17. DETERMINANTS

1. INTRODUCTION

Development of determinants took place when mathematicians were trying to solve a system of simultaneous linear equations.

E.g. $\left.\begin{matrix} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{matrix}\right] \Rightarrow \quad x=\dfrac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$ and $y=\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$

Mathematicians defined the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ as a determinant of order 2 and the four numbers arranged in row and column were called its elements. Its value was taken as $(a_1b_2-a_2b_1)$ which is the same as the denominator. If we write the coefficients of the equations in the following form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then such an arrangement is called a determinant. In a determinant, horizontal lines are known as rows and vertical lines are known as columns. The shape of every determinant is a square. If a determinant is of order n then it contains n rows and n columns.

E.g. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ are determinants of second and third order respectively.

Note: (i) No. of elements in a determinant of order n are n^2. (ii) A determinant of order 1 is the number itself.

1.1 Evaluation of the Determinant using SARRUS Diagram

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, the below diagram is a Sarrus Diagram obtained by adjoining the first two columns on the right and draw dark and dotted lines as shown.

The value of the determinant is $(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32})-(a_{13}a_{22}a_{31}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33})$.

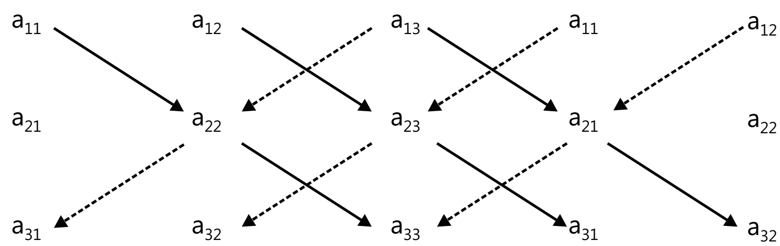

Figure 17.1

Illustration 1: Expand $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ by Sarrus rules. **(JEE MAIN)**

Sol: By using Sarrus rule i.e. $\Delta = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$ we can expand the given determinant.

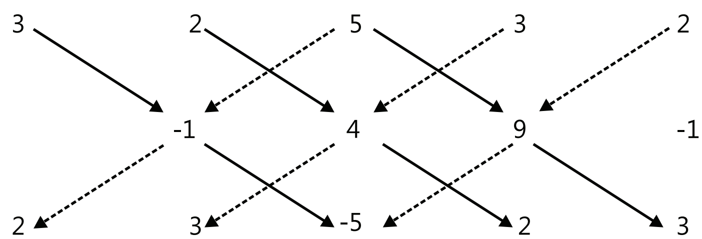

Figure 17.2

Here, $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix} \Rightarrow \Delta = 15 - 36 + 90 + 16 + 135 + 10 = 230$

Illustration 2: Evaluate the determinant : $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$ **(JEE MAIN)**

Sol: By using determinant expansion formula we can get the result.

we have, $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) = x^3 + x^2 - x^2 - x + x + 1 - x^2 + 1 = x^3 - x^2 + 2$

2. COFACTOR AND MINOR OF AN ELEMENT

Minor: Minor of an element is defined as the determinant obtained by deleting the row and column in which that element lies. e.g. in the determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, minor of a_{12} is denoted as $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactor: Cofactor of an element a_{ij} is related to its minor as $C_{ij} = (-1)^{i+j} M_{ij}$, where 'i' denotes the i^{th} row and 'j' denotes the j^{th} column to which the element a_{ij} belongs.

Now we define the value of the determinant of order three in terms of 'Minor' and 'Cofactor' as

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ or $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Note:

(a) A determinant of order 3 will have 9 minors and each minor will be a determinant of order 2 and a determinant of order 4 will have 16 minors and each minor will be determinant of order 3.

(b) $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, i.e. cofactor multiplied to different row/column elements results in zero value.

Row and Column Operations

(a) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$, when $i \neq j$; This notation is used when we interchange i^{th} row (or column) and j^{th} row (or column).

(b) $R_i \leftrightarrow C_i$; This converts the row into the corresponding column.

(c) $R_i \to Rk_i$ or $C_i \to kC_i$; $k \in R$; This represents multiplication of i^{th} row (or column) by k.

(d) $R_i \to R_ik + R_j$ or $Ci \to C_ik + C_j$; $(i \neq j)$; This symbol is used to multiply i^{th} row (or column) by k and adding the j^{th} row (or column) to it.

Illustration 3: Find the cofactor of a_{12} in the following $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ **(JEE MAIN)**

Sol: In this problem we have to find the cofactor of a_{12}, therefore eliminate all the elements of the first row and the second column and by obtaining the determinant of remaining elements we can calculate the cofactor of a_{12}.

