Delta C_1C_2C_3$

Thus, distance of point P from the points of contact

= In radius (r) of $\Delta C_1C_2C_3$

$$\text{ie, } r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

where $2s = 7 + 8 + 9$

$$\therefore s = 12$$

$$\text{Hence, } r = \sqrt{\frac{(12-7)(12-8)(12-9)}{12}} = \sqrt{\frac{5 \cdot 4 \cdot 3}{12}} = \sqrt{5}$$

$$\text{Sol 21: } \Delta = \frac{1}{2} ab \sin C \sin C = \frac{2\Delta}{ab} = \frac{2 \times 15\sqrt{3}}{6 \times 10} = \frac{\sqrt{3}}{2} C = 120^\circ$$

$$\Rightarrow c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

$$= \sqrt{6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ} = 14$$

$$r = \frac{\Delta}{s} \Rightarrow r^2 = \frac{225 \times 3}{\left(\frac{6+10+14}{2}\right)^2} = 3$$

Sol 22 : (B)

$$3 \sin P + 4 \cos Q = 6 \quad \dots (i)$$

$$4 \sin Q + 3 \cos P = 1 \quad \dots (ii)$$

From (1) and (2) $\angle P$ is obtuse,

$$(3 \sin P + 4 \cos Q)^2 + (4 \sin Q + 3 \cos P)^2 = 37$$

$$\Rightarrow 9 + 16 + 24(\sin P \cos Q + \cos P \sin Q) = 37$$

$$\Rightarrow 24 \sin(P+Q) = 12$$

$$\Rightarrow \sin(P+Q) = \frac{1}{2} \Rightarrow P+Q = \frac{5\pi}{6} \Rightarrow R = \frac{\pi}{6}$$

Sol 23: (A) Let $AB = x$

$$\tan(\pi - \theta - \alpha) = \frac{p}{x-q} \Rightarrow \tan(\theta + \alpha) = \frac{p}{q-x}$$

$$\Rightarrow q-x = p \cot(\theta + \alpha)$$

$$\Rightarrow x = q - p \cot(\theta + \alpha) = q - p \left(\frac{\cot \theta \cot \alpha - 1}{\cot \alpha + \cot \theta} \right)$$

$$= q - p \left(\frac{\frac{q}{p} \cot \theta - 1}{\frac{q}{p} + \cot \theta} \right) = q - p \left(\frac{q \cot \theta - p}{q + p \cot \theta} \right)$$

$$= q - p \left(\frac{q \cos \theta - p \sin \theta}{q \sin \theta + p \cos \theta} \right)$$

Sol 24: (A) $\tan 30^\circ = \frac{h}{AD} \Rightarrow AD = h\sqrt{3}$

$$BD = h; \quad CD = \frac{h}{\sqrt{3}}$$

$$\frac{AB}{BC} = \frac{AD - BD}{BD - CD} = \frac{\sqrt{3} - 1}{1 - \frac{1}{\sqrt{3}}} = \frac{3 - \sqrt{3}}{\sqrt{3} - 1} = \sqrt{3}$$

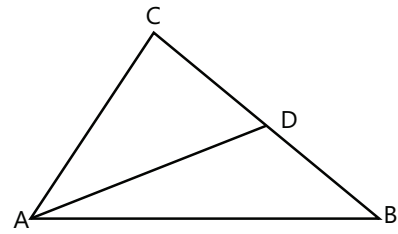
JEE Advanced/Boards

Exercise 1

Sol 1: $AB = 2, AC = 1, BC = a$

$$CD = \frac{1}{1+2}(a) = \frac{a}{3}, \quad DB = \frac{2a}{3}, \quad AD = \frac{2a}{3}$$

$$\text{Length of angle bisector} = \frac{2bc \cos \frac{A}{2}}{b+c} = \frac{2a}{3}$$



$$\Rightarrow \frac{2(2) \cos \frac{A}{2}}{3} = \frac{2a}{3} \Rightarrow \cos \frac{A}{2} = \frac{a}{2}$$

$$\Rightarrow \cos A = \frac{a^2}{2} - 1 = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5 - a^2}{4}$$

$$\Rightarrow a^2 = 3$$

$$\therefore 12\Delta^2 = 12 \times \frac{1}{2} bc \sin A = 9$$

Sol 2: $A = x, B = 2x, C = 4x$

$$\Rightarrow 7x = 180$$

$$x = \frac{180}{7}; \quad C \text{ is obtuse angle}$$

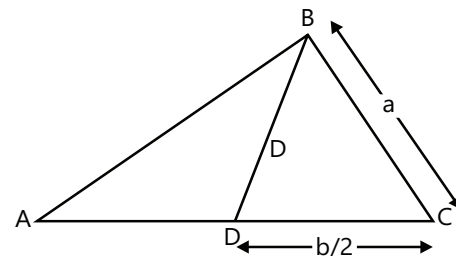
$$2R = 4 = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

$$\sin C = \sin 3A$$

$$-\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{4} \left[\frac{1}{\sin B} + \frac{1}{\sin C} - \frac{1}{\sin A} \right]$$

$$= \frac{1}{4} \left[\frac{1}{3A - 4\sin^3 A} + \frac{1}{\sin^2 A} - \frac{1}{\sin A} \right] = \frac{1}{4 \sin A}$$

Sol 3: Length of median $BD = \frac{\sqrt{3}}{4} \ell(AB)$



$$BD = \frac{\sqrt{3}}{4} c = \frac{1}{2} \sqrt{2a^2 + 2c^2 - b^2}$$

$$\frac{\sqrt{3}}{2} c = \sqrt{2a^2 + 2c^2 - b^2}$$

$$\frac{3}{4} c^2 = 2a^2 + 2c^2 - b^2 \Rightarrow b^2 = 2a^2 + \frac{5}{4} c^2$$

$$\Rightarrow 4b^2 = 8a^2 + 5c^2 \quad \dots (i)$$

$$BD = \frac{\sqrt{3}}{4}c = \sqrt{\frac{b^2}{4} - a^2}$$

$$\Rightarrow \frac{3}{16}c^2 = \frac{b^2}{4} - a^2 \Rightarrow 3c^2 = 4b^2 - 16a^2$$

$$4b^2 = 3c^2 + 16a^2 \quad \dots (ii)$$

From (1) & (2)

$$2c^2 = 8a^2; c = 2a$$

$$\Rightarrow 4b^2 = 12a^2 + 16a^2 \Rightarrow b = \sqrt{7}a$$

$$\cos \angle ABC = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + 4a^2 - 7a^2}{2a(2a)} = \frac{-3}{4}$$

$$\Rightarrow \angle ABC = 120^\circ$$

Sol 4: $OD = 2r$

$\sec A = ? =$ we need to find

$$OA = 2R \cos A$$

$$OA + OD = AD = \frac{a}{2} \cot \frac{A}{2}$$

$$2r + 2R \cos A = \frac{a}{2} \cot \frac{A}{2}$$

$$2 \left(R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) + 2R \cos A = \frac{2R \sin A}{2} \cot \frac{A}{2}$$

$$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 2 \cos A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}}$$

$$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 2 \cos A = 2 \cos^2 \frac{A}{2} - 1 + 1$$

$$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 2 \cos A = \cos A + 1$$

$$\cos A = 1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

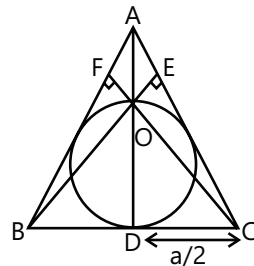
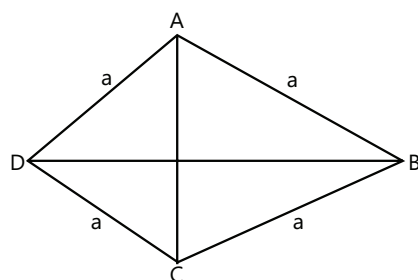
$$\therefore \sec A = 1 / (1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2})$$

Sol 5: $\angle DAB = A$

$$\angle DAC = \frac{A}{2}$$

$$DB = 2a \sin \frac{A}{2}$$

$$\angle BAC = 180 - A$$



$$AC = 2a \cos \frac{A}{2}$$

$$R_1(\text{of } \triangle ABD) = \frac{2a \sin \frac{A}{2}}{2 \sin A} = 2a \frac{a}{2 \cos \frac{A}{2}}$$

$$\cos \frac{A}{2} = \frac{a}{25}$$

$$R_2(\text{of } \triangle ACB) = 25$$

$$= \frac{2a \cos \frac{A}{2}}{2 \sin A} = \frac{a}{2 \sin \frac{A}{2}}$$

$$\sin \frac{A}{2} = \frac{A}{50}$$

$$a^2 \left[\frac{1}{(25)^2} + \frac{1}{(50)^2} \right] = 1$$

$$a^2 = (25)^2 \frac{4}{5} = 500$$

$$\text{Area of rhombus} = 2(\text{Area } \triangle ABD)$$

$$= 2 \frac{1}{2} a^2 \sin A = a^2 \sin A = 500 \left(\frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \right) = 400$$

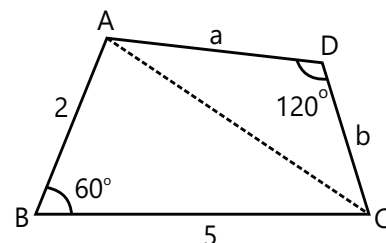
$$\text{Sol 6: } \frac{\cot C}{\cot A + \cot B} = \frac{\frac{\cos C}{\sin C}}{\frac{\cos A \sin B + \sin A \cos B}{\sin A \sin B}}$$

$$= \frac{\cos C}{\sin(A+B)} \cdot \frac{\sin A \sin B}{\sin C} = \frac{\cos C}{\sin C} \cdot \frac{\sin A \sin B}{\sin C}$$

Applying sine & cosine rule

$$= \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \frac{1}{c} \frac{ab}{c} = \frac{100c^2}{2ab} \left(\frac{ab}{c^2} \right) = 50$$

Sol 7: We know that in cyclic quadrilateral, sum of opposite angle is 180°



$$\angle ABC = 60^\circ, AD = a, CD = b$$

$$\angle ADC = 120^\circ$$

$$\text{Area } (\triangle ABC) = \frac{1}{2}(2)(5) \sin 60^\circ = \frac{5\sqrt{3}}{2}$$

$$\text{Total area} = 4\sqrt{3}$$

$$\text{Area } (\triangle ADC) = 4\sqrt{3} - \frac{5\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2} = \frac{1}{2}ab \sin 120^\circ$$

$$\Rightarrow ab = 6$$

$$\cos \angle ABC = \frac{(5)^2 + (2)^2 - (AC)^2}{2(5)(2)}$$

$$(AC)^2 = 19$$

$$\cos \angle ADC = \frac{a^2 + b^2 - 19}{2(a)(b)}$$

$$-\frac{1}{2} = \frac{a^2 + b^2 - 19}{2(6)}$$

$$a^2 + b^2 = 13$$

From (1) & (2)

$$a = 3\text{cm}, b = 2\text{cm}$$

$$\text{Sol 8: } \cos 30^\circ = \frac{(4)^2 + c^2 - 6}{2(4)(c)} = \frac{\sqrt{3}}{2}$$

$$\frac{10 + c^2}{4c} = \sqrt{3}$$

$$c^2 - 4\sqrt{3}c + 10 = 0 \Rightarrow c = 2\sqrt{3} \pm \frac{\sqrt{48 - 40}}{2}$$

$$\Rightarrow c = 2\sqrt{3} \pm \sqrt{2}$$

Two triangles are possible.

$$\text{Sol 9: } a^2 = \frac{b^2 c^2}{2bc \cos A}$$

$$\cos A = \frac{bc}{2a^2} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow b^2 c^2 = a^2 b^2 + a^2 c^2 - a^4$$

$$\Rightarrow b^2(c^2 - a^2) - a^2(c^2 - a^2) = 0$$

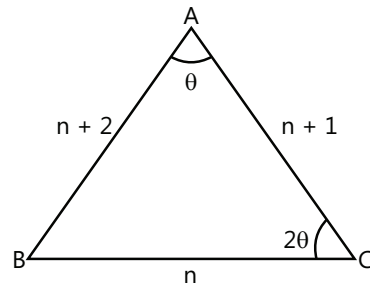
$$\Rightarrow (c^2 - a^2)(b^2 - a^2) = 0$$

$$\Rightarrow \text{either } c = a \text{ or } a = b$$

Hence triangle is isosceles.

