18. PROBABILITY

1. INTRODUCTION

There are various phenomena in nature leading to an outcome, which cannot be predicted beforehand. For example, tossing a coin may result into two outcomes- a head or a tail. Probability theory aims to measure the uncertainties of such outcomes. Consequently, probability is the measure of uncertainty of random experiments.

2. RANDOM EXPERIMENT

An experiment is said to be random if it has more than one possible known outcomes which cannot be predicted in advance. For example - Throwing of a die is a random experiment.

Sample Space: The set of all possible outcomes of a trial (random experiment) is called its sample space. It is generally denoted by S and each outcome of the trial is said to be a sample point.

For example - In throwing of a die, the sample space for the number that shows up on the top face would be:

$S = \{1, 2, 3, 4, 5, 6\}$

EVENT: Every subset of a sample space is called an event.

For example, in throwing a dice, the sample space

$S = \{1, 2, 3, 4, 5, 6\}$ and $n(S) = 6$

$E_1 = \{1, 3, 5\} \subset S$. So E_1 is an event and $n(E_1) = 3$.

The event $E_1 = \{1, 3, 5\}$ can also be expressed as the event of getting an odd number in throwing a dice.

(a) Simple event:A simple event or an elementary event is an event containing only a single sample point.

(b) Compound events: Compound events or decomposable events are those events that are obtained by combining together two or more elementary events.

For instance, the event of drawing a heart from a deck of cards is the subset A = {heart} of the sample space S = {heart, spade, club, diamond}. Therefore, A is a simple event. The event B of drawing a red card is a compound event since B = {heart U diamond} = {heart and diamond}.

(c) Mutually exclusive or disjoint events: Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the other events.

(d) Mutually non-exclusive events: The events which are not mutually exclusive are known as compatible events or mutually nonexclusive events.

(e) Independent events: Events are said to be independent, if the happening(or non-happening) of one event is not affected by the happening (or non-happening) of other events.

(f) **Dependent events:** Two or more events are said to be dependent, if the happening of one event affects (partially or totally) the other event.

The relationship between mutually exclusive and independent events

Mutually exclusive events can't happen at the same time. Mathematically put,

Independent events: P(A and B) = P(A) × P(B)

Mutually exclusive: P(A and B) = 0, where A and B are two events.

→ On comparing two definitions, we see that **the events can't be independent and Mutually exclusive at the same time.**

Equally likely events: Events which have the same chance of occurring are said to be equally likely events.

For example, in the experiment of tossing a coin,

where,

A: The event of getting a "HEAD" and

B: The event of getting a "TAIL"

Events "A" and "B" are said to be equally likely events.

[Both the events have the same chance of occurrence]

In the experiment of throwing a die,

where,

A: The event of getting 1

B: The event of getting 2

...

...

F: The event of getting 6

Events "A", "B", "C", "D", "E" and "F" are said to be equally likely events.

[All these events have the same chance of occurrence.]

3.COMPLEMENT OF EVENTS

The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\bar{A}$ or A^c

(a) The union $E_1 \cup E_2$ of events E_1 and E_2 is the event of at least one of the events E_1, E_2 happening.

(b) The intersection $E_1 \cap E_2$ of events E_1 and E_2 is the event of both the events E_1, E_2 happening.

e.g. Tossing of coin sample space s = {H, T}. Event of getting head in tossing of coin A = {H} $\Rightarrow$ $A^C = \{T\}$

MASTERJEE CONCEPTS

Terminology: Being closely familiar with the terminology of probability helps a lot in thinking with clearly. In particular, always think of outcomes as the most elementary results of an experiment, events as a set of outcomes, the sample space as the set of all possible outcomes and events as subsets of the sample space.

Ravi Vooda (JEE 2009, AIR 71)

4. ALGEBRA OF EVENTS

Verbal description of the event	Equivalent set theoretic notation
Not A	$\bar{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \bar{B}$
Neither A nor B	$\bar{A} \cap \bar{B}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \bar{B}) \cup (\bar{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C)$

5.PROBABILITY

If a random experiment results in n mutually exclusive, equally likely and exhaustive outcomes out of which m are favorable to the occurrence of an event A, then the probability of occurrence of A is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to A}}{\text{Number of total outcomes}}$$

It is obvious that $0 \le m \le n$. If an event A is certain to happen, then m = n, thus P (A) = 1.

If A is impossible to happen, then m = 0 and so P (A) = 0.

Hence we conclude that 0 £P (A) ≤ 1

MASTERJEE CONCEPTS

Working Rule to find probability

Step 1. For the given experiment, find out all possible outcomes n(S) of the sample space.

Step 2. Identify the outcomes n (A), which are favorable to the event A, whose probability is required.

Step 3. Apply the formula to find P (A),

$$P(A) = \frac{n(A)}{n(S)}$$

Equal Likelihood, the formula that we apply to calculate probability,

$\frac{\text{Number of favorable outcome}}{\text{Number of total outcome}}$, is valid only when all the cases have equal likelihood of occurrence.

This important point is overlooked a lot of times. For example, if a rolling die is not fair, then you cannot assign a probability of $\frac{1}{6}$ for each face showing up. Sometimes, the way you count the total cases and favorable cases can lead to a mistake. Consider a random experiment involving the rolling of two dice simultaneously. Suppose you have to evaluate the probability of getting a total of less than 6. The following argument has a mistake: "There are a total of 11 possible cases, namely {2,3,4,5,6,7,8,9,10,11,12}, out of which 4 are favorable, namely {2,3,4,5}, and thus the required probability is $\frac{4}{11}$."

MASTERJEE CONCEPTS

The mistake is that, the different cases do not have equal likelihood of occurrence. For example, the sum 6 is more likely to occur than the sum 2 (why?). The correct way to solve this problem would be to consider the 36 equally likely outcomes (x, y) where x and y can take integer values from 1 to 6, and then consider those outcomes from this set of 36 outcomes, which leads to a sum of less than 6. You can verify that there will be 10 such favorable outcomes. And now, it would be correct to apply the formula $\frac{\text{Number of favorable outcome}}{\text{Number of total outcome}}$ to obtain the required probability as $\frac{10}{36}$.

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 1: In a single case with two fair dice, find the chance of getting

(A) Two 4's (B) A doublet (C) Five-six (D) A sum of 7 **(JEE MAIN)**

Sol: Write all the possible outcomes and the favorable events in each case.