Here a_{12} = Element of first row and second column = –3

$$M_{12} = \text{Minor of } a_{12}(-3) = \begin{vmatrix} 2\cdots & -3 & \cdots 5 \\ & \vdots & \\ 6 & 0 & 4 \\ & \vdots & \\ 1 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = 6(-7) - 4(1) = -42 - 4 = -46.$$

$$\text{Cofactor of } (-3) = (-1)^{1+2}(-46) = -(-46) = 46$$

Illustration 4: Write the minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ **(JEE MAIN)**

Sol: By eliminating row and column of an element, the remaining is the minor of the element.

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$; M_{11} = Minor of element (2) = $\begin{vmatrix} 2\cdots & -4 \\ \vdots & \\ 0 & 3 \end{vmatrix} = 3$; Cofactor of $(2) = (-1)^{1+1}M_{11} = +3$

M_{12} = Minor of element (–4) = $\begin{vmatrix} 2\cdots & -4 \\ & \vdots \\ 0 & 3 \end{vmatrix} = 0$; Cofactor of $(-4) = (-1)^{1+2}M_{12} = (-1)0 = 0$

M_{21} = Minor of element (0) = $\begin{vmatrix} 2 & -4 \\ \vdots & \\ 0\cdots & 3 \end{vmatrix} = -4$; Cofactor of $(0) = (-1)^{2+1}M_{21} = (-1)(-4) = 4$

M_{22} = Minor of element (3) = $\begin{vmatrix} 2 & -4 \\ & \vdots \\ 0\cdots & 3 \end{vmatrix} = 2$; Cofactor of $(3) = (-1)^{2+2}M_{22} = +2$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$; M_{11} = Minor of element (a) = $\begin{vmatrix} a\cdots c & \\ & \vdots \\ b & d \end{vmatrix} = d$; Cofactor of $(a) = (-1)^{1+1}M_{11} = (-1)^2 d = d$

M_{12} = Minor of element (c) = $\begin{vmatrix} a\cdots c & \\ & \vdots \\ b & d \end{vmatrix} = b$; Cofactor of $(c) = (-1)^{1+2}M_{12} = (-1)^3 b = -b$

M_{21} = Minor of element (b) = $\begin{vmatrix} a & c \\ \vdots & \\ b\cdots d & \end{vmatrix} = c$; Cofactor of $(b) = (-1)^{2+1}M_{21} = (-1)^3 c = -c$

M_{22} = Minor of element (d) = $\begin{vmatrix} a & c \\ & \vdots \\ b\cdots d & \end{vmatrix} = a$; Cofactor of $(d) = (-1)^{2+2}M_{22} = (-1)^4 a = a$

Illustration 5: Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 2 & -2 & 3 \\ 1 & 4 & 5 \\ 2 & 1 & -3 \end{vmatrix}$. **(JEE ADVANCED)**

Sol: By eliminating the row and column of an element, the determinant of remaining elements is the minor of the element. i.e. $M_{i\times j}$ and by using formula $(-1)^{i+j}M_{i\times j}$ we will get the cofactor of the element.

The minors are $M_{11} = \begin{vmatrix} 4 & 5 \\ 1 & -3 \end{vmatrix} = -17$, $M_{12} = \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = -13$, $M_{13} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7$

$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 3$, $M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$, $M_{23} = \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6$

$M_{31} = \begin{vmatrix} -2 & 3 \\ 4 & 5 \end{vmatrix} = -22$, $M_{32} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$, $M_{33} = \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} = 10$

The cofactors are:

$A_{11} = (-1)^{1+1}M_{11} = M_{11} = -17$, $A_{12} = (-1)^{1+2}M_{12} = -M_{12} = 13$, $A_{13} = (-1)^{1+3}M_{13} = M_{13} = -7$

$A_{21} = (-1)^{2+1}M_{21} = -M_{21} = -3$, $A_{22} = (-1)^{2+2}M_{22} = M_{22} = -12$, $A_{23} = (-1)^{2+3}M_{23} = -M_{23} = 6$

$A_{31} = (-1)^{3+1}M_{31} = M_{31} = -22$, $A_{32} = (-1)^{3+2}M_{32} = -M_{32} = -7$, $A_{33} = (-1)^{3+3}M_{33} = M_{33} = 10$

3. PROPERTIES OF DETERMINANTS

Determinants have some properties that are useful as they permit us to generate the same results with different and simpler configurations of entries (elements).

(a) Reflection Property: The determinant remains unaltered if its rows are changed into columns and the columns into rows.

(b) All-zero Property: If all the elements of a row (or column) are zero, then the determinant is zero.