Sol 10: Let the sides of \triangle be $n, n + 1, n + 2$ where $n \in \mathbb{N}$.

$$\text{Let } a = n, b = n + 1, c = n + 2$$



Let the smallest, angle $\angle A = \theta$ then the greatest $\angle C = 2\theta$. In $\triangle ABC$ by applying Sine Law we get,

$$\dots \text{ (i) } \sin \theta/n = \sin 2\theta/n + 2$$

$$\Rightarrow \sin \theta/n = 2 \sin \theta \cos \theta/n + 2$$

$$\Rightarrow 1/n = 2 \cos \theta/n + 2 \text{ (as } \sin \theta \neq 0)$$

$$\Rightarrow \cos \theta = n + 2/2n \quad \dots \text{ (i)}$$

In $\triangle ABC$ by Cosine Law, we get

$$\cos \theta = \frac{(n + 1)^2 + (n + 2)^2 - n^2}{2(n + 1)(n + 2)} \quad \dots \text{ (ii)}$$

Comparing the values of $\cos \theta$ from (i) and (ii), we get

$$\frac{(n + 1)^2 + (n + 2)^2 - n^2}{2(n + 1)(n + 2)} = n + 2/2n$$

$$\Rightarrow (n + 2)^2 (n + 1) = n(n + 2)^2 + n(n + 1)^2 - n^3$$

$$\Rightarrow n(n + 2)^2 (n + 2) = n(n + 2)^2 + n(n + 1)^2 - n^3$$

$$\Rightarrow n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$$

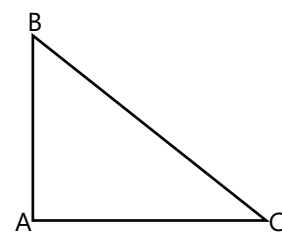
$$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow (n + 1)(n - 4) = 0$$

$$\Rightarrow n = 4 \text{ (as } n \neq -1)$$

\therefore Sides of \triangle are 4, 4 + 1, 4 + 2, i.e. 4, 5, 6.

$$\text{Sol 11: } r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\frac{R}{r_1} = \frac{\sqrt{2}}{\sqrt{3} + \sqrt{2}}$$



$$4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow 4 \frac{1}{\sqrt{2}} \sin \frac{B}{2} \sin \left(45^\circ - \frac{B}{2} \right) = \frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \sin \frac{B}{2} \left[\cos \frac{B}{2} - \sin \frac{B}{2} \right] = \frac{\sqrt{3} + \sqrt{2}}{4}$$

$$\Rightarrow \sin \frac{B}{2} \cos \frac{B}{2} - \sin^2 \frac{B}{2} = \frac{\sqrt{6}+2}{4}$$

$$\Rightarrow \sin B - 1 + \cos B = \frac{\sqrt{6}+2}{2}$$

$$\Rightarrow \sin B + \cos B = \frac{\sqrt{6}+4}{2}$$

$$\Rightarrow \sin(B + 45^\circ) = \frac{\sqrt{6}+4}{2\sqrt{2}}$$

$$\Rightarrow B = \frac{5\pi}{12}, C = \frac{\pi}{12}$$

$$\Rightarrow \frac{b}{c} = \frac{\sin B}{\sin C} = 2 + \sqrt{3}$$

Sol 12: $\left(\frac{a+c}{a}\right)\left(\frac{a}{b}\right) = 2$

$$a + c = 2b$$

$$\Rightarrow a(1-x^2) + 2bx + c(1+x^2) = 0$$

$$x^2(c-a) + 2bx + c + a = 0$$

$$\text{for equal roots } b^2 - c^2 + a^2 = 0$$

$$a^2 + b^2 = c^2$$

Hence it is right angle at $\angle C$

Putting value in equation (2) from equation (1)

$$\Rightarrow a^2 + b^2 = (2b-a)^2 \Rightarrow a^2 + b^2 = 4b^2 + a^2 - 4ab$$

$$\Rightarrow 3b^2 = 4ab \Rightarrow b = \frac{4}{3}a; c = \frac{5}{4}b \Rightarrow c = \frac{5}{3}a$$

$$\text{Hence } a = 3k, b = 4k, c = 5k$$

$$\sin A + \sin B + \sin C = \frac{3}{5} + \frac{4}{5} + 1 = \frac{12}{5}$$

Sol 13: $a = 7, b = 8, c = 5$

$$= \frac{\sin A + \sin B + \sin C}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{\frac{2\Delta}{bc} + \frac{2\Delta}{ac} + \frac{2\Delta}{ab}}{\frac{s}{s-c} \cot \frac{C}{2} + \cot \frac{C}{2}}$$

$$= \frac{2\Delta}{abc} \frac{(2)s(s-c)}{(a+b)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}}$$

$$= \frac{4\Delta}{abc} \sqrt{\frac{s(s-a)(s-b)(s-c)}{(a+b)^2}}$$

$$= \frac{4\Delta^2}{(a+b)abc}$$

Calculating value $p + q = 10$

Sol 14: $r_1 - r = r_2 + r_3$

$$\frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \frac{\Delta a}{s(s-a)} = \frac{\Delta a}{(s-b)(s-c)}$$

$$\Rightarrow s(s-a) = (s-b)(s-c) \Rightarrow \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = 1$$

$$\Rightarrow A = 90^\circ$$

Sol 15: $2(2R)^2 = a^2 + b^2 + c^2$

$$2 = \left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 + \left(\frac{c}{2R}\right)^2$$

$$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$$

$$\Rightarrow 3 - \cos 2A - \cos 2B - \cos 2C = 4$$

... (i)

$$\Rightarrow \cos 2A + \cos 2B + \cos 2C = -1$$

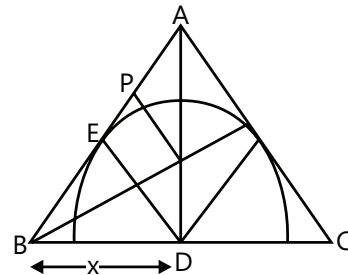
$$\Rightarrow 2\cos(A+B)[\cos(A-B) + 1] = -1$$

$$\Rightarrow \cos C \cos A \cos B = 0$$

Hence it is right angled triangle.

... (ii)

Sol 16: $\frac{3}{r} - \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$



$$OD = \frac{r}{\sin \frac{A}{2}}; AD = \frac{2bc \cos \frac{A}{2}}{b+c}$$

$$r_a = DE = \left(\frac{AD}{OD}\right) OP = \frac{2bc \cos \frac{A}{2} \sin \frac{A}{2} r}{(b+c)r} = \frac{\Delta}{b+c}$$

$$\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

$$\frac{1}{r} = \left(\frac{1}{r} - \frac{1}{r_a}\right) + \left(\frac{1}{r} - \frac{1}{r_b}\right) + \left(\frac{1}{r} - \frac{1}{r_c}\right) = \frac{s}{\Delta}$$

$$\frac{1}{r_a} = \frac{b+c}{\Delta}$$

$$\text{R.H.S.} = \frac{b+c+c+a+a+b}{\Delta} = \frac{2s}{\Delta} = \frac{2}{r}$$

Sol 17: $y^2 + 2 = 3y$

$y = 1, 2$

$AC = 1, AB = 2$

$$\frac{BD}{CD} = \frac{x^2 + 1}{2x}$$

$$\frac{BD + CD}{CD} = \frac{x^2 + 1 + 2x}{2x}$$

$$\frac{\sqrt{3}}{CD} = \frac{(x+1)^2}{2x}$$

$$CD = \frac{2\sqrt{3}x}{(x+1)^2} = \frac{b}{c+b}(\sqrt{3})$$

Similarly $BD = \frac{\sqrt{3}(x^2 + 1)}{(x+1)^2}$

$$\frac{2\sqrt{3}x}{(x+1)^2} = \frac{1}{\sqrt{3}}$$

$6x = (x + 1)^2 \Rightarrow x^2 + 1 - 4x = 0$

$\Rightarrow x_1 + x_2 = 4$

$$\tan \frac{A_1}{2} + \tan \frac{A_2}{2} = \frac{2x_1}{1} + \frac{2x_2}{1} = 8$$

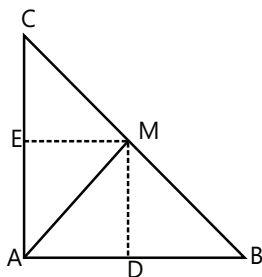
Sol 18: Area (ΔAMC) = Area(ΔAMB)

$$\frac{(BC)}{2} = AM = MB = \frac{a}{2}$$

$AC = 2(AE) = b, AD = DB = \frac{a}{2}$

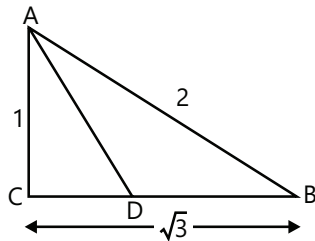
In radii of $\Delta AMB = \frac{\Delta}{s}$

$$= \frac{1}{2} \frac{\text{Area}(\Delta ABC)}{\left(\frac{a+c}{2}\right)} = \frac{\text{Area}(\Delta ABC)}{a+c} = \frac{bc}{a+c}$$



In radius of $\Delta AMC = \frac{1}{2} \frac{\text{Area}(\Delta ABC)}{\left(\frac{a+b}{2}\right)}$

$$= \frac{\text{Area}(\Delta ABC)}{a+b} = \frac{bc}{a+b}$$

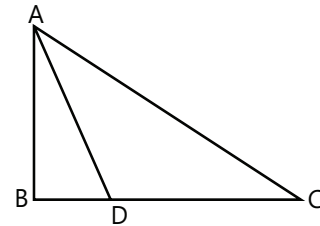


$$r_1 r_2 = \frac{bc}{a+c} \frac{bc}{a+b}$$

$$\frac{r_1}{r_2} = bc \left[\frac{1}{\frac{a}{c} + 1} \cdot \frac{1}{\frac{a}{b} + 1} \right] = bc \left[\frac{\sin C \sin B}{(1 + \sin C)(1 + \sin B)} \right]$$

