19. ELECTRIC POTENTIAL AND CAPACITANCE

1. INTRODUCTION

Associating a potential energy with a force allows us to apply the principle of the conservation of mechanical energy to closed systems involving the force. The principle allows us to calculate the results of experiments for which force calculations alone would be very difficult. In this chapter, we first define this type of potential energy and then put it to use. We will also discuss the behavior of conductors which can store the charges in it, in the presence of the electric field.

2. ELECTRIC POTENTIAL

An electric field at any point can be defined in two different ways:

(i) by the field strength $\vec{E}$, and

(ii) by the electric potential V at the point under consideration.

Both $\vec{E}$ and V are functions of position and there is a fixed relationship between these two. Of these the field strength $\vec{E}$ is a vector quantity while the electric potential V is a scalar quantity. The electric potential at any point in an electric field is defined as the potential energy per unit charge, and similarly, the field strength is defined as the force per unit charge. Thus,

$$V = \frac{U}{q_0} \qquad \text{or} \qquad U = q_0 V$$

The SI unit of potential is volt (V) which is equal to joule per coulomb. So,

$$1V = 1\text{volt} = 1J/C = 1\,\text{joule}/\text{coulomb}$$

According to the definition of potential energy, the work done by the electrostatic force in displacing a test charge q_0 from point a to point b in an electric field is defined as the negative of change in potential energy between them, or $\Delta U = -W_{a-b}$ $\quad \therefore U_b - U_a = -W_{a-b}$

Dividing this equation by q_o,

$$\frac{U_b}{q_o} - \frac{U_a}{q_o} = -\frac{W_{a-b}}{q_o} \quad \text{or} \quad V_a - V_b = \frac{W_{a-b}}{q_o} \ \text{since}\ V = \frac{U}{q_o}$$

Thus, the work done per unit charge by the electric force when a charge body moves from point a to point b is equal to the potential at point a minus the potential at point b. We sometimes abbreviate this difference as $V_{ab} = V_a - V_b$.

Another way to interpret the potential difference V_{ab} is that it is , equal to the work that must be done by an external force to move a unit positive charge from point b to point a against the electric force. Thus,

$$V_a - V_b = \frac{\left(W_{b-a}\right)_{\text{external force}}}{q_o}$$

MASTERJEE CONCEPTS

In the equation $V = \frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{r_i}$ or $V = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{r}$, if the whole charge is at equal distance r_o from the point where V is to be evaluated, then we can write, $V = \frac{1}{4\pi\varepsilon_o}.\frac{q_{net}}{r_o}$, where q_{net} is the algebraic sum of all the charges of which the system is made.

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 1: Consider a non-conducting rod of length ℓ having a uniform charge density λ. Find the electric potential at P at a perpendicular distance y above the midpoint of the rod **(JEE MAIN)**

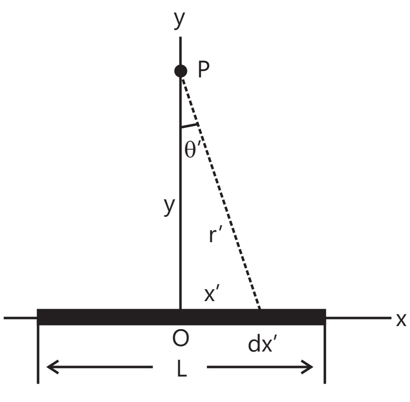


Figure 19.1

Sol: The potential due to the small length element dx of the rod at a point p at distance r apart is given by $dV = \frac{dq}{4\pi\varepsilon_o} \times \frac{1}{r}$ where dq is the charge on dx.

The electric field due to the rod is given by $E = -\frac{\partial V}{\partial r}$.

Consider a differential element of length dx' which carries a charge $dq = \lambda dx'$. As shown in Fig.19.1, the source element is located at (x', 0), while the field point P is located on the y-axis at (0,y).

The distance from dx' to P is $r = \left(x'^2 + y^2\right)^{1/2}$.

Its contribution to the potential is given by $dV = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_o}\frac{\lambda dx'}{\left(x'^2 + y^2\right)^{1/2}}$

Taking V to be zero at infinity, the total potential due to the entire rod is

$$V = \frac{\lambda}{4\pi\varepsilon_o}\int_{-\ell/2}^{\ell/2}\frac{dx'}{\sqrt{x'^2 + y^2}} = \frac{\lambda}{4\pi\varepsilon_o}\ln\left[x' + \sqrt{x'^2 + y^2}\right]\Bigg|_{-\ell/2}^{\ell/2}$$

$$= \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right]$$

Where we have used the integration formula

$$\int \frac{dx'}{\sqrt{x'^2 + y^2}} = \ln\left(x' + \sqrt{x'^2 + y^2}\right)$$

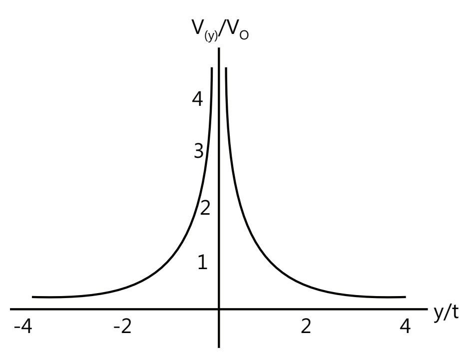


Figure 19.2

A plot of $V(y)/V_o$, where $V_o = \lambda/4\pi\varepsilon_o$, as a function of y/ℓ is shown in Fig. 19.2. (Electric potential along the axis that passes through the midpoint of a non-conducting rod.)

In the limit $\ell \gg y$, the potential becomes

$$V = \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}{-(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{1+\sqrt{1+(2y/\ell)^2}}{-1+\sqrt{1+(2y/\ell)^2}}\right]$$

$$\approx \frac{\lambda}{4\pi\varepsilon_o}\ln\left(\frac{2}{2y^2/\ell^2}\right) = \frac{\lambda}{4\pi\varepsilon_o}\ln\left(\frac{\ell^2}{y^2}\right) = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{\ell}{y}\right)$$

The corresponding electric field can be obtained as $\boxed{E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_o y}\frac{\ell/2}{\sqrt{(\ell/2)^2+y^2}}}$

Illustration 2: Consider a uniformly charged ring of radius R and charge density λ. What is the electric potential at a distance z from the central axis? **(JEE MAIN)**

The Fig. 19.3 shows a non-conducting ring of radius R with uniform charge density λ

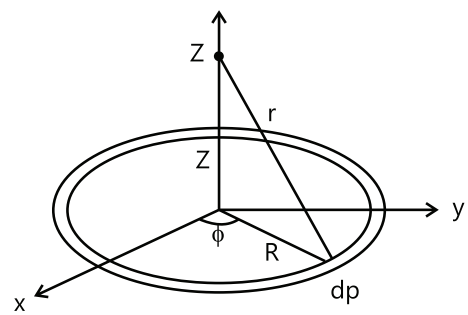


Figure 19.3

Sol: The point lies along the axis of the ring hence, the potential at the point due to the charge on the ring is given by $V = \int dV = \int \frac{dq}{4\pi\varepsilon_o r}$ where r is the distance of point from ring. As the point lies on the z axis, the field due to ring is given by $E_Z = -\frac{\partial V}{\partial z}$.

Consider a small differential element $d\ell = Rd\phi'$ on the ring. The element carries a charge $dq = \lambda d\ell = \lambda R d\phi'$, and its contribution to the electric potential at P is $dV = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_o}\frac{\lambda R d\phi'}{\sqrt{R^2+z^2}}$

The electric potential at P due to the entire ring is $V = \int dV = \frac{1}{4\pi\varepsilon_o}\frac{\lambda R}{\sqrt{R^2+Z^2}}\oint d\phi' = \frac{1}{4\pi\varepsilon_o}\frac{2\pi\lambda R}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{\sqrt{R^2+z^2}}$

Where we have substituted $Q = 2\pi R\lambda$ for the total charge on the ring. In the limit $z \gg R$,

The potential approaches its "point-charge" limit: $V \approx \frac{1}{4\pi\varepsilon_o}\frac{Q}{z}$

The z-component of the electric field may be obtained as $E_2 = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_o}\frac{Q}{\sqrt{R^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{Qz}{(R^2+z^2)^{3/2}}$

Illustration 3: Consider a uniformly charged disk of radius R and charge density σ lying in the xy-plane. What is the electric potential at a distance z from the central axis? **(JEE ADVANCED)**

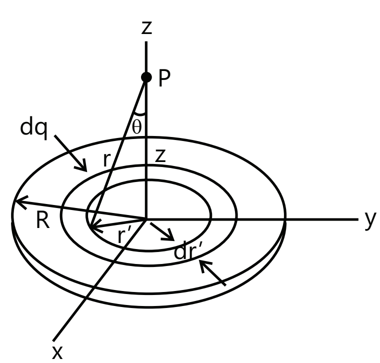


Figure 19.4

Sol: The disc can be assumed to be composed of many co-centric rings. Thus the potential due to small ring is $dV = \frac{dq}{4\pi\varepsilon_o r}$ where r is the distance from the surface of ring. As the point lie on the z axis the field due to ring is given by $E_Z = -\frac{\partial V}{\partial z}$.

Consider a small differential circular ring element of radius r′ and width dr′. The charge on the ring is $dq' = \sigma dA' = \sigma\left(2\pi r' dr'\right)$. The field at point P located along the z-axis a distance z from the plane of the disk is to be calculated. From the Fig. 19.4, we also see that the distance from a point on the ring to P is $r = \left(r'^2 + z^2\right)^{1/2}$.

Therefore, the contribution to the electric potential at P is $dV = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_o}\frac{\sigma\left(2\pi r' dr'\right)}{\sqrt{r'^2 + z^2}}$

By summing over all the rings that make up the disk, we have

$$V = \frac{\sigma}{4\pi\varepsilon_o}\int_o^R \frac{2\pi r' dr'}{\sqrt{r'^2 + z^2}} = \frac{\sigma}{2\varepsilon_o}\left[\sqrt{r'^2 + z^2}\right]_o^R = \frac{\sigma}{2\varepsilon_o}\left[\sqrt{R^2 + z^2} - |z|\right]$$

In the limit $|z| \gg R$, $\sqrt{R^2 + Z^2} = |z|\left(1 + \frac{R^2}{z^2}\right)^{1/2} = |z|\left(1 + \frac{R^2}{2z^2} + ...\right)$,

And the potential simplifies to the point-charge limit:

$$V \approx \frac{\sigma}{2\varepsilon_o} \cdot \frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_o}\frac{\sigma\left(\pi R^2\right)}{|z|} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{|z|}$$

As expected, at large distance, the potential due to a non-conducting charged disk is the same as that of a point charge Q. A comparison of the electric potentials of the disk and a point charge is shown in Fig. 19.5.

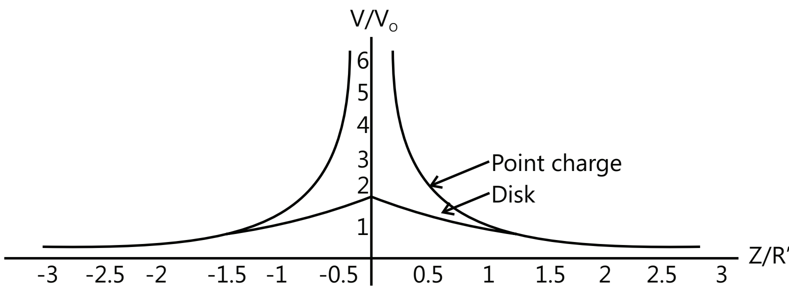


Figure 19.5

The electric potential is measured in terms of $V_o = Q / 4\pi\varepsilon_o R$.

Note that the electric potential at the center of the disk (z=0) is finite, and its value is

$$V_c = \frac{\sigma R}{2\varepsilon_o} = \frac{Q}{\pi R^2} \cdot \frac{R}{2\varepsilon_o} = \frac{1}{4\pi\varepsilon_o}\frac{2Q}{R} = 2V_o$$

This is the amount of work that needs to be done to bring a unit charge from infinity and place it at the center of the disk.

The corresponding electric field at P can be obtained as:

$$E_z = -\frac{\partial V}{\partial Z} = \frac{\sigma}{2\varepsilon_o}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}}\right]$$

In the limit $R \gg z$, the above equation becomes $E_2 = \sigma / 2\varepsilon_o$, which is the electric field for an infinitely large non-conducting sheet.

Illustration 4: Consider a metallic spherical shell of radius ‘a’ and charge Q, as shown in Fig. 19.6.

(a) Find the electric potential everywhere.

(b) Calculate the potential energy of the system. **(JEE MAIN)**

Sol: The sphere is symmetrical body. Thus the potential at the surface is constant while the potential changes outside of the sphere at a distance $a < r < \infty$.