(c) Proportionality (Repetition) Property: If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

(d) Switching Property: The interchange of any two rows (or columns) of the determinant changes its sign.

(e) Scalar Multiple Property: If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

(f) Sum Property: $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

(g) Property of Invariance: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$

(h) Factor Property: If a determinant Δ becomes zero when we put $x = \alpha$, then $(x - \alpha)$ is a factor of Δ.

(i) Triangle Property: If all the elements of a determinant above or below the main diagonal consist of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$$

(j) Determinant of cofactor matrix: $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$

where C_{ij} denotes the cofactor of the element a_{ij} in Δ.

MASTERJEE CONCEPTS

By interchanging two rows (or columns), the value of the determinant differs by a –ve sign.

If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^1 = 0$, else $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 6: Using properties of determinants, prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$ **(JEE MAIN)**

Sol: By using invariance and scalar multiple property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \quad [\text{Operating } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{Operating } (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)]$$

$$= (a+b+c)\{(c-b)(b-c)-(a-b)(a-c)\} = (a+b+c)(bc-b^2-c^2+bc-(a^2-ab-ac+bc)) = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

Illustration 7: Prove the following identity $\begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix} = 4\alpha^2\beta^2\gamma^2$ **(JEE MAIN)**

Sol: Take α, β, γ common from the L.H.S. and then by using scalar multiple property and invariance property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix}$$

Taking α, β, γ common from C_1, C_2, C_3 respectively $\Delta = \alpha\beta\gamma \begin{vmatrix} -\alpha & \alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & \gamma & -\gamma \end{vmatrix}$

Now taking α, β, γ common from R_1, R_2, R_3 respectively $\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

Now applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$ we have $\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$

Now expanding along C_1, $\Delta = \alpha^2 \times \beta^2(-1) \times \gamma^2(-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = \alpha^2\beta^2(-1)\gamma^2(0-4) = 4\alpha^2\beta^2\gamma^2$

Hence proved.

Illustration 8: Show that $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}$ **(JEE ADVANCED)**

Sol: Interchange the rows and columns across the diagonal using reflection property and then using the switching property of determinant we can obtain the required result.

L.H.S. $= \begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & \nu \end{vmatrix}$ (Interchanging rows and columns across the diagonal)

$$= (-1)\begin{vmatrix} \alpha & \lambda & \theta \\ \beta & \mu & \phi \\ \gamma & \nu & \psi \end{vmatrix} = (-1)^2 \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix} = \text{R.H.S.}$$

Illustration 9: If a, b, c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, prove that abc = –1. **(JEE ADVANCED)**

Sol: Split the given determinant using sum property. Then by using scalar multiple, switching and invariance properties of determinants, we can prove the given equation.

$$D = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^1\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad [C_1 \leftrightarrow C_3 \text{ in 1st det.}]$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad [C_2 \leftrightarrow C_3 \text{ in 1st det.}]$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (1 + abc)\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \quad (\text{expanding along } 1^{st} \text{ row}) = (1 + abc)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (1 + abc)(b – c)(c – a)(c + a – b – a) = (1 + abc)(b – a)(c – a)(c – b)$

$\Rightarrow D = (1 + abc)(a – b)(b – c)(c – a);$ But given $D = 0$

$\Rightarrow (1 + abc)(a – b)(b – c)(c – a) = 0;$ $\therefore (1 + abc) = 0$

[since a, b, c are different $a \neq b, b \neq c, c \neq a$]; Hence, $abc = – 1$

Illustration 10: Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$ **(JEE ADVANCED)**

Sol: Simply by using switching and scalar multiple property we can expand the L.H.S.

Given determinant $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + (C_2 + C_3)$, we obtain

$$\begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ given

$$2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} = 2(a+b+c)\cdot 1\{(b+c+a)(c+a+b) - (0 \times 0)\} = 2(a+b+c)^3$$

Hence proved.

Illustration 11: Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$ **(JEE ADVANCED)**

Sol: Expand the determinant $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$ by using scalar multiple and invariance property.

L.H.S.= $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$; Multiplying C_1, C_2, C_3 by a, b, c respectively

$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$; Now taking a, b, c common from R_1, R_2, R_3 respectively

$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$ $[C_1 \to C_1 + C_2 + C_3]$

$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

$= (1+a^2+b^2+c^2)\ (1.1.1) = 1+a^2+b^2+c^2$ = R.HS.

Hence proved.

MASTERJEE CONCEPTS

|AB| = |A||B|

The value of the determinant is the same when expanded by any row or any column. Using this property it is easier to expand determinant using a row or column in which most zeroes are involved.