(A) There are 6 × 6 equally likely cases (as any face of any die may turn up)

$\Rightarrow$ 36 possible outcomes. For this event, only one outcome (4 – 4) is favourable

$\therefore$ Probability = 1/36.

(B) A doublet can occur in six ways {(1, 1), (2, 2), (3, 3), (4, 4) (5, 5), (6, 6)}.

Therefore, probability of doublet = 6/36 = 1/6.

(C) Two favorable outcomes {(5, 6), (6, 5)}.Therefore, probability = 2/36 = 1/18.

Sample space = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)

(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)

(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)

(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)

(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)

(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

(D) A sum of 7 can occur in the following cases {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)} which are 6 in number. Therefore, probability = 6/36 = 1/6.

Illustration 2: Seven accidents occur in a week. What is the probability that they take place on the same day? **(JEE MAIN)**

Sol: Find the total number of ways accidents can happen. And clearly, all the accidents can take place on the same day in 7 ways.

Total no. of cases = Total no. of ways in which 7 accidents can happen in a week (or be distributed= 7^7

Favorable No. of cases out of these = number of those in which all 7 happen on one day (any the week)= 7

$\therefore$ Required probability = $\frac{7}{7^7} = \frac{1}{7^6}$

6. MUTUAL INDEPENDENCE AND PAIRWISE INDEPENDENCE

Three events A, B, C are said to be mutually independent if, $P(A \cap B) = P(A).P(B)$, $P(A \cap C) = P(A).P(C)$, $P(B \cap C) = P(B).P(C)$ and $P(A \cap B \cap C) = P(A).P(B).P(C)$

These events would be said to be pairwise independent if, $P(A \cap B \cap C) = \Rightarrow P(A \cap B) = P(A).P(B)$, $P(B \cap C) = P(B).P(C)$ and $P(A \cap C) = P(A).P(C)$

Thus, mutually independent events are pairwise independent but the converse may not be true.

Illustration 3: From a bag containing 5 white, 7 red and 4 black balls, a man draws 3 balls at random. Find the probability of them being all white. **(JEE MAIN)**

Sol: Use the principle of restricted combination.

Total number of balls in the bag = 5 + 7 + 4 = 16

Total number of ways in which 3 balls can be drawn is $^{16}C_3 = \dfrac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560$

Thus, the sample space S for this experiment has 560 outcomes i.e. n(S) = 560

Let E be the event of all the three balls being white. Total number of white balls is 5. So, the number of ways in which 3 white balls can be drawn = $^5C_3 = \dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$

Thus, E has 10 element of S, $\therefore$ n(E) = 10

$\therefore$ Probability of E, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{10}{560} = \dfrac{1}{56}$

MASTERJEE CONCEPTS

Exhaustive Event: A Set of events is said to be exhaustive if the performance of random experiments always results in the occurrence of at least one of them. For instance, consider an ordinary pack of cards. The events 'drawn card is heart', drawn card is diamond', 'drawn card is club' and 'drawn card is spade' is a set of events that is exhaustive. In other words all sample points put together (i.e. sample space itself) would give us an exhaustive event.

If 'E' is an exhaustive event then P (E) = 1.

Vaibhav Gupta (JEE 2009, AIR 54)

7. ODDS IN FAVOUR, ODDS AGAINST

(a) The odds in favor of the event E = $\dfrac{P(E)}{P(E')}$.

(b) The odds against the event E = $\dfrac{P(E')}{P(E)}$

(c) If odds in favor of the event E = a: b then $P(E) = \dfrac{a}{a+b}$

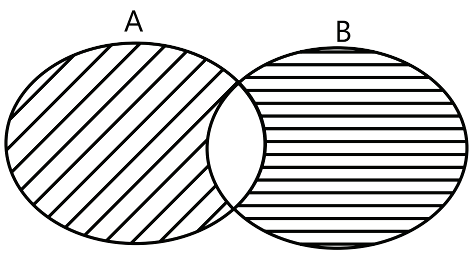
(d) If odds against the event E = a: b then $P(E) = \dfrac{b}{a+b}$

8. A FEW THEOREMS ON PROBABILITY

(a) If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

(b) If A is any event, then $P(A') = 1 - P(A)$

(c) If A and B are two events, then $P(A \cap B') = P(A) - P(A \cap B)$



(d) If A and B are two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(e) If A and B are two events, then

P(exactly one of A, B occurs)

$= P[(A \cap B') \cup (A' \cap B)] = P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$

Also, P(exactly one of A, B occurs)

$= P(A \cap B') + P(A' \cap B) = P(B') - P(A' \cap B') + P(A') - P(A' \cap B') = P(A') + P(B') - 2P(A' \cap B')$

$= P(A' \cup B') - P(A' \cap B')$

(f) If A and B are two events, $P(A' \cup B') = 1 - P(A \cap B)$ and $P(A' \cap B') = 1 - P(A \cup B)$

(g) If $A_1, A_2, \ldots, A_n$ are n events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n)$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} p(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap Ak) - \ldots + (-1)^{n-1} P(A_1 \cap A_2) \ldots \cap A_n)$$

(h) If A, B and C are three events, then

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + (A \cap B \cap C)$

(i) P(at least two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(ii) P(exactly two of A, B, C occur)$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iii) P(exactly one of A, B, C occurs)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

(i) If $A_1, A_2, \ldots, A_n$ are n events, then

(i) $P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n)$

(ii) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge 1 - P(A'_1) - P(A'_2) - \ldots - P(A'_n)$

(j) If $A_1, A_2, \ldots, A_n$ are n events, then $P(A_1 \cap A_2 \cap \ldots \cap A_n)^3 P(A_1) + P(A_2) + \ldots + P(A_n) - (n-1)$

(k) If A and B are two events, such that $A \subseteq B$, then $P(A) \le P(B)$

9. BOOLE'S INEQUALITY

(a) For any two events A and B

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(ii) $\therefore P(A \cup B) \le P(A) + P(B)$ $\{\because P(A \cap B) \ge 0\}$

(b) In general for any n events $A_1 A_2 \ldots A_n$

(i) $P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n)$

Illustration 4: Let A, B, C be three events. If the probability of the occurrence of one event out of A and B is 1 –a , out of B and C is 1 – 2a, out of C and A is 1 – a and that of occurrence ofthree events simultaneously is a^2, then prove that the probability that at least one event out of A, B, C will occur is greater than or equal to 0.5. **(JEE ADVANCED)**

Sol: Apply Boole's Inequality.

