4. BINOMIAL THEOREM

MATHEMATICAL INDUCTION

The technique of Induction is used to prove mathematical theorems. A variety of statements can be proved using this method. Mathematically, if we show that a statement is true for some integer value, say n = 0, and then we prove that the statement is true for some integer k+1 if it is true for the integer k (k is greater than or equal to 0), then we can conclude that it is true for all integers greater than or equal to 0.

The solution in mathematical induction consists of the following steps:

Step 1: Write the statement to be proved as P(n) where n is the variable.

Step 2: Show that P(n) is true for the starting value of n equal to 0(say).

Step 3: Assuming that P(k) is true for some k greater than the starting value of n, prove that P(k+1) is also true.

Step 4: Once P(k+1) has been proved to be true, we say that the statement is true for all values of the variable.

The following illustrations will help to understand the technique better.

Illustration 1: Prove that 1+2+3+...+n=n(n+1)/2 for all n, n is natural. **(JEE MAIN)**

Sol: Clearly, the statement P(n) is true for n = 1. Assuming P(k) to be true, add (k+1) on both sides of the statement.

P(n):1+2+3+...+n=n(n+1)/2

Clearly, P(1) is true as 1=1.2/2.

Let P(k) be true. That is, let 1+2+3+...+k be equal to k(k+1)/2

Now, we have to show that P(k+1) is true, or that

1+2+3+...+(k+1)=(k+1)(k+2)/2.

L.H.S = 1+2+3+...+(k+1)

= 1+2+3+...+k+(k+1) = k(k+1)/2 + (k+1) (As P(k) is true)

= (k+1) (k/2+1) = (k+1)(k+2)/2

= R.H.S

Illustration 2: Prove that (n+1)! $>2^n$ for all n>1. **(JEE MAIN)**

Sol: For n = 2, the given statement is true. Now assume the statement to be true for n = m and multiply (m+2) on both sides.

Let $(n+1)! > 2^n$ ... (i)

Putting n=2 in eq. (i), we get,

$3! > 2^2$

$3! > 4$

Since this is true,

Therefore the equation holds true for n=2.

Assume that equation holds true for n=m,

$(m+1)! > 2^m$ ... (ii)

Now, we have to prove that this equation holds true for n=m+1, i.e. $(m+2)! > 2^{m+1}$.

From equation 2, $(m+1)! > 2^m$.

Multiply above equation by m+2

$(m+2)! > 2^m (m+2)$

$> 2^{m+1} + 2^m.m$

$> 2^{m+1}$.

Hence proved.

Illustration 3: Prove that $n^2 + n$ is even for all natural numbers n. **(JEE MAIN)**

Sol: Consider $P(n) = n^2 + n$. It can written as a product of two consecutive natural numbers. Use this fact to prove the question.

Consider that P(n) $n^2 + n$ is even, P(1) is true as $1^2 + 1= 2$ is an even number.

Consider P(k) be true,

To prove : P(k + 1) is true.

P(k + 1) states that $(k + 1)^2 + (k + 1)$ is even.

Now, $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$

As P(k) is true, hence $k^2 + k$ is an even number and can be written as 2λ, where λ is sum of natural number.

$\therefore\ 2\lambda + 2k + 2 \Rightarrow 2(\lambda + k + 1)$ = a multiple of 2.

Thus, $(k + 1)^2 + (k + 1)$ is an even number.

Hence, P(n) is true for all n, where n is a natural number.

Illustration 4: Prove that exactly one among n+10, n+12 and n+14 is divisible by 3, considering n is always an natural number. **(JEE MAIN)**

Sol: We can observe here that

For n = 1,

n+10 = 11

n+12 = 13

n+14 = 15

Exactly one i.e 15 is divisible by 3.

Let us assume that that for n = m exactly one out of n+10, n+12, n+14 is divisible by 3

Without the loss of generality consider for n=m, m+10 was divisible by 3

Therefore, m+10 = 3k

m+12 = 3k+2

m+14 = 3k+4

We need to prove that for n=m+1 , exactly one among them is divisible by 3. Putting m+1 in place of n, we get

(m+1)+10 = m+11 = 3k + 1 (not divisible by 3)

(m+1)+12 = m+13 = 3k+3 = 3(k+1) (divisible by 3)

(m+1)+14 = m+15 = 3k+5 (not divisible by 3)

Therefore, for n=m+1 also exactly one among the three, n+10, n+12 and n+14 is divisible by 3.

Similarly we can prove that exactly one among three of these is divisible by 3 by considering cases when n+12 = 3k and n+14 = 3k.

BINOMIAL THEOREM

1. INTRODUCTION TO BINOMIAL THEOREM

1.1 Introduction

Consider two numbers a and b, then

$$\left(a+b\right)^2 = a^2 + 2ab + b^2$$

$$\left(a+b\right)^3 = \left(a+b\right)\left(a+b\right)^2 \qquad = \left(a+b\right)\left(a^2+2ab+b^2\right) \qquad = a^3+3a^2b+3ab^2+b^3$$

$$\left(a+b\right)^4 = \left(a+b\right)^2\left(a+b\right)^2 \qquad = \left(a^2+2ab+b^2\right)\left(a^2+2ab+b^2\right) \qquad = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

As the power increases, the expansion becomes lengthy, difficult to remember and tedious to calculate. A binomial expression that has been raised to a very large power (or degree), can be easily calculated with the help of Binomial Theorem.

1.2 Binomial Expression

A binomial expression is an algebraic expression which contains two dissimilar terms.

For example: $x+y,\ a^2+b^2,\ 3-x,\ \sqrt{x^2+1}+\dfrac{1}{\sqrt[3]{x^3+1}}$ etc.

1.3 Binomial Theorem

Let n be any natural number and x, a be any real number, then

$$\left(x+a\right)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}a^2 + \ldots\ldots + {}^nC_r\, x^{n-r}a^r + \ldots\ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e. $\left(x+a\right)^n = \sum_0^n {}^nC_r x^{n-r}a^r$ where ${}^nC_r = \dfrac{n!}{r!\left(n-r\right)!}$

and the co-efficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ldots\ldots$ and ${}^nC_n$ are known as binomial coefficient.

MASTERJEE CONCEPTS

(a) The total number of terms in the expansion of $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$, is (n + 1).

(b) The sum of the indices of x and a in each term is n.

(c) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients and also represented by C_0, C_1, C_2 and so on.

(i) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$ (ii) ${}^nC_r = {}^nC_{n-r}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (iv) ${}^nC_r = n/(n-r).{}^{n-1}C_r$

Vaibhav Gupta (JEE 2009, AIR 22)

Illustration 5: Expand the following binomials

(i) $(x - 2)^5$ (ii) $\left(1-\frac{3x^3}{2}\right)^4$ **(JEE MAIN)**

Sol: By using formula of binomial expansion.

(i) $(x-2)^5 = {}^5C_0x^5 + {}^5C_1x^4(-2)^1 + {}^5C_2x^3(-2)^2 + {}^5C_3x^2(-2)^3 + {}^5C_4x(-2)^4 + {}^5C_5(-2)^5$

$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

(ii) $\left(1-\frac{3x^3}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^3}{2}\right) + {}^4C_2\left(-\frac{3x^3}{2}\right)^2 + {}^4C_3\left(-\frac{3x^3}{2}\right)^3 + {}^4C_4\left(-\frac{3x^3}{2}\right)^4$

$= 1 - 6x^3 + \frac{27}{2}x^6 - \frac{27}{2}x^9 + \frac{81}{16}x^{12}$

2. DEDUCTIONS FROM BINOMIAL THEOREM

2.1 Results of Binomial Theorem

D-1 On replacing a by –a, in the expansion of $(x + a)^n$, we get

$$(x-a)^n = {}^nC_0\, x^na^0 - {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}.a^2 - \ldots + (-1)^r\, {}^nC_rx^{n-r}a^r + \ldots + (-1)^n\, {}^nC_nx^0a^n$$

i.e. $(x-a)^n = \sum_{r=0}^{n}(-1)^r\, {}^nC_rx^{n-r}a^r$

Therefore, the terms in $(x - a)^n$ are alternatively positive and negative, and the sign of the last term is positive or negative depending on whether n is even or odd.