Vaibhav Gupta (JEE 2009 AIR 54)

4. SYMMETRIC AND SKEW SYMMETRIC DETERMINANTS

4.1 Symmetric Determinant

A determinant is called Symmetric Determinant if $a_{ij} = a_{ji}, \forall$ i, j e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

4.2 Skew Symmetric Determinant

A determinant is called a skew symmetric determinant if $a_{ij} = -a_{ji}\ \forall$ i, j for every element.

E.g. $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Note: (i) det |A| = 0 $\Rightarrow$ A is singular matrix (ii) det $|A| \neq 0 \Rightarrow$ A is non-singular matrix

MASTERJEE CONCEPTS

The value of a skew symmetric determinant of an even order is always a perfect square and that of an odd order is always zero.

Vaibhav Krishnan (JEE 2009 AIR 22)

5. MULTIPLICATION OF TWO DETERMINANTS

(a) Multiplication of two second order determinants as follows: (as R to C method)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

(b) Multiplication of two third order determinants is defined.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \quad \text{(as R to C method)}$$

$$= \begin{vmatrix} a_1 l_1 + b_1 l_2 + c_1 l_3 & a_1 m_1 + b_1 m_2 + c_1 m_3 & a_1 n_1 + b_1 n_2 + c_1 n_3 \\ a_2 l_1 + b_2 l_2 + c_2 l_3 & a_2 m_1 + b_2 m_2 + c_2 m_3 & a_2 n_1 + b_2 n_2 + c_2 n_3 \\ a_3 l_1 + b_3 l_2 + c_3 l_3 & a_3 m_1 + b_3 m_2 + c_3 m_3 & a_3 n_1 + b_3 n_2 + c_3 n_3 \end{vmatrix}$$

Note:

(i) The two determinants to be multiplied must be of the same order.

(ii) To get the T_{mn} (term in the m^{th} row n^{th} column) in the product, Take the m^{th} row of the 1^{st} determinant and multiply it by the corresponding terms of the n^{th} column of the 2^{nd} determinant and add.

(iii) This method is the row by column multiplication rule for the product of 2 determinants of the n^{rd} order determinant.

(iv) If Δ' is the determinant formed by replacing the elements of a Δ of order n by their corresponding co-factors then $\Delta' = \Delta^{n-1}$. (Δ' is called the reciprocal determinant).

Illustration 12: Reduce the power of the determinant $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$ to 1. **(JEE MAIN)**

Sol: By multiplying the given determinant two times we get the determinant as required.

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} b^2 + c^2 & ab & ac \\ ba & c^2 + a^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix}$$

Illustration 13: Show that $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$. **(JEE ADVANCED)**

Sol: By replacing all elements of L.H.S. to their respective cofactors and using determinant property we will obtain the required result.

Let D = $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Co-factors of 1st row of D are $x^2+a^2, ab+cx, ac-bx$. Co-factors of 2nd row of D are ab – cx, x^2 + b^2, ax + bc and co-factors of 3rd row of D are $ac+bx, bc-ax$, x^2+c^2

$\therefore$ Determinant of cofactors of D is

$$D^c = \begin{vmatrix} x^2+a^2 & ab+cx & ac-bx \\ ab-cx & x^2+b^2 & ax+bc \\ ac+bx & bc-ax & x^2+c^2 \end{vmatrix} = \begin{vmatrix} a^2+x^2 & ab-cx & ac-bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & ax+bc & x^2+c^2 \end{vmatrix} = D^2$$

(Row interchanging into columns) = $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$ ($D^c = D^2$, D is third order determinant)

Hence $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$

6. SOME STANDARD DETERMINANTS

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ (ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc\,(a-b)(b-c)(c-a)$; (iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)\,(a+b+c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -a^3-b^3-c^3+3abc$

Illustration 14: Evaluate the determinant $\Delta = \begin{vmatrix} \sqrt{p}+\sqrt{q} & 2\sqrt{r} & \sqrt{r} \\ \sqrt{qr}+\sqrt{2p} & r & \sqrt{2r} \\ q+\sqrt{pr} & \sqrt{qr} & r \end{vmatrix}$, where p, q and r are positive real numbers. **(JEE MAIN)**

Sol: Taking $\sqrt{r}$ common from C_2 and C_3 of the given determinant using scalar multiple property and then expanding it using the invariance property we can evaluate the given problem.