Probability that exactly one event out of A and B occur is $P(A) + P(B)=2P(A \cap B)$ and probability that exactly one event out of B and C occur is $P(B) + P(C) - 2P(B \cap C)$ and so on.

Now, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$= \frac{1}{2}[P(A)+P(B) - 2P(A\cap B) + P(B) + P(C) - 2P(B\cap C) + P(C) + P(A) - 2P(A\cap C)] + P(A\cap B\cap C)$

$\Rightarrow \frac{1-a+1-2a+1-a}{2} + a^2 = a^2 - 2a + \frac{3}{2}$. Let,$a^2 - 2a + \frac{3}{2} = y \Rightarrow a^2 - 2a + \frac{3}{2} - y = 0$

Since a is real, so $4 - 4\left(\frac{3}{2} - y\right) \ge 0 \Rightarrow y \ge \frac{1}{2}$

Illustration 5: From a pack of 52 cards, two cards are drawn at random. Find the probability of the following events:

(A) Both cards are of spade.

(B) One card is of spade and one card is of diamond. **(JEE MAIN)**

Sol: Use combination to calculate the number of favorable ways and the total number of ways in both cases.

The total number of ways in which 2 cards can be drawn $= {}^{52}C_2 = \frac{52\times 51}{1\times 2} = 26 \times 51 = 1326$

$\therefore$ Number of elements in the space S are n(S) = 1326

(A) Let the event that both cards are of spade be denoted by E_1.Then,$n(E_1)$ = Number of elements in E_1 = Number of ways in which 2 cards can be selected out of 13 cards of spade $= {}^{13}C_2 = \frac{13\times 12}{1\times 2} = 78$.

$\therefore$ Probability of $E_1 = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{78}{1326} = \frac{1}{17}$.

(B) Let E_2 be the event that one card is of spade and one is of diamond. Then, $n(E_2)$ = number of elements in E_2 = number of ways in which one card of spade can be selected out of 13 spade cards and one card of diamond can be selected out of 13 diamond cards. $= {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 = 169$

$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{169}{1326} = \frac{13}{102}$.

Illustration 6: Two numbers x and y are chosen at random from the set {1, 2, 3,..., 3n}. Find the probability that $x^2 - y^2$ is divisible by 3. **(JEE ADVANCED)**

Sol: Divide the above given set in three subsets such that the difference of any two elements in any of these three sets is divisible by 3. Use this partition of set to find the answer.

$x^2 - y^2 = (x + y)(x - y)$ and 3 is a prime number.

$\therefore$ $x^2 - y^2$ is divisible by 3 if x + y or x – y is divisible by 3.

Now, $\{1, 2, 3,..., 3n\} = \{3, 6, 9,..., 3n\} \cup \{1, 4, 7,..., 3n - 2\} \cup \{2, 5, 8,..., 3n - 1\} = A \cup B \cup C$ (say).

Clearly, if x, y are selected from A or B or C then x + y or x – y are divisible by 3; and, if x, y are selected one from B and the other from C then x + y is divisible by 3.

$\therefore$ The probability of $x^2 - y^2$ is divisible by 3

= Probability of selecting both x, y from A or B or C + probability of selecting x, y one from B and the other from C

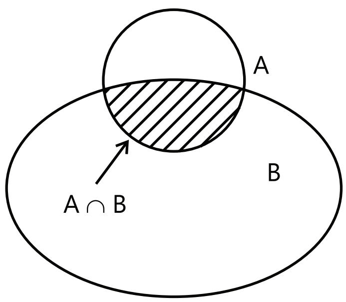
$$= \frac{{}^{n}C_2}{{}^{3n}C_2} \times 3 + \frac{{}^{n}C_1 \times {}^{n}C_1}{{}^{3n}C_2} = \frac{3n(n-1)}{3n(3n-1)} + \frac{2n^2}{3n(3n-1)} = \frac{3n-3}{3(3n-1)} + \frac{2n}{3(3n-1)} = \frac{5n-3}{3(3n-1)}.$$

10. CONDITIONAL PROBABILITY

The probability of occurrence of an event A, given that B has already occurred is called the conditional probability of occurrence of A. It is denoted by P(A | B).If the event B has already occurred, then the sample space reduces to B. Not the outcome favorable to the occurrence of A (given that B has already occurred) are those that are common to both A and B, that is, those which belong to $A \cap B$.

Thus, $P(A \mid B) = \frac{N_{A \cap B}}{N_B}$

where $N_{A \cap B}$ is the number of elements in $A \cap B$ and $N_B \neq 0$ is the number of elements in B and N the total number of elements in S. $\Rightarrow P(A \mid B) = \frac{N_{A \cap B}/N}{N_B/N} = \frac{P(A \cap B)}{P(B)}$



$$\text{Hence, } P(A \cap B) = \begin{cases} P(B)\,P(A \mid B) & \text{if } P(B) \neq 0 \\ P(A)\,P(B \mid A) & \text{if } P(A) \neq 0 \end{cases}$$

MASTERJEE CONCEPTS

Trick to solve conditional probability

The trick is to identify when a probability is a conditional probability (in a word problem)

When dealing with conditional probability, the difference is that, we know for certain that something else has already happened.

This means that in our definition of probability that says

$$P(E) = \frac{\text{Number of ways for something to Happen}}{\text{Total Number of Ways}}$$

$$P(A/(B) = \frac{P(A \cap B)}{P(B)}$$

Consider events A and B.

If A and B are independent events, then P(A/B) = P(A) and P(B/A) = P(B). Therefore, $P(A \cap B) = P(A)\,P(B)$

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)P(B) = 1 - (1 - P(A) - P(B) + P(A)\,P(B)] = 1 - [(1 - P(\bar{A}))\,(1 - P(\bar{B}))] = 1 - P(\bar{A})\,P(\bar{B})$

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 7: If m is a natural such that $m \leq 5$, then the probability that the quadratic equation $x^2 + mx + \frac{1}{2} + \frac{m}{2} = 0$ has real roots is **(JEE MAIN)**

(A) 1/5 (B) 2/3 (C) 3/5 (D) 2/5

Sol: Apply Discriminant ≥ 0.

Discriminant D of the quadratic equation$x^2 + mx + \frac{1}{2} + \frac{m}{2} = 0$

$$D = m^2 - 4\left(\frac{1}{2} + \frac{m}{2}\right) = m^2 - 2m - 2 = (m-1)^2 - 3$$

Now, $D \geq 0 \quad \Leftrightarrow \quad (m-1)^2 \geq 3$

This is possible for m = 3, 4 and 5. Also, the total number of ways of choosing m is 5.