D-2 Putting x = 1 and a = x in the expansion of $(x + a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

$$\Rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$$

This is the expansion of $(1 + x)^n$ in ascending powers of x.

D-3 Putting a = 1 in the expansion of $(x + a)^n$, we get

$$(x+1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + ... + {}^nC_rx^{n-r} + ... + {}^nC_{n-1}x + {}^nC_n \Rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}$$

This is the expansion of $(1 + x)^n$ in descending powers of x.

D-4 Putting x = 1 and a = –x in the expansion of $(x + a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2\,x^2 - {}^nC_3x^3 + ... + (-1)^r\,{}^nC_r\,x^r + ... + (-1)^n\,{}^nC_nx^n$$

D-5 From the above expansions, we can also deduce the following

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + ...\right]$$

and $(x+a)^n - (x-a)^n = 2\left[{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}\,a^3 + ...\right]$

MASTERJEE CONCEPTS

If n is odd then $\left\{(x+a)^n + (x-a)^n\right\}$ and $\left\{(x+a)^n - (x-a)^n\right\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ where as if n is even, then $\left\{(x+a)^n + (x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

Nikhil Khandelwal (JEE 2009, AIR 94)

2.2 Properties of Binomial Coefficients

Using binomial expansion, we have

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_rx^r + ... + {}^nC_nx^n$$

Also, $(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + ... + {}^nC_rx^{n-r} + ... + {}^nC_{n-1}x + {}^nC_n$

Let us represent the binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_{n-1}, {}^nC_n$ by $C_0, C_1, C_2, ..., C_{n-1}, C_n$ respectively. Then the above expansions become

$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ i.e. $(1+x)^n = \sum_{r=0}^{n} C_rx^r$

Also, $(1+x)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_rx^{n-r} + ... + C_{n-1}x + C_n$ i.e. $(1+x)^n = \sum_{r=0}^{n} C_rx^{n-r}$

The binomial coefficients C_0, C_1, C_2,........C_{n-1}, and C_n posses the following properties:

Property-I In the expansion of $(1 + x)^n$, the coefficients of terms equidistant from the beginning and the end are equal.

Property-II The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n.

i.e. $C_0 + C_1 + C_2 + ... + C_n = 2^n$ or, $\sum_{r=0}^{n} {}^nC_r = 2^n$.

Property-III The sum of the coefficient of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficient of the even terms and each is equal to 2^{n-1}.

i.e. $C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$

Property-IV ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$ and so on.

Property-V $C_0 - C_1 + C_2 - C_3 + C_4 - \ldots\ldots + (-1)^n C_n = 0$

i.e. $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$

MASTERJEE CONCEPTS

(a) ${}^{(n+1)}C_r = {}^nC_r + {}^nC_{r-1}$ (b) $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ (c) $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

(d) When n is even, $(x+a)^n + (x-a)^n = 2\left(x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + \ldots\ldots + {}^nC_n a_n\right)$

When n is odd, $(x+a)^n + (x-a)^n = 2\left(x^n + {}^nC_2 x^{n-2} a^2 + \ldots\ldots + {}^nC_{n-1} x a^{n-1}\right)$

When n is even $(x+a)^n - (x-a)^n = 2\left({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots\ldots + {}^nC_{n-1} x a^{n-1}\right)$

When n is odd $(x+a)^n - (x-a)^n = 2\left({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots\ldots + {}^nC_n a^n\right)$

Saurabh Gupta (JEE 2010, AIR 443)

Illustration 6: If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots + C_n x^n$, then show that **(JEE MAIN)**

(i) $C_0 + 4C_1 + 4^2 C_2 + \ldots\ldots\ldots\ldots + 4^n C_n = 5^n$

(ii) $C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots\ldots + (n+1)C_n = 2^{n-1}(n+2)$

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Sol: By using properties of binomial coefficients and methods of summation, differentiation, and integration we can easily prove given equations.

(i) $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots + C_n x^n$

Putting x=4, we have $C_0 + 4C_1 + 4^2 C_2 + \ldots\ldots\ldots\ldots + 4^n C_n = 5^n$

(ii) $C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots\ldots + (n+1)C_n = 2^{n-1}(n+2)$

Method 1: By Summations

r^{th} term in the series is given by $(r+1).{}^nC_r$

Therefore, L.H.S = ${}^nC_0 + 2.{}^nC_1 + 3.{}^nC_2 + \ldots\ldots\ldots + (n+1).{}^nC_n = \sum_{r=0}^{n} (r+1).{}^nC_r$

$$= \sum_{r=0}^{n} r.{}^nC_r + \sum_{r=0}^{n} {}^nC_r = n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \;=\; n.2^{n-1} + 2^n = 2^{n-1}(n+2) \;=\; \text{R.H.S}$$

Method 2: By Differentiation

$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Multiplying x on both sides, $x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$

On differentiating, we have $(1+x)^n + xn(1+x)^{n-1} = C_0 + 2.C_1x + 3.C_2x^2 + ... + (n+1)C_nx^n$

Putting x = 1, we get $C_0 + 2.C_1 + 3.C_2 + ... + (n+1)C_n = 2^n + n.2^{n-1}$

$C_0 + 2.C_1 + 3.C_2 + ... + (n+1)C_n = 2^{n-1}(n+2)$

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Method 1: By Summations

r^{th} term in the series is given by $(-1)^r.\frac{{}^nC_r}{r+1}$

Therefore, L.H.S. = $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n.\frac{C_n}{n+1} \;=\; \sum_{r=0}^{n}(-1)^r.\frac{{}^nC_r}{r+1}$

$$= \frac{1}{n+1}\sum_{r=0}^{n}(-1)^r\,{}^{n+1}C_{r+1} \qquad \left\{\text{using } \frac{n+1}{r+1}.{}^nC_r = {}^{n+1}C_{r+1}\right\} \;=\; \frac{1}{n+1}\left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - .... + (-1)^n.{}^{n+1}C_{n+1}\right]$$

Adding and subtracting the term ${}^{n+1}C_0$, we have

$$= \frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n.{}^{n+1}C_{n+1} + {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1} \qquad \text{as}\left[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n.{}^{n+1}C_{n+1} = 0\right] = \text{R.H.S.}$$

Method 2: By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$.

On integrating both sides within the limits –1 to 0, we have

$$\int_{-1}^{0}(1 + x)^n\,dx = \int_{-1}^{0}\left(C_0 + C_1x + C_2x^2 + ... + C_nx^n\right)dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ... + (-1)^{n+1}\frac{C_n}{n+1}\right] \Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} + ... + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$$

Illustration 7: If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1 . C_0^2 + 3. C_1^2 + 5 . C_2^2 + ... + (2n+1) . C_n^2 . = 2n . {}^{2n-1}C_n + {}^{2n}C_n$ **(JEE ADVANCED)**

Sol: In the expansion of $(1+x)^{2n}$, (i) and (ii) can be proved by comparing the coefficients of x^n and x^{n-2} respectively. The third equation can be proved by two methods - the method of summation and the methods of differentiation.

(i) $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ ....(i)

Also, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_nx^0$ .... (ii)

Multiplying equation (i) and (ii)

$$(1+x)^{2n} = \left(C_0 + C_1x + C_2x^2 + ... + C_nx^n\right)\left(C_0x^n + C_1x^{n-1} + ... + C_nx^0\right) \quad .... \text{(iii)}$$

On comparing the coefficients of x^n both sides, we have

$\Rightarrow {}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + ... + C_n{}^2$ Hence, Proved.