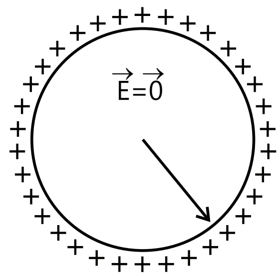


Figure 19.6

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{r}, & r > a \\ 0, & r < a \end{cases}$$

The electric potential may be calculated by $V_B - V_A = -\int_A^B \vec{E}.\vec{ds}$

For r>a, we have $V(r) - V(\infty) = -\int_\infty^r \dfrac{Q}{4\pi\varepsilon_o r'^2} dr' = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q}{r} = k_e \dfrac{Q}{r}$

We have chosen $V(\infty) = 0$ as our reference point. On the other hand, for r<a, the potential becomes

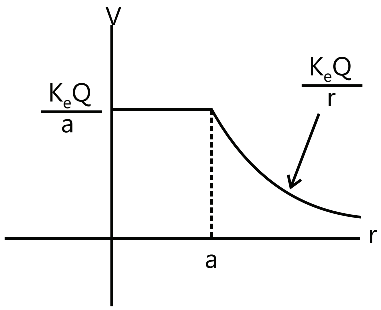


Figure 19.7

$$V(r) - V(\infty) = -\int_\infty^a drE(r > a) - \int_a^r drE(r < a)$$

$$= -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_o r^2} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{a} = k_e \frac{Q}{a}$$

A plot of the electric potential is shown in Fig. 19.7. Note that potential V is constant inside a conductor.

Illustration 5: An insulated solid sphere of radius a has a uniform charge density ρ. Compute the electric potential everywhere. **(JEE MAIN)**

Sol:

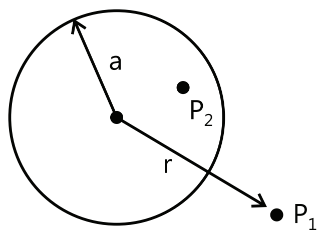


Figure 19.8

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{r}, & r > a \\ \dfrac{Qr}{4\pi\varepsilon_o r^3}\hat{r}, & r < a \end{cases}$$

For r > a,

$$V_1(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_o r'^2} dr' = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} = k_e \frac{Q}{r}$$

On the other hand, the electric potential at P_2 inside the sphere is given by

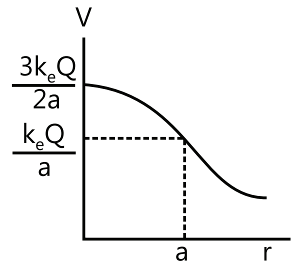


Figure 19.9

$$V_2(r) - V(\infty) = -\int_\infty^a drE(r > a) - \int_a^r E(r < a) = -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_o r^2} - \int_a^r dr' \frac{Qr}{4\pi\varepsilon_o a^3} r'$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_o}\frac{Q}{a^3}\frac{1}{2}\left(r^2 - a^2\right) = \frac{1}{8\pi\varepsilon_o}\frac{Q}{a}\left(3 - \frac{r^2}{a^2}\right)$$

$$= k_e \frac{Q}{2a}\left(3 - \frac{r^2}{a^2}\right)$$

A plot of electric potential as a function of r is given in Fig. 19.9.

2.1 Deriving Electric Field from the Electric Potential

In previous equations, we established the relation between $\vec{E}$ and V. If we consider two points which are separated by a small distance $d\vec{s}$, the following differential form is obtained:

$$dV = -\vec{E}.d\vec{s}$$

In Cartesian coordinates, $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_2\hat{k}$ and $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, we have

$$dV = \left(E_x\hat{i} + E_y\hat{j} + E_2\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = E_x dx + E_y dy + E_z dz$$

Which implies

$$\boxed{E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}}$$

By introducing a differential quantity called the "del (gradient) operator"

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

The electric field can be written as

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_2\hat{k} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V = -\nabla V$$

$$\vec{E} = -\nabla V$$

3. EQUIPOTENTIAL SURFACES

The equipotential surfaces in an electric field have the same basic idea as topographic maps used by civil engineers or mountain climbers. On a topographic map, contour lines are drawn passing through the points having the same elevation. The potential energy of a mass m does not change along a contour line as the elevation is same everywhere.

By analogy to contour lines on a topographic map, an equipotential surface is a three dimensional surface on which the electric potential V is the same at every point on it. An equipotential surface has the following characteristics.

(a) Potential difference between any two points in an equipotential surface is zero.

(b) If a test charge q_0 is moved from one point to the other on such a surface, the electric potential energy q_0V remains constant.

(c) No work is done by the electric force when the test charge is moved along this surface.

(d) Two equipotential surfaces can never intersect each other because otherwise the point of intersection will have two potentials which is of course not acceptable.

(e) As the work done by electric force is zero when a test charge is moved along the equipotential surface, it follows that $\vec{E}$ must be perpendicular to the surface at every point so that the electric force $q_o\vec{E}$ will always be perpendicular to the displacement of a charge moving on the surface, causing the work done to be 0. Thus, field lines and equipotential surfaces are always mutually perpendicular. Some equipotential surfaces are shown in Fig. 19.10.

The equipotential surfaces are a family of concentric spheres for a point charge or a sphere of charge and are a family of concentric cylinders for a line of charge or cylinder of charge. For a special case of a uniform field, where the field lines are straight, parallel and equally spaced the equipotential are parallel planes perpendicular to the field lines.

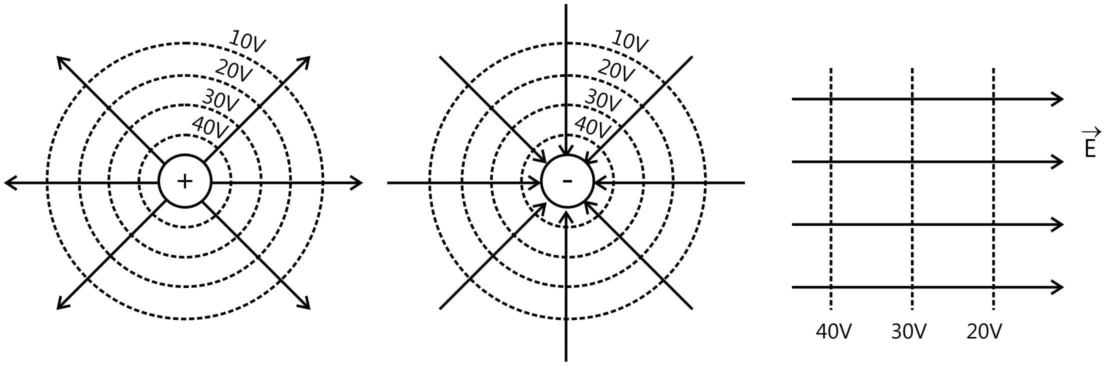


Figure 19.10

MASTERJEE CONCEPTS

While drawing the equipotential surfaces we should keep in mind the two main points.

- These are perpendicular to field lines at all places.
- Field lines always flow from higher potential to lower potential.

Anurag Saraf (JEE 2011 AIR 226)

Illustration 6: Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as $V(x,y,z) = Ax^2y^2 + Bxyz$ where A,B and C are constants. What is the associated electric field?

(JEE MAIN)

Sol: The field is given by $E_r = -\frac{\partial V}{\partial r}$ where r is the respective Cartesian co-ordinate.

$$E_x = -\frac{\partial V}{\partial x} = -2Axy^2 - Byz \ ; \ E_y = -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz \ ; E_z = -\frac{\partial V}{\partial z} = -Bxy$$

Therefore, the electric field is $\vec{E} = \left(-2Axy^2 - Byz\right)\hat{i} - \left(2Ax^2y + Bxz\right)\hat{j} - Bxy\hat{k}$

4. ELECTRIC POTENTIAL ENERGY

The electric force between two charges is directed along the line of the charges and is proportional to the inverse square of their separation, the same as the gravitational force between two masses. Like the gravitational force, the electric force is conservative, so there is a potential energy function U associated with it.

When a charged particle moves in an electric field, the field exerts a force that can do work on the particle. This work can always be expressed in terms of electric potential energy. Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field. When a force $\vec{F}$ acts on a particle that moves from point a to point b, the work $W_{a\to b}$ done by the force is given by, $W_{a\to b} = \int_a^b \vec{F}.d\vec{s} = \int_a^b F\cos\theta\, ds$

Here $W_{a\to b}$ is the work done in displacing the particle from a to b by the conservative force (here electrostatic) not by us. Moreover we can see from Eq. (i) that if $W_{a\to b}$ is positive, the change in potential energy ΔU is negative and the potential energy decreases. So, whenever the work done by a conservative force is positive, the potential energy of the system decreases and vice-versa. That's what happens when a particle is thrown upwards, the work done by gravity is negative, and the potential energy increases.

4.1 Potential Energy in a System of Charges

If a system of charges is assembled by an external agent, then $\Delta U = -W = +W_{ext}$.

That is, the change in potential energy of the system is the work that must be put in by an external agent to assemble the configuration. A simple example is lifting a mass m through a height h. The work done by an external agent you, is +mgh (The gravitational field does work -mgh). The charges are brought in from infinity without acceleration i.e. they are at rest at the end of the process. Let's start with just two charges q_1 and q_2. Let the potential due to q_1 at a point P be V_1 (See Fig. 19.11).

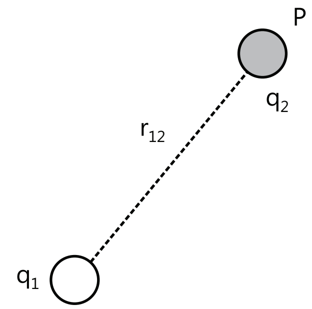


Figure 19.11

The work W_2 done by an agent in bringing the second charge q_2 from infinity to P is then $W_2 = q_2V_1$. (No work is required to set up the first charge and $W_1 = 0$). Since $V_1 = q_1 / 4\pi\varepsilon_0 r_{12}$, where q_1 and r_{12} is the distance measured from q_1 to P, we have

$$U_{12} = W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}$$

If q_1 and q_2 have the same sign, positive work must be done to overcome the electrostatic repulsion and the potential energy of the system is positive, $U_{12} > 0$. On the other hand, if the signs are opposite, then $U_{12} < 0$ due to the attractive force between the charges. To add a third charge q_3 to the system, the work required is

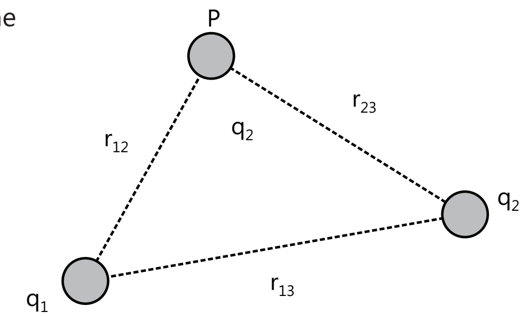


Figure 19.12

$$W_3 = q_3\left(V_1 + V_2\right) = \frac{q_3}{4\pi\varepsilon_o}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

The potential energy of this configuration is then

$$U = W_2 + W_3 = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) = U_{12} + U_{13} + U_{23}$$

The equation shows that the total potential energy is simply the sum of the contributions from distinct pairs. Generalizing to a system of N charges, we have $U = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\sum_{\substack{j=1\\j>i}}^{N}\frac{q_iq_j}{r_{ij}}$

Where the constraint j>i is placed to avoid double counting each pair. Alternatively, one may count each pair twice and divide the result by 2. This leads to

$$U = \frac{1}{8\pi\varepsilon_o}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N}q_i\left(\frac{1}{4\pi\varepsilon_o}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{N}q_iV\left(r\right)_i$$

Where $V\left(r_i\right)$, the quantity in the parenthesis is the potential at $\vec{r}_i$ (location of q_i) due to all the other charges.

4.2 Continuous Charge Distribution

If the charge distribution is continuous, the potential at a point P can be found by summing over the contributions from individual differential elements of charge dq.

Consider the charge distribution shown in Fig. 19.13. Taking infinity as our reference point with zero potential, the electric potential at P due to dq is $dV = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r}$

Summing over contributions from all differential elements, we have $\boxed{V = \frac{1}{4\pi\varepsilon_o}\int\frac{dq}{r}}$

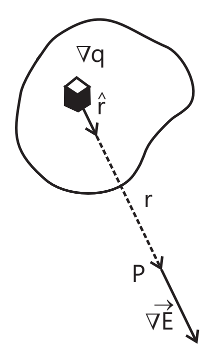


Figure 19.13

Illustration 7: Find the potential due to a uniformly charged sphere of radius R and charge per unit volume ρ at different points in space. **(JEE MAIN)**

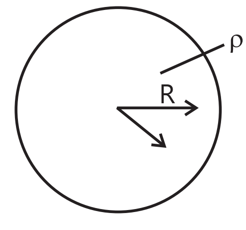


Figure 19.14

Sol: The field due to uniformly charged solid sphere is $E=\frac{q}{4\pi\varepsilon_o r^2}=\frac{\rho R^3}{3\varepsilon_o r^2}$. The potential inside the sphere is always constant while outside sphere at a distance $R<r<\infty$ is given by $\left[V_1\right]_R^r=\int_R^r E\,dl$

For a point outside the sphere, we can integrate the electric field outside to obtain an expression for the electric field outside.

$$\int V=-\int_\infty^r \vec{E}.\vec{dl}=-\int_\infty^r \frac{\rho R^3}{3\varepsilon_o x^2}dx=\frac{\rho R^3}{3\varepsilon_o r}$$

The dots represent a spherically symmetric distribution of charge of radius R, whose volume charge density ρ is a constant.

For finding the potential at an inside point, we integrate as before. Keep in mind that the lower limit in this integration cannot be infinity, as infinity does not lie inside the sphere. But lower limit can be assumed to be R where potential is not zero but a known value.