∴ Probability of the required event = 3/5.

11. PROBABILITY OF AT LEAST ONE OF THE N INDEPENDENT EVENTS

If $P_1, P_2, \ldots P_n$ are the probabilities of n independent events $A_1, A_2, A_3 \ldots A_n$ then the probability that at least one of these events will happen is $1 - [(1 - P_1)(1 - P_2) \ldots (1 - P_n)]$

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = 1 - P(\bar{A}_1)\,P(\bar{A}_2)\,P(\bar{A}_3 \ldots P(\bar{A}_n)$$

Illustration 8: A mathematics problem is given to three students A, B and C whose chances of solving it are 1/2, 1/3, 1/4 respectively. Then the probability that the problem is solved is **(JEE MAIN)**

Sol: Apply the principle of probability for independent events.

Obviously, the events of solving the problem by A, B and C are independent. Therefore required probability

$$= 1 - \left[\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\right] = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{3}{4}$$

(1) Multiplication theorems on probability

(i) If A and B are two events associated with a random experiment:

then $P(A \cap B) = P(A).P(B/A)$, If $P(A) \neq 0$ or $P(A \cap B) = P(B).P(A/B)$, if $P(B) \neq 0$.

(ii) Extension of multiplication theorem: If $A_1, A_2, \ldots, A_n$ are n events related to a random experiment, then

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2 / A_1)\,P(A_3 / A_1 \cap A_2)$

$\ldots P(A_n / A_1 \cap A_2 \cap \ldots \cap A_{n-1})$where $P(A_i / A_1 \cap A_2 \cap \ldots \cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already happened.

(iii) Multiplication theorem for independent events: If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A).P(B)$ i.e., the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A \cap B) = P(A).P(B/A)$. Since A and B are independent events, therefore $P(B/A) = P(B)$. Hence, $P(A \cap B) = P(A).P(B)$.

(iv) Extension of multiplication theorem for independent events: If $A_1, A_2, \ldots A_n$ are independent events associated with a random experiment, then $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$.

By multiplication theorem, we have

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2 / A_1)\,P(A_3 / A_1 \cap A_2) \ldots P(A_n / A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

Since $A_1, A_2, \ldots, A_{n-1}, A_n$ are independent events, therefore

$$P(A_2 / A_1) = P(A_2), P(A_3 / A_1 \cap A_2) = P(A_3), \ldots\ldots,$$

$$P(A_n / A_1 \cap A_2 \cap \ldots\ldots \cap A_{n-1}) = P(A_n)$$

Hence, $P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n) = P(A_1)P(A_2)\ldots\ldots P(A_n)$

Illustration 9: The probability that a married man watches a certain T.V. show is 0.4 and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find **(JEE MAIN)**

(A) the probability that married couples watch the show

(B) the probability that a wife watches the show given that her husband does.

(C) the probability that at least one person of a married couple will watch the show.

Sol: Refer to Multiplication theorems on probability.

Let 'H' be the event that a married man watches the show and 'W' be the probability that a married woman watches the show,

$\Rightarrow P(H) = 0.4,\ P(W) = 0.5,\ P(H/W) = 0.7$

(A) $P(H \cap W) = P(W).P(H/W) = 0.5 \times 0.7 = 0.35$

(B) $P(W/H) = \dfrac{P(H \cap W)}{P(H)} = \dfrac{0.35}{0.4} = \dfrac{7}{8}$

(C) $P(H \cup W) = P(H) + P(W) - P(H \cap W) = 0.4 + 0.5 - 0.35 = 0.55$

Illustration 10: Consider the sample space 'S' representing the adults in a small town who have completed the requirements for a college degree. They have been categorized according to sex and employment as under: **(JEE MAIN)**

	Employed	Unemployed
Male	460	40
Female	140	260

An employed person is selected at random. Find the probability that the chosen person is male.

Sol: Same as previous illustration.

Let M be the event that a man is chosen and E be the event that the chosen one is employed.

From the concept of reduced sample space we immediately get, $P(M/E) = \dfrac{460}{600} = \dfrac{23}{30}$

Also, $P(E) = \dfrac{600}{900} = \dfrac{2}{3}$; $\quad P(E \cap M) = \dfrac{460}{900} = \dfrac{23}{45} \Rightarrow P(M/E) = \dfrac{23/45}{2/3} = \dfrac{23}{30}$

Illustration 11: A bag contains 3 white balls and 2 black balls, another contains 5 white and 3 blackballs. If a bag is chosen at random and a ball is drawn from it, what is the probability that it is white? **(JEE MAIN)**

Sol: Consider two cases. Case I – When the ball is chosen from the first bag and Case II – When the ball is chosen from the second bag.

The probability that the first bag is chosen is 1/2 and the chance of drawing a white ball from it is 3/5.

$\therefore$ Chance of choosing the first bag and drawing a white ball is 1/2, 3/5 respectively

Similarly the chance that the second bag is chosen and a white ball is drawn is 1/2, 5/8 respectively

$\therefore$ The chance of randomly choosing a bag and drawing a white ball is

$= \dfrac{1}{2}.\dfrac{3}{5} + \dfrac{1}{2}.\dfrac{5}{8}$ (Mutually exclusive cases) = 49/80.

Illustration 12: Find the probability that a year chosen at random has 53 Sundays. **(JEE MAIN)**

Sol: Divide the solution in two parts, when the year is a leap year and otherwise.

Let P(L) $\rightarrow$ be the probability that a year chosen at random is a leap year P(L) = 1/4.

$\therefore$ $P(\bar{L}) = 3/4$

Let P(S) $\rightarrow$ be the probability that a year chosen at random has 53 Sundays.

$\therefore$ $P(S) = P(L) . P(S/L) + P(\bar{L}). P(S/\bar{L})$

Now, P(S/L) is the probability that a leap year has 53 Sundays.