(ii) From (iii), on comparing the coefficients of x^{n-2} or x^{n+2}, we have

$C_0C_1 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1 . C_0^2 + 3. C_1^2 + 5 . C_2^2 + ... + (2n+1) . C_n^2 . = 2n . {}^{2n-1}C_n + {}^{2n}C_n$

Method 1: By Summation

r^{th} term in the series is given by $(2r+1)\,{}^nC_r^2$

L.H.S. $= 1.C_0^2 + 3.C_1^2 + 5.C_2^2 + ... + (2n+1)C_n^2 \qquad = \sum_{r=0}^{n}(2r+1)\,{}^nC_r^2$

$$= \sum_{r=0}^{n} 2.r.\left({}^nC_r\right)^2 + \sum_{r=0}^{n}\left({}^nC_r\right)^2 = 2\sum_{r=1}^{n}.n.{}^{n-1}C_{r-1}\,{}^nC_r + {}^{2n}C_n$$

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$ .... (i)

$(x+1)^{n-1} = {}^{n-1}C_0x^{n-1} + {}^{n-1}C_1x^{n-2} + ... + {}^{n-1}C_{n-1}x^0$ .... (ii)

Multiplying (i) and (ii) and comparing coefficients of x^n, we have

$${}^{2n-1}C_n = {}^{n-1}C_0.{}^nC_1 + {}^{n-1}C_1.{}^nC_2 + ... + {}^{n-1}C_{n-1}.{}^nC_n$$

i.e. $\sum_{r=1}^{n} {}^{n-1}C_{r-1}\,{}^nC_r = {}^{2n-1}C_n$

Hence, required summation is $2n.\,{}^{2n-1}C_n + {}^{2n}C_n$

Method 2: By Differentiation

$$\left(1+x^2\right)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + ... + C_nx^{2n}$$

Multiplying x on both sides

$$x\left(1+x^2\right)^n = C_0x + C_1x^3 + C_2x^5 + ... + C_nx^{2n+1}$$

Differentiating both sides

$$x.n\left(1+x^2\right)^{n-1}.2x+\left(1+x^2\right)^n=C_0+3.C_1x^2+5.C_2x^4+......+\left(2n+1\right)C_nx^{2n} \quad .... (i)$$

$$\left(x^2+1\right)^n=C_0x^{2n}+C_1x^{2n-2}+C_2x^{2n-4}+.......+C_n \quad .... (ii)$$

On multiplying (i) and (ii), we have

$$2nx^2\left(1+x^2\right)^{2n-1}+\left(1+x^2\right)^{2n}=\left(C_0+3C_1x^2+5C_2x^4+.....+\left(2n+1\right)C_nx^{2n}\right)\left(C_0x^{2n}+C_1x^{2n-2}+......+C_n\right)$$

Comparing coefficient of x^{2n},

$$2n.^{2n-1}C_{n-1}+^{2n}C_n=C_0^2+3C_1^2+5C_2^2+.....+\left(2n+1\right)C_n^2$$

$$\therefore C_0^2+3C_1^2+5C_2^2+.....+\left(2n+1\right)C_n^2=2n.^{2n-1}C_n+^{2n}C_n$$

Illustration 8: If $\left(1+x\right)^n=C_0+C_1x+C_2x^2+......+C_nx^n$,

Prove that $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n=\dfrac{2n!}{\left(n-r\right)!\left(n+r\right)!}$ **(JEE MAIN)**

Sol: Clearly the differences of lower suffixes of binomial coefficients in each term is r.

By using properties of binomial coefficients we can easily prove given equations.

Given $\left(1+x\right)^n=C_0+C_1x+C_2x^2+........+C_{n-r}x^{n-r}+.........+C_nx^n$ .... (i)

Now $\left(x+1\right)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+........+C_rx^{n-r}+C_{r+1}x^{n-r-1}+.........+C_n$ ....(ii)

Multiplying (i) and (ii), we get

$$\left(x+1\right)^{2n}=\left(C_0+C_1x+C_2x^2+.....+C_{n-r}x^{n-r}+.........+C_nx^n\right)$$

$$\times\left(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+.....+C_rx^{n-r}+C_{r+1}x^{n-r-1}+C_{r+2}x^{n-r-2}+......+C_n\right) \quad .... (iii)$$

Now coefficient of x^{n-r} on L.H.S. of (iii) = $^{2n}C_{n-r}=\dfrac{2n!}{\left(n-r\right)!\left(n+r\right)!}$

and coefficient of x^{n-r} on R.H.S. of (iii) = $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n$

But (iii) is an identity, therefore, of x^{n-r} in R.H.S. = Coefficient of x^{n-r} in L.H.S.

$$\Rightarrow C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n=\frac{2n!}{\left(n-r\right)!\left(n+r\right)!}$$

Hence, Proved.

Illustration 9: Prove that $^nC_0\cdot{}^{2n}C_r-{}^nC_1\cdot{}^{2n-2}C_r+.......=2^{2n-r}.{}^nC_{r-n}$ if r > n and 0 if r < n. **(JEE MAIN)**

Sol: By comparing coefficient of x^r in L.H.S. and R.H.S. in the expansion of $\left[\left(1+x\right)^2-1\right]^n$ we can prove it.

$$\left[\left(1+x\right)^2-1\right]^n={}^nC_0\left(1+x\right)^{2n}-{}^nC_1\left(1+x\right)^{2n-2}+{}^nC_2\left(1+x\right)^{2n-4}+... \quad .... (i)$$

Coefficient of x^r in R.H.S. = $^nC_0.{}^{2n}C_r-{}^nC_1.{}^{2n-2}C_r+............$ ... (ii)

L.H.S. = $[(1+x)^2-1]^n=[2x+x^2]^n=x^n(2+x)^n$

$\therefore$ Coefficient of x^r in $x^n (2 + x)^n$

= Coefficient of x^{r-n} in $(2 + x)^n = {}^nC_{r-n} 2^{2n-r}$ if $r > n$

= 0 if $r < n$ (Since lower suffix cannot be negative)

But (i) is an identity, therefore coefficient of x^r in R.H.S. = coefficient of x^r in L.H.S.

Hence ${}^nC_0.{}^{2n}C_r - {}^nC_1.{}^{2n-2}C_r + ...$ $= {}^nC_{r-n}2^{2n-r}$ if $r > n$

$= 0$ if $r < n$.

Illustration 10: Show that $C_0.{}^{2n}C_n - C_1.{}^{2n-1}C_n + C_2.{}^{2n-2}C_n - C_3.{}^{2n-3}C_n + ... + (-1)^n C_n.{}^nC_n = 1$ **(JEE ADVANCED)**

Sol: Observe the pattern in the terms on the LHS. The first term $C_0.{}^{2n}C_n$ is the co-efficient of x^n in the expansion of $C_0(1+x)^{2n}$. Similarly, $C_1.{}^{2n-1}C_n$ is the co-efficient of x^n in $C_0(1+x)^{2n}$ and so on. On adding all the coefficients of x^n we can prove the given equation.

Note that $C_0.{}^{2n}C_n - C_1.{}^{2n-1}C_n + C_2.{}^{2n-2}C_n - C_3.{}^{2n-3}C_n + ... + (-1)\ {}^nC_n.{}^nC_n$

= Coefficient of x^n in $\left[C_0(1+x)^{2n} - C_1(1+x)^{2n-1} + C_2(1+x)^{2n-2} - C_3(1+x)^{2n-3} + ...(-1)^n C_n(1+x)^n\right]$

= Coefficient of x^n in $(1+x)^n\left[C_0(1+x)^n - C_1(1+x)^{n-1} + C_2(1+x)^{n-2} - C_3(1+x)^{n-3} + ...(-1)^n C_n\right]$

= Coefficient of x^n in $(1+x)^n\left[(1+x)-1\right]^n$

= Coefficient of x^n in $(1+x)^n(x)^n$

= Coefficient of the constant terms in $(1 + x)^n = 1$

3. TERMS IN BINOMIAL EXPANSION

3.1 General Term in Binomial Expansion

We have, $(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} x^2 + ... + {}^nC_r x^{n-r} a^r + ... + {}^nC_n x^0 a^n$

$(r+1)^{th}$ term is given by ${}^nC_r\, x^{n-r} a^r$

Thus, if T_{r+1} denotes the $(r+1)^{th}$ term, then $T_{r+1} = {}^nC_r x^{n-r} a^r$

This is called the general term of the binomial expansion.

