Binomial Theorem

PROBLEM-SOLVING TACTICS

Summation of series involving binomial coefficients

For $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$, the binomial coefficients are ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$. A number of series may be formed with these coefficients figuring in the terms of a series.

Some standard series of the binomial coefficients are as follows:

(a) By putting x = 1, we get $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$ ...(i)

(b) By putting x =-1, we get $\quad {}^nC_0 - {}^nC_1 + {}^nC_2 - ...... + (-1)^n . {}^nC_n = 0$ ...(ii)

(c) On adding (i) and (ii), we get $\quad {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}$ ...(iii)

(d) On subtracting (ii) from (i), we get $\quad {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$ ...(iv)

(e) ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + ... + {}^{2n}C_{n-1} + {}^{2n}C_n = 2^{2n-1}$

Proof: From the expansion of $(1+x)^{2n}$, we get ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + ... + {}^{2n}C_{2n-1} + {}^{2n}C_{2n} = 2^{2n}$

$$\Rightarrow 2\left({}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + ... + {}^{2n}C_{n-1}\right) + {}^{2n}C_n = 2^{2n} \; [\because {}^{2n}C_0 = {}^{2n}C_{2n}, {}^{2n}C_1 = {}^{2n}C_{2n-1} \text{ and so on.}]$$

(f) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n = 2^{2n}$

Proof: (as above)

(g) Sum of the first half of ${}^nC_0 + {}^nC_1 + ... + {}^nC_n$ = Sum of the last half of ${}^nC_0 + {}^nC_1 + ... + {}^nC_n = 2^{n-1}$

(h) **Bino-geometric series:** ${}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n = (1+x)^n$

(i) **Bino-arithmetic series:** $a\,{}^nC_0 + (a+d)\,{}^nC_1 + (a+2d)\,{}^nC_2 + ... + (a+nd)\,{}^nC_n$

Consider an AP-a, (a+d), (a+2d), ... , (a+nd)

Sequence of Binomial Co-efficient - ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$

A **bino-arithmetic** series is nothing but the sum of the products of corresponding terms of the sequences. It can be added in two ways.

(i) By elimination of r in the multiplier of binomial coefficient from the $(r+1)^{th}$ term of the series (By using $r.{}^nC_r = n\,{}^{n-1}C_{r-1}$)

(ii) By differentiating the expansion of $x^d(1+x^d)^n$.

(j) Bino-harmonic series: $\frac{{}^nC_0}{a}+\frac{{}^nC_1}{a+d}+\frac{{}^nC_2}{a+2d}+......+\frac{{}^nC_n}{a+nd}$

Consider an HP - $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},...,\frac{1}{a+nd}$

Sequence of Binomial Co-efficient - ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$

It is obtained by the sum of the products of corresponding terms of the sequences. Such series are calculated in two ways :

(i) By elimination of r in the multiplier of binomial coefficient from the (r + 1)th term of the series $\left(\text{By using}\frac{1}{r+1}{}^nC_r=\frac{1}{n+1}{}^{n+1}C_{r+1}\right)$

(ii) By integrating suitable expansion.

For explanation see illustration 2

(k) Bino-binomial series: ${}^nC_0.{}^nC_r+{}^nC_1.{}^nC_{r+1}+{}^nC_2.{}^nC_{r+2}+.......+{}^nC_{n-r}.{}^nC_r$

or, ${}^mC_0.{}^nC_r+{}^mC_1.{}^nC_{r-1}+{}^mC_2.{}^nC_{r-2}+.....+{}^mC_r.{}^nC_0$

As the name suggests such series are obtained by multiplying two binomial expansion, one involving the first factors as coefficient and the other involving the second factors as coefficient. They can be calculated by equating coefficients of a suitable power on both sides.

For explanation see illustration 4

FORMULAE SHEET

Binomial theorem for any positive integral index:

$$(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+........+{}^nC_rx^{n-r}a^r+......+{}^nC_na^n=\sum_{r=0}^{n}{}^nC_rx^{n-r}a^r$$

(a) General term – $T_{r+1}={}^nC_rx^{n-r}a^r$ is the (r + 1)th term from beginning.

(b) (m + 1)th term from the end = (n – m + 1)th from beginning = T_{n-m+1}

(c) Middle term

(i) If n is even then middle term = $\left(\frac{n}{2}+1\right)^{th}$ term

(ii) If n is odd then middle term = $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$

Binomial coefficient of middle term is the greatest binomial coefficient.

To determine a particular term in the given expansion:

Let the given expansion be $\left(x^{\alpha}\pm\frac{1}{x^{\beta}}\right)^{th}$, if x^n occurs in T_{r+1} (r + 1)$^{th\ term}$ then r is given by $n\alpha-r(\alpha+\beta)=m$ and for $x^0, n\alpha-r(\alpha+\beta)=0$

Properties of Binomial coefficients:

For the sake of convenience the coefficients ${}^nC_0, {}^nC_1, {}^nC_2 ...... {}^nC_r .... {}^nC_n$ are usually denoted by $C_0, C_1, ... C_r .... C_n$ respectively.

$$C_0 + C_1 + C_2 + ... + C_n = 2^n$$

$$C_0 - C_1 + C_2 - C_3 + ... + C_n = 0$$

$$C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$$

$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} \text{ and so on........}$$

$${}^{2n}C_{n+r} = \frac{2n!}{(n-r)!(n+r)!}$$

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$C_1 + 2C_2 + 3C_3 + ... + {}^nC_n = n.2^{n-1}$$

$$C_1 - 2C_2 + 3C_3 .... = 0$$

$$C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$$

$$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$$

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... = \begin{cases} 0, \text{ if n is odd} \\ (-1)^{n/2}\,{}^nC_{n/2}, \text{ if n is even} \end{cases}$$

Note: ${}^{2n+1}C_0 + {}^{2n+1}C_1 + ... + {}^{2n+1}C_n = {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ... + {}^{2n+1}C_{2n+1} = 2^{2n}$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1};\ C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} ...... + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$$

(a) Greatest term:

(i) If $\frac{(n+1)a}{x+a} \in Z$ (integer) then the expansion has two greatest terms. These are k^{th} and $(k + 1)^{th}$ where x and a are +ve real numbers.

(ii) If $\frac{(n+1)a}{x+a} \notin Z$ then the expansion has only one greatest term. This is $(K + 1)^{th}$ term $k = \left[\frac{(n+1)a}{x+a}\right]$ denotes greatest integer less than or equal to x}

(b) Multinomial theorem:

Generalized $(x_1 + x_2 + ... + x_k)^n = \sum_{r_1 + r_2 + ... r_k = n} \frac{n!}{r_1!r_2!......r_k!} x_1^{r_1} x_2^{r_2} .....x_k^{r_k}$

(c) Total no. of terms in the expansion $(x_1 + x_2 + ...x_n)^m$ is ${}^{m+n-1}C_{n-1}$