Sol 19: $AD = (AB) \sin B = C \sin B$

$P_1 = AD = 2R \sin C \sin B$



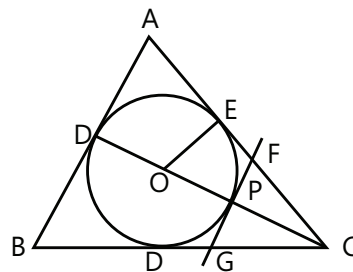
$$\sum \frac{1}{A} = \frac{(\sin A + \sin B + \sin C)}{2R \sin A \sin B \sin C}$$

$$= \frac{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2R \left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right) \left(2 \sin \frac{B}{2} \cos \frac{B}{2} \right) \left(2 \sin \frac{C}{2} \cos \frac{C}{2} \right)}$$

$$= \frac{1}{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{1}{r} = \text{R.H.S.}$$

$$= \frac{1}{r} = \frac{\Delta}{s} = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Sol 20: Area of $\Delta ABC = \text{Area of } \Delta ABGF + \text{Area of } \Delta GFC$



$$\frac{1}{2} cP_3 = \frac{z+c}{2} (2r) + \frac{1}{2} z(P_3 - 2r)$$

$$\Rightarrow zP_3 + 2cr = cP_3 \Rightarrow \frac{z}{c} = 1 - \frac{2r}{P_3}$$

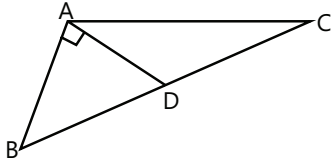
$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 - 2r \left(\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} \right)$$

$$= 3 - 2 = 1$$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $BD = \frac{1}{2}BC$



Pythagoras theorem in $\triangle BAD$

$$c^2 + (AD)^2 = (BD)^2$$

$$c^2 + \frac{1}{4}(2b^2 + 2c^2 - a^2) = \frac{a^2}{4}$$

$$\Rightarrow a^2 = b^2 + 3c^2$$

$$\frac{\tan A}{\tan B} = \frac{\sin A}{\sin B} \frac{\cos B}{\cos A}$$

$$= \frac{a}{b} \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \left(\frac{2bc}{b^2 + c^2 - a^2} \right) = \frac{a^2 + c^2 - b^2}{b^2 + c^2 - a^2} = -2$$

Sol 2: (A) Put $a = 2R \sin A$

$$= \frac{2(\sin A \cos A + \cos B \sin B + \sin C \cos C)}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{\sin 2A + \sin 2B + \sin 2C}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{\sin C [\cos(A-B) - \cos(A+B)]}{(\sin A + \sin B + \sin C)}$$

$$= \frac{2 \sin A \sin B \sin C}{\sin A + \sin B + \sin C}$$

$$= \frac{2 \sin A \sin B \sin C}{2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + 2 \sin \frac{C}{2} \cos \frac{C}{2}}$$

$$= \frac{2 \sin A \sin B \sin C}{2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right]}$$

$$= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{R}$$

Sol 3: (C) $r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

R.H.S. =

$$\frac{64R^3 \cos \frac{C}{2} \left[\sin \frac{A+B}{2} \right] \left[\cos \frac{B}{2} \sin \frac{A+C}{2} \right] \left[\cos \frac{A}{2} \sin \frac{B+C}{2} \right]}{\sum 16R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \left[\sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \right]}$$

$$= \frac{4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{\sum_{i=1}^3 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \frac{4R \frac{s \sqrt{s(s-a)(s-b)(s-c)}}{abc}}{\sum \frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc} (s-c)}$$

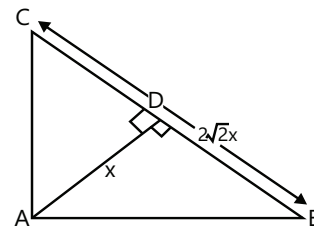
$$= \frac{\left(\frac{4RS}{abc} \Delta \right)}{[(s-c) + (s-b) + (s-a)] \frac{\Delta}{abc}} = 4R \Rightarrow k = \frac{1}{4}$$

Sol 4: (B) $AC = \frac{x}{\sin \angle ACD} = \frac{x}{\sin C}$

$$AB = \frac{x}{\sin B}$$

$$x^2 \left(\frac{1}{\sin^2 C} + \frac{1}{\sin^2 B} \right) = 8x^2$$

$$\sin^2 B + \sin^2 C = 8 \sin^2 C \sin^2 B$$



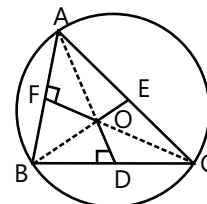
$$1 = 8 \sin^2 C \cos^2 C$$

$$(\sin 2C) = \frac{1}{\sqrt{2}} \Rightarrow C = \frac{\pi}{8} \Rightarrow B = \frac{3\pi}{8}$$

Sol 5: (A) $OB = R$

$\angle AOC = 2A, \angle BOD = A$

$$f = OD = R \cos A$$



$$\text{L.H.S.} = \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{a}{R \cos A} + \frac{b}{R \cos B} + \frac{c}{R \cos C}$$

$$\text{Put } R = \frac{a}{2 \sin A} = 2(\tan A + \tan B + \tan C) = 2 \tan A \tan B \tan C$$

$$\text{R.H.S.} = \lambda \frac{abc}{R^3 \cos A \cos B \cos C}$$

$$\text{Put } R = \frac{a}{2 \sin A} = 8 \lambda \tan A \tan B \tan C$$

$$\Rightarrow \lambda = \frac{1}{4}$$

Sol 6: (C) $\angle AOB = 2\angle C = 2C$

$$OA = OB = R$$

$$\frac{AB}{\sin \angle AOB} = 2R_1$$

$$R_1 = \frac{C}{2 \sin 2C}$$

$$\text{L.H.S.} = 2[\sin 2A + \sin 2B + \sin 2C]$$

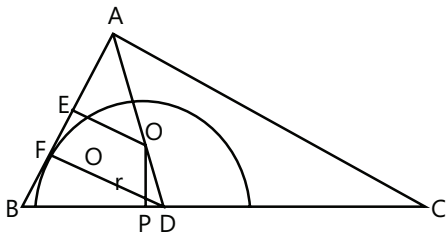
$$= 2[2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C]$$

$$= 4 \sin C [\cos(A-B) \pm \cos(A+B)] = 8 \sin A \sin B \sin C$$

$$= 8 \left(\frac{1}{2} ab \sin C \right) \left(\frac{1}{2} bc \sin A \right) \left(\frac{1}{2} ca \sin B \right)$$

$$= \frac{8a}{a^2 b^2 c^2} = \frac{64 \Delta^3}{a^2 b^2 c^2} = \frac{4 \Delta}{R^2}$$

Sol 7: (A) $DF = ?$; $OE = r$; $OD = \frac{r}{\sin \frac{A}{2}}$



$$AD = \frac{2bc \cos \frac{A}{2}}{b+c}$$

$$FD = \left(\frac{AD}{OD} \right)$$

$$FD = \frac{2bc \cos \frac{A}{2}}{r(b+c)} r \sin \frac{A}{2} = \frac{bc \sin A}{b+c} = \frac{2 \Delta}{b+c}$$

$$\text{If } C \text{ is the base} = \frac{2 \Delta}{a+b}$$

Sol 8: (C) From sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Given situation

$$\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C} = k$$

$$2R \sin A = k \cos A$$

$$\tan A = \tan B = \tan C$$

Hence it is equilateral triangle.

Sol 9: (A) $\cos A + \cos B = 2(1 - \cos C)$

$$\cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) = 2 \sin^2 \frac{C}{2}$$

$$\Rightarrow \cos \left(\frac{A-B}{2} \right) = 2 \sin \frac{C}{2} = 2 \cos \left(\frac{A+B}{2} \right)$$

$$\Rightarrow \frac{\cos \left(\frac{A-B}{2} \right)}{\cos \left(\frac{A+B}{2} \right)} = 2$$

$$\Rightarrow \frac{\cos \left(\frac{A-B}{2} \right) + \cos \left(\frac{A+B}{2} \right)}{\cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right)} = \frac{2+1}{2-1}$$

$$\Rightarrow \cot \frac{A}{2} \cot \frac{B}{2} = 3 = \frac{5}{5-c}$$

$$a + b + c = 3a + 3b - 3c \Rightarrow 2c = a + b$$

Sol 10: (B) Arithmetic mean = $\frac{a+b+c}{3}$

$$= \frac{2R}{3} (\sin A + \sin B + \sin C)$$

$$\text{Length of altitude} = b \sin C = 2R \sin B \sin C$$

$$\text{Harmonic mean} = \frac{3}{\sum \frac{1}{2R \sin B \sin C}}$$

$$= \frac{6R \sin A \sin B \sin C}{\sin A + \sin B + \sin C}$$

$$\text{Product} = (6R) \left(\frac{2R}{3} \right) (\sin A \sin B \sin C)$$

$$= 4R^2 \sin A \sin B \sin C$$

$$= \left(\frac{a}{\sin A} \right) \left(\frac{b}{\sin B} \right) \sin A \sin B \sin C = 2 \Delta$$

Sol 11: (A) $\frac{\sin A \sin(B-C)}{\sin C \sin(A-B)} = 1$

L.H.S. = $\frac{\sin(B+C)\sin(B-C)}{\sin(A+B)\sin(A-B)} = \frac{\cos 2B - \cos 2C}{\cos 2A - \cos 2B} = 1$

$\Rightarrow 2\sin^2 B = \sin^2 A + \sin^2 C \Rightarrow 2b^2 = a^2 + c^2$

Sol 12: (B) $\sin A = \frac{5}{13}$ A = acute angle

$\sin B = \frac{99}{101} \sim 1$

B ~ 90

C will be acute angle.

$\cos C > 0$

$\cos C = \sqrt{1 - \sin^2 C} = \sqrt{1 - \sin^2(A+B)}$

$= \sqrt{1 - [\sin A \cos B + \cos A \sin B]^2}$

$\Rightarrow \cos B = \sqrt{1 - \left(\frac{99}{101}\right)^2} = \frac{20}{101}$

$\Rightarrow \cos A = \frac{12}{13}$

$\Rightarrow \cos C = \sqrt{1 - \left[\frac{5}{12}\left(\frac{20}{101}\right) + \frac{12}{13}\left(\frac{99}{101}\right)\right]^2} = \frac{255}{1313}$

Multiple correct choice type

Sol 13: (A, C) Hypotenuse

$= \sqrt{[(\cos 2\alpha + \cos 2\beta + 2\cos(\alpha + \beta))]^2 + [\sin 2\alpha + \sin 2\beta + 2\sin(\alpha + \beta)]^2}$

$\cos 2\alpha + \cos 2\beta + 2\cos(\alpha + \beta)$
 $= \cos^2\alpha - \sin^2\alpha + \cos^2\beta - \sin^2\beta + 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$
 $= (\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2$

$\sin 2\alpha + \sin 2\beta + 2\sin(\alpha + \beta)$
 $= 2\sin\alpha\cos\alpha + 2\sin\beta\cos\beta + 2\sin\alpha\cos\beta + 2\cos\alpha\sin\beta$
 $= 2\sin\alpha[\cos\alpha + \cos\beta] + 2\sin\beta[\cos\beta + \cos\alpha]$
 $= 2(\cos\alpha + \cos\beta)(\sin\alpha + \sin\beta)$

Hypotenuse

$= (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$
 $= 2 + 2\cos^2\alpha\cos\beta + 2\sin\beta\cos\beta$
 $= 2 + 2\cos(\alpha - \beta) = 4\cos^2\frac{\alpha - \beta}{2}$