A leap year has 366 days, 52 weeks + the remaining 2 days may be Sunday-Monday, M-T, T-W, W-Th, Th-F, F-Sat or Sat –Sunday,

Out of the 7 possibilities, 2 are favorable

$\therefore$ $P(S/L) = \frac{2}{7}$. Similarly $P(S/\bar{L}) = \frac{1}{7}$

$\therefore$ $P(S) = \frac{1}{4}.\frac{2}{7} + \frac{3}{4}.\frac{1}{7} = \frac{5}{28}$

Theorem of total probability: If $E_1, E_2, \ldots E_n$ are mutually exclusive and exhaustive events such that $P(E_i) \neq 0$ for each i and A is an event, then $P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + P(E_n)\, P(A|E_n)$

Bayes' Theorem: If $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events such that

$P(E_i) > 0$ $(1 \le i \le n)$ and A is an event, then for $1 \le k \le n$,

$$P(E_k|A) = \frac{P(E_k)P(A|E_k)}{P(E_i)P(A|E_i) + P(E_2)P(A|E_2) + \ldots + P(E_n)P(A|E_n)}$$

The probabilities $P(E_j)$ $(1 \le j \le n)$ are called '*a priori probabilities*' and conditional probabilities

$P(E_j|A)$ are known as '*posteriori probabilities*'. Two events are said to be independent if occurrence (non-occurrence) of one does not affect the probability of occurrence (non occurrence) of the other i.e. $P(B|A) = P(B)$

Bayes' Theorem: The probability of event A, given that event B has subsequently occurred, is

$$P(A|B) = \frac{P(A).P(B|A)}{[P(A).P(B|A)] + [P(\bar{A}).P(B|\bar{A})]}$$

This is a direct result from condition probability and theorem of total probability. In general we can write Bayes' theorem as $P(A_i|B) = \dfrac{P(B|A_i) \times P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)}$

MASTERJEE CONCEPTS

Generally, Bayes' theorem is remembered as a formula, and whenever students encounter an inverse probability problem, they try to apply that formula without in-depth analysis of that problem. In our opinion, in any such problem, you should always draw a probability tree corresponding to the situation described. This will always give you more insight into the problem than a direct application of the formula and it may even prevent you from obtaining wrong results.

The method of probability tree diagrams has been discussed later.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 13: One bag contains four white balls and three black balls. The second bag contains three white balls and five black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black? **(JEE ADVANCED)**

Sol: Use Total Probability theorem. Consider two events

A_1 – When white ball is transferred from bag I to II, and

A_2 – When black ball is transferred from bag I to II and proceed.

Bag-I	Bag-II
4W	3W
3B	5B

Let A_1 be the event that a white ball is transferred from bag-I to bag-II and A_2 be the event that a black ball is transferred from bag-I to bag-II.

$P(A_1) = \frac{4}{7}$, $P(A_2) = \frac{3}{7}$

Let 'A' be the probability that finally a black ball is drawn from the second bag

$P(A/A_1) = \frac{5}{9}$, $P(A/A_2) = \frac{6}{9}$

Now from total probability theorem we get, $P(A) = P(A_1).P(A/A_1) + P(A_2).P(A/A_2) = \frac{4}{7}.\frac{5}{9}+\frac{3}{7}.\frac{6}{9}=\frac{38}{63}$

Illustration 14: A real estate man has eight master keys to open several new homes. Only one master key will open any given house. If 40% of these homes are usually left unlocked, what is the probability that the real estate man can get into a specific home if he selects three master keys at random before leaving the office? **(JEE ADVANCED)**

Sol: Use Total Probability theorem. Let A_1 and A_2 be the events that the specific home is left unlocked and is left locked respectively

$\Rightarrow P(A_1) = 0.4$, $P(A_2) = 0.6$

Let 'A' be the event that the real estate man get into the specific home $P(A/A_1) = 1$,

$P(A/A_2) = \frac{^7C_2}{^8C_3}=\frac{3}{8} \Rightarrow P(A) = P(A_1)\ P(A/A_1) + P(A_2)\ P(A/A_2) = (0.4)(1) + (0.6)\ (3/8) = \frac{4}{10}+\frac{18}{80}=\frac{5}{8}.$

Illustration 15: A bag 'A' contains 2 white balls and 3 red balls, a bag 'B' contains 4 white and 5 blackballs. A bag is selected at random and a ball is drawn from it. Drawn ball is observed to be white. Find the probability that bag 'B' was selected. **(JEE ADVANCED)**

Sol: Take two cases, when bag A is selected and another when bag B is selected.

Bag A	Bag B
2W, 3R	4W, 5B

Let A_1 be the event that bag 'A' is selected and A be the event that bag B is selected

$P(A_1) = P(A_2) = 1/2$

Let 'A' be the event that a white ball is drawn from the selected bag.

$\Rightarrow P(A/A_1) = 2/5$, $P(A/A_2) = \frac{4}{9}$

$P(A) = P(A_1)\ .\ P(A/A_1) + P(A_2)\ .P(A/A_2)= \frac{1}{2}.\frac{2}{5}+\frac{1}{2}.\frac{4}{9}=\frac{1}{2}\left(\frac{2}{5}+\frac{4}{9}\right)=\frac{38}{90}$

$= \text{Finally, } P(A_2/A) = \dfrac{P(A_2).P(A/A_2)}{P(A)} = \dfrac{(1/2).(4/9)}{(38/90)} = \dfrac{90\times 4}{18\times 38} = \dfrac{10}{19}$

Illustration 16: A card from a pack of 52 cards is lost. From the remaining cards, two cards are drawn and are found to be spades. Find the probability that the missing card is also a spade. **(JEE ADVANCED)**

Sol: Take two cases, when the missing card is a spade or a non-spade. Let A_1 be the event that missing card is spade and A_2 be event that missing card is non-spade.

$\Rightarrow P(A_1) = \dfrac{1}{4} . P(A_2) = \dfrac{3}{4}$

Let 'A' be the event that 2 spade cards are drawn from the remaining cards,

$$P\left(\frac{A}{A_1}\right) = \frac{{}^{12}C_2}{{}^{51}C_2} \text{ and } P\left(\frac{A}{A_2}\right) = \frac{{}^{13}C_2}{{}^{51}C_2};\ P(A) = P(A_1) . P\left(\frac{A}{A_1}\right) + P(A_2).P\left(\frac{A}{A_2}\right)$$

$$= \frac{1}{4}\frac{{}^{12}C_2}{{}^{51}C_2} + \frac{3}{4}\frac{{}^{13}C_2}{{}^{51}C_2} = \frac{1}{4.{}^{51}C_2}\left[{}^{12}C_2 + 3.{}^{13}C_2\right]$$

$$\text{Now, } P\left(\frac{A_1}{A}\right) = \frac{P(A_1)P\left(\frac{A}{A_1}\right)}{P(A)} = \frac{\frac{1}{4}\frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4.{}^{51}C_2}\left[{}^{12}C_2 + 3.{}^{13}C_2\right]} = \frac{11}{50}$$

12. BINOMIAL DISTRIBUTIONS

Binomial distributions occur in relation to those experiments that are binary in nature, i.e. whose outcomes can be grouped into two classes, say, success and failure, or, say 1 and 0. For example, when you toss a coin, there are only two outcomes possible: Heads (which you may call success) and Tails (which then becomes Failure).

