Quadratic Equations and Inequalities

PROBLEM-SOLVING TACTICS

Some hints for solving polynomial equations:

(a) To solve an equation of the form $\left(x-a\right)^4+\left(x-b\right)^4=A$; Put y = $x-\frac{a+b}{2}$

In general to solve an equation of the form $\left(x-a\right)^{2n}+\left(x-b\right)^{2n}=A$, where $n\in Z^+$, put y = $x-\frac{a+b}{2}$

(b) To solve an equation of the form, $a_0f\left(x\right)^{2n}+a_1\left(f\left(x\right)\right)^n+a_2=0$ we put $\left(f\left(x\right)\right)^n=y$ and solve $a_0y^2+a_1y+a_2=0$ to obtain its roots y_1 and y_2.

Finally, to obtain the solution of (1) we solve, $\left(f\left(x\right)\right)^n=y_1$ and $\left(f\left(x\right)\right)^n=y_2$

(c) An equation of the form $\left(ax^2+bx+c_1\right)\left(ax^2+bx+c_2\right).....\left(ax^2+bx+c_n\right)=A$. Where c_1, c_2, ...c_n, $A\in R$, can be solved by putting $ax^2+bx=y$.

(d) An equation of the form (x – a)(x – b)(x – c)(x – d) = ⇒ Awhere ab = cd, can be reduced to a product of two quadratic polynomials by putting y = x + $\frac{ab}{2}$.

(e) An equation of the form (x – a) (x – b)(x – c)(x – d) = A where a < b < c < d, b – a = d – c can be solved by a change of variable $y=\frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}=x-\frac{1}{4}(a+b+c+d)$

(f) A polynomial f(x, y) is said to be symmetric if f(x, y) = f(y, x) ∀ x, y. A symmetric polynomial can be represented as a function of x + y and xy.

Solving equations reducible to quadratic

(a) To solve an equation of the type $ax^4+bx^2+c=0$, put $x^2=y$.

(b) To solve an equation of the type $a(p(x))^2+bp(x)+c=0$ (p(x) is an expression of x), put p(x) = y.

(c) To solve an equation of the type $ap(x)+\frac{b}{p(x)}+c=0$ where p(x) is an expression of x, put p(x)= y

This reduces the equation to $ay^2+cy+b=0$

(d) To solve an equation of the form $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, put $x+\frac{1}{x}=y$

and to solve $a\left(x^2+\frac{1}{x^2}\right)+b\left(x-\frac{1}{x}\right)+c=0$, put $x-\frac{1}{x}=y$

(e) To solve a reciprocal equation of the type $ax^4+bx^3+cx^2+bx+a=0,\ a\neq 0,$

we divide the equation by $\frac{d^2y}{dx^2}$ to obtain $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, and then put $x+\frac{1}{x}=y$

(f) To solve an equation of the type (x + a)(x + b)(x + c)(x + d) + k = 0 where a+b =c+d, put x^2 +(a+ b)x = y

(g) To solve an equation of the type $\sqrt{ax+b}=cx+d$ or $\sqrt{ax^2+bx+c}=dx+e$, square both the sides.

(h) To solve an equation of the type $=\sqrt{ax+b}\pm\sqrt{cx+d}=e$, proceed as follows.

Step 1: Transfer one of the radical to the other side and square both the sides.

Step 2: Keep the expression with radical sign on one side and transfer the remaining expression on the other side

Step 3: Now solve as in 7 above.

FORMULAE SHEET

(a) **A quadratic equation is represented as :** $ax^2+bx+c=0,\ a\neq 0$

(b) **Roots of quadratic equation:** $x=\frac{-b\pm\sqrt{D}}{2a}$, where D(discriminant) = b^2-4ac

(c) **Nature of roots:** **(i)** D > 0 ⇒ roots are real and distinct (unequal)

(ii) D = 0 ⇒ roots are real and equal (coincident)

(iii) D < 0 ⇒ roots are imaginary and unequal

(d) The roots $(\alpha+i\beta)$, $(\alpha-i\beta)$ and $(\alpha+\sqrt{\beta}),(\alpha-\sqrt{\beta})$ are the conjugate pair of each other.

(e) **Sum and Product of roots :** If α and β are the roots of a quadratic equation, then

(i) $S=\alpha+\beta=\frac{-b}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ **(ii)** $P=\alpha\beta=\frac{c}{a}=\frac{\text{constant term}}{\text{Coefficient of } x^2}$

(f) **Equation in the form of roots:** $x^2-(\alpha+\beta)x+(\alpha.\beta)=0$

(g) In equation $ax^2+bx+c=0,\ a\neq 0$ If

(i) b = 0 $\Rightarrow$ roots are of equal magnitude but of opposite sign.

(ii) c = 0 $\Rightarrow$ one root is zero and other is –b/a

(iii) b = c = 0 $\Rightarrow$ both roots are zero.

(iv) a = c $\Rightarrow$ roots are reciprocal to each other.

(v) a > 0, c < 0 or a < 0, c > 0 $\Rightarrow$ roots are of opposite signs.

(vi) a > 0, b > 0, c > 0 or a < 0, b < 0, c < 0 $\Rightarrow$ both roots are –ve.

(vii) a > 0, b < 0, c > 0 or a < 0,b > 0, c < 0 $\Rightarrow$ both roots are +ve.

(h) The equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ have

(i) One common root if $\frac{b_1c_2-b_2c_1}{c_1a_2-c_2a_1}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

(ii) Both roots common if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(i) In equation $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right]$

(i) If a>0, the equation has minimum value $=\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no maximum value.

(ii) If a < 0, the equation has maximum value $\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no minimum value.

(j) For cubic equation $ax^3+bx^2+cx+d=0$,

(i) $\alpha+\beta+\gamma=\frac{-b}{a}$

(ii) $\alpha\beta+\beta\gamma+\lambda\alpha=\frac{c}{a}$

(iii) $\alpha\beta\gamma=\frac{-d}{a}$ ... where α,β,γ are its roots.