PROBLEM SOLVING TACTICS

Following are some extra methods which may be useful to solve probability questions:

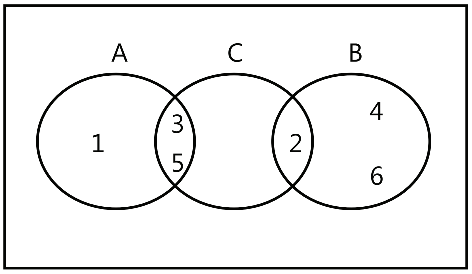
Venn Diagrams: It is a diagram in which the sample space is represented by a rectangle and the element of the sample space by points within it. Subsets (or events) of the sample space are represented by the region within the rectangle, usually using circles.

For example, consider the following events when a die is thrown,

A = {odd numbers} = {1, 3, 5}

B = {even numbers} = {2, 4, 6}

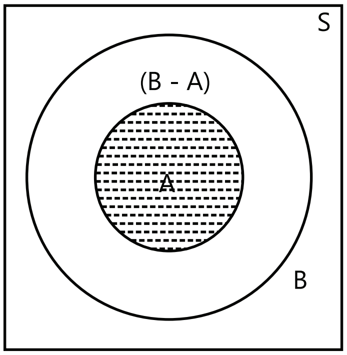
C = {prime numbers} = {2, 3, 5}



Let us see how Venn diagrams are to be applied by using them to prove some results as follows:

Theorem 1: For any two events A and B, $A \subseteq B \Rightarrow P(A) \le P(B)$.

Proof. From the adjoining diagram, we have



$A \cup (B - A) = B$ and $A \cap (B - A) = f$

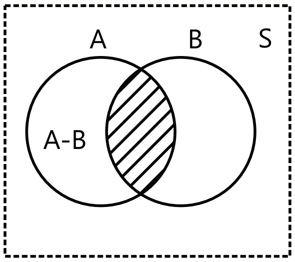
$\therefore P(B) = P[A \cup (B - A)]$ [$\because$ $A \cap (B - A) = f$]

$\Rightarrow P(B) = P(A) + P(B - A)$ [$\because$ $P(B - A) \ge 0$]

$\Rightarrow P(A) \le P(B)$ **Proved**

Theorem 2: For any two events A and B, $P(A - B) = P(A) - P(A \cap B)$

Proof: Let A and B be two compatible events. Then $A \cap B \ne \phi$. From the adjoining Venn diagram. it is clear that:



$(A - B) \cap (A \cap B) = f$and $(A - B) \cup (A \cap B) = A$

$\Rightarrow P\left(A - B\right) + P(A \cap B) = P(A)$

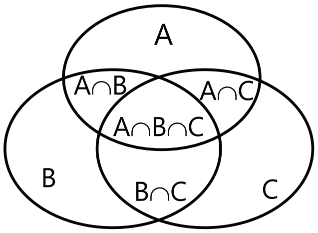
$\Rightarrow P(A - B) = P(A) - P(A \cap B)$ **Proved**

Remarks: This result may be expressed as

$P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Theorem 3: For any three events A, B, C



$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Proof: We have ,$P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

$= P(A \cup B) + P(C) - P[(A \cap C) \cup (B \cap C)]$ (Distributive Law)

$= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap B) \cup (C \cap C)]$ [Addition law]

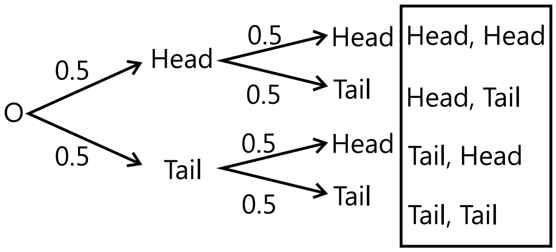
$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)]$

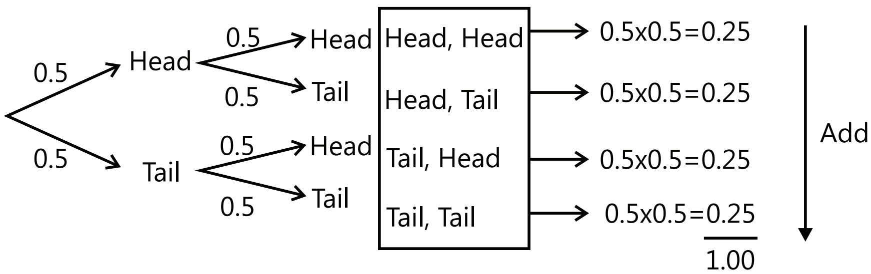
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) + P(B \cap C) + P[A \cap B \cap C]$ **Proved**

Probability Tree Diagrams: Calculating probabilities can be hard. Sometimes you add them, sometimes you multiply them and often, it is hard to figure out what to do. That's when **tree diagrams come to the rescue!**

Here is a tree diagram for two tosses of a coin:



How do you calculate the overall probabilities?