(a) The general term in the expansion of $(x - a)^n$, is given by $T_{r+1} = (-1)^r.{}^nC_r x^{n-r} a^r$

(b) The general term in the expansion of $(1 + x)^n$, is given by $T_{r+1} = {}^nC_r\, x^r$

(c) The general term in the expansion of $(1 - x)^n$, is given by $T_{r+1} = (-1)^r\, {}^nC_r x^r$

(d) In the binomial expansion of $(x + a)^n$, the r^{th} term from the end is $((n + 1) - r + 1)^{th}$ term i.e. $(n - r + 2)^{th}$ term from the beginning.

Illustration 11: The number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **(JEE MAIN)**

Sol: As we know that number of dissimilar terms in the expansion of $(1 - x)^n$ is n+1. Rewrite the given expression in the form of $(1 - x)^n$.

$(1 - 3x + 3x^2 - x)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is 61.

Illustration 12: Find (i) 28^{th} term of $(5x + 8y)^{30}$ (ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$ **(JEE MAIN)**

Sol: Here in this problem, by using $T_{r+1} = {}^nC_r x^{n-r} a^r$ we can easily obtain $(r+1)^{th}$ term of given expansion.

(i) 28^{th} term of $(5x + 8y)^{30}$

$$T_{28} = T_{27+1} = {}^{30}C_{27}(5x)^{30-27}(8y)^{27} = \frac{30!}{3!.27!}(5x)^3.(8y)^{27}$$

(ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$$T_7 = T_{6+1} = {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$$

Illustration 13: Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$. **(JEE ADVANCED)**

Sol: In this problem, by using $T_{r+1} = {}^nC_r x^{n-r} a^r$ we can easily obtain $(r+1)^{th}$ term of given expansion and after that by using the conditions of rational number we can obtain number of rational terms.

The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{1/6}\right)^r = {}^{1000}C_r 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

T_{r+1} will be rational if the power of 3 and 2 are integers. It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers.

Therefore the possible set of values of r is {0, 2, 4… … 1000}. Hence, number of rational terms is 501.

3.2 Middle Term in Binomial Expansion

(a) If n is even, then the number of terms in the expansion i.e. (n + 1) is odd, therefore, there will be only one middle term which is $\left(\frac{n+2}{2}\right)^{th}$ term i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

So middle term = $\left(\frac{n}{2}+1\right)^{th}$ term i.e. $T_{\left(\frac{n}{2}+1\right)} = {}^nC_{\frac{n}{2}} x^{\frac{n}{2}} a^{\frac{n}{2}}$

(b) If n is odd, then the number of terms in the expansion i.e. (n+1) is even, therefore there will be two middle terms which are

$$= \left(\frac{n+1}{2}\right)^{th} \text{and} \left(\frac{n+3}{2}\right)^{th} \text{ term i.e. } T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\left(\frac{n-1}{2}\right)} x^{\frac{n+1}{2}} a^{\frac{n-1}{2}} \text{ and } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)} x^{\frac{n-1}{2}} a^{\frac{n+1}{2}}$$

MASTERJEE CONCEPTS

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest Binomial coefficient.

Neeraj Toshniwal (JEE 2009, AIR 21)

Illustration 14: Find the middle term(s) in the expansion of (i) $\left(1-\frac{x^2}{2}\right)^{14}$ (ii) $\left(3a-\frac{a^3}{6}\right)^9$ **(JEE MAIN)**

Sol: By using appropriate formula of finding middle term(s) i.e. $\left(\frac{n}{2}+1\right)^{th}$ when n is even and $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ when n is odd, we can obtain the middle terms of given expansion.

(i) $\left(1-\frac{x^2}{2}\right)^{14}$ Since, n is even, therefore middle term is $\left(\frac{14}{2}+1\right)^{th}$ term.

$\therefore T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}$

(ii) $\left(3a-\frac{a^3}{6}\right)^9$

Since, n is odd therefore, the middle terms are $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+1}{2}+1\right)^{th}$.

$\therefore\ T_5 = {}^9C_4\left(3a\right)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}a^{17}$ and $T_6 = {}^9C_5\left(3a\right)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}a^{19}$.

3.3 Determining a Particular Term

In the expansion of $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$n\alpha - r\left(\alpha+\beta\right) = m \qquad \Rightarrow r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term i.e. the term independent of x, occurs in T_{r+1} then r is determined by

$n\alpha - r\left(\alpha+\beta\right) = 0 \qquad \Rightarrow r = \frac{n\alpha}{\alpha+\beta}$

Illustration 15: The term independent of x in the expansion of $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$ is **(JEE MAIN)**

Sol: By using the result proved above i.e. $r = \frac{n\alpha}{\alpha+\beta}$, we can obtain the term independent of x. Here, α and β are obtained by comparing given expansion to $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$.

On comparing $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$ with $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, we get $\alpha = 2, \beta = 1, n = 9$

i.e. $r = \frac{9(2)}{2+1} = 6$ $\therefore$ (6 + 1) = 7th term is independent of x.

Illustration 16: The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is **(JEE MAIN)**

Sol: Here in this problem, by using standard formulas of finding general term and term independent of x we can obtain the required ratio.

General term in the expansion is $T_{r+1} = {}^{15}C_r \left(x^2\right)^{15-r}\left(\frac{2}{x}\right)^r$ i.e., ${}^{15}C_r x^{30-3r}.2^r$

For x^{15}, $30 - 3r = 15 \Rightarrow 3r = 15 \quad \Rightarrow r = 5$

$\therefore T_6 = T_{5+1} = {}^{15}C_5\left(x^2\right)^{15-5}\left(\frac{2}{x}\right)^5$ i.e., ${}^{15}C_5 x^{15}.2^5$

$\therefore$ Coefficient of x^{15} is ${}^{15}C_5\, 2^5$ (r = 5)

For the constant term $30 - 3r = 0 \Rightarrow r = 10$.

$\therefore\ T_{11} = T_{10+1} = {}^{15}C_{10}\left(x^2\right)^{15-10}\left(\frac{2}{x}\right)^{10}$ i.e., ${}^{15}C_{10}2^{10}$

$\therefore$ Coefficient of constant term is ${}^{15}C_{10}2^{10}$.

Hence, the required ratio is 1 : 32.

Illustration 17: The term independent of x in the expansion of $\left(\sqrt[6]{x} - \frac{1}{\sqrt[3]{x}}\right)^9$ is equal to **(JEE MAIN)**

Sol: By using the formula $T_{r+1} = {}^nC_r x^{n-r} a^r$ we can solve it.

$$T_{r+1} = {}^9C_r\left(\sqrt[6]{x}\right)^{9-r}\left(-\frac{1}{\sqrt[3]{x}}\right)^r = {}^9C_r(-1)^r x^{\frac{9-r}{6}-\frac{r}{3}} = {}^9C_r(-1)^r x^{\left(\frac{9-3r}{6}\right)}$$

$$\Rightarrow \frac{9-3r}{6} = 0 \quad \Rightarrow r = 3 \quad \therefore T_4 = T_{3+1} = -\ {}^9C_3$$

Illustration 18: If the second, third and fourth terms in the expansion of $(b+a)^n$ are 135, 30 and 10/3 respectively, then n is equal to **(JEE MAIN)**

Sol: In this problem, by using the formula of finding general term we will get the equation of given terms and by taking ratios of these terms we can get the value of n.