Match the columns type

Sol 14: A → s; B → r; C → p; D → q

(A) Centroid divides the triangle into three equal part so 1 : 1 : 1

(B) P is incentre

PD = r

BC = a

Area (ΔPBC) = ar

hence ratio = a : b : c = sinA : sinB : sinC

(C) P is orthocentre

Area of (ΔPBC) = $\frac{1}{2} \times PD \times BC = \frac{a}{2} \times (AD + AP)$

$= \frac{a}{2} (2R \sin B \sin C - 2R \cos A)$

$= aR(\sin B \sin C - \cos A) = aR[\sin B \sin C + \cos(B + C)]$

$= 2R \cos B \cos C = 2R \sin A \cos B \cos C$

Hence ratio tanA : tanB : tanC

(D) If P is circumcentre

PB = R

BD = $\frac{a}{2}$

PD = $\sqrt{R^2 - \frac{a^2}{4}}$

Area ΔPBC = $\frac{1}{2} \times PD \times BC$

$= \frac{a}{2} \sqrt{R^2 - \frac{a^2}{4}} = \frac{a}{2} \sqrt{4\sin^2 2A - \frac{a^2}{4}}$

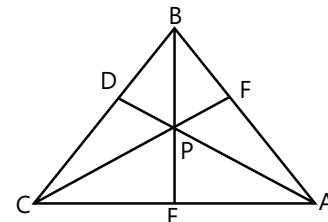
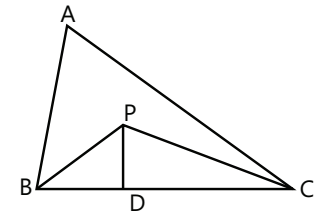
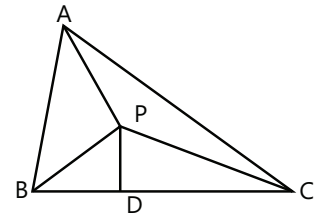
$= \frac{a^4 \cos A}{4 \sin B} = (2R)^2 \sin 2A$

Ratio = sin2A : sin2B : sin2C

Sol 15: A → q; B → r; C → p; D → q

CA = b = $1 + \sqrt{3}$

BC = a = 2, $\cos C = \frac{1}{2}$



$$(AB) = \sqrt{6} = C$$

$$\Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(2)^2 + (6) - (\sqrt{3}+1)^2}{2(2)(\sqrt{6})}$$

$$\Rightarrow \cos B = \frac{6 - 2\sqrt{3}}{4\sqrt{6}} = \frac{3 - \sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3} - 1}{2}$$

$$B = 75^\circ$$

$$A = 45^\circ$$

$$\Rightarrow (BE) = a \sin C = 2 \left(\frac{\sqrt{3}}{2} \right) = \sqrt{3}$$

$$\Rightarrow DA = C \sin B = \sqrt{6} (\sin 75^\circ)$$

$$\Rightarrow CF = b \sin A = (\sqrt{3} + 1) \frac{1}{\sqrt{2}}$$

$\triangle DEF$ is pedal triangle of $\triangle ABC$

$$\angle DEB = \frac{\pi}{2} - A \Rightarrow \angle DEF = \pi - 2A$$

$$(A) \text{ Circum scribing circle} = \frac{EF}{\sin(\angle EDF)}$$

$$= \frac{a \cos A}{2 \sin(\pi - 2A)} = \frac{R}{2} = \frac{a}{\Delta \sin A} = \frac{1}{\sqrt{2}}$$

$$(B) \text{ Area} = \frac{1}{2} \times (DE)(DF) \times \sin(\angle EDF)$$

$$= \frac{1}{2} (c \cos C \times b \cos B) \times \sin(\pi - 2A)$$

$$= bc \sin A \cos A \cos B \cos C$$

$$= (\sqrt{3} + 1) \sqrt{6} \frac{1}{\sqrt{2}} \left(\frac{\sqrt{3} - 1}{2} \right) \left(\frac{1}{2} \right) = \frac{2\sqrt{6}}{8} = \frac{\sqrt{3}}{4}$$

$$(C) r = \frac{\Delta}{s} = \frac{\sqrt{3}/4}{\Sigma a \cos A} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Previous Years' Questions

Sol 1: (D) Here, for

(a) If $a, \sin A, \sin B$ are given, then we can determine $b = \frac{a}{\sin A} \sin B$, $c = \frac{a}{\sin A} \sin C$. So, all the three sides are unique.

So, option (A) is incorrect.

(b) The three sides can uniquely make an acute angled triangle. So, option (B) is incorrect.

(c) If $a, \sin B, R$ are given, then we can determine $b = 2R \sin B$, $\sin A = \frac{a \sin B}{b}$. So, $\sin C$ can be determined.

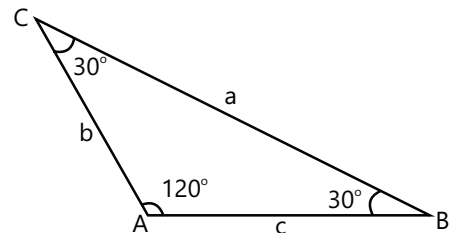
Hence, side c can also be uniquely determined.

(d) If $a, \sin A, R$ are given, then

$$\frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

But this could not determine the exact values of b and c .

Sol 2: (A)



Given, ratio of angles are $4 : 1 : 1$

$$\Rightarrow 4x + x + x = 180^\circ$$

$$\Rightarrow x = 30^\circ$$

$$\therefore \angle A = 120^\circ, \angle B = \angle C = 30^\circ$$

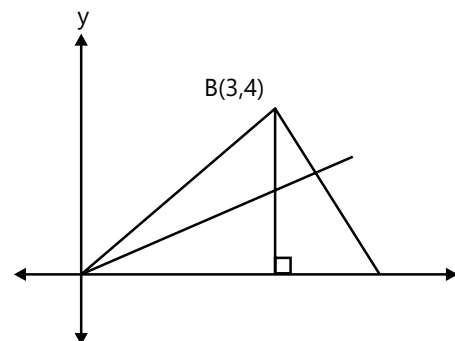
Thus, the ratio of longest side to perimeter = $\frac{a}{a+b+c}$

Let $b = c = x$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Rightarrow a^2 = 2x^2 - 2x^2 \cos A = 2x^2(1 - \cos)$$

$$\Rightarrow a^2 = 4x^2 \sin^2 A/2 \Rightarrow a = 2x \sin A/2$$



$$\Rightarrow a = 2x \sin 60^\circ = \sqrt{3}x$$

Thus, required ratio is

$$\frac{a}{a+b+c} = \frac{\sqrt{3}}{x+x+\sqrt{3}x} = \frac{\sqrt{3}}{2+\sqrt{3}}$$

Sol 3: (C) To find orthocentre of the triangle formed by

(0, 0), (3, 4) and (4, 0).

Let H be the orthocentre of ΔOAB

\therefore (Slope of OP ie, OH) \cdot (slope of BA) = -1

$$\Rightarrow \left(\frac{y-0}{3-0}\right) \cdot \left(\frac{4-0}{3-4}\right) = -1 \Rightarrow -\frac{4}{3}y = -1 \Rightarrow y = \frac{3}{4}$$

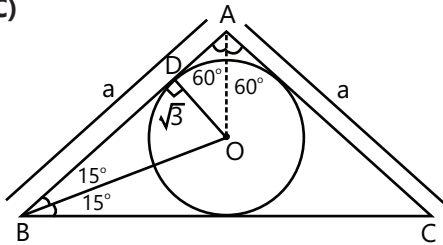
\therefore Required orthocentre = $(3, y) = \left(3, \frac{3}{4}\right)$

Sol 4: (D) Let a, b, c are the sides of triangle ABC.

$$\text{Now, } \frac{b+c}{a} = \frac{k(\sin B + \sin C)}{k \sin A} = \frac{2 \sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right)}{2 \sin\frac{A}{2} \cos\frac{A}{2}}$$

$$\Rightarrow \frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} \quad \text{Also, } \frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$$

Sol 5: (C)



Let $AB = AC = a$ and $\angle A = 120^\circ$

$$\therefore \text{Area of triangle} = \frac{1}{2} a^2 \sin 120^\circ$$

where, $a = AD = BD$

$$= \sqrt{3} \tan 30^\circ + \sqrt{3} \cot 15^\circ$$

$$= 1 + \frac{\sqrt{3}}{\tan(45^\circ - 15^\circ)} = 1 + \sqrt{3} \left(\frac{1 + \tan 45^\circ \tan 30^\circ}{\tan 45^\circ - \tan 30^\circ} \right)$$

$$= 1 + \sqrt{3} \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) \quad \therefore a = 4 + 2\sqrt{3}$$

$$\Rightarrow \text{Area of triangle} = \frac{1}{2} (4 + 2\sqrt{3})^2 \left(\frac{\sqrt{3}}{2} \right) = 12 + 7\sqrt{3}$$

Sol 6: (D) Since, A, B, C are in AP

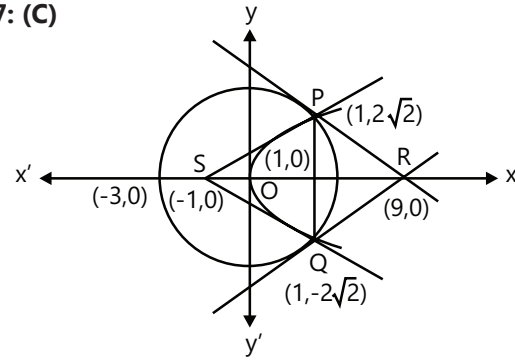
$$\Rightarrow 2B = A + C \text{ ie, } \angle B = 60^\circ$$

$$\therefore \frac{a}{c} (2 \sin C \cos C) + \frac{a}{c} (2 \sin A \cos A) = 2k (a \cos C + c \cos A)$$

$$\text{Using, } \left[\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k} \right]$$

$$= 2k (b) = 2 \sin B \text{ [using, } b = a \cos C + c \cos A] = \sqrt{3}$$

Sol 7: (C)



Coordinates of P and Q are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$

$$\text{Now, } PQ = \sqrt{(4\sqrt{2})^2 + 0^2} = 4\sqrt{2}$$

$$\text{Area of } \Delta PQR = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}$$

$$\text{Area of } \Delta PQS = \frac{1}{2} \cdot 4\sqrt{2} \cdot 2 = 4\sqrt{2}$$

Ratio of area of triangle PQS and PQR is 1 : 4

Sol 8: (B) Equation of circumcircle of ΔPRS is

$$(x + 1)(x + 9) + y^2 + \lambda y = 0$$

It will pass through $(1, 2\sqrt{2})$, then

$$-16 + 8 + \lambda \cdot 2\sqrt{2} = 0$$

$$\Rightarrow \lambda = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$$

\therefore Equation of circumcircle is

$$x^2 + y^2 - 8x + 2\sqrt{2}y - 9 = 0$$

Hence, its radius is $3\sqrt{3}$.

Alternate Solution

$$\text{Let } \angle PSR = \theta \Rightarrow \sin \theta = \frac{2\sqrt{2}}{2\sqrt{3}} \quad \therefore \sin \theta = \frac{PR}{2R}$$

$$\Rightarrow PR = 6\sqrt{2} = 2R \cdot \sin \theta \Rightarrow R = 3\sqrt{3}$$

Sol 9: (D) Radius of incircle is, $r = \frac{\Delta}{s}$

$$\text{Since, } \Delta = 16\sqrt{2}$$

$$\text{Now, } s = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$$

$$\therefore r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Sol 10: (A, B, C, D) Since, $\Delta ABC = \Delta ABD + \Delta ACD$

$$\Rightarrow \frac{1}{2}bc \sin A = \frac{1}{2}c \cdot AD \sin \frac{A}{2} + \frac{1}{2}b \cdot AD \sin \frac{A}{2}$$

$$\Rightarrow AD = \frac{2bc}{b+c} \cos \frac{A}{2}$$

Again, $AE = AD \sec \frac{A}{2} = \frac{2bc}{b+c} \Rightarrow AE$ is HM of b and c .