We get $\Delta = r\begin{vmatrix} \sqrt{p}+\sqrt{q} & 2 & 1 \\ \sqrt{qr}+\sqrt{2p} & \sqrt{r} & \sqrt{2} \\ q+\sqrt{pr} & \sqrt{q} & \sqrt{r} \end{vmatrix}$

Applying $C_1 \to C_1 - \sqrt{q}C_2 - \sqrt{p}C_3$

We get $D = r\begin{vmatrix} -\sqrt{q} & 2 & 1 \\ 0 & \sqrt{r} & \sqrt{2} \\ 0 & \sqrt{q} & \sqrt{r} \end{vmatrix} = -r\sqrt{q}(r-\sqrt{2q}) = r\left(q\sqrt{2}-r\sqrt{q}\right)$.

Illustration 15: Let a, b, c be positive and not equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

(JEE ADVANCED)

Sol: By applying invariance and scalar multiple properties to the given determinant we can get the required result.

$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$; then $D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$ $[C_1 \to C_1 + C_2 + C_3]$

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ [Taking (a + b+ c) common from the first column]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$= (a+b+c+)\,[(c-b)(b-c)-(a-b)(a-c)] = (a+b+c+)\,[bc+ca+ab-a^2-b^2-c^2]$

$= -(a+b+c)\,(a^2+b^2+c^2-bc-ca-ab) = -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2bc-2ca-2ab)$

$= -\frac{1}{2}(a+b+c)\,[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ac)]$

$= -\frac{1}{2}(a+b+c)\,[(a-b)^2+(b-c)^2+(c-a)^2]$ ... ... (i)

$\because$ a, b, c, are positive $\Rightarrow$ a + b + c > 0

$\because$ a, b, c are unequal $\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2 > 0$ ... ...(ii)

$\therefore$ From (i) and (ii) , $\Delta < 0$.

Illustration 16: Show that $\Delta = \begin{vmatrix} 1 & \cos^2(\alpha-\beta) & \cos^2(\alpha-\gamma) \\ \cos^2(\beta-\alpha) & 1 & \cos^2(\beta-\gamma) \\ \cos^2(\gamma-\alpha) & \cos^2(\gamma-\beta) & 1 \end{vmatrix} = 2\sin^2(\beta-\gamma)\sin^2(\gamma-\alpha)\sin^2(\alpha-\beta)$ **(JEE ADVANCED)**

Sol:.By Putting $\beta-\gamma = A$, $\gamma-\alpha = B$, $\alpha-\beta = C$ and then by using switching and invariance properties we can prove the above problem.

We can write Δ as, $\Delta = \begin{vmatrix} 1 & \cos^2 C & \cos^2 B \\ \cos^2 C & 1 & \cos^2 A \\ \cos^2 B & \cos^2 A & 1 \end{vmatrix}$ (Note that A + B + C = 0.)

Using $C_2 \to C_2 - C_1,\ C_1 \to C_3 - C_1$ we get

$$\Delta = \begin{vmatrix} 1 & -\sin^2 C & -\sin^2 B \\ \cos^2 C & \sin^2 C & \cos^2 A - \cos^2 C \\ \cos^2 B & \cos^2 A - \cos^2 B & \sin^2 B \end{vmatrix} \begin{vmatrix} 1 & -\sin^2 C & -\sin^2 B \\ \cos^2 C & \sin^2 C & \sin B \sin(C-A) \\ \cos^2 B & \sin C \sin(B-A) & \sin^2 B \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & \sin^2 C & \sin^2 B \\ \cos^2 C & -\sin^2 C & \sin B \sin(-A) \\ \cos^2 B & \sin C \sin(B-A) & -\sin^2 B \end{vmatrix}$$

$[\because\ \cos^2 A - \cos^2 B = \sin(A+B)\sin(B-A),\ A+B = -C, C+A = -B\ ];$ $\qquad = \sin C \sin B\ [\Delta_1]$

where $\quad \Delta_1 = \begin{vmatrix} 1 & \sin^2 C & \sin B \\ \cos^2 C & -\sin^2 C & \sin(C-A) \\ \cos^2 B & \sin(B-A) & -\sin B \end{vmatrix}$ Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$\Delta_1 = \begin{vmatrix} 1 & \sin C & \sin B \\ -\sin^2 C & -2\sin^2 C & \sin(C-A) - \sin B \\ -\sin^2 B & \sin(B-A) - \sin C & -2\sin^2 B \end{vmatrix}$$

But sin (C – A) – sin B = sin (C – A) + sin (C + A) = 2 sin C cos A and sin (B – A) – sin C = 2 sin B cos A

Therefore, $\Delta_1 = \sin C \sin B \Delta_2$ where $\Delta_2 = \begin{vmatrix} 1 & \sin C & \sin B \\ \sin C & 2 & -2\cos A \\ \sin B & -2\cos A & 2 \end{vmatrix}$