So, there you go. When in doubt, draw a tree diagram, multiply along the branches and add the columns. Make sure all probabilities add to 1 and you are good to go!

FORMULAE SHEET

(a) Mathematical definition of probability:

$$\text{Probability of an event} = \frac{\text{Number of favorable cases to event A}}{\text{Total Number of cases}}$$

Note: (i) $0 \le P(A) \le 1$

(ii) Probability of an impossible event is zero

(iii) Probability of a sure event is one.

(iv) $P(A) + P(\text{Not } A) = 1$ i.e. $P(A) + P(\bar{A}) = 1$

(b) Odd for an event: If $P(A) = \frac{m}{n}$ and $P(\bar{A}) = \frac{n-m}{n}$

Then odds in favor of A = $\frac{P(A)}{P(\bar{A})} = \frac{m}{n-m}$ and odd in against of A = $\frac{p(\bar{A})}{P(A)} = \frac{n-m}{m}$

(c) Set theoretical notation of probability and some important results:

(i) $P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

(iii) $P(A \cup B) = P(A \cap B) + P(\bar{A} \cap B) + P(A \cap \bar{B})$

(iv) $A \subseteq B \Rightarrow P(A) \le P(B)$

(v) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(vi) $P(A \cap B) \le P(A)\ P(B) \le P(A \cup B) \le P(A) + P(B)$

(vii) P(Exactly one event) = $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(viii) $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = P(A) + P(B) - 2P\,(A \cap B) = P(A + B) - P(A \cap B)$

(ix) P(neither A nor B) = $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

(x) When a coin is tossed n times or n coins are tossed once, the probability of each simple event is $\frac{1}{2^n}$

(xi) When a dice is rolled n times or n dice are rolled once, the probability of each simple event is $\frac{1}{6^n}$

(xii) When n cards are drawn ($1 \le n \le 52$) from well shuffled deck of 52 cards, the probability of each simple event is $\frac{1}{{}^{52}C_n}$.

(xiii) If n cards are drawn one after the other with replacement, the probability of each simple event is $\frac{1}{(52)^n}$

(xiv) P(none) = 1 – P (at least one)

(xv) Playing cards

- Total cards: 52 (26 red, 26 black)
- Four suits: Heart, diamond, spade, club (13 cards each)
- Court (face) cards: 12 (4 kings, 4 queens, 4 jacks)
- Honor cards: 16 (4 Aces, 4 kings, 4 queens, 4 Jacks)

(xvi) Probability regarding n letters and their envelopes:

If n letters corresponding to n envelopes are placed in the envelopes at random, then

- Probability that all letters are in the right envelopes = $\frac{1}{n!}$
- Probability that all letters are not in the right envelopes = $1 - \frac{1}{n!}$
- Probability that no letter is in the right envelope = $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + ... + (-1)^n \frac{1}{n!}$
- Probability that r letters are in the right envelope = $\frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + ... + (-1)^{n-r} \frac{1}{(n-r)!}\right]$

(d) Addition Theorem of Probability:

(i) When events are mutually exclusive

i.e. $n(A \cap B) = 0 \quad \Rightarrow P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

(ii) When events are not mutually exclusive i.e. $P(A \cap B) \ne 0$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or $P(A + B) = P(A) + P(B) - P(AB)$

(iii) When events are independent i.e. $P(A \cap B) = P(A)\ P(B)$

$\therefore P(A + B) = P(A) + P(B) - P(A)\ P(B)$

(e) Conditional probability:

P(A/B) = Probability of occurrence of A, given that B has already happened = $\frac{P(A \cap B)}{P(B)}$

P(B/A) = Probability of occurrence of B, given that A has already happened = $\frac{P(A \cap B)}{P(A)}$

Note: If the outcomes of the experiment are equally likely, then

$$P(A/B) = \frac{\text{Number of sample points in } A \cap B}{\text{Number of points in } B}$$

(i) If A and B are independent events, then P(A/B) = P(A) and P(B/A) = P(B)

(ii) Multiplication Theorem:

$P(A \cap B) = P(A/B).\ P(B),\ P(B) \neq 0$ or $P(A \cap B) = P(B/A)\ P(A),\ P(A) \neq 0$

Generalized: $P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n)$

$= P(E_1)\ P(E_2/E_1)\ P(E_3/E_1 \cap E_2)\ P(E_4/E_1 \cap E_2 \cap E_3) \ldots$ If events are independent, then

$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1)\ P(E_2) \ldots P(E_n)$

(f) Probability of at least one of the n Independent events: If $P_1, P_2, \ldots P_n$ are the probabilities of n independent events $A_1, A_2, \ldots A_n$ then the probability that at least one of these events will happen is $1 - [(1 - P_1)(1 - P_2) \ldots (1 - P_n)]$

or $P(A_1 + A_2 + \ldots + A_n) = 1 - P(\bar{A}_1)\,P(\bar{A}_2) \ldots P(\bar{A}_n)$