$T_2 = {}^nC_1\, ab^{n-1} = 135$ ...(i)

$T_3 = {}^nC_2\, a^2b^{n-2} = 30$ ...(ii)

$T_4 = {}^nC_3\, a^3b^{n-3} = \frac{10}{3}$ ...(iii)

On dividing (i) by (ii), we get

$$\frac{{}^nC_1 ab^{n-1}}{{}^nC_2 a^2b^{n-2}} = \frac{135}{30} \qquad \Rightarrow \frac{n}{\frac{n(n-1)}{2}}\frac{b}{a} = \frac{9}{2} \quad \text{...(iv)}$$

$$\therefore \frac{b}{a} = \frac{9}{4}(n-1) \quad \text{...(v)}$$

Dividing (ii) and (iii), we get

$$\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)(n-2)}{3.2}}\cdot\frac{b}{a}=\frac{30\times 3}{10}=9 \qquad \Rightarrow \frac{3}{(n-2)}\frac{b}{a}=9 \qquad \text{...(vi)}$$

Eliminating a and b from (v) and (vi) $\Rightarrow$ n = 5

Illustration 19: If a, b, c and d are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, n being positive integer, show that $\frac{a}{a+b}+\frac{c}{c+d}=\frac{2b}{b+c}$ **(JEE MAIN)**

Sol: Consider four consecutive terms and use ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$.

The $(r + 1)^{th}$ term is $T_{r+1}={}^nC_rx^r$

$\therefore$ The coefficient of term $T_{r+1}={}^nC_r$

$\therefore$ Now take four consecutive terms as (r – 1)th, rth, (r + 1)th and (r + 2)th

$\therefore$ We get $a={}^nC_{r-2}, b={}^nC_{r-1}, c={}^nC_r, d={}^nC_{r+1}$

$$a+b={}^nC_{r-2}+{}^nC_{r-1}={}^{n+1}C_{r-1}$$

$$b+c={}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$$

$$c+d={}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$$

$$\therefore \frac{a}{a+b}=\frac{{}^nC_{r-2}}{{}^{n+1}C_{r-1}}=\frac{n!}{(r-2)!(n-r+2)!}\times\frac{(r-1)!(n-r+2)!}{(n+1)!}=\frac{r-1}{n+1}$$

$$\frac{b}{b+c}=\frac{{}^nC_{r-1}}{{}^{n+1}C_r}=\frac{n!}{(r-1)!(n-r+2)!}\times\frac{r!(n-r+1)!}{(n+1)!}=\frac{r}{n+1}$$

$$\frac{c}{c+d}=\frac{{}^nC_r}{{}^{n+1}C_{r+1}}=\frac{n!}{n!(n-r)!}\times\frac{(r+1)!\,(n-r)!}{(n+1)!}=\frac{r+1}{n+1}$$

$$\therefore \frac{a}{a+b}+\frac{c}{c+d}=\frac{r-1}{n+1}+\frac{r+1}{n+1}=\frac{2r}{n+1}=2\left(\frac{r}{n+1}\right)=\frac{2b}{b+c}$$

3.4 Finding a Term from the End of Expansion

In the expansion of $(x + a)^n$, $(r + 1)^{th}$ term from end = $(n - r + 1)^{th}$ term from beginning i.e. $T_{r+1}(E)=T_{n-r+1}(B)$

$\therefore T_r(E)=T_{n-r+2}(B)$

Illustration 20: The 4^{th} term from the end in the expansion of $(2x - 1/x^2)^{10}$ is **(JEE MAIN)**

Sol: By using $T_r(E) = T_{n-r+2}(B)$ we will get the fourth term from the end in the given expansion.

Required term = $T_{10-4+2}=T_8={}^{10}C_7(2x)^3\left(-\frac{1}{x^2}\right)^7=-960\,x^{-11}$

3.5 Greatest Term in the Expansion

Let T_{r+1} and T_r be (r+1)th and rth terms respectively in the expansion of $(x+a)^n$. Then, $T_{r+1} = {}^nC_r x^{n-r} a^r$ and $T_r = {}^nC_{r-1} x^{n-r+1} a^{r-1}$.

$$\therefore \frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}} \quad = \frac{n!}{(n-r)!r!} \times \frac{(r-1)!(n-r+1)!}{n!} \cdot \frac{a}{x} \quad = \frac{n-r+1}{r} \cdot \frac{a}{x}$$

Now, $T_{r+1} >=< T_r \quad \Rightarrow \frac{T_{r+1}}{T_r} >=< 1 \quad \Rightarrow \frac{n-r+1}{r} \cdot \frac{a}{x} >=< 1 \quad \Rightarrow \left\{\left(\frac{n+1}{r}\right) - 1\right\} \frac{a}{x} >=< 1$

$$\Rightarrow \frac{n+1}{r} - 1 >=< \frac{x}{a} \quad \Rightarrow \frac{n+1}{r} >=< \left(1 + \frac{x}{a}\right) \quad \Rightarrow \frac{n+1}{1+\frac{x}{a}} >=< r$$

Thus, $T_{r+1} >=< T_r$ according as $\left(\frac{n+1}{1+\frac{x}{a}}\right) >=< r$ ...(i)

Now, two cases arise

Case-I: When $\frac{n+1}{1+\frac{x}{a}}$ is an integer Let $\frac{n+1}{1+\frac{x}{a}} = m$, Then, from (i), we have

$T_{r+1} > T_r$, for $r = 1,2,3,.....(m-1)$ ....(ii)

$T_{r+1} = T_r$, for $r = m$ ...(iii)

and, $T_{r+1} < T_r$, for $r = m+1,....n$ ....(iv)

$\therefore T_2 > T_1, T_3 > T_2, T_4 > T_3, ...T_m > T_{m-1}$ [From (ii)]

$T_{m+1} = T_m$ [From (iii)]

and, $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, T_{n+1} < T_n$ [From (iv)]

$\Rightarrow T_1 < T_2 < ..... < T_{m-1} < T_m = T_{m+1} > T_{m+2} .... > T_n$

This shows that m^{th} and $(m + 1)^{th}$ terms are greatest terms.

Case-II: When $\left[\frac{n+1}{1+\frac{x}{a}}\right] = m$. Then, from (i), we have

$T_{r+1} > T_r$ for $r = 1,2,.....,m$ ....(v)

and $T_{r+1} < T_r$ for $r = m+1, m+2,.....,n$ ....(vi)

$\therefore T_2 > T_1, T_3 > T_2, ..., T_{m+1} > T_m$ [From (v)]

and, $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ..., T_{n+1} < T_n$ [From (vi)]

$\Rightarrow T_1 < T_2 < T_3 < ..... < T_m < T_{m+1} > T_{m+2} > T_{m+3} ...... < T_{n+1}$

$\Rightarrow$ $(m + 1)^{th}$ term is the greatest term.

Following algorithm may be used to find the greatest term in a binomial expansion.

3.6 Algorithm to Find Greatest Term

Step I: From the given expansion, get T_{r+1} and T_r

Step II: Find $\frac{T_{r+1}}{T_r}$

Step III: Put $\frac{T_{r+1}}{T_r} > 1$

Step IV: Simplify the inequality obtained in step III, and write it in the form of either $r < m$ or $r > m$.

Step V: If m is an integer, then m^{th} and $(m+1)^{th}$ terms are the greatest terms and they are equal.

If m is not an integer, then $([m]+1)^{th}$ term is the greatest term, where [m] means the integral part of m.

3.7 Greatest Coefficient

Case-I When n is even, we have

$$\frac{^nC_r}{^nC_{r+1}} = \frac{n!}{(n-r)!r!} \times \frac{(r+1)!(n-r-1)!}{n!} = \frac{r+1}{n-r} \qquad \text{....(i)}$$

Now, for $0 \le r \le \frac{n}{2} - 1$ $\qquad \Rightarrow 1 \le r + 1 \le \frac{n}{2}$ and $\frac{n}{2} + 1 < n - r \le n$

$\Rightarrow \frac{r+1}{n-r} < 1$ $\qquad$ [Using (i)] $\Rightarrow \frac{^nC_r}{^nC_{r+1}} < 1 \Rightarrow {}^nC_r < {}^nC_{r+1}$

Putting $r = 0,1,2,......,\left(\frac{n}{2}-1\right)$, we get ${}^nC_0 < {}^nC_1, {}^nC_1 < {}^nC_2, {}^nC_2 < {}^nC_3 ..... < {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}}$

$$\Rightarrow {}^nC_0 < {}^nC_1 < {}^nC_2 < ..... < {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}} \qquad \text{...(ii)}$$

Since ${}^nC_{n-r} = {}^nC_r$

$\therefore {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}, ..., {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}}$

Substituting these values in (ii), we get

$$ {}^nC_n < {}^nC_{n-1} < {}^nC_{n-2} < ..... < {}^nC_{\frac{n}{2}+1} < {}^nC_{\frac{n}{2}} \qquad \text{...(iii)}$$

From (ii) and (iii), we refer that the maximum value of ${}^nC_r$ is ${}^nC_{n/2}$.