MASTERJEE CONCEPTS

Note that an experiment need not have only two outcomes for it to be called binary. For example, consider the experiment of rolling a die.If youmake the following definitions-

Success: Numbers 1, 2 or 3

Failure: Numbers 4, 5 and 6

Then, with respect to this definition, the experiment is binary. Thus, an experiment needs to have two **classes of outcomes** for it to be called binary.

Let us consider a binomial experiment which has been repeated 'n' times. Let the probability of success and failure in any trial be p and q respectively. We are interested in the probability of occurrence of exactly 'r' successes in these n trials. Now, number of ways of choosing 'r' success in 'n' trials = ${}^nC_r$.

Probability of 'r' successes and (n-r) failures is $p^r.q^{n-r}$. Thus probability of having exactly r successes = ${}^nC_r . p^r.q^{n-r}$

Let 'X' be a random variable representing the number of successes, then

$P(X = r) = {}^nC_r . p^r.q^{n-r} (r = 0, 1, 2, ..., n)$

$1 = (p + q)^n = {}^nC_0\, p^0q^n + {}^nC_1\, p^1\, q^{n-1} + {}^nC_2\, p^2q^{n-2} + ... + {}^nC_r\, p^r\, q^{n-r} + ... + {}^nC_r\, p^nq^0$

$X \rightarrow$ Number of successes 0, 1, 2, ... r, ... N

MASTERJEE CONCEPTS

- Probability of at most 'r' successes in n trials = $\sum_{r=0}^{r} {}^nC_r p^r q^{n-r}$
- Probability of at least 'r' successes in n trials = $\sum_{r=r}^{n} {}^nC_r p^r . q^{n-r}$
- Probability of having 1st success at the rth trial = $p.q^{r-1}$

Shivam Aggarwal (JEE 2009, AIR 27)

13. BINOMIAL PROBABILITY DISTRIBUTION

(a) A probability distribution spells out how a total probability is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by $\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i$ (Since $\Sigma p_i = 1$)

(c) Variance of a random variable is given by, $\sigma^2 = \Sigma(x_i - \mu)^2 p_i$

$\therefore \sigma^2 = \Sigma p_i x_i^2 - \mu^2$ (Note that SD = $+\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variable 'X' is given by $P(X = r) = {}^nC_r\, p^r\, q^{n-r}$ where $p(X = r)$ is the probability of r successes.

The recurrence formula $\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}.\frac{p}{q}$ is very helpful for computing P(1). P(2) . P(3) etc. quickly, if P(0) is known.

(e) Mean of BPD = np ; Variance of BPD = npq.

(f) If P represents a person's chance of success in any venture and 'M' represents the sum of only what he will receive in case of success, then his expectations of probable value = PM.

14. MODE AND MEDIAN

Usually the mode of a binomial B(n, p) distribution is equal to $\left[(n+1)p\right]$, where[.] is the greater integer function. However when (n + 1)p is an integer and p is neither 0 nor 1, then the distribution has two modes: (n + 1)p and (n + 1)p − 1. When p is equal to 0 or 1, the mode will be 0 and n correspondingly. These cases can be summarized as follows:

$$\text{mode} = \begin{cases} \left[(n+1)p\right] & \text{if } (n+1)p \text{ is 0 or a non integer,} \\ (n+1)p \text{ and } (n+1)p-1 & \text{if } (n+1)p \in \{1,\ldots,n\}, \\ n & \text{if } (n+1)p = n+1 \end{cases}$$

In general, there is no single formula to find the median for a binomial distribution and it may even be non-unique. However several special results have been established:

(a) If np is an integer, then the mean, median, and mode coincide and equal np.

(b) When p = 1/2 and n is odd, any number m in the interval $\frac{1}{2}(n-1) \le m \le 1/2(n+1)$ is a median of the binomial distribution. If p = 1/2 and n is even, then m = n/2 is the unique median.

PROBLEM SOLVING TACTICS

Following are some extra methods which may be useful to solve probability questions:

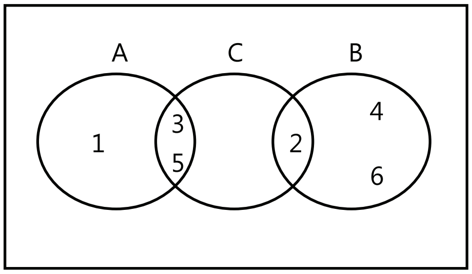
Venn Diagrams: It is a diagram in which the sample space is represented by a rectangle and the element of the sample space by points within it. Subsets (or events) of the sample space are represented by the region within the rectangle, usually using circles.

For example, consider the following events when a die is thrown,

A = {odd numbers} = {1, 3, 5}

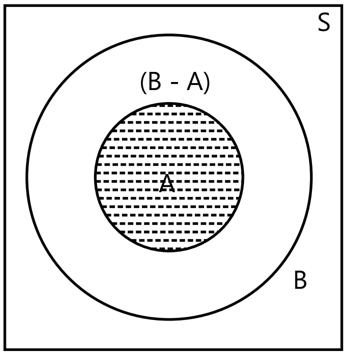
B = {even numbers} = {2, 4, 6}

C = {prime numbers} = {2, 3, 5}



Let us see how Venn diagrams are to be applied by using them to prove some results as follows:

Theorem 1: For any two events A and B, $A \subseteq B \Rightarrow P(A) \leq P(B)$.