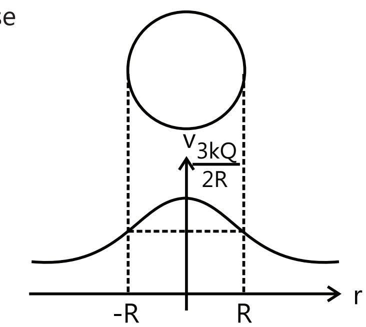


Figure 19.15

$$\int_{V_R}^{V_{in}} dV=-\int_R^r \vec{E}.\vec{dl}=-\int_R^r \frac{\rho x}{3\varepsilon_o}dx=\frac{\rho\left(R^2-r^2\right)}{6\varepsilon_o}\;;V_R=\frac{\rho R^2}{3\varepsilon_o}\Rightarrow V_r=\frac{\rho\left(3R^2-r^2\right)}{6\varepsilon_o}.$$

This potential is plotted in Fig. 19.15.

Illustration 8: A non-conducting disc of radius a and uniform positive surface charge density σ is placed on the ground with its axis vertical. A particle of mass m and positive charge q is dropped, along the axis of the disc from a height H with zero initial velocity. The particle has $q/m=4\varepsilon_o g/\sigma$

(a) Find the value of H if the particle just reaches the disc.

(b) Sketch the potential energy of the particle as a function of its height and find its equilibrium position. **(JEE ADVANCED)**

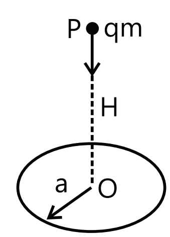


Figure 19.16

Sol: It is known that, for non conducting disc the potential at a point situated at height H above the disc is given by $V_p=\frac{\sigma}{2\varepsilon_o}\left[\sqrt{a^2+H^2}-H\right]$ where a is the radius of the disc. When the charge of the particle of mass m falls along the axis of the disc, the change in the gravitational portential energy is equal to gain in its electric potential energy. As we required minimum H, the kinetic energy, will be zero.

As we have derived in the theory, $V_p=\frac{\sigma}{2\varepsilon_o}\left[\sqrt{a^2+H^2}-H\right]$

Potential at centre, (O) will be $V_o=\frac{\sigma a}{2\varepsilon_o}$ (H=0)

(a) Particle is released from P and it just reaches point O. Therefore, from observation of mechanical energy

Decrease in gravitational potential energy = increase in electrostatic potential energy ($\Delta KE=0$ because $K_i=K_f=0$)

$$\therefore mgH=q\left[V_o-V_\rho\right] \text{ or } \quad gH=\left(\frac{q}{m}\right)\left(\frac{\sigma}{2\varepsilon_o}\right)\left[a-\sqrt{a^2+H^2}+H\right] \quad \text{... (i)}$$

$\frac{q}{m} = \frac{4\varepsilon_o g}{\sigma} \qquad \therefore \frac{q\sigma}{2\varepsilon_o m} = 2g$

Substituting in Eq. (i), we get

$gH = 2g\left[a + H - \sqrt{a^2 + H^2}\right]$ or $\frac{H}{2} = (a+H) - \sqrt{a^2+H^2}$ or $\sqrt{a^2+H^2} = a + \frac{H}{2}$

or $a^2 + H^2 = a^2 + \frac{H^2}{4} + aH$ or $\frac{3}{4}H^2 = aH$ or $H = \frac{4}{3}a$ and $H = 0$ $\therefore H = (4/3)a$

(b) Potential energy of the particle at height H = Electrostatic potential energy + gravitational potential energy

$U = qV + mgH$

Here V=potential at height H

$$U = \frac{\sigma q}{2\varepsilon_o}\left[\sqrt{a^2+H^2} - H\right] + mgH \qquad \text{... (ii)}$$

At equilibrium position, $F = \frac{-dU}{dH} = 0$

Differentiating E.q. (ii) w.r.t.H

$$\text{or } mg + \frac{\sigma q}{2\varepsilon_o}\left[\left(\frac{1}{2}\right)(2H)\frac{1}{\sqrt{a^2+H^2}} - 1\right] = 0 \quad \left[\frac{\sigma q}{2\varepsilon_o} = 2mg\right]$$

$$\therefore mg + 2mg\left[\frac{H}{\sqrt{a^2+H^2}} - 1\right] = 0 \text{ or } 1 + \frac{2H}{\sqrt{a^2+H^2}} - 2 = 0$$

$$\frac{2H}{\sqrt{a^2+H^2}} = 1 \text{ or } \frac{H^2}{a^2+H^2} = \frac{1}{4} \text{ or } 3H^2 = a^2 \quad \text{or } H = \frac{a}{\sqrt{3}}$$

From Eq. (ii), we can see that,

$U = 2mga$ at $H = 0$ and $U = U_{min} = \sqrt{3}mga$ at $H = \frac{a}{\sqrt{3}}$

Therefore, U-H graph will be as shown.

Note that at $H = \frac{a}{\sqrt{3}}$, U is minimum.

Therefore, $H = \frac{a}{\sqrt{3}}$ is stable equilibrium position.

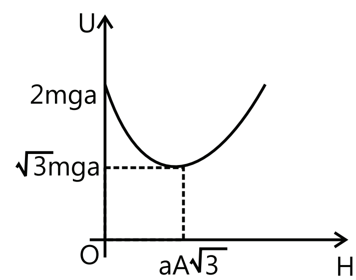


Figure 19.17

5. ELECTRIC DIPOLE

An electric dipole is a system of equal and opposite charges separated by a fixed distance. Every electric dipole is characterized by its electric dipole moment which is a vector $\vec{P}$ directed from the negative to the positive charge.

The magnitude of dipole moment is, P=(2a)q; Here, 2a is the distance between the two charges.

5.1 Electric Potential and Field Due to an Electric Dipole

Consider an electric dipole lying along positive y-direction with its center at origin.

$\vec{P} = 2aq\hat{j}$

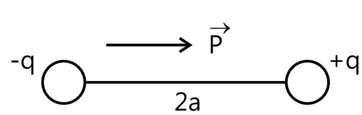


Figure 19.18

The electric potential due to this dipole at point A(x, y, z) as shown is simply the sum of the potentials due to the two charges. Thus,

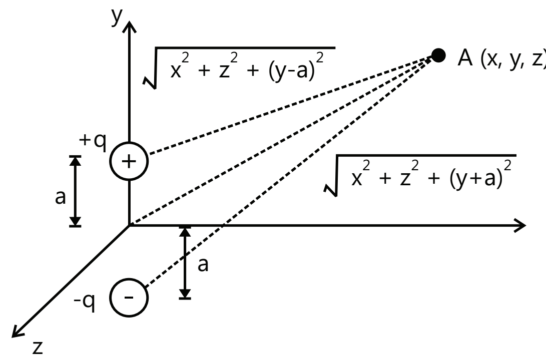


Figure 19.19

$$V = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2 + (y-a)^2 + z^2}} - \frac{q}{\sqrt{x^2 + (y+a)^2 + z^2}}\right]$$

By differentiating this function, we obtain the electric field of the dipole.

$$E_x = \frac{\partial V}{\partial x} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{x}{\left[x^2 + (y-a)^2 + z^2\right]^{3/2}} - \frac{x}{\left[x^2 + (y+a)^2 + z^2\right]^{3/2}}\right\}$$

$$E_y = \frac{\partial V}{\partial y} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{y-a}{\left[x^2 + (y-a)^2 + z^2\right]^{3/2}} - \frac{y+a}{\left[x^2 + (y+a)^2 + z^2\right]^{3/2}}\right\}$$

$$E_z = \frac{\partial V}{\partial z} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{z}{\left[x^2 + (y-a)^2 + z^2\right]^{3/2}} - \frac{x}{\left[x^2 + (y+a)^2 + z^2\right]^{3/2}}\right\}$$

Special Cases

(i) On the axis of the dipole (say, along y-axis) X=0, z=0

$$\therefore V = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{y-a} - \frac{1}{y+a}\right] = \frac{2aq}{4\pi\varepsilon_o\left(y^2 - a^2\right)}$$

or $V = \dfrac{p}{4\pi\varepsilon_o\left(y^2 - a^2\right)}$ (as 2aq = p) i.e., at a distance r from the Centre of the dipole(y=r)

$$V \approx \frac{p}{4\pi\varepsilon_o\left(r^2 - a^2\right)} \quad \text{or} \quad V_{axis} = \frac{p}{4\pi\varepsilon_o r^2} \quad (\text{for } r \gg a)$$

V is positive when the point under consideration is towards positive charge and negative if it is towards negative charge.

Moreover the components of electric field are as under,

$E_x = 0, E_z = 0$ (as x = 0, z = 0)

and $E_y = \dfrac{q}{4\pi\varepsilon_o}\left[\dfrac{1}{(y-a)^2} - \dfrac{1}{(y+a)^2}\right] = \dfrac{4ayq}{4\pi\varepsilon_o\left(y^2 - a^2\right)^2}$ or $E_y = \dfrac{1}{4\pi\varepsilon_o}\dfrac{2py}{\left(y^2 - a^2\right)^2}$

Note that E_y is along positive y-direction or parallel to $\vec{P}$. Further, at a distance r from the centre of the dipole (y=r).

$$E_y = \frac{1}{4\pi\varepsilon_o}\frac{2pr}{\left(r^2 - a^2\right)^2} \quad \text{or} \quad E_{axis} \approx \frac{1}{4\pi\varepsilon_o}\cdot\frac{2p}{r^3} \quad \text{for } r \gg a$$

(ii) On the perpendicular bisector of dipole

Say along x-axis (it may be along z-axis also). y = 0, z = 0

$$\therefore V = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2+a^2}} - \frac{q}{\sqrt{x^2+a^2}}\right] = 0 \quad \text{or} \quad V_{\perp \text{bisector}} = 0$$

Moreover the components of electric field are as under, $E_x = 0 \;\; E_z = 0$

$$\text{and } E_y = \frac{q}{4\pi\varepsilon_o} = \left\{\frac{-a}{\left(x^2+a^2\right)^{3/2}} - \frac{a}{\left(x^2+a^2\right)^{3/2}}\right\} = \frac{-2aq}{4\pi\varepsilon_o\left(x^2+a^2\right)^{3/2}}\,;\; E_y = \frac{1}{4\pi\varepsilon_o}\cdot\frac{P}{\left(x^2+a^2\right)^{3/2}}$$

Here, negative sign implies that the electric field is along negative y-direction or antiparallel to $\vec{P}$. Further, at a distance r from the Centre of dipole (x = r), the magnitude of electric field is,

$$E = \frac{1}{4\pi\varepsilon_o}\frac{P}{\left(r^2+a^2\right)^{3/2}} \quad \text{or} \quad E_{\perp \text{bisector}} \approx \frac{1}{4\pi\varepsilon_o}\cdot\frac{P}{r^3} \;\; (\text{for } r \gg a)$$

5.2 Force on Dipole

Suppose an electric dipole of dipole moment $\left|\vec{P}\right| = 2aq$ is placed in a uniform electric field $\vec{E}$ at an angle θ. Here, θ is the angle between $\vec{P}$ and $\vec{E}$. A force $\vec{F}_1 = q\vec{E}$ will act on positive charge and $\vec{F}_2 = -q\vec{E}$ on negative charge. Since, $\vec{F}_1$ and $\vec{F}_2$ are equal in magnitude but opposite in direction. Hence, $\vec{F}_1 + \vec{F}_2 = 0$ or $\vec{F}_{net} = 0$

Thus, net force on a dipole in uniform electric field is zero. While in non-uniform electric field it may or may not be zero.

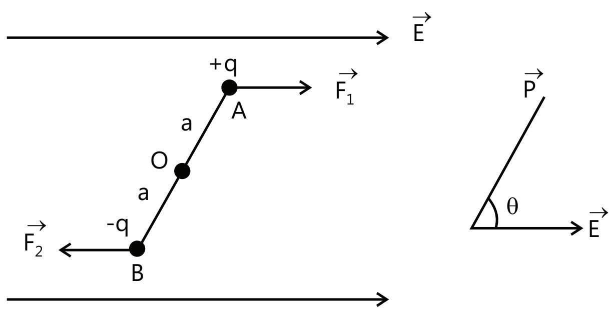


Figure 19.20

5.3 Torque on Dipole

The torque of $\vec{F}_1$ about O, $\vec{\tau}_1 = \overrightarrow{OA} \times \vec{F}_1 = q\left(\overrightarrow{OA} \times \vec{E}\right)$

And torque of $\vec{F}_2$ about O is, $\vec{\tau}_2 = \overrightarrow{OB} \times \vec{F}_2 = -q\left(\overrightarrow{OB} \times \vec{E}\right) = q\left(\overrightarrow{BO} \times \vec{E}\right)$

The net torque acting on the dipole is,

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = q\left(\overrightarrow{OA} \times \vec{E}\right) + q\left(\overrightarrow{BO} \times \vec{E}\right) = q\left(\overrightarrow{OA} \times \overrightarrow{BO}\right) \times \vec{E} = q\left(\overrightarrow{BA} \times \vec{E}\right) \quad \text{or } \vec{\tau} = \vec{P} \times \vec{E}$$

Thus, the magnitude of torque is $\tau = PE\sin\theta$. The direction of torque is perpendicular to the plane of paper inwards. Further this torque is zero at $\theta = 0^o$ or $\theta = 180^o$, i.e., when the dipole is parallel or antiparallel to $\vec{E}$ and maximum at $\theta = 90^o$.