Case-II When n is odd

We have, $\frac{^nC_r}{^nC_{r+1}} = \frac{r+1}{n-r}$ $\qquad$ ...(i)

Now, $0 \le r < \frac{n-3}{2}$ $\qquad \Rightarrow 0 < r + 1 < \frac{n-1}{2}$ and $\frac{n-1}{2} \le n - r \le n$

$\Rightarrow \frac{r+1}{n-1} < 1 \Rightarrow \frac{{}^nC_r}{{}^nC_{r+1}} < 1$ [Using (i)] $\Rightarrow {}^nC_r < {}^nC_{r+1}$

Putting r = 0, 1, 2, ……., $\frac{n-3}{2}$

We get ${}^nC_0 < {}^nC_1, {}^nC_1 < {}^nC_2, {}^nC_2 < {}^nC_3, \ldots, < {}^nC_{\frac{n-3}{2}} < {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$

$\Rightarrow {}^nC_0 < {}^nC_1 < {}^nC_2 < {}^nC_3 < \ldots < {}^nC_{\frac{n-3}{2}} < {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n-1}{2}}$ ...(ii)

Since ${}^nC_{n-r} = {}^nC_r$. Therefore,

$\therefore {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}, \ldots, {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ ...(iii)

From (ii) and (iii), it follows that the maximum value of ${}^nC_r$ is ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$

Illustration 21: Find the numerically greatest term in the expansion of $(3 - 4x)^{15}$, when $x = \frac{1}{4}$. **(JEE MAIN)**

Sol: Follow the algorithm mentioned above.

Let r^{th} and $(r + 1)^{th}$ be two consecutive terms in the expansion of $(3 - 4x)^{15}$
$T_{r+1} > T_r$

$${}^{15}C_r 3^{15-r}\left(|-4x|\right)^r > {}^{15}C_{r-1} 3^{15-(r-1)}\left(|-4x|\right)^{r-1}$$

$$\frac{(15)!}{(15-r)!r!}|-4x| > \frac{3.(15!)}{(16-r)!(r-1)!} \quad \Rightarrow 5.\frac{1}{5}(16-r) > 3r \quad \Rightarrow 16 - r > 3r$$

$\Rightarrow 4r < 16 \qquad \Rightarrow r < 4$

Hence, we have $T_1 < T_2 < T_3 < T_4$.

Similarly, if we simplify $T_{r+1} = T_r$, we get r=4.

Therefore the numerically greatest term is T_4 and T_5.

4. APPLICATION OF BINOMIAL THEOREM

4.1 Divisibility Test

Illustration 22: Show that $7^{2n} + 7$ is divisible by 8, where n is a positive integer. **(JEE MAIN)**

Sol: Write $7^{2n} + 7$ in the form of $8\lambda + c$, where c is a constant. If c = 0 then we can conclude that $7^{2n} + 7$ is divisible by 8.

$7^{2n} + 7 = (8 - 1)^{2n} + 7 \qquad = {}^{2n}C_0 8^{2n} - {}^{2n}C_1.8^{2n-1} + {}^{2n}C_2.8^{2n-2} - \ldots + {}^{2n}C_{2n} + 7$

$= 8^{2n}.{}^{2n}C_0 - 8^{2n-1}.{}^{2n}C_1 + \ldots - 8.{}^{2n}C_{2n-1} + 8 \qquad = 8\lambda$ where λ is a positive integer

Hence, $7^{2n} + 7$ is divisible by 8.

Illustration 23: Prove that $13^{99} - 19^{57}$ is divisible by 162. **(JEE ADVANCED)**

Sol: Reduce $13^{99} - 19^{57}$ into the form of $162\lambda + C$ using binomial expansion and If C = 0 then $13^{99} - 19^{57}$ is divisible by 162.

Let the given number be called S. Hence, $S = 13^{99} - 19^{57} = (1 + 3\times4)^{99} - (1 + 9\times2)^{57}$

$$S = \left\{1 + {}^{99}C_1.(3\times4) + {}^{99}C_2.(3\times4)^2 + {}^{99}C_3.(3\times4)^3 + \ldots\ldots + {}^{99}C_{99}.(3\times4)^{99}\right\}$$

$$- \left\{1 + {}^{57}C_1.(9\times2) + {}^{57}C_2.(9\times2)^2 + {}^{57}C_3.(9\times2)^3 + \ldots\ldots + {}^{57}C_{57}.(9\times2)^{57}\right\}$$

$$S = \left\{1 + {}^{99}C_1.(3\times4) + (3^4\times2)k_1\right\} - \left\{1 + {}^{57}C_1.(9\times2) + (3^4\times2)k_2\right\}$$

All terms like $\left\{{}^{99}C_1.(3\times4)^2, {}^{99}C_2.(3\times4)^3, \ldots\ldots, {}^{99}C_{99}.(3\times4)^{99}\right\}$ and

$\left\{{}^{57}C_2.(9\times2)^2, {}^{57}C_3.(9\times2)^3, \ldots\ldots, {}^{57}C_{57}.(9\times2)^{57}\right\}$ have a common factor of $(3^4.2 = 162)$.

Hence they can be written as $(3^4.2)\,k_1$ and $(3^4.2)\,k_2$ respectively, where k_1 and k_2 are integers.

Therefore, $S = 1 + {}^{99}C_0.(3\times4) - 1 - {}^{57}C_1.(9\times2) + (162)(k_1 - k_2)$

$= (1188 - 1026) + \{162 \times (k_1 - k_2)\}$ = (162 × some integer)

Hence the given number S is exactly divisible by 162.

4.2 Finding Remainder

Illustration 24: What is the remainder when 5^{2015} is divisible by 13. **(JEE MAIN)**

Sol: In this problem, we can obtain required remainder by reducing 5^{2015} into the form of $13\lambda + a$, where λ and a are integers.

$5^{2015} = 5.5^{2014} = 5.(25)^{1007}$

$$= 5(26-1)^{1007} = 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + \ldots\ldots + {}^{1007}C_{1006}(26)^1 - {}^{1007}C_{1007}(26)^0\right]$$

$$= 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + \ldots\ldots + {}^{1007}C_{1006}(26)^1 - 1\right]$$

$$= 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + \ldots\ldots + {}^{1007}C_{1006}(26)^1 - 13\right] + 60$$

$= 13(k) + 52 + 8$ = 13 × (some integer) + 8.

4.3 Finding Digits of a Number

Illustration 25: Find the last two digits of the number $(13)^{10}$. **(JEE MAIN)**

Sol: Write $(13)^{10}$ in the form of $(x-1)^n$, such that x is a multiple of 10. Then using expansion formula we will get last two digits.

$$(13)^{10} = (169)^5 = (170-1)^5 = {}^5C_0(170)^5 - {}^5C_1.(170)^4 + \ldots\ldots + {}^5C_4(170)^1 - {}^5C_5(170)^0$$

$= {}^5C_0\left(170\right)^5 - {}^5C_1.\left(170\right)^4 + ... + {}^5C_3\left(170\right)^2 + 5\times 170 - 1 =$ A multiple of 100 + 849

Therefore, the last two digits are 49

Illustration 26: Find the last three digits of 13^{256}. **(JEE MAIN)**

Sol: Similar to above problem..