Proof. From the adjoining diagram, we have

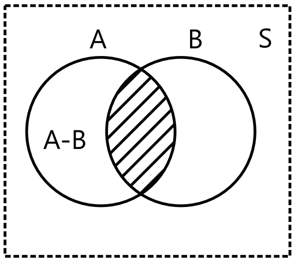
$A \cup (B - A) = B$ and $A \cap (B - A) = f$

$\therefore P(B) = P[A \cup (B - A)]$ [$\because$ $A \cap (B - A) = f$]

$\Rightarrow P(B) = P(A) + P(B - A)$ [$\because$ $P(B - A) \geq 0$]

$\Rightarrow P(A) \leq P(B)$ **Proved**

Theorem 2: For any two events A and B, $P(A - B) = P(A) - P(A \cap B)$



Proof: Let A and B be two compatible events. Then $A \cap B \neq \phi$. From the adjoining Venn diagram. it is clear that:

$(A - B) \cap (A \cap B) = f$and $(A - B) \cup (A \cap B) = A$

$\Rightarrow P(A - B) + P(A \cap B) = P(A)$

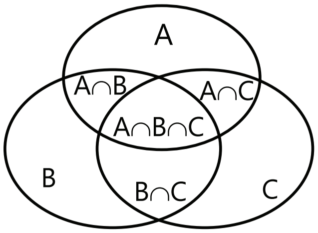
$\Rightarrow P(A - B) = P(A) - P(A \cap B)$ **Proved**

Remarks: This result may be expressed as

$P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Theorem 3: For any three events A, B, C



$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Proof: We have ,$P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

$= P(A \cup B) + P(C) - P[(A \cap C) \cup (B \cap C)]$ (Distributive Law)

$= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap B) \cup (C \cap C)]$ [Addition law]

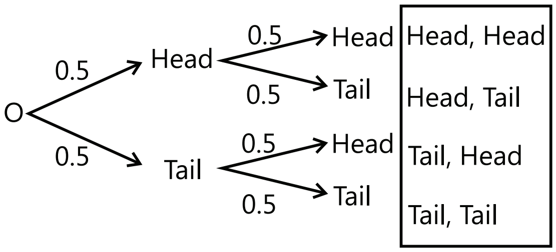
$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)]$

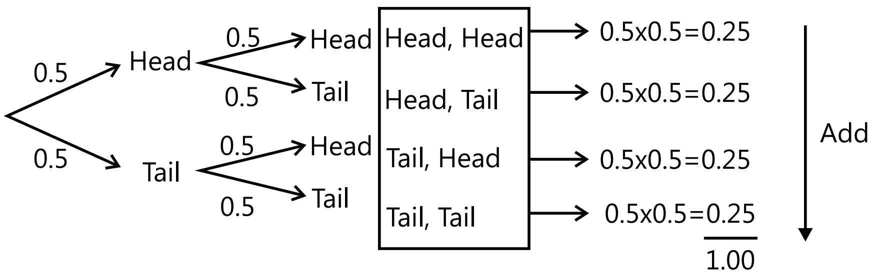
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) + P(B \cap C) + P[A \cap B \cap C]$ **Proved**

Probability Tree Diagrams: Calculating probabilities can be hard. Sometimes you add them, sometimes you multiply them and often, it is hard to figure out what to do. That's when **tree diagrams come to the rescue!**

Here is a tree diagram for two tosses of a coin:



How do you calculate the overall probabilities?



So, there you go. When in doubt, draw a tree diagram, multiply along the branches and add the columns. Make sure all probabilities add to 1 and you are good to go!

FORMULAE SHEET

(a) Mathematical definition of probability:

$$\text{Probability of an event} = \frac{\text{Number of favorable cases to event A}}{\text{Total Number of cases}}$$

Note: (i) $0 \leq P(A) \leq 1$

(ii) Probability of an impossible event is zero

(iii) Probability of a sure event is one.

(iv) $P(A) + P(\text{Not } A) = 1$ i.e. $P(A) + P(\bar{A}) = 1$

(b) Odd for an event: If $P(A) = \frac{m}{n}$ and $P(\bar{A}) = \frac{n-m}{n}$

Then odds in favor of A $= \frac{P(A)}{P(\bar{A})} = \frac{m}{n-m}$ and odd in against of A $= \frac{p(\bar{A})}{P(A)} = \frac{n-m}{m}$

(c) Set theoretical notation of probability and some important results:

(i) $P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

(iii) $P(A \cup B) = P(A \cap B) + P(\bar{A} \cap B) + P(A \cap \bar{B})$

(iv) $A \subseteq B \Rightarrow P(A) \le P(B)$

(v) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(vi) $P(A \cap B) \le P(A)\ P(B) \le P(A \cup B) \le P(A) + P(B)$

(vii) P(Exactly one event) = $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(viii) $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = P(A) + P(B) - 2P(A \cap B) = P(A + B) - P(A \cap B)$

(ix) P(neither A nor B) = $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

(x) When a coin is tossed n times or n coins are tossed once, the probability of each simple event is $\frac{1}{2^n}$

(xi) When a dice is rolled n times or n dice are rolled once, the probability of each simple event is $\frac{1}{6^n}$

(xii) When n cards are drawn ($1 \le n \le 52$) from well shuffled deck of 52 cards, the probability of each simple event is $\frac{1}{{}^{52}C_n}$.

(xiii) If n cards are drawn one after the other with replacement, the probability of each simple event is $\frac{1}{(52)^n}$

(xiv) P(none) = 1 – P (at least one)

(xv) Playing cards

- Total cards: 52 (26 red, 26 black)
- Four suits: Heart, diamond, spade, club (13 cards each)
- Court (face) cards: 12 (4 kings, 4 queens, 4 jacks)
- Honor cards: 16 (4 Aces, 4 kings, 4 queens, 4 Jacks)

(xvi) Probability regarding n letters and their envelopes:

If n letters corresponding to n envelopes are placed in the envelopes at random, then

- Probability that all letters are in the right envelopes = $\frac{1}{n!}$
- Probability that all letters are not in the right envelopes = $1 - \frac{1}{n!}$
- Probability that no letter is in the right envelope = $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots.. + (-1)^n \frac{1}{n!}$
- Probability that r letters are in the right envelope = $\frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$

(d) Addition Theorem of Probability:

(i) When events are mutually exclusive

i.e. $n(A \cap B) = 0 \qquad \Rightarrow P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

(ii) When events are not mutually exclusive i.e. $P(A \cap B) \ne 0$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or $P(A + B) = P(A) + P(B) - P(AB)$

(iii) When events are independent i.e. $P(A \cap B) = P(A)\ P(B)$

$\therefore P(A + B) = P(A) + P(B) - P(A)\ P(B)$

(e) Conditional probability:

$P(A/B)$ = Probability of occurrence of A, given that B has already happened $= \dfrac{P(A \cap B)}{P(B)}$

$P(B/A)$ = Probability of occurrence of B, given that A has already happened $= \dfrac{P(A \cap B)}{P(A)}$