5.4 Potential Energy of Dipole

When an electric dipole is placed in an electric field $\vec{E}$, a torque $\vec{\tau} = \vec{P} \times \vec{E}$ acts on it.

If we rotate the dipole through a small angle $d\theta$, the work done by the torque is,

$dW = \tau d\theta$; $\quad dW = -PE\sin\theta d\theta$

The work is negative as the rotation $d\theta$ is opposite to the torque. The change in electric potential energy of the dipole is therefore.

$dU = -dW = PE\sin\theta d\theta$

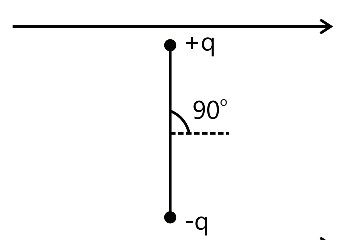


Figure 19.21

Now, at angle $\theta = 90°$, the electric potential energy of the dipole may be assumed to be zero as net work done by the electric forces in bringing the dipole from infinity to this position will be zero.

Integrating, $dU = PE\sin\theta d\theta$

From 90^0 to θ, we have $\int_{90^o}^{\theta} dU = \int_{90^o}^{\theta} PE\sin\theta d\theta$ or $U(\theta) - U(90^o) = PE[-\cos\theta]_{90^o}^{\theta}$

$\therefore\ U(\theta) = -PE\cos\theta = -\vec{P}\cdot\vec{E}$

If the dipole is rotated from an angle θ_1 to θ_2 then

Work done by external forces = $U(\theta_2) - U(\theta_1)$ or $W_{ext.forces} = -PE\cos\theta_2 - (-PE\cos\theta_1)$

Or $W_{ext.forces} = PE(\cos\theta_1 - \cos\theta_2)$ and work done by electric forces, $W_{electric\,force} = -W_{ext.force} = PE(\cos\theta_2 - \cos\theta_1)$

Illustration 9: Figure 19.22 is a graph of E_x, the x component of the electric field, versus position along the x axis. Find and graph V(x). Assume V = 0; V at x = 0m. **(JEE MAIN)**

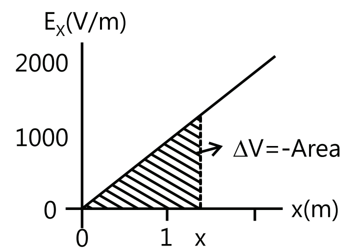


Figure 19.22

Sol: The potential V(x) is given by V=E·dx=Area of the shaded region of the graph.

The potential difference is the negative of the area under the curve.

E_x is positive throughout this region of space, meaning that $\vec{E}$ points in the positive x direction.

If we integrate from x-0, then $V_i = V(x-0) - 0$. The potential for x>0 is the negative of the triangular area under the E_x curve. We can see that $E_x = 1000xV/m$, where x is in meters(m). Thus,

$V_t = V(x) = 0 -$ (Area under the E_x curve)

$\frac{1}{2}\times \text{base}\times\text{height} = -\frac{1}{2}(x)(1000x) = -500x^2V.$

Figure 19.23 shows that the electric potential in this region of space is parabolic, decreasing from 0 V at x=0 m to -2000V at x=2m.

The electric field points in the direction in which V is decreasing. We'll soon see that as this is a general rule.

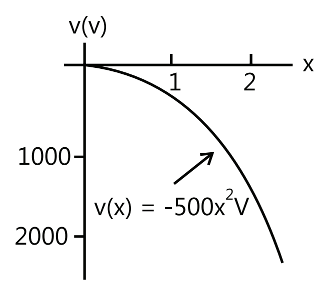


Figure 19.23

Illustration 10: The electric potential at any point on the central axis of a uniformly charged disk is given by $V = \frac{\sigma}{2\varepsilon_o}\left(\sqrt{z^2+R^2} - z\right)$.

Starting with this expression, derive an expression for the electric field at any point on the axis of the disk. **(JEE MAIN)**

Sol: As the point lie on the z axis the field due to ring is given by $E_Z = -\frac{\partial V}{\partial z}$.

Conceptualize/Classify: We want the electric field $\vec{E}$ as a function of distance z along the axis of the disk. For any value of z, the direction of $\vec{E}$ must be along that axis because the disk has circular symmetry about that axis. Thus, we want the component E_z of $\vec{E}$ in the direction of z. This component is the negative of the rate at which the electric potential changes with distance z.

Compute: Thus, from the previous equations, we can write

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_o}\frac{d}{dz}\left(\sqrt{z^2+R^2} - z\right); = \frac{\sigma}{2\varepsilon_o}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right).$$

6. CAPACITOR

A capacitor is a combination of two conductors placed close to each other. It is used to store energy electrostatically in an electric field. The 'non-conducting' dielectric acts to increase the capacitor's charge capacity. Capacitors are widely used as parts of electrical circuits in many common electrical devices. Unlike a resistor, a capacitor does not dissipate energy. Instead, a capacitor stores energy in the form of an electrostatic field between its plates.

The physics of capacitors can be utilized to any scenario involving electric fields. For example, Earth's atmospheric electric field is analyzed by meterologists as being produced by a huge spherical capacitor that partially discharges via lightning. The charge that is collected as they slide along snow can be modeled as being stored in a capacitor that frequently discharges as sparks.

7. CAPACITANCE

There are 2 conductors in a capacitor. One conductor has positive charge (positive plate) and the other has an equal and opposite negative charge (negative plate). The charge on the positive plate is called the charge on the capacitor and the potential difference between the plates is called the potential of the capacitor. For a given capacitor, the charge Q on the capacitor is proportional to the potential difference V between the plates Thus, $Q \propto V$ or, $Q=CV$.

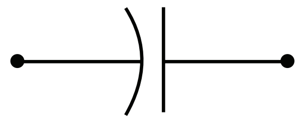


Figure 19.24

The proportionality constant C is called the capacitance of the capacitor. It depends on the shape, size and geometrical placing of the conductors and the medium between them.

The SI unit of capacitance is coulomb per volt which is written as Farad. The symbol F is used for it.

Illustration 12: A Capacitor gets a charge of 60μC when it is connected to a battery of emf 12V. Calculate the capacitance of the capacitor. **(JEE MAIN)**

Sol: The capacitance is given by $C = \frac{Q}{V}$

The potential difference between the plates is the same as the emf of the battery which is 12V.

Thus, the capacitance is $C = \frac{Q}{V} = \frac{60\mu C}{12V} = 5\mu F.$

8. CALCULATING CAPACITANCE

To calcutate the capacitance of a capacitor once we know its geometry:-

(a) Assume a charges q and –q on the plates ;

(b) Calculate the electric field $\bar{E}$ between the plates in terms of this charge, using Gauss' law;

(c) Knowing $\bar{E}$, calculate the potential difference V between the plates.

(d) Calculate C.

9. TYPES OF CAPACITORS

9.1 Parallel Plate Capacitor

A parallel-plate capacitor contains two large plane plates parallel to each other with a small separation between them. Suppose, the area of each of the surfaces is A and the separation between the two plates is d. Also, assume that vacuum fills the space between the plates.

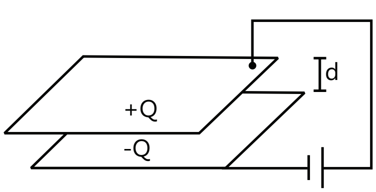


Figure 19.25

The magnitude of charge density on each of these surfaces is given by $\sigma=\frac{Q}{A}$

Let us draw a small area A parallel to the plates and in between them, draw a cylinder with A as a cross-section and terminate it by another symmetrically situated area A' inside the positive plate. The flux through A' and through the curved part inside the plate is zero as the electric field is zero inside a conductor. The flux through the curved part outside the plates is also zero as the direction of the field E is parallel to this surface.

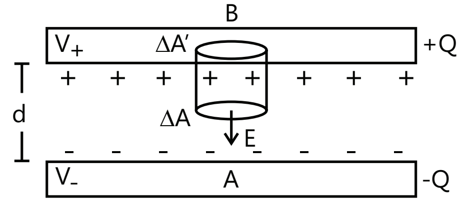


Figure 19.26

The flux through A is $\phi=\vec{E}.\Delta\vec{A}=E\Delta A$. The only charge inside the Gaussian surface is $\Delta Q=\sigma\Delta A=\frac{Q}{A}\Delta A$.

From Gauss's law, $\oint \vec{E}.d\vec{S}=Q_{in}/\varepsilon_0$ or, $E\Delta A=\frac{Q}{\varepsilon_0 A}\Delta A$; or, $E=\frac{Q}{\varepsilon_0 A}$.

The potential difference between the plates is $V=V_+-V_-=-\int_A^B \vec{E}.d\vec{r}$.

$\vec{E}.d\vec{r}=-Edr$ and $V=\int_A^B E\,dr=Ed=\frac{Qd}{\varepsilon_0 A}$

The capacitance of the parallel-plate capacitor is $C=\frac{Q}{V}=\frac{Q\varepsilon_0 A}{Qd}=\frac{\varepsilon_0 A}{d}$.

Illustration 13: Calculate the capacitance of a parallel-plate capacitor having 20cm x 20cm square plates separated by a distance of 1.0mm. **(JEE MAIN)**

Sol: The capacitance is given by $C=\frac{\varepsilon_0 A}{d}$.

The capacitance is $C=\frac{\varepsilon_0 A}{d}=\frac{8.85\times10^{-12}Fm^{-1}\times400\times10^{-4}m^2}{1\times10^{-3}m}=3.54\times10^{-10}F=350pF$.

9.2 Cylindrical Capacitor

A cylindrical capacitor consists of a solid or a hollow cylindrical conductor surrounded by another coaxial hollow cylindrical conductor. Let l be the length of the cylinders and R_1 and R_2 be the radii of the inner and outer cylinders respectively. If the cylinders are long as compared to the separation between them, the electric field at a point between the cylinders will be radial and its magnitude will depend only on the distance of the point from the axis.

To calculate the electric field at the point P, at a distance r, draw a coaxial cylinder of length x through the point. A Gaussian surface is made by the cylinder and its two cross sections. The flux through the cross sections is zero as the electric field is radial wherever it exists and hence is parallel to the cross sections.

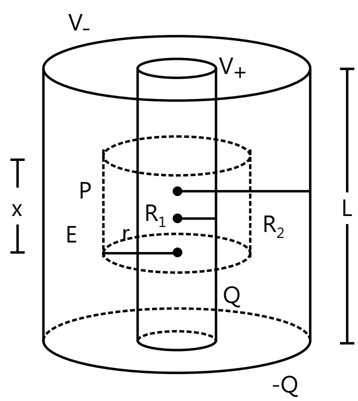


Figure 19.27

The flux through the curved part is $\phi=E\int dS=E2\pi rx \Rightarrow \phi=\oint\vec{E}.d\vec{S}=\int EdS$.

The charge enclosed by the Gaussian surface is $Q_{in}=\frac{Q}{l}x$, $E2\pi rx=\left(\frac{Q}{l}x\right)/\varepsilon_0$ or, $E=\frac{Q}{2\pi\varepsilon_0 rl}$

The potential difference between the cylinders is $V=V_+-V_-=-\int_A^B\vec{E}.d\vec{r}=-\int_{R_2}^{R_1}\frac{Q}{2\pi\varepsilon_0 rl}dr=\frac{Q}{2\pi\varepsilon_0 l}\ln\frac{R_2}{R_1}$.

The capacitance is $C=\frac{Q}{V}=\frac{2\pi\varepsilon_0 l}{\ln\left(R_2/R_1\right)}$

9.3 Spherical Capacitor

A Spherical Capacitor consists of two concentric spherical shells, of radii a and b. As a Gaussian surface we draw a sphere of radius r concentric with the two shells $q=\varepsilon_0 EA=\varepsilon_0 E\left(4\pi r^2\right)$, in which the area of the spherical Gaussian surface is given by $4\pi r^2$. We solve equation for E, obtaining $E=\frac{1}{4\pi\varepsilon_0 r^2}$, which we recognize as the expression for the electric field due to a uniform spherical charge distribution

$$V=\int_{-}^{+}E\,ds=-\frac{q}{4\pi\varepsilon_0}\int_b^a\frac{dr}{r^2}=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)=\frac{q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right),\text{ Substituted }-dr\text{ for }ds$$

$$C=4\pi\varepsilon_0\frac{ab}{b-a}\text{ (Spherical capacitor).}$$

Isolated Sphere

We can assign a capacitance to a single isolated spherical conductor of radius R by assuming that the "missing plate" is a conducting sphere of infinite radius. $C=4\pi\varepsilon_0\frac{a}{1-a/b}$.

If we then let b ∞ and substitute R for a, we find $C=4\pi\varepsilon_0 R$ (isolated sphere)

MASTERJEE CONCEPTS

- Earth can be considered as a capacitor with infinite capacitance.
- When a conductor is grounded and charge flows into the earth, the ground is still at potential zero.