We have $13^2 = 169 = 170 - 1$

Now, $13^2 = \left(13^2\right)^{128} = \left(170 - 1\right)^{128}$

$= {}^{128}C_0\left(170\right)^{128} - {}^{128}C_1.\left(170\right)^{127} + {}^{128}C_2.\left(170\right)^{126} - ... + {}^{128}C_{126}\left(170\right)^2 - {}^{128}C_{127}\left(170\right) + 1$

= 1000 m + (128) (170) (10794) + 1 (where m is a positive integer)

= 1000 m + 234877440 + 1 = 1000m+234877441

Thus, the last three digits of 13^{256} are 441.

4.4 Relation between Two Numbers

Illustration 27: Which number is smaller $(1.01)^{1000000}$ or 10,000 **(JEE MAIN)**

Sol: By reducing $(1.01)^{1000000}$ into the form of $\left(1 + 0.01\right)^n$ and solve it by using expansion formula we can obtain the value of $(1.01)^{1000000}$.

$(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$= 1 + {}^{1000000}C_1\left(0.01\right) + {}^{1000000}C_2\left(0.01\right)^2 + {}^{1000000}C_3\left(0.01\right)^3 + ...$

$= 1 + 1000000 \times (0.01) +$ some positive terms

= 1 + 10000 + some positive terms

Hence 10,000 < $(1.01)^{1000000}$.

5. MULTINOMIAL THEOREM

Using binomial theorem, we have

$$\left(x + a\right)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r, \quad n \in N$$

$$= \sum_{r=0}^{n} \frac{n!}{\left(n-r\right)!r!} x^{n-r} a^r \qquad = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r, \qquad \text{where } s = n - r$$

Let us now consider the expansion of $\left(x_1 + x_2 + x_3\right)^n$

$$\left(x_1 + x_2 + x_3\right)^n = \sum_{k=0}^{n} {}^nC_k\, x_1^{n-k}\left(x_2 + x_3\right)^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} x_1^{n-k}\left(\sum_{p=0}^{k} \frac{k!}{(k-p)!\,p!} x_2^{k-p} x_3^p\right)$$

$$= \sum_{k=0}^{n}\sum_{p=0}^{k} \frac{n!}{(n-k)!\,(k-p)!\,p!} x_1^{n-k} x_2^{k-p} x_3^p \qquad = \sum_{p+q+r=n} \frac{n!}{r!\,q!\,p!} x_1^r\, x_2^q\, x_3^p \quad \text{where, } k - p = q,\ n - k = r.$$

And so on, if we want to generalize for n terms, we get

$$\left(x_1 + x_2 + ... + x_k\right)^n = \sum_{r_1+r_2+...+r_k=n} \frac{n!}{r_1!r_2!.....r_k!} x_1^{r_1} x_2^{r_2} ....x_k^{r_k}$$

Therefore, general term in the expansion of $\left(x_1 + x_2 + ... + x_k\right)^n$ is $\dfrac{n!}{r_1!r_2!r_3!.....r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} ....x_k^{r_k}$

The number of terms is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$, because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$.

Number of terms for the following expansions

(a) $\left(x+y+z\right)^n = \sum\limits_{r+s+t=n} \dfrac{n!}{r!s!t!} x^r y^s z^t$ The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms.

(b) $\left(x+y+z+u\right)^n = \sum\limits_{p+q+r+s=n} \dfrac{n!}{p!q!r!s!} x^p y^q z^r u^s$. There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ term in the above.

MASTERJEE CONCEPTS

The greatest coefficient in the expansion of $\left(x_1 + x_2 + ... + x_m^n\right)$ is $\dfrac{n!}{(q!)^{m-r}\left[(q+1)!\right]^r}$, where q and r are the quotient and remainder respectively when n is divided by m.

Aman Gour (JEE 2012, AIR 230)

6. BINOMIAL THEOREM FOR ANY INDEX

Let n be a rational number and x be a real number such that |x| < 1, then

$$\left(1+x\right)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r + ... + \text{terms upto } \infty$$

The general term in the expansion of $(1 + x)^n$ is $\dfrac{n(n-1)(n-2).........(n-r+1)}{r!} x^r$ and is represented by T_{r+1}.

MASTERJEE CONCEPTS

The above result is also true for complex x, n.

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 28: If x is very large and n is a negative integer or a proper fraction, then an approximate value of $\left(\dfrac{1+x}{x}\right)^n$ is equal to__________ **(JEE MAIN)**

Sol: Since x is very large therefore $\dfrac{1}{x}$ will be very small. Neglect the terms containing three and higher powers of $\dfrac{1}{x}$ in the expansion to obtain the approximate value of $\left(\dfrac{1+x}{x}\right)^n$.

$$\left(1+\frac{1}{x}\right)^n = 1+\frac{n}{x}+\frac{n(n-1)}{1.2}\left(\frac{1}{x}\right)^2+\ldots\ldots$$ Since x is very large, we can ignore terms after the 2nd term.

Illustration 29: If $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to a + bx for small values of x, then (a, b) is equals to. **(JEE MAIN)**

Sol: Calculate the value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ and equate it to a + bx.

Using the binomial expansion for any rational index, we have

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{2\left[1-\frac{x}{4}\right]^{1/2}} = \frac{\left[1+\frac{1}{2}(-3x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2+\ldots\ldots\right]+\left[1+\frac{5}{3}(-x)+\frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2+\ldots\ldots\right]}{2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2+\ldots\ldots\right]}$$

$$=\frac{\left[1-\frac{19}{12}x+\frac{53}{144}x^2-\ldots\ldots\ldots\right]}{\left[1-\frac{x}{8}-\frac{1}{8}x^2-\ldots\ldots\right]}=1-\frac{35}{24}x+\ldots\ldots$$

Neglecting the higher powers of x, $\Rightarrow a+bx=1-\frac{35}{24}x \quad \Rightarrow a=1, b=-\frac{35}{24}$

Illustration 30: Find the coefficient of $a^3b^2c^4d$ in the expansion of $(a - b - c + d)^{10}$ **(JEE ADVANCED)**

Sol: Expand $(a - b - c + d)^{10}$ using multinomial theorem and by using coefficient property we can obtain the required result.

Using multinomial theorem, we have

$$(a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10}\frac{(10)!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$$

We want to get coefficient of $a^3b^2c^4$, this implies that $r_1 = 3$, $r_2 = 2$, $r_3 = 4$, $r_4 = 1$

$\therefore$ Coefficient of $a^3b^2c^4d$ is $\frac{(10)!}{3!2!4!}(-1)^2(-1)^4=12600$

Illustration 31: In the expansion of $\left(1+x+\frac{5}{x}\right)^{11}$ find the term independent of x. **(JEE ADVANCED)**

Sol: By expanding $\left(1+x+\frac{5}{x}\right)^{11}$ using multinomial theorem and obtaining the coefficient of x^0 we will get the term independent of x.

$$\left(1+x+\frac{5}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11}\frac{(11)!}{r_1!r_2!r_3!}(1)^{r_1}(x)^{r_2}\left(\frac{5}{x}\right)^{r_3}$$

The exponent 11 is to be divided in such a way that we get x^0. Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2) (5, 3, 3), (3, 4, 4), (1, 5, 5) Hence the required term is

$$\frac{(11)!}{(11)!}\left(5^0\right)+\frac{(11)!}{9!1!1!}5^1+\frac{(11)!}{7!2!2!}5^2+\frac{(11)!}{5!3!3!}5^3+\frac{(11)!}{3!4!4!}5^4+\frac{(11)!}{1!5!5!}5^5$$

$$=1+\frac{(11)!}{9!2!}.\frac{2!}{1!1!}5^1+\frac{(11)!}{7!4!}.\frac{4!}{2!2!}5^2+\frac{(11)!}{5!6!}.\frac{6!}{3!3!}5^3+\frac{(11)!}{3!8!}.\frac{8!}{4!4!}5^4+\frac{(11)!}{1!10!}.\frac{(10)!}{5!5!}5^5$$

$$=1+{}^{11}C_2\times{}^2C_1\times5^1+{}^{11}C_4\times{}^4C_2\times5^2+{}^{11}C_6\times{}^6C_3\times5^3+{}^{11}C_8\times{}^8C_4\times5^4+{}^{11}C_{10}\times{}^{10}C_5\times5^5$$

$$=1+\sum_{r=1}^{5}{}^{11}C_{2r}.{}^{2r}C_r\times5^r$$

PROBLEM-SOLVING TACTICS

Summation of series involving binomial coefficients

For $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+......+{}^nC_nx^n$, the binomial coefficients are ${}^nC_0,{}^nC_1,{}^nC_2,...,{}^nC_n$. A number of series may be formed with these coefficients figuring in the terms of a series.