Applying $R_2 \to R_2 - \sin C\ R_1$ and $R_3 \to -\sin B\ R_1$ we get

$$\Delta_2 = \begin{vmatrix} 1 & \sin C & \sin B \\ 0 & 2 - \sin^2 C & -2\cos A - \sin B \sin C \\ 0 & -2\cos A - \sin B \sin C & 2 - \sin^2 B \end{vmatrix} \quad = (2 - \sin^2 B)(2 - \sin C) - (2\cos A + \sin B \sin C)^2$$

$$= 4 - 2\sin^2 B - 2\sin^2 C + \sin^2 B \sin^2 C - [4\cos^2 A + 4\cos A \sin B \sin C + \sin^2 B \sin^2 C]$$

$$= 4\sin^2 A - 2\sin^2 B - 2\sin^2 C - 4\cos A \sin B \sin C$$

$$= 2\sin^2 A - 2\,[\sin^2 B + \sin^2 C - \sin^2 A + 2\cos A \sin B \sin C]$$

But A + B + C = 0 implies; $\quad \sin^2 B + \sin^2 C - \sin^2 A = -2\cos A \sin B \sin C$

$\therefore \quad \Delta_2 = 2\sin^2 A;$ $\qquad$ Hence, $\quad D = \sin C \sin B \Delta_1 = \sin^2 C \sin^2 B \Delta_2$

$= 2\sin^2 A \sin^2 B \sin^2 C = 2\sin^2(\alpha - \beta)\sin^2(\beta - \gamma)\sin^2(\gamma - \alpha)$.

Illustration 17: Prove that the following determinant vanishes if any two of x; y; z are equal

(JEE ADVANCED)

$$\Delta = \begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix}$$

Sol: Taking $\cos x$, $\cos y$, and $\cos z$ common from first, second and third column using scalar multiple and then using the invariance property we can prove the given statement.

Here, $\Delta = \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y & \tan z \\ 1 & 1 & 1 \\ \cos^2 x & \cos^2 y & \cos^2 z \end{vmatrix}$

$$= \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y - \tan x & \tan z - \tan y \\ 1 & 0 & 0 \\ \cos^2 x & \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix} \quad (C_3 \to C_3 - C_2,\ C_2 \to C_2 - C_1)$$

Expanding along R_2; $\quad \Delta = -\cos x \cos y \cos z \begin{vmatrix} \tan y - \tan x & \tan z - \tan y \\ \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix}$

$$= -\cos x \cos y \cos z \begin{vmatrix} \dfrac{\sin(y-x)}{\cos x \cos y} & \dfrac{\sin(z-y)}{\cos y . \cos z} \\ \sin^2 x - \sin^2 y & \sin^2 y - \sin^2 z \end{vmatrix} = \begin{vmatrix} \cos z . \sin(x-y) & \cos x . \sin(y-z) \\ \sin(x+y) . \sin(x-y) & \sin(y+z) . \sin(y-z) \end{vmatrix} \quad \text{... (i)}$$

$$= \sin(x-y)\sin(y-z) \begin{vmatrix} \cos z & \cos x \\ \sin(x+y) & \sin(y+z) \end{vmatrix} = \sin(x-y)\sin(y-z)[\sin(y+z)\cos z - \sin(x+y)\cos x]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)[\{\sin(y+2z) + \sin y\} - \{\sin(y+2x\} + \sin y\}]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)[\sin(y+2z) - \sin(y+2x)] \ = \frac{1}{2}\sin(x-y)\sin(y-z)2\cos(x+y+z)\sin(z-x)$$

$$= \sin(x-y)\sin(y-z)\sin(z-x)\cos(x+y+z)$$

Clearly, Δ is zero when any two of x, y, z are equal or x + y + z = $\frac{\pi}{2}$.

Hence proved.