GV Abhinav (JEE 2012 AIR 329)

Illustration 14: Three identical metallic plates are kept parallel to one another at a separation of a and b. The outer plates are connected by a thin conducting wire and a charge Q is placed on the central plate. Find final charges on all the six surfaces. **(JEE ADVANCED)**

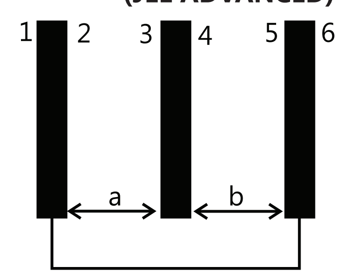


Figure 19.28

Sol: The charge on any plate is computed using $q=CV$ where C is the capacitance of the plate and V is the potential difference applied on it. The chrge induced in plates A and C are such that total $q(A)+q(C)=0$. Plates A and C are at the same potential.

Let the charge distribution in all the six faces be as shown in Fig. 19.28. While distributing the charge on different faces, we have used the fact that opposite faces have equal and opposite charges on them.

Net charge on plates A and C is zero.

Hence, $q_2-q_1+q_3+q_1-Q=0$ Or $q_2+q_3=Q$ ..... (i)

Further A anc C at same potentials. Hence, $V_B-V_A=V_B-V_C$ or $E_1a=E_2b$

$$\frac{q_1}{A\varepsilon_0}.a=\frac{Q-q_1}{A\varepsilon_0}.b\quad\text{(A= Area of plates); }q_1a=\left(Q-q_1\right)b\ \therefore q_1=\frac{Qb}{a+b}\qquad\text{..... (ii)}$$

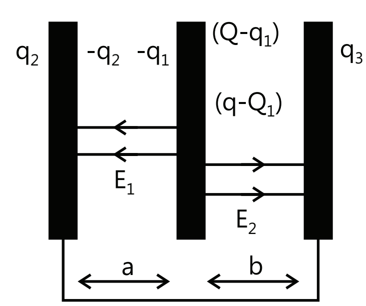


Figure 19.29

Electric field inside any conducting plate (say inside C) is zero. Therefore,

$$\frac{q_2}{2A\varepsilon_0}-\frac{q_1}{2A\varepsilon_0}+\frac{q_1}{2A\varepsilon_0}+\frac{Q-q_1}{2A\varepsilon_0}+\frac{q_1-Q}{2A\varepsilon_0}-\frac{q_3}{2A\varepsilon_0}=0$$

Solving these three equations, we get $q_1 = \frac{Qb}{a+b}, q_2 = q_3 = \frac{Q}{2}$

10. COMBINATION OF CAPACITORS

The desired value of capacitance can be obtained by combining more than one capacitor. We shall discuss two ways in which more then one capacitor can be connected.

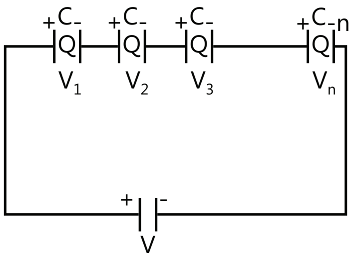


Figure 19.30

10.1 Series Combination of capacitors:

There is equal amount of charge Q deposited on each capacitor; but the potential difference between their plates is different (if the capacitance of each of the capacitors is different)

$$V = V_1 + V_2 + V_3 + ... + V_n \quad = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + ... + \frac{Q}{C_n} \text{ (from } C = \frac{Q}{V}\text{)};$$

$$\therefore \frac{V}{Q} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... + \frac{1}{C_n}$$

If C is the resultant capacitance of C_1, C_2......, C_n which are connected

In series then $\frac{V}{Q} = \frac{1}{C}$; $\quad \therefore \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... + \frac{1}{C_n}$

MASTERJEE CONCEPTS

The value of C is smaller than the smallest of the values of the capacitors which are connected in series.

B Rajiv Reddy (JEE 2012 AIR 111)

10.2 Parallel Combination of Capacitors

In this arrangement of the capacitors the charge accumulated on each of the capacitors is different while the potential difference between them is the same and is equal to potential difference between the common points A and B of such a connection.

Now, $Q_1 = C_1V, Q_2 = C_2V, Q_3 = C_3V$

The total electric charge, $Q = Q_1 + Q_2 + Q_3 = (C_1 + C_2 + C_3)V$

In general, $C = \frac{Q}{V} = C_1 + C_2 + C_3 + ...,$

Hence, by connecting capacitors in a parallel combination, a resultant capacitance can be obtained, whose value is equal to the sum of all capacitances connected in parallel.

- When a potential difference V is applied across several capacitors connected in parallel, it is applied across each capacitor. The total charge q stored on the capacitors is the sum of the charges stored on all the capacitors.
- Capacitors connected in parallel can be replaced with an equivalent capacitor that has the same total charge q and the same potential difference V as the actual capacitors.
- When a potential difference V is applied across serveral capacitors connected in series, the capacitors have identical charge q. The sum of the potential differences across all the capacitors is equal to the applied potential difference V.

Illustration 15: Three capacitors each of capacitance 9pF are connected in series.

(a) What is the total capacitance of the combination?

(b) What is the potential difference across each capacitor if the combination is connected to a 120V supply?

(JEE MAIN)

Sol: The equivalent capacitance in series combination is given by $C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)^{-1}$.

The potential difference applied across each capacitor is given by $V = \frac{q}{C}$ where C is the capacitance of the capacitor.

(a) Here, $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \quad = \frac{1}{C} + \frac{1}{C} + \frac{1}{C} = 3x\frac{1}{C}$

i.e. $\frac{1}{C_{eq}} = 3x\frac{1}{9x10^{-12}} \quad = \frac{1}{3x10^{-12}}$ or $C_{eq} = 3x10^{-12}F = 3pF$.

(b) Here $V_1 + V_2 + V_3 = 120$ i.e. $\frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = 120$ $\left[\therefore C = \frac{q}{V} \text{ or } V = \frac{q}{C}\right]$

i.e. $\frac{3q}{C} = 120$ i.e. $\frac{3xq}{9x10^{-12}} = 120$ or $q = 360x10^{-12}F$

$\therefore$ P.D. across a capacitor $= \frac{q}{C} = \frac{360x10^{-12}}{9x10^{-12}} = 40V$

Short cut

Since three equal capacitors are across 120V, so p.d. across each capacitor $= \frac{120}{3} = 40V$

Illustration 16: Calculate the charge on each capacitor shown in Fig. 19.31. **(JEE MAIN)**

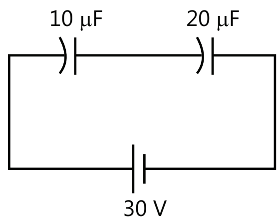


Figure 19.31

Sol: The equivalent capccitance is given by $C = \frac{C_1C_2}{C_1 + C_2}$

The two capacitors are joined in series. Their equivalent capacitance

Is given by $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ or, $C = \frac{C_1C_2}{C_1 + C_2} = \frac{(10\mu F)(20\mu F)}{30\mu F} = \frac{20}{3}\mu F.$

The charge supplied by the battery is Q = CV

In series combination, each capacitor has equal charge and this charge equals the charge supplied by the battery.

Thus, each capacitor has a charge of 200 $\mu C. = \left(\frac{20}{3}\mu F\right)(30V) = 200\mu F$

Illustration 17: Find the equivalent capacitance of the combination shown in Fig. 19.32 between the points P and N. **(JEE MAIN)**

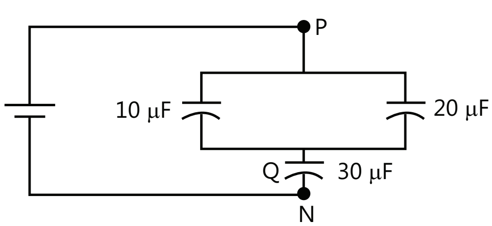


Figure 19.32

Sol: The circuit is the combination of series and parallel connection of capacitors. The equivalent capacitance of series combination is given by $C = \frac{C_{eq}C_2}{C_{eq} + C_2}$ whereas the equivalent capacitance of the parallel connection is given by $C_{eq} = C_1 + C_3$.

The 10μF and 20μF capacitors are connected in parallel. Their equivalent capacitance is 10μF +20μF = 30μF.

We can replace the 10μF and the 20μF capacitors by a single capacitor of capacitance 30μF between P and Q.

This is connected in series with the given 30μF capacitor. The equivalent capacitance C of this combination is given by $\frac{1}{C}=\frac{1}{30\mu F}+\frac{1}{30\mu F}$ or, $C = 15\mu F$.

11. ENERGY STORED IN AN ELECTRIC FIELD OF A CAPACITOR

The work that needs to be done by an external force to charge a capacitor is visualized as electric potential energy U stored in the electric field between the plates. This energy can be recovered by discharging the capacitor in a circuit.

Suppose that, at a given instant, a charge q′ has been transferred from one plate of a capacitor to the other. The potential difference V′ between the plates at that instant is q′/C. The work required to increase the charge by dq′ i

$$dW = V'dq' = \frac{q}{C}dq'$$

The work required to bring the total capacitor charge up to a final value q is $w=\int dW=\frac{1}{C}\int_0^q q'dq'=\frac{q^2}{2C}$.

This work is stored as potential energy U in the capacitor, so that we can also write this as $U=\frac{q^2}{2C}$ (Potential energy).

- The potential energy of a charged capacitor may be viewed as being stored in the electrc field between its plates.

Illustration 18: A parallel plate air capacitor is made using two plates 0.2 m square, spaced 1 cm apart. It is connected to a 50V battery. **(JEE ADVANCED)**

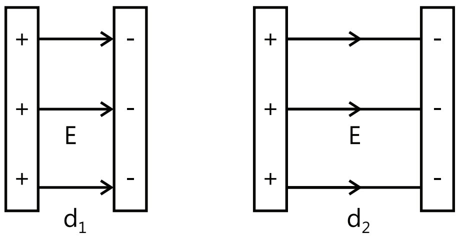


Figure 19.33

(a) What is the capacitance?

(b) What is the charge on each plate?

(c) What is the energy stored in the capacitor?

(d) What is the electric field between the plates?

(e) If the battery is disconnected and then the plates are pulled apart to a separation of 2cm, what are the answers to the above parts?

Sol: The capacitance, charge on capacitor, energy stored in capacitor, the electric field between plates of capacitor, are given by $C=\frac{\varepsilon_0 A}{d}$; $Q = CV$; $U=\frac{1}{2}CV^2$ and $E=\frac{V}{d}$ respectively.

(a) $C_0=\frac{\varepsilon_0 A}{d_0}=\frac{8.85X10^{-12}X0.2X0.2}{0.01}$; $C_0 = 3.54X10^{-5}\mu F$

(b) $Q_0 = C_0V_0 = (3.54X10^{-5}X50)\mu C = 1.77X10^{-3}\mu C$

(c) $U_0=\frac{1}{2}C_0V_0^2 = 1/2(3.54X10^{-11})(50)^2$; $U_0 = 4.42X10^{-8}J.$

(d) $E_0=\frac{V_0}{d_0}=\frac{50}{0.01}=5000V/m.$

(e) If the battery is disconnected, the charge on the capacitor plates remain constant while the potential difference between plates can change.

$C = \frac{A\varepsilon_0}{2d_0} =: 1.77X10^{-5}\mu F$; $Q = Q_0 = 1.77X10^{-3}\mu C$; $V = \frac{Q}{C} = \frac{Q_0}{C_0/2} = 2V_0 = 100 Volts.$;

$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q_0^2}{(C_0/2)} = 2U_0 = 8.84X10^{-8} J$, $E = \frac{2V_0}{2d_0} = E_0 = 5000V/m.$

Work has to be against the attraction of plates when they are separated. This gets stored in the energy of the capacitor.

Illustration 19: Find the energy stored in a capacitor of capacitance $100\,\mu F$ when it is charged to a potential difference of 20V. **(JEE MAIN)**

Sol: The energy stored in the capacitor is given by $U = \frac{1}{2}CV^2$.

The energy stored in the capacitor is $U = \frac{1}{2}CV^2 = \frac{1}{2}(100\mu F)(20V)^2 = 0.02J$

Illustration 20: Prove that in charging a capacitor half of the energy supplied by the battery is dissipated in the form of heat. **(JEE ADVANCED)**

Sol: When the swich S is closed, q=CV charge is stored in the capacitor. Charge transferred from the battery is also q. Hence, energy supplied by the battery $= qV = (CV)(V) = CV^2$ Half, of its Energy, i.e., $\frac{1}{2}CV^2$ is stored in the capacitor and the remaining 50% or $\frac{1}{2}CV^2$ is dissipated as heat.

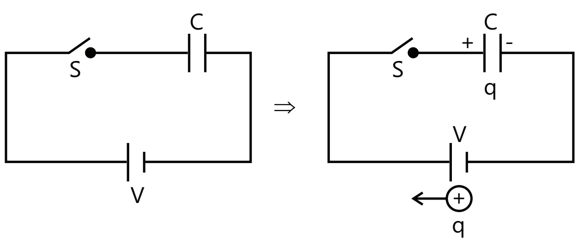


Figure 19.34

11.1 Energy Density

Neglecting fringing, the electric field in a parallel-plate capacitor has same value at all points between the plates. The energy density u, which is the potential energy per unit volume between the plates should also be uniform. We can find u by dividing the total potential energy by the volume Ad of the space between the plates $u = \frac{U}{Ad} = \frac{CV^2}{2Ad}$, $C = \varepsilon_0 A/d$, this result becomes $u = \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2$

V/d equals the electric field magnitude E; so $u = \frac{1}{2}\varepsilon_0 E^2$ (Energy density)

Although we derived this result for the special case of a parallel-plate capacitor, it holds generally, whatever may be the source of the electric field.