Note: If the outcomes of the experiment are equally likely, then

$$P(A/B) = \frac{\text{Number of sample points in } A \cap B}{\text{Number of points in } B}$$

(i) If A and B are independent events, then $P(A/B) = P(A)$ and $P(B/A) = P(B)$

(ii) Multiplication Theorem:

$P(A \cap B) = P(A/B).\ P(B),\ P(B) \neq 0$ or $P(A \cap B) = P(B/A)\ P(A),\ P(A) \neq 0$

Generalized: $P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n)$

$= P(E_1)\ P(E_2/E_1)\ P(E_3/E_1 \cap E_2)\ P(E_4/E_1 \cap E_2 \cap E_3) \ldots$ If events are independent, then

$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1)\ P(E_2) \ldots P(E_n)$

(f) Probability of at least one of the n Independent events: If $P_1, P_2, \ldots P_n$ are the probabilities of n independent events $A_1, A_2, \ldots A_n$ then the probability that at least one of these events will happen is $1 - [(1 - P_1)\ (1 - P_2) \ldots (1 - P_n)]$

or $P(A_1 + A_2 + \ldots + A_n) = 1 - P\ (\bar{A}_1)\ P(\bar{A}_2) \ldots P(\bar{A}_n)$

(g) Total probability: Let $A_1, A_2, \ldots A_n$ be n mutually exclusive & set of exhaustive events. If event A can occur through any one of these events, then the probability of occurrence of A

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \ldots + P(A \cap A_n) = \sum_{r=1}^{n} P(A_r) P(A / A_r)$$

(h) Bayes' Rule: Let A_1, A_2, A_3 be any three mutually exclusive & exhaustive events (i.e. $A_1 \cup A_2 \cup A_3$ = sample space & $A_1 \cap A_2 \cap A_3 = \phi$) of a sample space S and B is any other event on sample space then,

$$P(A_i/B) = \frac{P(B / A_i)(P(A_i)}{P(B / A_1)\ P(A_1) + P(B / A_2)P(A_2) + P(B / A_3)P(A_3)},\ i = 1, 2, 3$$

(i) Probability distribution:

(i) If a random variable x assumes values $x_1, x_2, \ldots x_n$ with probabilities $P_1, P_2, \ldots P_n$ respectively then

- $P_1 + P_2 + P_3 + \ldots + P_n = 1$
- Mean $E(x) = \Sigma P_i x_i$
- Variance $= \sum x^2 P_i - (\text{mean})^2 = \sum (x^2) - (E(x))^2$

(ii) Binomial distribution: If an experiment is repeated n times, the successive trials being independent of one another, then the probability of r success is ${}^nC_r\ P^r\ q^{n-r}$at least r success is $\sum_{k=r}^{n} {}^nC_k P^k q^{n-k}$ where p is probability of success in a single trial, $q = 1 - p$

- Mean $E(x) = np$
- $E(x2) = npq + n2p2$
- Variance $E(x2) - (E(x))2 = npq$
- Standard deviation $= \sqrt{npq}$

(j) Truth of the statement:

(i) If two persons A and B speak the truth with probabilities P_1& P_2 respectively and if they agree on a statement, then the probability that they are speaking the truth will be given by $\dfrac{P_1P_2}{P_1P_2+(1-P_1)(1-P_2)}$.

(ii) If A and B both assert that an event has occurred, the probability of occurrence of which is α, then the probability that the event has occurred $\dfrac{\alpha P_1P_2}{\alpha P_1P_2+(1-\alpha)(1-P_1)(1-P_2)}$ given that the probability of A & B speaking truth is $p_{1,}p_2$ respectively.

(iii) If in the second part, the probability that their lies coincide is β, then from the above case, the required probability will be $\dfrac{\alpha P_1P_2}{\alpha P_1P_2+(1-\alpha)(1-P_1)(1-P_2)\beta}$

Solved Examples

JEE Main/Boards

Example 1: If there are two events A and B such that P(A′) = 0.3 P(B) = 0.5 and P(A ∩ B) = 0.3, then P(B|A ∪ B′) is:

(A) 3/8 (B) 2/3 (C) 5/6 (D) 1/4

Sol: Use set theory and probability of complimentary events to calculate P(A ∪ B′)

We have P(A ∪ B′)

= P(A) + P(B′) – P(A ∩ B′)

= [1 – P(A′)] + [1 – P(B)] – [P(A) – P(A ∩ B)]

= (1 – 0.3) + (1 – 0.5) – (0.7 – 0.3) = 0.8

Now, $P(B|A \cup B') = \dfrac{P[B\cap(A\cup B')]}{P(A\cup B')}$

$= \dfrac{P[(B\cap A)\cup(B\cap B')]}{P(A\cup B')} = \dfrac{P(A\cap B)}{P(A\cup B')} = \dfrac{0.3}{0.8} = \dfrac{3}{8}$

Example 2: Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals:

(A) 1/2 (B) 7/15 (C) 2/15 (D) 1/3

Sol: Each black balls can be arranged in between any two white balls. Us e this idea to find the number of ways in which no two black balls are together.

The number of ways of placing 3 black balls at 10 places is $^{10}C_3$. The number of ways in which two black balls are not together is equal to the number of ways of choosing 3 places marked with X out of eight places

X W X W X W X W X W X W X W X

This can be done in $^{8}C_3$ ways. Thus, probability of the required event is $\dfrac{^8C_3}{^{10}C_3} = \dfrac{8\times7\times6}{10\times9\times8} = \dfrac{7}{15}$

Example 3: A group of 2n boys and 2n girls is randomly divided into two equal groups. The probability that catch contains the same number of boys and girls is:

(A) 1/2 (B) 1/n

(C) 1/2n (D) None of these

Sol: If one group is selected the second group automatically gets created. Hence, select n boys and n girls from the given group.

Total number of ways of choosing a group is $^{4n}C_{2n}$ The number of ways in which each group contains equal number of boys and girls is $(^{2n}C_n)$ $(^{2n}C_n)$

∴ Required probability = $\dfrac{(^{2n}C_n)^2}{^{4n}C_{2n}}$.

Example 4: Let A and B be two events such that P(A) = 0.3 and P(A ∪ B) = 0.8. If A and B are independent events, then P(B) is:

(A) 3/7 (B) 4/7 (C) 5/7 (D) 6/7

Sol: If say A and B are two independent events then P(A ∩ B) = P(A) × P(B)