(g) Total probability: Let $A_1, A_2, \ldots A_n$ be n mutually exclusive & set of exhaustive events. If event A can occur through any one of these events, then the probability of occurrence of A

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \ldots + P(A \cap A_n) = \sum_{r=1}^{n} P(A_r)P(A / A_r)$$

(h) Bayes' Rule: Let A_1, A_2, A_3 be any three mutually exclusive & exhaustive events (i.e. $A_1 \cup A_2 \cup A_3$ = sample space & $A_1 \cap A_2 \cap A_3 = \phi$) of a sample space S and B is any other event on sample space then,

$$P(A_i/B) = \frac{P(B / A_i)(P(A_i)}{P(B / A_1)\,P(A_1) + P(B / A_2)P(A_2) + P(B / A_3)P(A_3)},\ i = 1, 2, 3$$

(i) Probability distribution:

(i) If a random variable x assumes values $x_1, x_2, \ldots x_n$ with probabilities $P_1, P_2, \ldots P_n$ respectively then

- $P_1 + P_2 + P_3 + \ldots + P_n = 1$
- Mean $E(x) = \Sigma P_i x_i$
- Variance $= \sum x^2 P_i - (\text{mean})^2 = \sum (x^2) - (E(x))^2$

(ii) Binomial distribution: If an experiment is repeated n times, the successive trials being independent of one another, then the probability of r success is ${}^nC_r\, P^r\, q^{n-r}$ at least r success is $\sum_{k=r}^{n} {}^nC_k P^k q^{n-k}$ where p is probability of success in a single trial, q = 1 – p

- Mean E(x) = np
- E(x2) = npq + n2p2
- Variance E(x2) – (E(x))2 = npq
- Standard deviation = $\sqrt{npq}$

(j) Truth of the statement:

(i) If two persons A and B speak the truth with probabilities P_1 & P_2 respectively and if they agree on a statement, then the probability that they are speaking the truth will be given by $\frac{P_1P_2}{P_1P_2+(1-P_1)(1-P_2)}$.

(ii) If A and B both assert that an event has occurred, the probability of occurrence of which is α, then the probability that the event has occurred $\frac{\alpha P_1P_2}{\alpha P_1P_2+(1-\alpha)(1-P_1)(1-P_2)}$ given that the probability of A & B speaking truth is p_1, p_2 respectively.

(iii) If in the second part, the probability that their lies coincide is β, then from the above case, the required probability will be $\frac{\alpha P_1P_2}{\alpha P_1P_2+(1-\alpha)(1-P_1)(1-P_2)\beta}$

Solved Examples

JEE Main/Boards

Example 1: If there are two events A and B such that $P(A') = 0.3$ $P(B) = 0.5$ and $P(A \cap B) = 0.3$, then $P(B|A \cup B')$ is:

(A) 3/8 (B) 2/3 (C) 5/6 (D) 1/4

Sol: Use set theory and probability of complimentary events to calculate $P(A \cup B')$

We have $P(A \cup B')$

$= P(A) + P(B') - P(A \cap B')$

$= [1 - P(A')] + [1 - P(B)] - [P(A) - P(A \cap B)]$

$= (1 - 0.3) + (1 - 0.5) - (0.7 - 0.3) = 0.8$

Now, $P(B|A \cup B') = \frac{P[B \cap (A \cup B')]}{P(A \cup B')}$

$= \frac{P[(B \cap A) \cup (B \cap B')]}{P(A \cup B')} = \frac{P(A \cap B)}{P(A \cup B')} = \frac{0.3}{0.8} = \frac{3}{8}$

Example 2: Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals:

(A) 1/2 (B) 7/15 (C) 2/15 (D) 1/3

Sol: Each black balls can be arranged in between any two white balls. Us e this idea to find the number of ways in which no two black balls are together.

The number of ways of placing 3 black balls at 10 places is ${}^{10}C_3$. The number of ways in which two black balls are not together is equal to the number of ways of choosing 3 places marked with X out of eight places

X W X W X W X W X W X W X W X

This can be done in ${}^{8}C_3$ ways. Thus, probability of the required event is $\frac{{}^{8}C_3}{{}^{10}C_3} = \frac{8\times7\times6}{10\times9\times8} = \frac{7}{15}$

Example 3: A group of 2n boys and 2n girls is randomly divided into two equal groups. The probability that catch contains the same number of boys and girls is:

(A) 1/2 (B) 1/n

(C) 1/2n (D) None of these

Sol: If one group is selected the second group automatically gets created. Hence, select n boys and n girls from the given group.

Total number of ways of choosing a group is ${}^{4n}C_{2n}$ The number of ways in which each group contains equal number of boys and girls is $({}^{2n}C_n)$ $({}^{2n}C_n)$

$\therefore$ Required probability $= \frac{({}^{2n}C_n)^2}{{}^{4n}C_{2n}}$.

Example 4: Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then P(B) is:

(A) 3/7 (B) 4/7 (C) 5/7 (D) 6/7

Sol: If say A and B are two independent events then $P(A \cap B) = P(A) \times P(B)$