12. FORCE BETWEEN THE PLATES OF A CAPACITOR

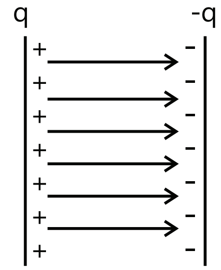


Figure 19.35

Consider a parallel plate capacitor with plate area A. Suppose a positive charge q is given to one plate and a negative charge – q to the other plate. The electric field on the negative plate due to positive charge is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{q}{2A\varepsilon_0} \quad \because \sigma = \frac{q}{A}$$

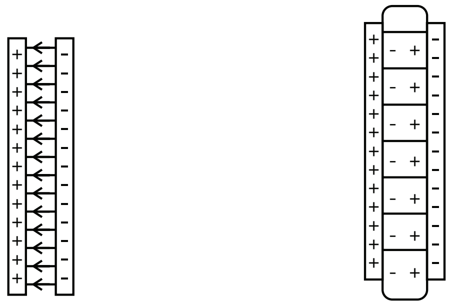
The magnitude of force on the charge in negative plate is $F = qE = \frac{q^2}{2A\varepsilon_0}$

This is the force with both the plates attract each other. Thus, $F = \frac{q^2}{2A\varepsilon_0}$

13. DIELECTRICS

If the medium between the plates of a capacitor is filled with an insulating substance (dielectric), the electric field due to the charges plates induces a net dipole moment in the dielectric. This effect called polarization, gives rise to a field in the opposite direction. Due to this, the potential difference between the plates is reduced. Consequently, the capacitance C increases from its value C_0 when there is no medium (vacuum). The dielectric constant K is defined by the relation $K = \frac{C_{dielectric}}{C_{vacuum}} = \frac{C}{C_0}$

Where C_{vacuum} = capacity of a capacitor when there is vacuum between the plates. Thus, the dielectric constant of a substance is the factor by which the capacitance increases from its vacuum value, when dielectric is inserted fully between the plates.



(a) Electric field lines with vacuum between the plates

(b) The induced charges on the faces of the dielectric reduce the electric field

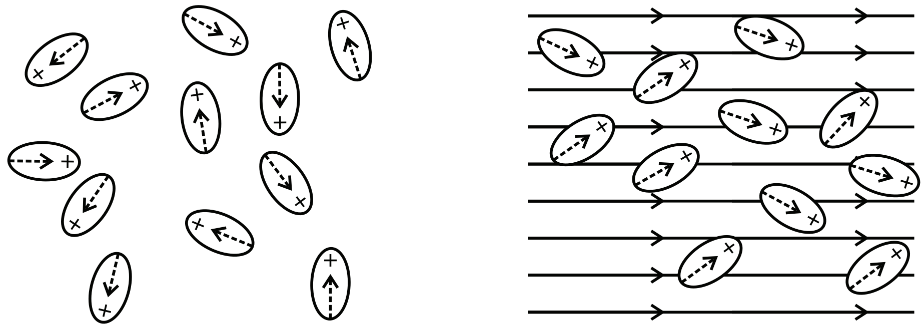
Figure 19.36

13.1 Dielectrics at an Atomic View

What happens, in atomic and molecular terms, when we put a dielectric in an electric field? There are two possibilities, depending on the type of molecule:-

(a) **Polar Dielectrics**: The molecules of some dielectrics, like water, have permanent electric dipole moments. In such cases, the electric dipoles tend to line up with an external electric field because the molecules are continuously jostling each other as a result of their random thermal motion. This alignment becomes more complete as the magnitude of the applied field is increased (or as the temperature, and thus the jostling, are decreased). The alignment of the electric dipoles produces an electric field that is directed opposite to the applied field and is smaller in magnitude.

(b) **Nonpolar moments:** Even nonpolar molecules acquire dipole moments by induction when placed in an external electric field. This occurs because the external field tends to "stretch" the molecules, slightly separating the centres of negative and positive charge.

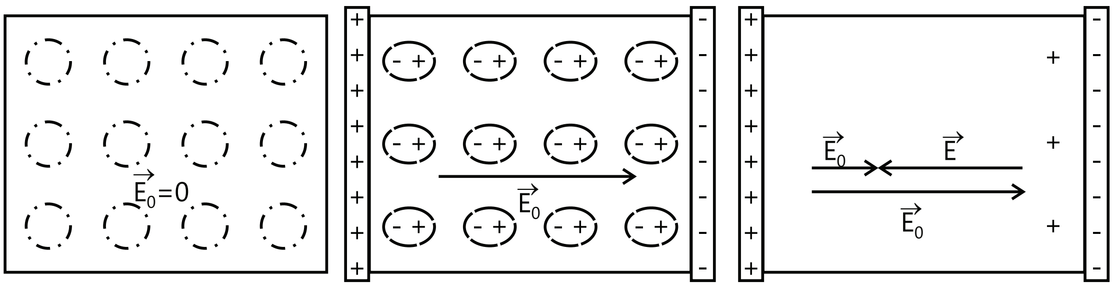


(a) (b)

Figure 19.37

Molecules with a permanent electric dipole moment, showing their random orientation in the absence of an external electric field.

An electric field is applied producing partial alignment of the dipoles. Thermal agitation prevents complete alignment



(a) (b) (c)

Figure 19.38

(i) A nonpolar dielectric slab. The circles represent the electrically neutral atoms within the slab.

(ii) An electric field is applied via charged capacitor plates; the field slightly stretches the atoms, separating the centres of positive and negative charge.

(iii) The separation produces surface charges on the slab faces. These charges set up a field $\vec{E}'$, which opposes the applied field $\vec{E}_0$. The resultant field $\vec{E}$ inside the dielectric (the vector sum of $\vec{E}_0$ and $\vec{E}'$) has the same direction as $\vec{E}_0$ but a smaller magnitude.

MASTERJEE CONCEPTS

Real Capacitors

- Real capacitors used in circuits are parallel plate capacitors with a dielectric material inserted in between the plates and rolled into a cylinder.
- This model reduces the size to get maximum capacitance.

Nitin Chandrol (JEE 2012 AIR 134)

Illustration 21: A parallel plate capacitor is to be designed with a voltage rating 1kV, using a material of dielectric constant 3 and dielectric strength about 10^7 Vm^{-1} (Dielectric strength is the maximum electric field a material can tolerate without breakdown, i.e. without starting to conduct electricity through partial ionization.) For safety, we should like the field never to exceed, say 10% of the dielectric strength. What minimum area of the plates is required to have a capacitance of 50pF? **(JEE ADVANCED)**

Sol: The area of the capacitor is given by $A = \frac{C \cdot D}{\varepsilon}$ where $\varepsilon = \varepsilon_0 \cdot \varepsilon_r \lim_{x\to\infty}$ is the permittivity of the medium.

10% of the given field i.e. 10^7 Vm^{-1}. Given E=0.1x10^7 Cm^{-1}

Using $E = \frac{-dV}{dr}$; i.e. $E = \frac{V}{r}$, we get; $r = \frac{V}{E} = \frac{1000}{0.1x10^7} = 10^{-3}m$

Using $C = \frac{\varepsilon_0 \varepsilon_r A}{d}$, we get; $A = \frac{Cd}{\varepsilon_0 \varepsilon_r} = \frac{Cr}{\varepsilon_0 \varepsilon_r} = \frac{(450x10)^{-12}(10^{-13})}{8.854x10^{-3} \times 3} = 19cm^2$

Illustration 22: A parallel plate capacitor with air between the plates has a capacitance of 8pF (1pF = 10^{-12} F.) What will be the capacitance if the distance between the plates is reduced by half, and the space between them is filled with a substance of dielectric constant 6? **(JEE MAIN)**

Sol: The capacitance is $C = \frac{k\varepsilon A}{d}$ where k is the dielectric constant. If d reduces to ½ and k changes to 6 then new capacitance is 12 times the original capacitance.

Using C′ = $C = \frac{\varepsilon_0 \varepsilon_r A}{d}$, C' $\in_r$ C = 12(8 × 10^{-12}) = 96 × 10^{-12}F = 96pF

Illustration 23: Two parallel-plate capacitors, each of capacitance $40\,\mu F$, are connected in series. The space between the plates of one capacitor is filled with a dielectric material of dielectric constant K=4. Find the equivalent capacitance of the system. **(JEE ADVANCED)**

Sol: The equivalent capacitance of series connection is $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$.

The capacitance of the capacitor with the dielectric is $C_1 = KC_0 = 4 \times 40\ \mu F = 160\ \mu F$.

The other capacitor has capacitance $C_2 = 40\ \mu F$.

As they are connected in series, the equivalent capacitance is $C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(160\mu F)(40\mu F)}{200\mu F} = 32\mu F$

13.4 Dielectrics and Gauss law

From parallel plate capacitor with a dielectric equation we can write Gauss’ law in the form $\varepsilon_0 \oint K\ \vec{E}.d\vec{A} = q$ (Gauss’ law with dielectric)

This equation, although derived for a parallel-plate capacitor, is true generally and is the most general form in which Gauss’ law can be written.

Some more information about capacitors

(a) If charge is held constant, i.e. battery disconnected and dielectric is inserted between plates.

(i) Charge remains unchanged, i.e., $q = q_0$, as in an isolated system, charge is conserved.

(ii) Capacity increases, i.e. $C = KC_0$, as by presence of a dielectric capacity becomes K times.

(iii) P.D. between the plate decreases, i.e., $V=(V_0/K)$ $V = \frac{q}{C} = \frac{q_0}{KC}$ $[\text{as } q = q_0 \text{ and } c = KC_0]$ as,

$V = \frac{q}{C} = \frac{q_0}{KC}$ $\left[\text{as } q = q_0 \text{ and } c = KC_0\right]$

(iv) E between the plates decreases, i.e., $E = (E_0 / K)$, as, $E = \frac{V}{d} = \frac{V_0}{Kd} = \frac{E_0}{K}$ $\left[\text{as } V = \frac{V_0}{K} \text{ and } E_0 = \frac{V_0}{d}\right]$

(v) Energy stored in the capacitor decreases, i.e. $U = (U_0 / K)$.

$U = \frac{q^2}{2C} = \frac{q^2{}_0}{Kd} = \frac{U_0}{K}$ $\left[\text{as } q = q_0 \text{ and } C = KC_0\right]$

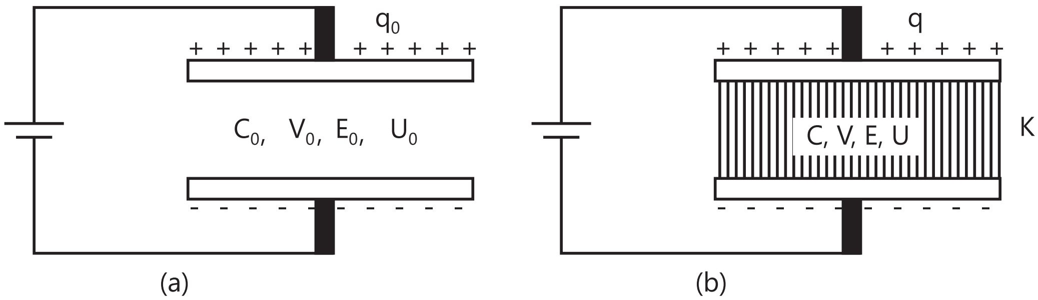


Figure 19.39

(b) If potential is held constant, i.e., battery remains attached and dielectric in inserted between plates.

(i) PD remains constant, i.e. $V = V_0$, as battery is a source of constant potential difference.

(ii) Capacity increases, i.e., $C = KC_0$, as by presence of a dielectric capacity becomes K times.

(iii) Charge on capacitor increases, i.e., $q = Kq_0$ as $q = CV = (KC_0)V = Kq_0$ $\left[\text{as } q_0 = C_0V\right]$

(iv) Electric field remains unchanged, i.e., $E = E_0$, $E = \left[\frac{V}{d}\right] = \frac{V_0}{d} = E_0$ $\left[\text{as } V = V_0 \text{ and } \frac{V_0}{d} = E_0\right]$

(v) Energy stored in the capacitor increases, i.e., $U = KU_0$

As, $U = \frac{1}{2}CV^2 = \frac{1}{2}\left((KC_0)(V_0)^2\right); = \frac{1}{2}KU_0 \left[\text{as } C = KC_0 \text{ and } U_0 = \frac{1}{2}C_0V_0^2\right]$

Illustration 24: A parallel-plate capacitor has plate area A and plate separation d. The space between the plates is filled up to a thickness x(<d) with a dielectric of dielectric constant K. Calculate the capacitance of the system. **(JEE ADVANCED)**

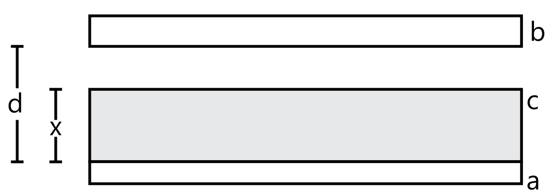


Figure 19.40

Sol: As the distance between the plates is filled with dielectric material upto a distance x<d, this system represent two capacitors connected in series. Thus equivalent capacitance is $C_{eq} = \frac{C_1C_2}{C_1 + C_2}$ where each capacitance depends on the dielectric constant K and distance from the plate of capacitor.