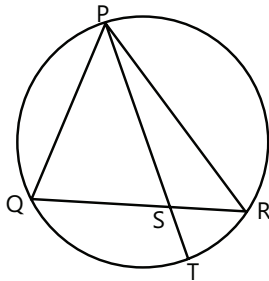
$$EF = ED + DF = 2DE = 2AD \tan \frac{A}{2}$$

$$= 2 \frac{2bc}{b+c} \cos \frac{A}{2} \tan \frac{A}{2} = \frac{4bc}{b+c} \sin \frac{A}{2}$$

Since, $AD \perp EF$ and $DE = DF$ and AD is bisector.

$\Rightarrow \Delta AEF$ is isosceles.

Sol 11: (B, D)



Let a straight line through the vertex P of a given ΔPQR intersects the side QR at the point S and the circumcircle of ΔPQR at the point T .

Points P, Q, R, T are concyclic, then $PS \cdot ST = QS \cdot SR$

$$\text{Now, } \frac{PS+ST}{2} > \sqrt{PS \cdot ST} \quad (\text{AM} > \text{GM})$$

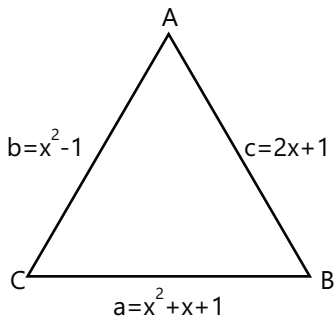
$$\text{and } \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{PS \cdot ST}} > \frac{2}{\sqrt{QS \cdot SR}}$$

$$\text{Also, } \frac{SQ+QR}{2} > \sqrt{SQ \cdot SR}$$

$$\Rightarrow \frac{QR}{2} > \sqrt{SQ \cdot SR} \Rightarrow \frac{1}{\sqrt{SQ \cdot SR}} > \frac{2}{QR} \Rightarrow \frac{2}{\sqrt{SQ \cdot SR}} > \frac{4}{QR}$$

$$\text{Hence, } \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \cdot SR}} > \frac{4}{QR}$$

Sol 12: (B)



$$\text{Using, } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2 + x + 1)^2 + (x^2 - 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\Rightarrow (x+2)(x+1)(x-1)x + (x^2 - 1)^2 = \sqrt{3} (x^2 + x + 1)(x^2 - 1)$$

$$\Rightarrow x^2 + 2x + (x^2 - 1) = \sqrt{3} (x^2 + x + 1)$$

$$\Rightarrow (2 - \sqrt{3})x^2 + (2 - \sqrt{3})x - (\sqrt{3} + 1) = 0$$

$$\Rightarrow x = -(2 + \sqrt{3}) \text{ and } x = 1 + \sqrt{3}$$

But, $x = -(2 + \sqrt{3}) \Rightarrow c$ is negative

$\therefore x = 1 + \sqrt{3}$ is the only solution.

Sol 13: In ΔADC , $\frac{AD}{b} = \sin 23^\circ$

$$\Rightarrow AD = b \sin 23^\circ$$

$$\text{But } AD = \frac{abc}{b^2 - c^2} \text{ (given)}$$

$$\Rightarrow \frac{abc}{b^2 - c^2} = b \sin 23^\circ \Rightarrow \frac{a}{b^2 - c^2} = \frac{\sin 23^\circ}{c} \quad \dots(i)$$

Again, in ΔABC ,

$$\frac{\sin A}{a} = \frac{\sin 23^\circ}{c}$$

$$\Rightarrow \frac{\sin A}{a} = \frac{a}{b^2 - c^2} \text{ [from Eq. (i)]}$$

$$\Rightarrow \sin A = \frac{a^2}{b^2 - c^2} \Rightarrow \sin A = \frac{k^2 \sin^2 A}{k^2 \sin^2 B - k^2 \sin^2 C}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin^2 B - \sin^2 C}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin(B+C)\sin(B-C)}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin A \cdot \sin(B-C)}$$

$$\Rightarrow \sin(B-C) = 1$$

($\sin A \neq 0$)

$$\Rightarrow \sin(B-23^\circ) = \sin 90^\circ \Rightarrow B-23^\circ = 90^\circ$$

$$\Rightarrow B = 113^\circ$$

Sol 14: We have, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$

$$\frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{s}{\Delta} = \frac{abc \cdot s}{4\Delta^2} = \frac{abc}{4(s-a)(s-b)(s-c)}$$

But $a : b : c = 4 : 5 : 6$ (given)

$$\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k(\text{let})$$

$$\Rightarrow a = 4k, b = 5k, c = 6k$$

$$\text{Now, } s = \frac{1}{2}(a + b + c) = \frac{1}{2}(4k + 5k + 6k) = \frac{15k}{2}$$

$$\begin{aligned} \therefore \frac{R}{r} &= \frac{(4k)(5k)(6k)}{4\left(\frac{15k}{2} - 4k\right)\left(\frac{15k}{2} - 5k\right)\left(\frac{15k}{2} - 6k\right)} \\ &= \frac{30k^3}{k^3\left(\frac{15-8}{2}\right)\left(\frac{15-10}{2}\right)\left(\frac{15-12}{2}\right)} = \frac{30 \cdot 8}{7 \cdot 5 \cdot 3} = \frac{16}{7} \end{aligned}$$

Sol 15: In $\triangle ADC$, we have $\cos C = \frac{AC}{CD}$

$$\cos C = \frac{2b}{a} \quad \dots (i)$$

Applying cosine formula in $\triangle ABC$, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{2b}{a}$$

$$\Rightarrow a^2 + b^2 - c^2 = 4b^2$$

$$\Rightarrow a^2 - c^2 = 3b^2 \quad \dots (iii)$$

$$\begin{aligned} \text{Now, } \cos A \cos C &= \frac{b^2 + c^2 - a^2}{2bc} \cdot \frac{2b}{a} = \frac{b^2 + c^2 - a^2}{ac} \\ &= \frac{3b^2 + 3(c^2 - a^2)}{3ac} \\ &= \frac{(a^2 - c^2) + 3(c^2 - a^2)}{3ac} = \frac{2(c^2 - a^2)}{3ac} \end{aligned}$$

Sol 16: Let a, b, c are the sides of a $\triangle ABC$.

$$\begin{aligned} \text{Given, } \cos A + \cos B + \cos C &= \frac{3}{2} \\ \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab} &= \frac{3}{2} \\ \Rightarrow ab^2 + ac^2 - a^3 + ba^2 + bc^2 - b^3 + ca^2 + cb^2 - c^3 &= 3abc \\ \Rightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2 & \\ = \frac{(a + b + c)}{2} [(a - b)^2 + (b - c)^2 + (c - a)^2] & \\ \Rightarrow (a + b - c)(a - b)^2 + (b + c - a)(b - c)^2 + (c + a - b) & \end{aligned}$$

$$(c - a)^2 = 0$$

(as we know, $a + b - c > 0, b + c - a > 0, c + a - b > 0$)

\therefore Each term on the left of equation has positive coefficient multiplied by perfect square, each term must be separately zero.

$$\Rightarrow a = b = c$$

\therefore Triangle is an equilateral.

Sol 17: Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \lambda$

$$\Rightarrow (b + c) = 11\lambda, c + a = 12\lambda, a + b = 13\lambda \quad \dots (i)$$

$$\Rightarrow 2(a + b + c) = 36\lambda$$

$$\Rightarrow a + b + c = 18\lambda \quad \dots (ii)$$

On solving the Eqs. (i) and (ii), we get

$$a = 7\lambda, b = 6\lambda \text{ and } c = 5\lambda$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36\lambda^2 + 25\lambda^2 - 49\lambda^2}{2(30)\lambda^2} = \frac{1}{5}$$

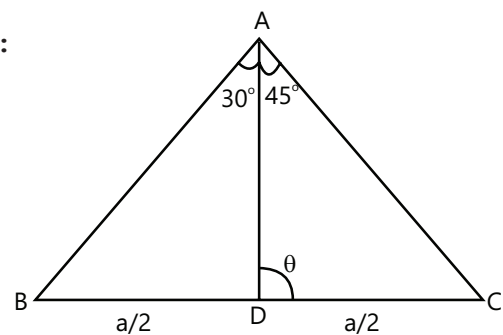
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{49\lambda^2 + 25\lambda^2 - 36\lambda^2}{70\lambda^2} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49\lambda^2 + 36\lambda^2 - 25\lambda^2}{84\lambda^2} = \frac{5}{7}$$

$$\therefore \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$$

Sol 18:



Let AD be the median to the base $BC = a$ of $\triangle ABC$, and let $\angle ADC = \theta$, then

$$\left(\frac{a}{2} + \frac{a}{2}\right) \cot \theta = \frac{a}{2} \cot 30^\circ - \frac{a}{2} \cot 45^\circ \Rightarrow \cot \theta = \frac{\sqrt{3} - 1}{2}$$

Applying sine rule in $\triangle ADC$, we get

$$\frac{AD}{\sin(\pi - \theta - 45^\circ)} = \frac{DC}{\sin 45^\circ}$$

$$\Rightarrow \frac{AD}{\sin(\theta + 45^\circ)} = \frac{\frac{a}{2}}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}} (\sin 45^\circ \cos \theta + \cos 45^\circ \sin \theta)$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}} \left(\frac{\cos \theta + \sin \theta}{\sqrt{2}} \right) = \frac{a}{2} (\cos \theta + \sin \theta)$$

$$\Rightarrow \frac{1}{\sqrt{11 - 6\sqrt{3}}} = \frac{a}{2} \left(\frac{\sqrt{3} - 1}{\sqrt{8 - 2\sqrt{3}}} + \frac{2}{\sqrt{8 - 2\sqrt{3}}} \right)$$

$$\Rightarrow a = \frac{2\sqrt{8 - 2\sqrt{3}}}{\sqrt{44 - 24\sqrt{3} + 22\sqrt{3} - 36}} = 2 \frac{\sqrt{8 - 2\sqrt{3}}}{\sqrt{8 - 2\sqrt{3}}} = 2$$

Sol 19: Given,

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

$$\Rightarrow \sin C = \frac{1 - \cos A \cos B}{\sin A \sin B} \quad \dots (i)$$

$$\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1 \quad (\sin C \leq 1)$$

$$\Rightarrow 1 - \cos A \cos B \leq \sin A \sin B$$

$$\Rightarrow 1 \leq \cos(A - B)$$

$$\Rightarrow \cos(A - B) \geq 1$$

$$\Rightarrow \cos(A - B) = 1 \quad [\text{as } \cos(\theta) \leq 1]$$

$$\Rightarrow A - B = 0$$

On putting $A = B$ in Eq. (i), we get

$$\sin C = \frac{1 - \cos^2 A}{\sin^2 A}$$

$$\Rightarrow \sin C = 1 \Rightarrow C = \frac{\pi}{2}$$

$$\text{Now, } A + B + C = \pi$$

$$\Rightarrow A + B = \frac{\pi}{2}$$

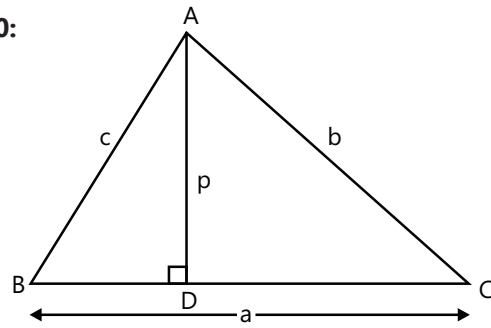
$$\Rightarrow A = \frac{\pi}{4} \quad (\because A = B \text{ and } C = \frac{\pi}{2})$$

$$\therefore \sin A : \sin B : \sin C$$

$$= \sin \frac{\pi}{4} : \sin \frac{\pi}{4} : \sin \frac{\pi}{2}$$

$$\Rightarrow a : b : c = \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1 = 1 : 1 : \sqrt{2}$$

