Permutations and Combinations

14. MULTINOMIAL THEOREM

(a) If there are l objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of ways of choosing r objects out these (i.e., l + m + n ...) is the coefficient of x^r in the expansion of

$(1 + x + x^2 + x^3 + \ldots + xl)\,(1 + x + x^2 + x^3 + \ldots + x^m)\,(1 + x + x^2 + x^3 + \ldots + x^n)$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of

$(x + x^2 + x^3 + \ldots + xl)\,(x + x^2 + x^3 + \ldots + x^m)\,(x + x^2 + x^3 + \ldots + x^n)\ldots$

(b) If there are objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of possible arrangements/permutations of r objects out of these object (i.e., l + m + m + ...) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots..\frac{x^f}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\frac{x^n}{n!}\right)$$

Illustration 49: In an examination, the maximum marks for each of three papers is n and that for the fourth paper is 2n. Prove that the number of ways in which candidate can get 3n marks is

$\frac{1}{6}(n + 1)(5n^2 + 10n + 6)$. **(JEE ADVANCED)**

Sol: The maximum marks in the four papers are n, n, n and 2n. Consider a polynomial $(1 + x + x^2 + \ldots + x^n)^3 (1 + x + \ldots + x^{2n})$. The number of ways of securing a total of 3n is equal to the co-efficient of the term containing x^{3n}.

The number of ways of getting 3n marks

= coefficient of x^{3n} in $(1 + x + x^2 + \ldots + x^n)^3 (1 + x + \ldots + x^{2n})$

= coefficient of x^{3n} in $(1 - x^{n+1})^3 (1 - x^{2n+1}) (1 - x)^{-4}$

= coefficient of x^{3n} in $(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}) (1 - x^{2n+1}) \times (1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + \ldots.)$

= coefficient of x^{3n} in $(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2}) (1 + {}^4C_1 x + {}^5C_2 x^2 \ldots)$

$= {}^{3n+3}C_{3n} - 3.{}^{2n+2}C_{2n-1} + 3\,.\,{}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$

$$= \frac{(3n+3)!}{3!(3n)!} - 3\,.\,\frac{(2n+2)!}{3!(2n-1)!} + 3\,\frac{(n-1)!}{3!(n-2)!} - \frac{(n+2)!}{3!(n-1)!}$$

$= 1/6\,(n + 1)\,(27n^2 + 27n + 6 - 24n^2 - 12n + 3n^2 - 3n - n^2 - 2n) = 1/6\,(n + 1)\,(5n^2 + 10n + 6)$

PROBLEM-SOLVING TACTICS

In any given problem, first try to understand whether it is a problem of permutations or combinations. Now, think if repetition is allowed and then try solving problem.

A simple method to solve these problems where repetition is not allowed is as follows -

First draw series of dashes representing the number of places you want to fill or number of items you want to select.

Now start filling dashes by the number of objects available to choose from and multiply the numbers. This is the final answer for a permutations problem.

If it is a combination problem then divide the answer with the factorial or number of items.

This calculation becomes complex if repetition is allowed.

FORMULAE SHEET

(a) Permutation (Arrangement of Objects): Each of the different arrangement, which can be made by taking some or all of a number of objects is called permutation.

(i) The number of permutations of n different objects taken r at a time is $^nP_r = \frac{n!}{(n-r)!}$.

(ii) The number of all permutations of n distinct objects taken all at a time is n!.

Permutation with Repetition: The number of permutations of n different objects taken r at a time when each object may be repeated any number of times is n^r.

Permutation of Alike Objects: The number of permutations of n objects taken all at a time in which, p are alike objects of one kind, q are alike objects of second kind & r are alike objects of a third kind and the rest (n – (p + q + r)) are all different, is $\frac{n!}{p!q!r!}$.

Permutation under Restriction: The number of permutations of n different objects, taken all at a time, when m specified objects always come together is m! × (n – m + 1)!.

(b) Combination (Selection of Objects): Each of the different groups or selection which can be made by some or all of a number of given objects without reference to the order of the objects in each group is called a combination.

The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or $^nC_r = \frac{n!}{r!(n-r)!}$ $(0 \le r \le n) = \frac{^nP_r}{r!}$

Note:

(a) The number of ways of selecting r objects out of n objects, is the same as the number of ways in which the remaining (n - r) can be selected and rejected.

(b) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ^nC_r is the combination of n elements chosen r at a time.

(c) (i) $^nC_r = {^nC_{n-r}}$

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iii) $^nC_x = {^nC_y} \Rightarrow x = y$ or $x + y = n$

(iv) If n is even, then the greatest value of ^nC_r is $^nC_{n/2}$

(v) If n is odd, then the greatest value of ^nC_r is $^nC_{(n+1)/2}$

(vi) $^nC_0 + {^nC_r} + \ldots\ldots\ldots\ldots + {^nC_n} = 2^n$

(vii) $^nC_n + {^{n+1}C_n} + {^{n+2}C_n} + \ldots\ldots\ldots\ldots + {^{2n-1}C_n} = {^{2n}C_{n+1}}$

Combinations under Restrictions

(a) The number of ways of choosing r objects out of n different objects if p particular objects must be excluded $= {}^{(n-p)}C_r$

(b) The number of ways of choosing r objects out of n different objects if p particular objects must be included $(p \le r) = {}^{n-p}C_{r-p}$

(c) The total number of combinations of n different objects taken one or more at a time = $2^n - 1$.