The situation is shown in Fig. 19.40. The given system is equivalent to the series combination of two capacitors, one between a and c and the other between c and b.

Here c represents the upper surface of the dielectric. This is because the potential at the upper surface of the dielectric is constant and we can imagine a thin metal plate being placed there.

The capacitance of the capacitor between a and c is $C_1 = \frac{K\varepsilon_0 A}{x}$ and that between c and b is $C_2 = \frac{\varepsilon_0 A}{d-x}$.

The equivalent capacitance is $C = \frac{C_1C_2}{C_1+C_2} = \frac{K\varepsilon_0 A}{Kd - x(K-1)}$

Illustration 25: In the situation shown in Fig. 19.41 the area of the plates is A. The dielectric slab is released from rest. Prove that the slab will execute periodic motion and find its time period. Mass of the slab is m. **(JEE ADVANCED)**

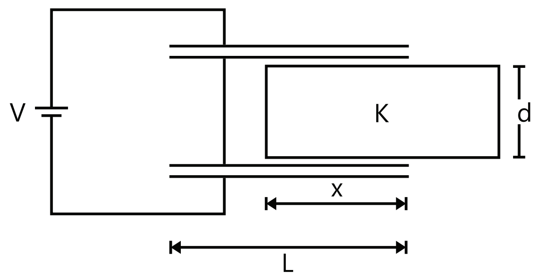


Figure 19.41

Sol: As the slab is moving in constant electric field applied between the plates of capacitors, the force acting on it is constant. The acceleration is obtained by $a = \frac{F}{m}$.When the slab moves completly out side the plates the electric force pulls the slab slides inside the plates of capacitor. The time period of oscillation is given by $t = \sqrt{\frac{2s}{a}}$ where s is the distance travelled by the slab. After using, $F = -\frac{dU}{dx}$, where $U \rightarrow$ Potential energy. Constant force, $F = \frac{\varepsilon_0 bV^2(K-1)}{2d}$

Here, $b =$ width of plate $= \frac{A}{L}$

$\therefore$ Acceleration of slab $a = \frac{F}{m}$ or $a = \frac{\varepsilon_0 (A/l)V^2(K-1)}{2md}$

The equilibrium position of the slab is, at the instant when the slab is fully inside the plates. So, the slab will execute oscillations in the phases as shown in Fig. 19.42.

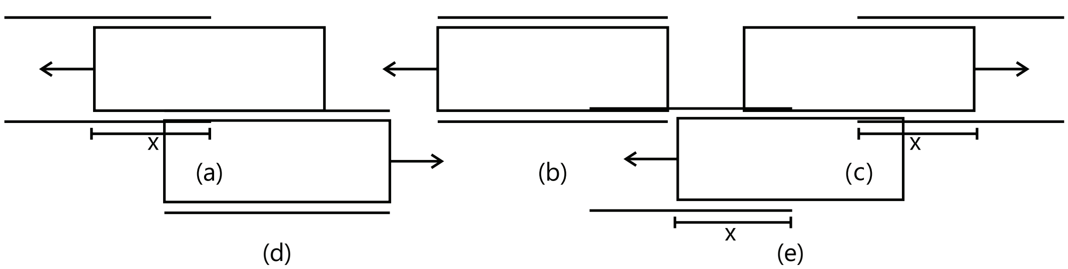


Figure 19.42

$\frac{T}{4}$ = time taken to reach from position (a) to position (b) = t (say) ; using $S = \frac{1}{2}at^2$

We have $t = \sqrt{\frac{2s}{a}}$; $(S = l - x) = \sqrt{\frac{2(l-x)}{\frac{\varepsilon_0 (A/l)V^2(K-1)}{2md}}} = \sqrt{\frac{4(l-x)mdl}{\varepsilon_0 AV^2(K-1)}}$

The desired time period is, $T = 4t = 8\sqrt{\frac{(L-x)mdl}{\varepsilon_0 AV^2(K-1)}}$

Illustration 26: Three concentric conducting shells A, B and C of radii a, b and c are a shown in Fig. 19.43. A dielectric of dielectric constant K is filled between A and B, find the capacitance between A and C. **(JEE ADVANCED)**

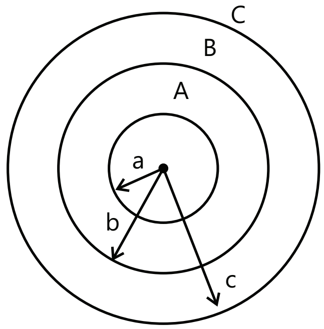


Figure 19.43

Sol: The potential difference in the region *dr* between the shells is given by $\Delta V = \int E \cdot dr$ where E is the electric field. Capacitance is the ratio of charge and potential difference. Let the sphere A have a charge q. When the dielectric is filled between A and B, the electric field will change in this region. Therefore the potential difference and hence the capacitance of the system will change. So, first find the electric field E(r) in the region $a \le r \le c$. Then find the potential difference (V) between A and C and finally the capacitance of the system is $C = \frac{q}{V}$, $E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$ for $a \le r \le b$ $= \frac{q}{4\pi\varepsilon_0 r^2}$ for $b \le r \le c$. Using, $dv = -\int \vec{E}.\vec{dr}$

Here potential difference between A and C is, $V = V_A - V_C = \int_a^b \frac{q}{4\pi\varepsilon_0 K r^2}.dr - \int_a^b \frac{q}{4\pi\varepsilon_0 r^2} dr$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{K}\left(\frac{1}{a}-\frac{1}{b}\right)+\left(\frac{1}{b}-\frac{1}{c}\right)\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{(b-a)}{Kab}+\frac{(c-b)}{bc}\right] = \frac{q}{4\pi\varepsilon_0 kabc}\left[c(b-a)+Ka(c-b)\right]$$

The desired capacitance is $C = \frac{q}{V} = \frac{4\pi\varepsilon_0 Kabc}{Ka(c-b)+c(b-a)}$

14. R-C CIRCUITS

To understand the charging of a capacitor in C-R circuit, let us first consider the charging of a capacitor without resistance.

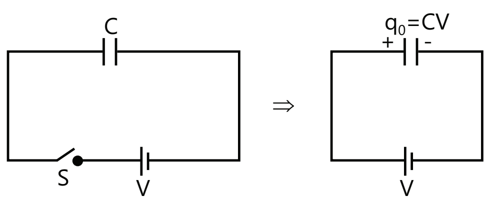


Figure 19.44

Consider a capacitor connected to a battery of emf V through a switch S. When we close the switch, the capacitor gets charged immediately. Charging takes no time. A charge q_0 = CV comes in the capacitor as soon as switch is closed and the q-t graph in this case is a straight line parallel to t-axis as shown.

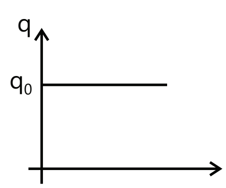


Figure 19.45

But if there is some resistance in the circuit charging takes some time. Because resistance opposes the charging (or current flow in the circuit).

Final charge (called steady state charge) is still q_0 but it is acquired after a long period of time.

The q-t equation in this case is, $q = q_0\left(l - e^{-t/\tau_c}\right)$

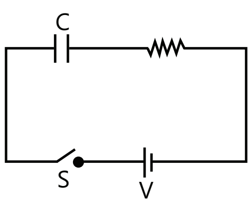


Figure 19.46

Here, q_0 = CV and τ_c = CR = time constant.

q-t graph is an exponentially increasing graph. The charge q increases exponentially from 0 to q_0.

From the graph and equation we see that at t = 0, q = 0 and at t = ∞, q = q_0

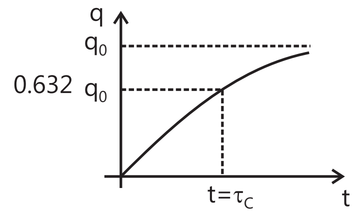


Figure 19.47

14.1 Charging

When a capacitor C is connected to a battery through a resistance R, the plates of a capacitor will acquire equal and opposite charge and the potential difference across it becomes equal to the emf of the battery. The process (called charging) takes sometime and during this time there is an electric current through the resistance. If at any time t, I is the current through the resistance R and q is the charge on capacitor C, the equation of emf for the circuit will be $V_C + V_R = E$, i.e., $V + IR = E$

But $I = (dq/dt)$ and $q = CV$

so, $R\frac{dq}{dt} + \frac{q}{C} = E$ or $\int_0^q \frac{dq}{(CE - q)} = -\int_0^t \frac{dt}{CR}$

Which on solving for q gives $q = q_0\left(l - e^{-t/CR}\right)$ with $q_0 = CE(\text{for } t = \infty)$ .....(i)

This is the required result and from this it is clear that:

(a) During charging, charge on the capacitor increases from 0 to q_0 (=CE) non-linearly.

(b) The density CR is called capacitive time constant τ of the circuit [as it has dimensions of time] and physically represents the time in which charge on the capacitor reaches 0.632 times of its maximum value during charging.

(c) During charging current at any time t in the circuit will be

$$I = \frac{dq}{dt} = \frac{d}{dt}\left[q_0\left(l - e^{-t/CR}\right)\right] = I_0 e^{-t/CR} \text{ with } I_0 = \frac{E}{R}(\text{at } t = 0)$$

i.e., initially it acts as short circuit or as a simple conducting wire. If $t \to \infty$, $I \to 0$, i.e., it acts as open circuit or as a broken wire.

14.2 Discharging

If a charged capacitor C having charge q_0 is discharged through a resistance R, then at any time t,

V = IR but as $I = (-dq/dt)$ and $q = CV$

$R\frac{dq}{dt} + \frac{q}{C} = 0$ i,e., $\int_{q_0}^{q} \frac{dq}{dt} = -\int_0^t \frac{dt}{CR}$ or $q = q_0 e^{-t/CR}$

This is the required result and from this it is clear that

(a) During discharging, the charge on capacitor decreases exponentially from q_0 to 0

(b) The capacitive time constant $\tau = CR$ is the time in which charge becomes (l/e), i.e., 0.368 times of its initial value (q_0)

(c) During discharging current at any time t in the circuit:

$$I = -\frac{dq}{dt} = -\frac{d}{dt}\left(q_0 e^{-t/CR}\right) = I_0 e^{-t/CR} \text{ with } I_0 = \frac{E}{R}$$ and its direction is opposite to that of charging.

(d) As in discharging of a capacitor through a resistance $q = q_0 e^{-t/CR}$ i.e., $R = \dfrac{t}{C\log_e(q_0/q)}$

So resistance R can be determined from the value of t and (q/q_0), i.e. (V_0/V). Using this concept in laboratory we determine the value of high resistances $(\sim M\Omega)$ by the so called 'Leakage method'.

Definition of τ_c At t = τ_c, $q = q_0(1-e^{-1}) = 0.632\, q_0$

Hence, τ_c can be defined as the time in which 63.2% charging is over in a C-R circuit. Note that τ_c is the time. Hence, $[\tau_c] = [\text{time}]$ or $[CR] = [M^0L^0T]$.

15. VAN DE GRAAFF GENERATOR

Van de Graaff generator is a device that can generate a potential difference of a few million volts. The highly intense electric field produced in this matching is used to accelerate charged particles, which can then be used to study the composition of matter at the microscopic level.

Principle: let us presume (See Fig. 19.48) that charge Q is residing on an isolated conducting shell, having radius R.

There is another conducting sphere having charge q and radius equal to r at the centre of the above mentioned spherical shell (r<R)

The electric potential on the spherical shell of radius R is equal to $V_R = \dfrac{kQ}{R} + \dfrac{kq}{R}$

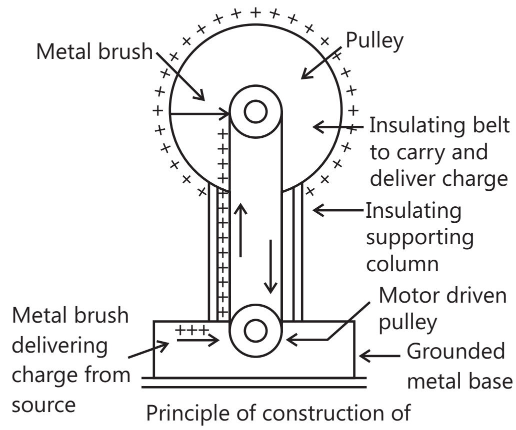


Figure 19.48

The electric potential on the surface of the sphere of radius r is, $V_r = \dfrac{kQ}{R} + \dfrac{kq}{r}$

Therefore, the potential difference between the surfaces of the two sphere is, $V_r - V_R = \dfrac{kQ}{R} + \dfrac{kq}{r} - \dfrac{kQ}{R} - \dfrac{kq}{R} = kq\left(\dfrac{1}{r} - \dfrac{1}{R}\right)$

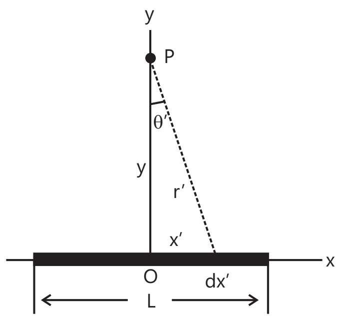
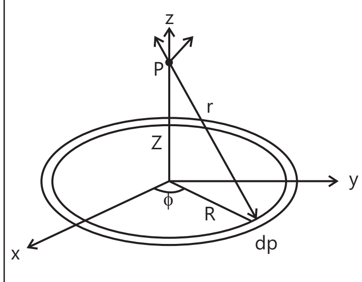
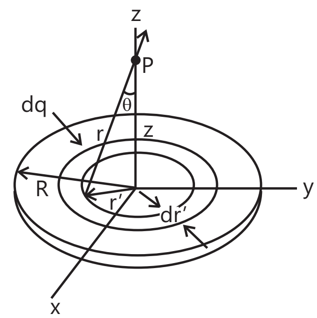
The above equation shows that the smaller sphere is at a higher electric potential in comparison to the larger spherical shell. If the smaller sphere is brought in contact with the larger spherical shell then electric charge will flow from the smaller sphere towards the larger spherical shell. Note if charge can be continuously transferred to the smaller sphere in some way, it will keep getting accumulated on the larger shell, thereby increasing its electric potential to a very high value.