Sol 20:



Let ABC be a triangle with base $BC = a$ and altitude $AD = p$

$$\text{then, Area of } \Delta ABC = \frac{1}{2} bc \sin A$$

$$\text{Also, area of } \Delta ABC = \frac{1}{2} ap$$

$$\therefore \frac{1}{2} ap = \frac{1}{2} bc \sin A$$

$$\Rightarrow p = \frac{bc \sin A}{a} \Rightarrow p = \frac{abc \sin A}{a^2}$$

$$\Rightarrow p = \frac{abc \sin A \cdot (\sin^2 B - \sin^2 C)}{a^2 (\sin^2 B - \sin^2 C)}$$

$$\Rightarrow p = \frac{abc \sin A \cdot \sin(B + C) \sin(B - C)}{(b^2 \sin^2 A - c^2 \sin^2 A)}$$

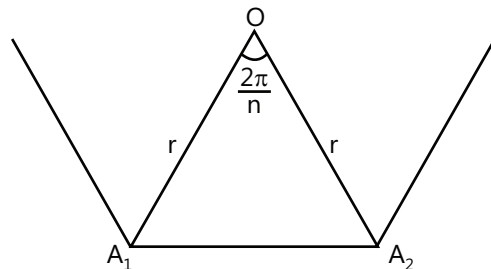
$$\left(\because \text{sin rule: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right)$$

$$= \frac{abc \sin^2 A \cdot \sin(B - C)}{(b^2 - c^2) \cdot \sin^2 A} = \frac{abc \sin(B - C)}{b^2 - c^2}$$

$$= \frac{ab^2 r \sin(B - C)}{b^2 - b^2 r^2} = \frac{ar \sin(B - C)}{1 - r^2}$$

$$\Rightarrow p \leq \frac{ar}{1 - r^2} \quad [\sin(B - C) \leq 1]$$

Sol 21:



Let O be the centre and r be the radius of the circle passing

through the vertices A_1, A_2, \dots, A_n

Then, $\angle A_1 O A_2 = \frac{2\pi}{n}$, also $OA_1 = OA_2 = r$

Again by cos formula we know that,

$$\cos\left(\frac{2\pi}{n}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2(OA_1)(OA_2)}$$

$$\Rightarrow \cos\left(\frac{2\pi}{n}\right) = \frac{r^2 + r^2 - A_1A_2^2}{2(r)(r)}$$

$$\Rightarrow 2r^2 \cos\left(\frac{2\pi}{n}\right) = 2r^2 - A_1A_2^2$$

$$\Rightarrow A_1A_2^2 = 2r^2 - 2r^2 \cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow A_1A_2^2 = 2r^2 \left(1 - \cos\left(\frac{2\pi}{n}\right)\right)$$

$$\Rightarrow A_1A_2^2 = 2r^2 \cdot 2\sin^2\left(\frac{\pi}{n}\right)$$

$$\Rightarrow A_1A_2^2 = 4r^2 \sin^2\left(\frac{\pi}{n}\right) \Rightarrow A_1A_2 = 2r \sin\left(\frac{\pi}{n}\right)$$

Similarly, $A_1A_3 = 2r \sin\left(\frac{2\pi}{n}\right)$

and $A_1A_4 = 2r \sin\left(\frac{3\pi}{n}\right)$

Since, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$ (given)

$$\Rightarrow \frac{1}{2r \sin(\pi/n)} = \frac{1}{2r \sin(2\pi/n)} + \frac{1}{2r \sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)} = \frac{1}{\sin(2\pi/n)} + \frac{1}{\sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)} = \frac{\sin\left(\frac{3\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)}{\sin(2\pi/n) \sin(3\pi/n)}$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right) \cdot \sin\left(\frac{3\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{3\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right) \left[\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right) \right] = \sin\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right) \left[\left\{ 2 \cos\left(\frac{3\pi + \pi}{2n}\right) \sin\left(\frac{3\pi - \pi}{2n}\right) \right\} \right]$$

$$= \sin\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow 2 \sin\left(\frac{2\pi}{n}\right) \cdot \cos\left(\frac{2\pi}{n}\right) \cdot \sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow 2 \sin\left(\frac{2\pi}{n}\right) \cos\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \Rightarrow \frac{7\pi}{n} = \pi \Rightarrow n = 7$$

Sol 22 : It is given that a, b, c and area of Δ are rational.

We have, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-b)} = \frac{\Delta}{s(s-b)}$$

Again, a, b, c are rational given, $s = \frac{a+b+c}{2}$ are rational,

Also, (s - b) is rational since, Δ is rational, therefore, we get

$$\tan\left(\frac{B}{2}\right) = \frac{\Delta}{s(s-b)} \text{ is rational.}$$

Similarly, $\tan\left(\frac{C}{2}\right) = \frac{\Delta}{s(s-c)}$ is rational,

Therefore a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are rational.

Which shows that (i) \Rightarrow (ii)

Again, it is given that

a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are rational, then

$$\tan\frac{A}{2} = \tan\left(\frac{\pi}{2} - \frac{B+C}{2}\right)$$

$$\Rightarrow \cot\left(\frac{B+C}{2}\right) = \frac{1}{\tan\left(\frac{B+C}{2}\right)} = \frac{1 - \tan\left(\frac{B}{2}\right) \cdot \tan\left(\frac{C}{2}\right)}{\tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right)}$$

Since, $\tan(B/2)$ and $\tan(C/2)$ are rational numbers, we get $\tan(A/2)$ is a rational number.

Now, $\sin A = \frac{2 \tan A/2}{1 + \tan^2 A/2}$ as $\tan(A/2)$ is a rational

number, $\sin A$ is a rational number. Similarly, $\sin B$ and $\sin C$ are rational numbers. Thus, a, $\sin A$, $\sin B$, $\sin C$ are rational numbers therefore (ii) \Rightarrow (iii)

Now again, a, $\sin A$, $\sin B$, $\sin C$ are rational.

By the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A}$$

since a, $\sin A$, $\sin B$ and $\sin C$ are rational numbers,

\Rightarrow b and c are also rational.

Also, $\Delta = \frac{1}{2}bc\sin A$

As b, c and $\sin A$ are rational numbers, Δ is a rational numbers.

Therefore, a, b, c and Δ are rational numbers.

Therefore (iii) \Rightarrow (i).

Sol 23: Since, $A + B + C = \pi$

$\Rightarrow B + C = \pi - \pi/4 = 3\pi/4$... (i) ($A = \pi/4$, given)

$\therefore 0 < B, C < 3\pi/4$.

Also, given $\tan B \cdot \tan C = p$

$\Rightarrow \frac{\sin B \cdot \sin C}{\cos B \cdot \cos C} = \frac{p}{1}$... (i)

$\Rightarrow \frac{\sin B \cdot \sin C + \cos B \cdot \cos C}{\sin B \cdot \sin C - \cos B \cdot \cos C} = \frac{p+1}{p-1}$

$\Rightarrow \frac{\cos(B-C)}{\cos(B+C)} = \frac{1+p}{1-p}$

$\Rightarrow \cos(B-C) = -\frac{(1+p)}{\sqrt{2}(1-p)}$... (ii) ($B+C = 3\pi/4$)

Since, B or C can vary from 0 to $3\pi/4$

$0 \leq B - C < 3\pi/4$

$-\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1$... (iii)

From Eqs. (ii) and (iii), we get

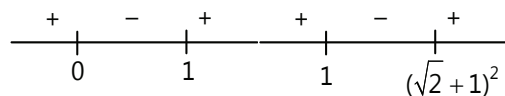
$-\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \leq 1$

$\Rightarrow -\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)}$ and $\frac{1+p}{\sqrt{2}(p-1)} \leq 1$

$\Rightarrow \frac{1+p}{p-1} + 1 \geq 0$ and $\frac{1+p-\sqrt{2}p+\sqrt{2}}{\sqrt{2}(p-1)} \leq 0$

$\Rightarrow \frac{2p}{p-1} \geq 0$ and $\frac{(1-\sqrt{2})\left(p-\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)}{\sqrt{2}(p-1)} \leq 0$

$\Rightarrow \frac{2p}{p-1} > 0$ and $\frac{(p-(\sqrt{2}+1)^2)}{(p-1)} \geq 0$



$\Rightarrow (p < 0 \text{ or } p > 1)$

and $(p < 1 \text{ or } p > (\sqrt{2} + 1)^2)$

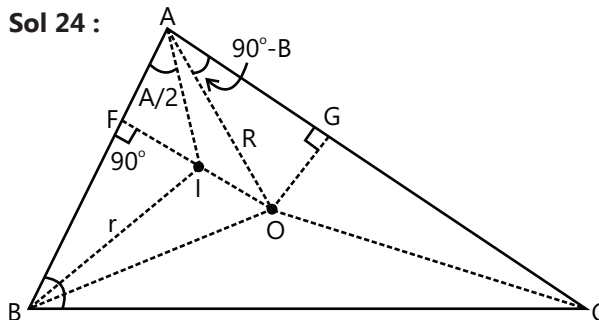
Combing above expressions;

$p < 0$ or $p \geq (\sqrt{2} + 1)^2$

ie, $P \in (-\infty, 0) \cup [(\sqrt{2} + 1)^2, \infty)$

or $P \in (-\infty, 0) \cup [3+2\sqrt{2}, \infty)$

Sol 24 :



It is clear from figure that $OA = R$.