Combinations of Alike Objects

(a) The number of combinations of n identical objects taking $(r \le n)$ at a time is 1.

(b) The number of ways of selecting r objects out of n identical objects is n + 1.

(c) If out of (p + q + r + s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then total number of combinations is $(p + 1)(q + 1)(r + 1)2^s - 1$

(d) The number of ways in which r objects can be selected from a group of n objects of which p are identical, is

$\sum_{0}^{t} {}^{n-p}C_r$, if $r \le p$ and $\sum_{r=p}^{t} {}^{n-p}C_r$ if $r > p$

Division into Groups

(a) The number of ways in which (m + n) different objects can be divided into two unequal groups containing m and n objects respectively is $\frac{(m+n)!}{m!n!}$.

If m = n, the groups are equal and in this case the number of divisions is $\frac{(2n)!}{n!n!2!}$; as it is possible to interchange the two groups without obtaining a new distribution.

(b) However, if 2n objects are to be divided equally between two persons then the number of ways

$= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!}$

(c) The number of ways in which (m + n + p) different objects can be divided into three unequal groups containing m , n and p objects respectively is $= \frac{(m+n+p)}{m!n!p!}$, $m \neq n \neq p$

If m = n = P then the number of groups = $\frac{(3n)!}{n!n!n!3!}$. However, if 3n objects are to be divided equally among three persons then the number of ways = $\frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3}$

(d) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is not important is $\frac{mn!}{(n!)^m m!}$

(e) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is important is $\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$

Circular Permutation

(a) The number of circular permutations of n different objects taken r at a time

${}^nP_r/r$, when clockwise and anticlockwise orders are treated as different.

${}^nP_r/2r$, when clockwise and anticlockwise orders are treated as same.

(b) The number of circular permutations of n different objects altogether

${}^nP_n/n = (n-1)!$, when clockwise and anticlockwise order are treated as different

${}^{n}P_n/2n = 1/2(n-1)!$, when above two orders are treated as same

The number of non-negative integral solutions of equation $x_1 + x_2 + \ldots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get zero or one or more objects = ${}^{n+r-1}C_{r-1}$

The number of positive integral solutions for the equation $x_1 + x_2 + \ldots. + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get at least one object = ${}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

(c) For given n different digits $a_1, a_2, a_3 \ldots a_n$ the sum of the digits in the units place of all the numbers formed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + \ldots + a_n)(n-1)!$ i.e. (sum of the digits) $(n-1)!$

(d) The sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \ldots\ldots a_n$ is

$(a + a_2 + a_3 + \ldots + a_n)(n-1)!(111. \ldots\ldots\ldots n$ times)

Factors of Natural Numbers

Let $N = p^a . q^b . r^c \ldots$ where p, q, r … are distinct primes & a, b, c … are natural numbers, then:

(a) The total number of divisors of N including 1 and N are = $(a+1)(b+1)(c+1)\ldots$

(b) The sum of these divisors is = $(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) The number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1) & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(a+1)(b+1)(c+1)\ldots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N

Exponent of a Prime P in N! = $\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$

Inclusion-Exclusion Principle: The principle of inclusion-exclusion states that for finite sets $A_1, \ldots A_n$. One has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{1 \le i < j \le n}|A_i \cap Aj| + \sum_{1 \le i < j < k \le n}|A_i \cap Aj \cap A_k| - \ldots + (-1)^{n-1}|A_1 \cap \ldots \cap A_n|.$$

This can be compactly written as $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1 \le i_1 < \ldots\ldots\ldots < i_k \le n}|A_{i_1} \cap \ldots \cap A_{i_k}|\right)$

Derangements Theorem: The number of ways in which letters n can be placed in n envelopes (one letter in each envelope) so that no letter is placed in the correct envelope is $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$

If n objects are arranged at n places then the number of ways to rearrange exactly r objects at right places is =

$$\frac{n!}{r}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

Some Important results

(a) The number of totally different straight lines formed by joining n points on a plane of which $m(<n)$ are collinear is ${}^{n}C_2 - {}^{m}C_2 + 1$.

(b) The number of total triangles formed by joining n points on a plane of which $m(<n)$ are collinear is ${}^{n}C_3 - {}^{m}C_3$.

(c) The number of diagonals in a polygon of n sides is ${}^{n}C_2 - n$.

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed are ${}^{m}C_2 \times {}^{n}C_2$.

(e) Given n points on the circumference of a circle, then

the number of straight lines between these points are ${}^{n}C_2$

the number of triangles between these points are ${}^{n}C_3$

the number of quadrilaterals between these points are ${}^{n}C_4$

(f) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane is = 1 + Sn