Construction: S is a large spherical conducting shell with a radius of a few meters. This is erected to suitable height over the insulating pillars, C_1 and C_2. A long narrow belt of insulating material moves continuously between 2 pulleys P_1 and P_2 as shown in the Fig. 19.48. B_1 and B_2 are two sharply pointed brushes fixed near pulleys, P_1 and P_2 respectively, such that B_2 touches the belt near the pulley P_2. B_1 is called the spray brush and B_2 is called the collector brush.

Working: The spray brush is given a positive potential w.r.t. the earth by high tension source known as E.H.T. around the sharp points of B_1, the high potential causes the air to ionize and this sprays positive charges on the belt. As the belt moves, and reaches the sphere S, the collecting brush B_2, which touches the belt near pulley P_2, collects the positive charge, which spreads on the outer surface of S. the uncharged belt returns down and again collects the positive charge from B_1. As the belt moves continuously between P_1 and P_2 the positive charge starts accumulating on sphere S and the charge due to ionization is minimized by enclosing the metallic shell in an earth connected steel tank filled with air at high pressure. The charge particles may accelerate in this large potential to high kinetic energies of the order of more than 2 MeV.

PROBLEM-SOLVING TACTICS

Below we illustrate how the above methodologies can be employed to compute the electric potential for a line of charge, a ring of charge and a uniformly charged disk.

	Charged Rod	**Charged Ring**	**Charged disk**
Figure	 **Figure 19.49**	 **Figure 19.50**	 **Figure 19.51**
(2) Express dq in terms of charge density	$dq = \lambda dx'$	$dq = \lambda dl$	$dq = \sigma dA$
(3) Substitute dq into expression for dV	$dV = k_e \dfrac{\lambda dx'}{r}$	$dV = k_e \dfrac{\lambda dl}{r}$	$dV = k_e \dfrac{\sigma dA}{r}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $r = \sqrt{x'^2 + y^2}$	$dl = Rd\phi$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r' dr'$ $r = \sqrt{r'^2 + z^2}$
(5) Rewrite dV	$dV = k_e \dfrac{\lambda dx'}{(x'^2 + y^2)^{1/2}}$	$dV = k_e \dfrac{\lambda R d\phi'}{(R^2 + z^2)^{1/2}}$	$dV = k_e \dfrac{2\pi\sigma r' dr'}{(r'^2 + z^2)^{1/2}}$
(6) Integrate to get V	$V = \dfrac{\lambda}{4\pi\varepsilon_0} \int_{-\ell/2}^{\ell/2} \dfrac{dx^1}{\sqrt{x'^2 + y^2}}$ $= \dfrac{\lambda}{4\pi\varepsilon_0} \ln$ $\left[\dfrac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right]$	$V = k_e \dfrac{R\lambda}{(R^2 + z^2)^{1/2}} \oint d\phi'$ $= k_e \dfrac{(2\pi R\lambda)}{\sqrt{R^2 + z^2}}$ $= k_e \dfrac{Q}{\sqrt{R^2 + z^2}}$	$V = k_e 2\pi\sigma \int_0^R \dfrac{r' dr'}{(r'^2 + z^2)^{1/2}}$ $= 2k_e\pi\sigma\left(\sqrt{z^2 + R^2} - \lvert z\rvert\right)$ $= \dfrac{2k_e Q}{R^2}\left(\sqrt{z^2 + R^2} - \lvert z\rvert\right)$
Derive E from V	$E_y = -\dfrac{\partial V}{\partial y}$ $= \dfrac{\lambda}{2\pi\varepsilon_0 y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{k_e Q_Z}{\left(R^2 + z^2\right)^{3/2}}$	$E_z = -\dfrac{\partial V}{\partial z}$ $= \dfrac{2k_e Q}{R^2}\left(\dfrac{z}{\lvert z\rvert} - \dfrac{z}{\sqrt{z^2 + R^2}}\right)$
Point- charge limit for E	$E_y \approx \dfrac{k_e Q}{y^2}$ $\quad y >> \ell$	$E_z \approx \dfrac{k_e Q}{z^2}$ $z >> R$	$E_z \approx \dfrac{k_e Q}{z^2}$ $z >> R$

For any given combination, one may proceed as follows:

Step 1: Identify the two points between which the equivalent capacitance is to be calculated. Call any one of them as P and the other as N.

Step 2: Connect (mentally) a battery between P and N with the positive terminal connected to P and the negative terminal to N. Send a charge +Q from the positive terminal of the battery.

Step 3: Write the charges appearing on each of the plates of the capacitors. The charge conservation principle may be used. The facing surfaces of a capacitor will always have equal and opposite charges. Assume variables Q_1, Q_2 ..., etc., for charges wherever needed.

Step 4: Take the potential of the negative terminal N to be zero and that of the positive terminal P to be V. Write the potential of each of the plates. If necessary, assume variables V_1, V_2....

Step 5: Write the capacitor equation Q = CV for each capacitor. Eliminate Q_1, Q_2 ...and V_1, V_2...., etc., to obtain the equivalent capacitance C=Q/V.

FORMULAE SHEET

S. No	FORMULA
1.	$q = CV$
2.	$\varepsilon_0 \oint \vec{E}.d\vec{A} = q.$
3.	$V_1 - V_2 = \int_i^f \vec{E}.d\vec{S}.$
4.	$V = \int_-^+ Eds = E\int_0^d ds = Ed$
5.	$q = \varepsilon_0 EA.$
6.	$C = \frac{\varepsilon_0 A}{d}$ (parallel-plate capacitor)
7.	$C = 2\pi\varepsilon_0 \frac{L}{In(b/a)}$ (cylindrical capacitor)
8.	$C = 4\pi\varepsilon_0 \frac{ab}{b-a}$ (spherical capacitor)
9.	$C = 4\pi\varepsilon_0 R$ (isolated sphere)
10.	$C_{eq} = \sum_{j=1}^{n} C_J$ (n capacitors in parallel)
11.	$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_J}$ (n capacitors in series)

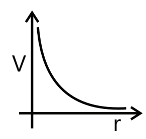
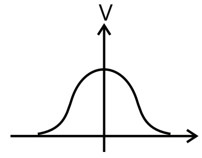
S. No	FORMULA
12.	$U = \frac{1}{2}CV^2$ (potential energy)
13.	$U = \frac{q^2}{2C}$ (potential energy)
14.	$u = \frac{1}{2}\varepsilon_0 E^2$ (energy density)
15.	$E = \frac{q}{4\pi K\varepsilon_0 r^2}$
16.	$\varepsilon_0 \oint K\vec{E}.d\vec{A} = q$ (Gauss' law with dielectric).
17.	Force on a Dielectric Slab inside a Capacitor $F = \frac{\varepsilon_0 bV^2(K-1)}{2d}$

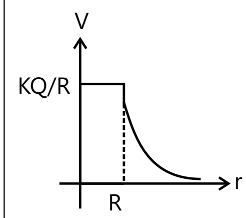
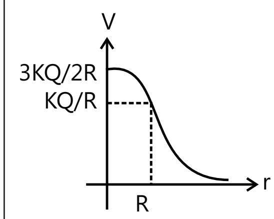
Electric Potential Formulae

S. No	Term	Description
1	Electric Potential energy	ΔU = -W Where ΔU = Change in potential energy and W= Work done by the electric lines of force. For a system of two particles $U(r) = q_1q_2 / 4\pi\varepsilon r$ Where r is the separation between the charges. We assume U to be zero at infinity. Similarly for a system of n charges U = Sum of potential energy of all the distinct pairs in the system For example for three charges $U = (1/4\pi\varepsilon)(q_1q_2/r_{12} + q_2q_3/r_{23} + q_1q_3/r_{13})$
2	Electric PE of a charge	= qV where V is the potential.
3	Electric Potential	Like Electric field intensity is used to define the electric field; we can also use Electric Potential to define the field. Potential at any point P is equal to the work done per unit test charge by the external agent in moving the test charge from the reference point (without Change in KE) $V_p = W_{ext} / q$. So for a point charge V0 = $Q / 4\pi\varepsilon r$ Where r is the distance of the point from charge.

S. No	Term	Description
4.	Some points about Electric potential	1. It is scalar quantity 2. Potential at a point due to system of charges will be obtained by the summation of potential of each charge at that point $V = V_1 + V_2 + V_3 + V_4$ 3. Electric forces are conservative forcew so work done by the electric force between two points is independent of the path taken 4. $V_2 - V_1 = -\int E.dr$
		5. In Cartesian coordinates system $dV = -E.dr;\ dV = -(E_x\,dx + E_y\,dy + E_z\,dz)$ So $E_x = \partial V/\partial x,\ E_y = \partial V/\partial y$ and $E_z = \partial V/\partial z,$ Also $E = \left[(\partial V/\partial x)i + (\partial V/\partial y)i + (\partial V/\partial z)k\right]$ 6. Surface where electric potential is same everywhere is called equipotential surface. Electric field components parallel to equipotential surface are always zero.
5	Electric dipole	A combination of two charges +q and –q separated by a distance d has a dipole moment p = qd, where d is the vector joining negative to positive charge.
6	Electric potential due to dipole	$V = (1/4\pi\varepsilon)\times(p\cos\theta/r^2)$ Where r is the distance from the center and θ is angle made by the line from the axis of dipole.
7	Electric field due to dipole	$E_\theta = (1/4\pi\varepsilon)(p\sin\theta/r^3);\ E_r = (1/4\pi\varepsilon)\times(2p\cos\theta/r^3)$ $\text{Total } E = \sqrt{E_\theta^{\,2} + E_r^{\,2}} = (p/4\pi\varepsilon r^3)(\sqrt{(3\cos^2\theta + 1})$ Torque on dipole = $p \times E$ Potential Energy $U = -p.E$
8	Few more points	1. $\int E.dl$ over closed path is zero 2. Electric potential in the spherical charge conductor is $Q/4\pi\varepsilon R$ where R is the radius of the shell and the potential is same everywhere in the conductor. 3. Conductor surface is a equipotential surface

Electric potential due to various charge distributions

Name/Type	Formula	Note	Graph
Point Charge	$\frac{Kq}{r}$	• q is source charge • r is the distance of the point from the point charge.	
Ring (uniform/non uniform charge distribution)	At centre, $\frac{KQ}{R}$ At axis, $\frac{KQ}{\sqrt{R^2 + x^2}}$	• Q is source charge • x is distance of the point from centre.	

Name/Type	Formula	Note	Graph
Uniformly charged hollow conducting/ non - conducting sphere or solid conducting sphere	For $r \geq R, V = \frac{KQ}{R}$ For $r \leq R, V = \frac{KQ}{R}$	• R is radius of sphere • r is distance of the point from centre of the sphere • Q is total charge $((= \sigma 4\pi R^2)$	
Uniformly charged solid non - conducting sphere (insulating material)	For $r \geq R$, $V = \frac{KQ}{r}$ For $r \leq R$, $V = \frac{KQ(3R^2 - r^2)}{2R^3}$ $= \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)$	• R is radius of sphere • r is distance of point from centre of the sphere. • Q is total charge $(= \rho \frac{4}{3}\pi R^3)$ • $V_{centre} = \frac{3}{2} V_{surface}$ • Inside sphere potential varies parabolically. • Outside potential varies hyperbolically.	
Line charge	Not defined	• Absolute potential is not defined • Potential difference between two points is given by formula $V_B - V_A = -2K\lambda \ln(r_B / r_A)$, where lambda is the charge per unit length	
Infinite nonconducting thin sheet	Not defined	• Absolute potential Is not defined • Potential difference between two points is given by formula $V_B - V_A = -\frac{\sigma}{2\varepsilon_0}(r_B - r_A)$, where sigma is the charge density	
Infinite charged conducting thin sheet	Not defined	• Absolute potential is not defined • Potential difference between two point is given by formula $V_B - V_A = -\frac{\sigma}{\varepsilon_0}(r_B - r_A)$, where sigma is the charge density	

Electric dipole moment: $\vec{p} = qd\hat{z}$, where two charges of charge $\pm q$ are placed along the z axis at $z = \pm\frac{d}{2}$

Electric dipole field: Along the z axis (z>>d): $\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{p}{|z|^3}\hat{z}$, in the +z direction.

Along the x axis(x>>d): $\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{p}{|x|^3}\hat{x}$ in the +x direction.

Along the x axis(y>>d): $\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{p}{|y|^3}\hat{y}$ in the +y direction.