$AI = \frac{IF}{\sin(A/2)}$

$\therefore \Delta AIF$ is right angled triangle = $\frac{r}{\sin(A/2)}$

But $r = 4R \sin(A/2) \sin(B/2) \sin(C/2)$

$\therefore AI = 4R \sin(B/2) \sin(C/2)$

Again, $\angle GOA = B \Rightarrow \angle OAG = 90^\circ - B$

Therefore, $\angle IAO = \angle IAC - \angle OAC$

$= A/2 - (90^\circ - B) = \frac{1}{2}(A + 2B - 180^\circ)$

$= \frac{1}{2}(A + 2B - A - B - C) = \frac{1}{2}(B - C)$

In ΔOAI , $OI^2 = OA^2 + AI^2 - 2(OA)(AI)\cos(\angle IAO)$

$= R^2 + [4R \sin(B/2)\sin(C/2)]^2 - 2R \cdot [4R\sin(B/2) \sin(C/2)]$

$\cos\left(\frac{B-C}{2}\right)$

$= [R^2 + 16R^2 \sin^2(B/2) \sin^2(C/2) - 8R^2 \sin(B/2) \sin(C/2)] \cos\left(\frac{B-C}{2}\right)$

$= R^2 [1 + 16 \sin^2(B/2) \sin^2(C/2) - 8 \sin(B/2) \sin(C/2)] \cos\left(\frac{B-C}{2}\right)$

$= R^2 [1 + 8 \sin(B/2)$

$\sin(C/2) \left\{ 2 \sin(B/2) \sin(C/2) \cos\left(\frac{B-C}{2}\right) \right\}]$

$= R^2 [1 + 8 \sin(B/2)$

$$\begin{aligned} & \sin(C/2) \left\{ \cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{B+C}{2}\right) - \cos\left(\frac{B-C}{2}\right) \right\} \\ &= R^2 \left[1 - 8 \sin(B/2) \sin(C/2) \cos\left(\frac{B+C}{2}\right) \right] \\ &= R^2 \left[1 - 8 \sin(B/2) \sin(C/2) \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) \right] \\ & \left[\because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \right] \\ &= R^2 [1 - 8 \sin(A/2) \sin(B/2) \sin(C/2)] \\ &= R^2 \left[1 - 8 \left(\frac{r}{4R} \right) \right] = R^2 - 2Rr \end{aligned}$$

Now, in right $\triangle BIO$

$$\begin{aligned} \Rightarrow OB^2 &= BI^2 + IO^2 \\ \Rightarrow R^2 &= BI^2 + R^2 - 2Rr \\ \Rightarrow 2Rr &= BI^2 \\ \Rightarrow 2Rr &= r^2/\sin^2(B/2) \\ \Rightarrow 2R &= r/\sin^2(B/2) \\ \Rightarrow 2R \sin^2 B/2 &= r \\ \Rightarrow R(1-\cos B) &= r \end{aligned}$$

$$\Rightarrow \frac{abc}{4\Delta} (1-\cos B) = \frac{\Delta}{s}$$

$$\Rightarrow abc (1-\cos B) = \frac{4\Delta^2}{s}$$

$$\Rightarrow abc \left[1 - \frac{a^2 + c^2 - b^2}{2ac} \right] = \frac{4\Delta^2}{s}$$

$$\Rightarrow abc \left[\frac{2ac - a^2 - c^2 + b^2}{2ac} \right] = \frac{4\Delta^2}{s}$$

$$\Rightarrow b[b^2 - (a-c)^2] = \frac{4\Delta^2}{s}$$

$$\Rightarrow b[b^2 - (a-c)^2] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b\{[b - (a-c)] [b + (a-c)]\} = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b[(b+c-a)(b+a-c)] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b[(2s-2a)(2s-2c)] = 8(s-a)(s-b)(s-c)$$

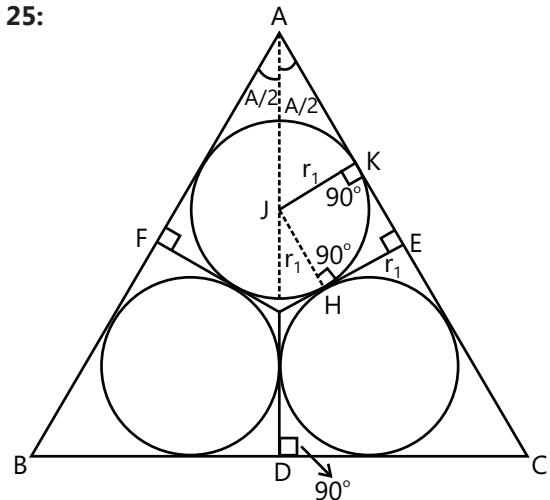
$$\Rightarrow b[2 \cdot 2 (s-a)(s-c)] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b = 2s - 2b$$

$$\Rightarrow 2b = a + c$$

Which shows that b is arithmetic mean between a and c .

Sol 25:



The quadrilateral HEKJ is a square because all four angles are right angle and $JK = JH$.

Therefore, $HE = JK = r_1$ and $IE = r$ (given)

$$\Rightarrow IH = r - r_1$$

Now, in right angled triangle IHJ,

$$\angle JIH = \pi/2 - A/2$$

[$\angle IEA = 90^\circ$, $\angle IAE = A/2$ and $\angle JIH = \angle AIE$] in triangle JIH

$$\tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \frac{r_1}{r - r_1} \Rightarrow \cot \frac{A}{2} = \frac{r_1}{r - r_1}$$

$$\text{Similarly, } \cot \frac{B}{2} = \frac{r_2}{r - r_2} \text{ and } \cot \frac{C}{2} = \frac{r_3}{r - r_3}$$

On adding above results, we get

$$\cot A/2 + \cot B/2 + \cot C/2 = \cot A/2 \cot B/2 \cot C/2$$

$$\Rightarrow \frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1 r_2 r_3}{(r - r_1)(r - r_2)(r - r_3)}$$

Sol 26: Given $\Delta \leq \frac{1}{4} \sqrt{(a+b+c)abc}$

$$\Rightarrow \frac{1}{4\Delta} \sqrt{(a+b+c)abc} \geq 1$$

$$\Rightarrow \frac{(a+b+c)abc}{16\Delta^2} \geq 1 \Rightarrow \frac{2s abc}{16\Delta^2} \geq 1$$

$$\Rightarrow \frac{s abc}{8 \cdot s(s-a)(s-b)(s-c)} \geq 1$$

$$\Rightarrow \frac{abc}{8(s-a)(s-b)(s-c)} \geq 1$$

$$\Rightarrow \frac{abc}{8} \geq (s-a)(s-b)(s-c)$$

Now, puts $-a = x \geq 0, s - b = y \geq 0, s - c = z \geq 0$

$$s - a + s - b = x + y$$

$$2s - a - b = x + y$$

$$c = x + y$$

Similarly, $a = y + z, b = x + z$

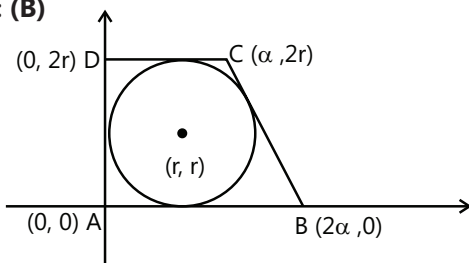
$$\Rightarrow \frac{(x+y)}{2} \cdot \frac{(y+z)}{2} \cdot \frac{(x+z)}{2} \geq xyz$$

which it true

Now quality will hold if

$x = y = z \Rightarrow a = b = c \Rightarrow$ triangle is equilateral.

Sol 27: (B)



$$18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$$

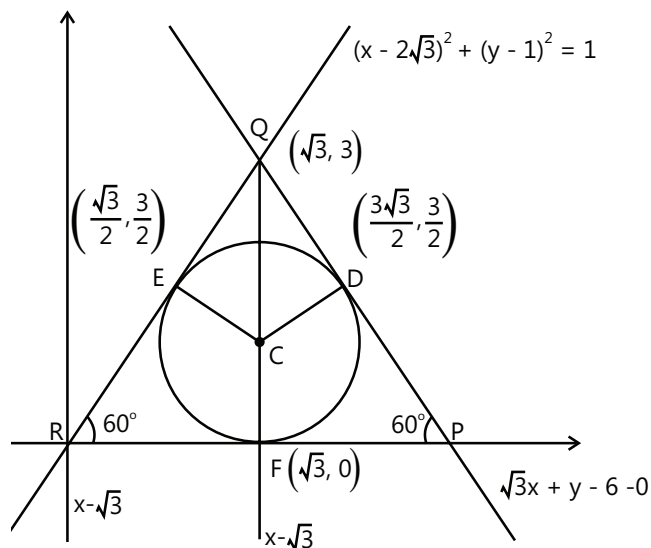
Line $y = -\frac{2r}{\alpha}(x - 2\alpha)$ is tangent to

$$(x - r)^2 + (y - r)^2 = r^2$$

$$2\alpha = 3r \text{ and } \alpha r = 6$$

$$r = 2.$$

Sol 28: (D)



$$\text{Equation of CD is } \frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = -1 \Rightarrow C \equiv (\sqrt{3}, 1)$$

$$\text{Equation of the circle is } (x - \sqrt{3})^2 + (y - 1)^2 = 1$$

Sol 29: (A) Since the radius of the circle is 1 and $C(\sqrt{3}, 1)$, coordinates of $F \equiv (\sqrt{3}, 0)$

$$\text{Equation of CE is } \frac{x - \sqrt{3}}{-\frac{\sqrt{3}}{2}} = \frac{y - 1}{\frac{1}{2}} = 1 \Rightarrow E \equiv \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$

Sol 30: (D) Equation of QR is $y - 3 = \sqrt{3}(x - \sqrt{3})$

$$\Rightarrow y = \sqrt{3}x$$

Equation of RP is $y = 0$.

Sol 31: (D) $X = \sin\theta + \sin 3\theta + \dots + \sin 29\theta$

$$2(\sin\theta)X = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \dots + \cos 28\theta - \cos 30\theta$$

$$X = \frac{1 - \cos 30\theta}{2 \sin\theta} = \frac{1}{4 \sin 2\theta}$$

Sol 32: (B, C)

$$2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) = 4 \sin^2 \frac{A}{2}$$

$$\cos\left(\frac{B-C}{2}\right) = 2 \sin(A/2)$$

$$\Rightarrow \frac{\cos\left(\frac{B-C}{2}\right)}{\sin A/2} = 2 \Rightarrow \frac{\sin B + \sin C}{\sin A} = 2$$

$$\Rightarrow b + c = 2a \text{ (constant)}$$

Sol 33: (C, D) Given solutions

$$\frac{1}{\sin(\pi/4)} \left[\frac{\sin(\theta + \pi/4 - \theta)}{\sin\theta \cdot \sin(\theta + \pi/4)} + \frac{\sin(\theta + \pi/2 - (\theta + \pi/4))}{\sin(\theta + \pi/4) \cdot \sin(\theta + \pi/2)} + \dots + \frac{\sin((\theta + 3\pi/2) - (\theta + 5\pi/4))}{\sin(\theta + 3\pi/2) \cdot \sin(\theta + 5\pi/4)} \right]$$

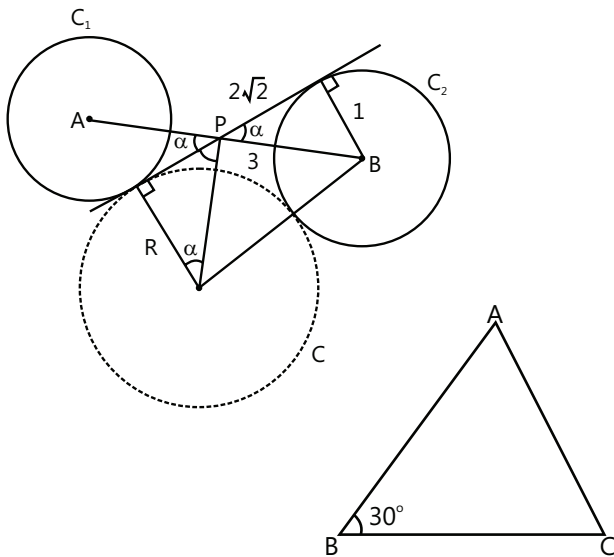
$$= 4\sqrt{2}$$

$$\Rightarrow \sqrt{2}[\cos\theta - \cot(\theta + \pi/4) + \cot(\theta + \pi/4)$$

$$- \cot(\theta + \pi/2) + \dots + \cot(\theta + 5\pi/4) - \cot(\theta + 3\pi/2)] = 4\sqrt{2}$$

Sol 34: $\cos \alpha = \frac{2\sqrt{2}}{3}$, $\sin \alpha = \frac{1}{3}$, $\tan \alpha = \frac{2\sqrt{2}}{R}$