Some standard series of the binomial coefficients are as follows:

(a) By putting x = 1, we get $\quad {}^nC_0+{}^nC_1+{}^nC_2+......+{}^nC_n=2^n$ ...(i)

(b) By putting x =-1, we get $\quad {}^nC_0-{}^nC_1+{}^nC_2-......+(-1)^n.{}^nC_n=0$ ...(ii)

(c) On adding (i) and (ii), we get $\quad {}^nC_0+{}^nC_2+{}^nC_4+......=2^{n-1}$ ...(iii)

(d) On subtracting (ii) from (i), we get $\quad {}^nC_1+{}^nC_3+{}^nC_5+......=2^{n-1}$ ...(iv)

(e) ${}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+......+{}^{2n}C_{n-1}+{}^{2n}C_n=2^{2n-1}$

Proof: From the expansion of $(1+x)^{2n}$, we get ${}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+......+{}^{2n}C_{2n-1}+{}^{2n}C_{2n}=2^{2n}$

$$\Rightarrow 2\left({}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+.......+{}^{2n}C_{n-1}\right)+{}^{2n}C_n=2^{2n}\ [\because {}^{2n}C_0={}^{2n}C_{2n},{}^{2n}C_1={}^{2n}C_{2n-1}\text{ and so on.}]$$

(f) ${}^{2n+1}C_0+{}^{2n+1}C_1+{}^{2n+1}C_2+.......+{}^{2n+1}C_n=2^{2n}$

Proof: (as above)

(g) Sum of the first half of ${}^nC_0+{}^nC_1+...+{}^nC_n$ = Sum of the last half of ${}^nC_0+{}^nC_1+...+{}^nC_n=2^{n-1}$

(h) **Bino-geometric series:** ${}^nC_0+{}^nC_1x+{}^nC_2x^2+.......+{}^nC_nx^n=(1+x)^n$

(i) **Bino-arithmetic series:** $a\,{}^nC_0+(a+d)\,{}^nC_1+(a+2d)\,{}^nC_2+.......+(a+nd)\,{}^nC_n$

Consider an AP-a, (a+d), (a+2d), ... , (a+nd)

Sequence of Binomial Co-efficient - ${}^nC_0,{}^nC_1,{}^nC_2,.......,{}^nC_n$

A **bino-arithmetic** series is nothing but the sum of the products of corresponding terms of the sequences. It can be added in two ways.

(i) By elimination of r in the multiplier of binomial coefficient from the (r+1)th term of the series $\left(\text{By using } r.{}^nC_r=n\,{}^{n-1}C_{r-1}\right)$

(ii) By differentiating the expansion of $x^d\left(1+x^d\right)^n$.

(j) Bino-harmonic series: $\frac{^nC_0}{a}+\frac{^nC_1}{a+d}+\frac{^nC_2}{a+2d}+......+\frac{^nC_n}{a+nd}$

Consider an HP - $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},...,\frac{1}{a+nd}$

Sequence of Binomial Co-efficient - $^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$

It is obtained by the sum of the products of corresponding terms of the sequences. Such series are calculated in two ways :

(i) By elimination of r in the multiplier of binomial coefficient from the (r + 1)th term of the series $\left(\text{By using}\frac{1}{r+1}{}^nC_r=\frac{1}{n+1}{}^{n+1}C_{r+1}\right)$

(ii) By integrating suitable expansion.

For explanation see illustration 2

(k) Bino-binomial series: $^nC_0.{}^nC_r+{}^nC_1.{}^nC_{r+1}+{}^nC_2.{}^nC_{r+2}+.......+{}^nC_{n-r}.{}^nC_r$

or, $^mC_0.{}^nC_r+{}^mC_1.{}^nC_{r-1}+{}^mC_2.{}^nC_{r-2}+.....+{}^mC_r.{}^nC_0$

As the name suggests such series are obtained by multiplying two binomial expansion, one involving the first factors as coefficient and the other involving the second factors as coefficient. They can be calculated by equating coefficients of a suitable power on both sides.

For explanation see illustration 4

FORMULAE SHEET

Binomial theorem for any positive integral index:

$$\left(x+a\right)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+........+{}^nC_rx^{n-r}a^r+......+{}^nC_na^n=\sum_{r=0}^{n}{}^nC_rx^{n-r}a^r$$

(a) General term – $T_{r+1}={}^nC_rx^{n-r}a^r$ is the (r + 1)th term from beginning.

(b) (m + 1)th term from the end = (n – m + 1)th from beginning = T_{n-m+1}

(c) Middle term

(i) If n is even then middle term = $\left(\frac{n}{2}+1\right)^{th}$ term

(ii) If n is odd then middle term = $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$

Binomial coefficient of middle term is the greatest binomial coefficient.

To determine a particular term in the given expansion:

Let the given expansion be $\left(x^\alpha\pm\frac{1}{x^\beta}\right)^{th}$, if x^n occurs in T_{r+1} (r + 1)$^{th\ term}$ then r is given by $n\alpha-r\left(\alpha+\beta\right)=m$ and for $x^0, n\alpha-r\left(\alpha+\beta\right)=0$

Properties of Binomial coefficients:

For the sake of convenience the coefficients ${}^nC_0, {}^nC_1, {}^nC_2 \ldots\ldots {}^nC_r \ldots\ldots {}^nC_n$ are usually denoted by $C_0, C_1, \ldots\ldots C_r \ldots\ldots C_n$ respectively.

$$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n$$

$$C_0 - C_1 + C_2 - C_3 + \ldots\ldots + C_n = 0$$

$$C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$$

$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} \text{ and so on} \ldots\ldots$$

$${}^{2n}C_{n+r} = \frac{2n!}{(n-r)!(n+r)!}$$

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$C_1 + 2C_2 + 3C_3 + \ldots\ldots + {}^nC_n = n.2^{n-1}$$

$$C_1 - 2C_2 + 3C_3 \ldots\ldots = 0$$

$$C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$$

$$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$$

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots = \begin{cases} 0, \text{ if n is odd} \\ (-1)^{n/2}\,{}^nC_{n/2}, \text{ if n is even} \end{cases}$$

Note: ${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots\ldots + {}^{2n+1}C_n = {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots\ldots + {}^{2n+1}C_{2n+1} = 2^{2n}$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1};\ C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots\ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$$

(a) Greatest term:

(i) If $\frac{(n+1)a}{x+a} \in Z$ (integer) then the expansion has two greatest terms. These are k^{th} and $(k + 1)^{th}$ where x and a are +ve real numbers.

(ii) If $\frac{(n+1)a}{x+a} \notin Z$ then the expansion has only one greatest term. This is $(K + 1)^{th}$ term $k = \left[\frac{(n+1)a}{x+a}\right]$ denotes greatest integer less than or equal to x}

(b) Multinomial theorem:

Generalized $(x_1 + x_2 + \ldots\ldots + x_k)^n = \sum_{r_1+r_2+\ldots\ldots r_k = n} \frac{n!}{r_1!r_2!\ldots\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots\ldots x_k^{r_k}$

(c) Total no. of terms in the expansion $(x_1 + x_2 + \ldots\ldots x_n)^m$ is ${}^{m+n-1}C_{n-1}$