$\Rightarrow R = \frac{2\sqrt{2}}{\tan \theta} = 8$ units



Sol 35:

$\cos \beta = \frac{a^2 + 16 - 8}{2 \times a \times 4} \Rightarrow \frac{\sqrt{3}}{2} = \frac{a^2 + 8}{8a}$

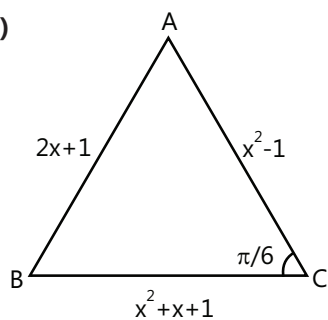
$\Rightarrow a^2 + 4\sqrt{3}a + 8 = 0$

$\Rightarrow a_1 + a_2 = 4\sqrt{3}$, $a_1 a_2 = 8$

$\Rightarrow |a_1 - a_2| = 4$

$\Rightarrow |\Delta_1 - \Delta_2| = \frac{1}{2} \times 4 \sin 30^\circ \times 4 = 4$

Sol 36: (A, B)



$\cos \frac{\pi}{6} = \frac{(x^2 - 1)^2 + (x^2 + x + 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2 - 1)^2 + (x^2 + 3x + 2)(x^2 - x)}{2(x^2 + x + 1)(x^2 - 1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2 - 1)^2 + (x + 1)(x + 2)x(x - 1)}{2(x^2 + x + 1)(x^2 - 1)}$

$\Rightarrow \sqrt{3} = \frac{x^2 - 1 + x(x + 2)}{x^2 + x + 1}$

$\Rightarrow \sqrt{3}(x^2 + x + 1) = 2x^2 + 2x - 1$

$\Rightarrow (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) = 0$

On solving

$\Rightarrow x^2 + x - (3\sqrt{3} + 5) = 0$

$x = \sqrt{3} + 1, -(2 + \sqrt{3})$

Sol 37: (A, C, D)

$2 \cos \theta (1 - \sin \theta) = \frac{2 \sin^2 \theta}{\sin \theta} \cos \theta - 1 = 2 \sin \theta \cos \theta - 1$

$2 \cos \theta - 2 \cos \theta \sin \theta = 2 \sin \theta \cos \theta - 1$

$2 \cos \theta + 1 = 2 \sin (\theta + \varphi)$

$\tan(2\pi - \theta) > 0 \Rightarrow \tan \theta < 0$ and $-1 < \sin \theta < -\frac{\sqrt{3}}{2}$

$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

$\frac{1}{2} < \sin (\theta + \varphi) < 1$

$\Rightarrow 2\pi + \frac{\pi}{6} < \theta + \varphi < \frac{5\pi}{6} + 2\pi$

$2\pi + \frac{\pi}{6} - \theta_{\max} < \varphi < 2\pi + \frac{5\pi}{6} - \theta_{\min}$

$\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$

Sol 38: (C)

$a = 2 = QR$, $b = \frac{7}{2} = PR$, $c = \frac{5}{2} = PQ$

$s = \frac{a + b + c}{2} = \frac{8}{4} = 4$

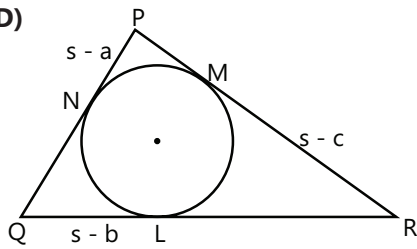
$\frac{2 \sin P - 2 \sin P \cos P}{2 \sin P + 2 \sin P \cos P} = \frac{2 \sin P (1 - \cos P)}{2 \sin P (1 + \cos P)}$

$= \frac{1 - \cos P}{1 + \cos P} = \frac{2 \sin^2 \frac{P}{2}}{2 \cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$

$= \frac{(s - b)(s - c)}{s(s - a)} = \frac{(s - b)^2 (s - c)^2}{\Delta^2}$

$= \frac{\left(4 - \frac{7}{2}\right)^2 \left(4 - \frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$

Sol 39: (B, D)



Let $s - a = 2k - 2$, $s - b = 2k$, $s - c = 2k + 2$, $k \in \mathbb{I}$, $k > 1$

Adding we get,

$$s = 6k$$

$$\text{So, } a = 4k + 2, b = 4k, c = 4k - 2$$

$$\text{Now, } \cos P = \frac{1}{3}$$

$$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{3}$$

$$\Rightarrow 3[(4k)^2 + (4k - 2)^2 - (4k + 2)^2] = 2 \times 4k(4k - 2)$$

$$\Rightarrow 3[16k^2 - 4(4k) \times 2] = 8k(4k - 2)$$

$$\Rightarrow 48k^2 - 96k = 32k^2 - 16k \Rightarrow 16k^2 = 80k \Rightarrow k = 5$$

So, sides are 22, 20, 18

Sol 40: (B) $x = a + b$, $y = ab$

$$x^2 - c^2 = y$$

$$\Rightarrow \frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{2} = \cos(120^\circ)$$

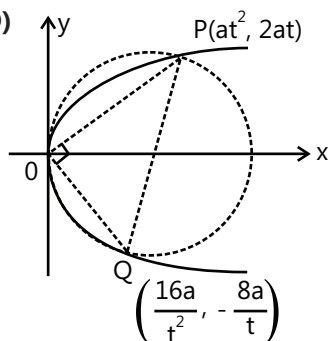
$$\Rightarrow \angle C = \frac{2\pi}{3}$$

$$\Rightarrow R = \frac{abc}{4\Delta}, r = \frac{\Delta}{s}$$

$$\Rightarrow \frac{r}{R} = \frac{4\Delta^2}{s(abc)} = \frac{4 \left[\frac{1}{2} ab \sin\left(\frac{2\pi}{3}\right) \right]^2}{\frac{x+c}{2} \cdot y \cdot c}$$

$$\frac{r}{R} = \frac{3y}{2c(x+c)}$$

Sol 41: (A, D)



$$P(at^2, 2at)$$

$$Q\left(\frac{16a}{t^2}, -\frac{8a}{t}\right)$$

$$\Delta OPQ = \frac{1}{2} OP \cdot OQ$$

$$\Rightarrow \frac{1}{2} \left| at\sqrt{t^2+4} \cdot \frac{a(-4)}{t} \sqrt{\frac{16}{t^2}+4} \right| = 3\sqrt{2}$$

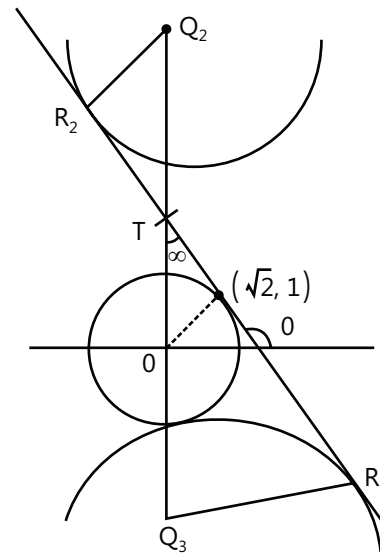
$$\Rightarrow t^2 - 3\sqrt{2}t + 4 = 0 \Rightarrow t = \sqrt{2}, 2\sqrt{2}$$

$$P(at^2, 2at) = P\left(\frac{t^2}{2}, t\right)$$

$$t = \sqrt{2} \Rightarrow P(1, \sqrt{2})$$

$$t = 2\sqrt{2} \Rightarrow P(4, 2\sqrt{2})$$

Sol 42: (A, B, C)



$$x^2 + y^2 = 3$$

$$x^2 = 2y$$

Intersection point is $P = (\sqrt{2}, 1)$

Equation of tangent is $\sqrt{2}x + y = 3$

$$\tan(\theta) = -\sqrt{2}$$

$$\tan(\alpha) = \tan(\theta - 90) = -\cot \theta = \frac{1}{\sqrt{2}}$$

$$\sin(\alpha) = \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{Q_3 T}$$

$$\Rightarrow Q_3 T = 6$$

$$\therefore Q_2Q_3 = 2Q_3T = 12$$

$$\tan(\alpha) = \frac{1}{\sqrt{2}} = \frac{2\sqrt{3}}{R_3T} \Rightarrow R_3T = 2\sqrt{6}$$

$$\therefore R_2R_3 = 2R_3T = 4\sqrt{6}$$

$$\perp \text{ distance of } o \text{ from } R_2R_3 \text{ is } \left| \frac{3}{\sqrt{(\sqrt{2})^2 + 1^2}} = \sqrt{3} \right|$$

$$\therefore \text{Area } (OR_2R_3) = \frac{1}{2} \times \sqrt{3} \times 4\sqrt{6} = 6\sqrt{2} \text{ square units}$$

$$\text{Similarly, Area } (PQ_2Q_3) = \frac{1}{2} \times \sqrt{2} \times 12 = 6\sqrt{2} \text{ square units}$$

Sol 43: (A, C, D)

$$\frac{s-X}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s-(x+y+z)}{9} = \frac{s}{9}$$

$$\therefore x = \frac{5s}{9}, y = \frac{2s}{3}, z = \frac{7s}{9}$$

$$A = \pi r^2 = \frac{8\pi}{3}$$

$$\Rightarrow \frac{\Delta}{s} = \sqrt{\frac{8}{3}} \Rightarrow \Delta^2 = \frac{8s^2}{3}$$

$$\Rightarrow s \cdot \frac{4s}{9} \cdot \frac{s}{3} \cdot \frac{2s}{9} = \frac{8}{3}s^2$$

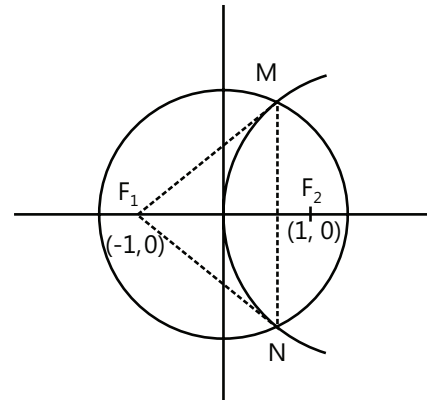
$$\Rightarrow s = 9$$

$$\therefore \Delta = \sqrt{\frac{8}{3}} \times 9 = 6\sqrt{6} \text{ square units}$$

$$R = \frac{xyz}{4\Delta} = \frac{\frac{5s}{9} \cdot \frac{2s}{3} \cdot \frac{7s}{9}}{4 \times 6\sqrt{6}} = \frac{35}{24}\sqrt{6}$$

$$\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\sin\left(\frac{z}{2}\right) = \frac{r}{4R} = \frac{\sqrt{\frac{8}{3}}}{4 \times \frac{35\sqrt{6}}{24}} = \frac{4}{35}$$

$$\sin^2\left(\frac{x+y}{z}\right) = \cos^2\left(\frac{z}{2}\right) = \frac{1+\cos(z)}{2} = \frac{3}{5}$$

Sol 44: (A)


$$a = 3$$

$$e = \frac{1}{3}$$

$$\therefore F_1 \equiv (-1, 0)$$

$$F_2 \equiv (1, 0)$$

So, equation of parabola is $y^2 = 4x$

Solving simultaneously, we get $\left(\frac{3}{2}, \pm\sqrt{6}\right)$

\therefore Orthocentre is $\left(\frac{-9}{10}, 0\right)$