7. SYSTEM OF EQUATIONS

7.1 Involving Two Variables

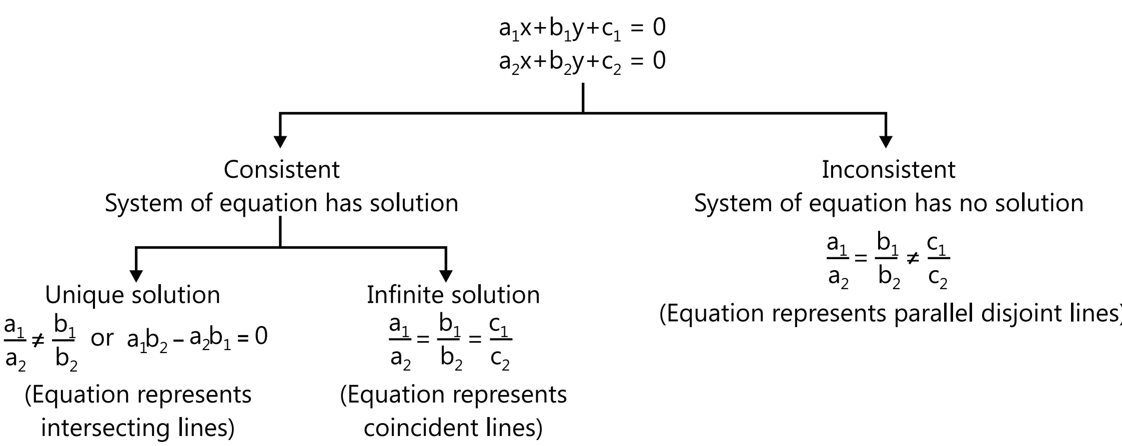

Figure 17.3

Solution to this system of equations is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$ or $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$; $y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

where $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

7.2 Involving Three Variables

$a_1x + b_1y + c_1z = d_1 \qquad a_2x + b_2y + c_2z = d_2 \qquad a_3x + b_3y + c_3z = d_3$

To solve this system we first define the following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is followed to solve the system (**CRITERION FOR CONSISTENCY**)

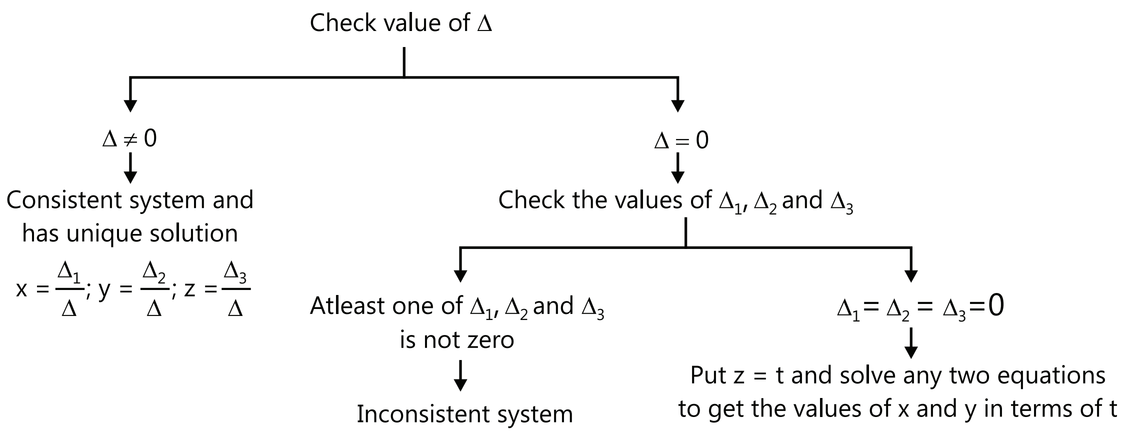

Figure 17.4

This method of finding solution to a system of equations is called Cramer's rule.

Note:

(a) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then system of equation may or may not be consistent:

(i) If the value of x, y and z in terms of t satisfy the third equation then system is said to be consistent and will have infinite solutions.

(ii) If the values of x, y, z don't satisfy the third equation, then system is said to be inconsistent and will have no solution.

(b) If $d_1 = d_2 = d_3 = 0$, then system of linear equations is known as Homogeneous linear equations, which always possess at least one solution i.e. (0, 0, 0). This is called trivial solution for homogeneous linear equations.

(c) If the system of homogeneous linear equations possess non-zero/nontrivial solutions, and $\Delta = 0$. In such case given system has infinite solutions.

We can also solve these solutions using the matrix inversion method.

We can write the linear equations in the matrix form as AX = B where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now, solution set is obtained by solving $X = A^{-1}B$. Hence the solution set exists only if the inverse of A exists.

Illustration 18: Solve the following equations by Cramer's rule x + y + z = 9, 2x + 5y + 7z = 52, 2x+ y – z = 0. **(JEE MAIN)**

Sol: Here in this problem define the determinants Δ, Δ_1, Δ_2 and Δ_3 and find out their value by using the invariance property and then by using Cramer's rule, we can get the values of x, y and z.

Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$ (Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$\therefore \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 2 & -1 & -3 \end{vmatrix} = 1\,(-9 + 5) = -4; \quad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$ (Applying $C_2 \to C_2 + C_3$)

$\therefore \quad \Delta_1 = \begin{vmatrix} 9 & 2 & 1 \\ 52 & 12 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1\,(108 - 104) = -4; \Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$ (Applying $C_1 \to C_1 + 2C_3$)

$\therefore \quad \Delta_2 = \begin{vmatrix} 3 & 9 & 1 \\ 16 & 52 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1\,(156 - 144) = -12$ and $\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 - 2C_2$)

$\therefore \quad \Delta_3 = \begin{vmatrix} -1 & 1 & 9 \\ -8 & 5 & 52 \\ 0 & 1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 - 2C_2$) $= -1(-52 + 72) = -20$

$\therefore$ By Cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3$ and $z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$

$\therefore$ x = 1, y = 3, z = 5

Illustration 19: Solve the following linear equations: $\frac{4}{x+5} + \frac{3}{y+7} = -1$ and $\frac{6}{x+5} - \frac{6}{y+7} = -5$ **(JEE MAIN)**

Sol: Here in this problem first put $\frac{1}{x+5}$ = a and $\frac{1}{y+7}$ = b and then define the determinants Δ, Δ_1 and Δ_2. Then by using Cramer's rule we can get the values of x and y.

Let us put $\frac{1}{x+5}$ = a and $\frac{1}{y+7}$ = b then the 2 linear equations become

4a + 3b = – 1 ...(i)

and 6a – 6b = –5 ...(ii);

Using Cramer's Rule, we get,

$$\frac{x}{\begin{vmatrix} -1 & 3 \\ -5 & -6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & -1 \\ 6 & -5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 6 & -6 \end{vmatrix}} \quad \Rightarrow \frac{a}{6+15} = \frac{b}{-20+6} = \frac{1}{-24-18}$$

$$\therefore \quad \frac{a}{21} = \frac{b}{-14} = \frac{1}{-42} \qquad \Rightarrow a = \frac{-1}{2} \text{ and } b = \frac{1}{3}$$

$$\therefore \quad a = -\frac{1}{2} \quad \Rightarrow \frac{1}{x+5} = -\frac{1}{2} \quad \Rightarrow 2 = -x - 5 \Rightarrow x = -7$$

$$b = \frac{1}{3} \quad \Rightarrow \frac{1}{y+7} = \frac{1}{3} \quad \Rightarrow 3 = y + 7 \quad \Rightarrow y = -4$$

Illustration 20: For what value of k will the following system of equations possess nontrivial solutions. Also find all the solutions of the system for that value of k.

$x + y - kz = 0; 3x - y - 2z = 0; x - y + 2z = 0.$ **(JEE ADVANCED)**

Sol: Here in this problem first define Δ. As we know that, for non-trivial solution $\Delta = 0$.

So by using the invariance property we can solve $\Delta = 0$ and will get the value of k.

For non-trivial solution, $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -k \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 0 \qquad \Rightarrow \begin{vmatrix} 2 & 0 & -k+2 \\ 2 & 0 & -4 \\ 1 & -1 & 2 \end{vmatrix} = 0 \quad [R_1 \to R_1 + R_3, R_2 \to R_2 - R_3]$$

Expanding along C_2. $\Rightarrow \; -(-1)[-8 - 2(2-k)] = 0 \;\Rightarrow\; 2k - 12 = 0 \;\Rightarrow\; k = 6$

Putting the value of k in the given equation, we get,

$x + y - 6z = 0$ ... (i)

$3x - y - 2z = 0$ ... (ii)

$x - y + 2z = 0$ ... (iii)

(i) + (ii) $\Rightarrow 4x - 8z = 0 \qquad \therefore\; z = \dfrac{x}{2}$

Putting the value of z in (i), we get $x + y - 3x = 0 \quad \therefore\; y = 2x$

Thus when k = 6, solution of the given system of equations will be $x = t, y = 2t, z = \dfrac{t}{2}$, when t is an arbitrary number.

Illustration 21: Solve the following equations by matrix inversion.

$2x + y + 2z = 0 \qquad 2x - y + z = 10 \qquad x + 3y - z = 5$ **(JEE ADVANCED)**

Sol: By writing the given equations into the form of AX = D and then multiplying both side by A^{-1} we will get the required value of x, y and z.

In the matrix form, the equations can be written as $\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$$\therefore \quad AX = D \text{ where } A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad A^{-1}(AX) = A^{-1}D \qquad \Rightarrow \quad X = A^{-1}D$ ... (i)

$$\text{Now } A^{-1} = \frac{\text{adj}A}{|A|}; \qquad |A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1-3) - 1(-2-1) + 2(6+1) = 13$$

The matrix of cofactors of |A| is $\begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & 4 \end{bmatrix}$. So, adj A$= \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$; $A^{-1} = \dfrac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}$.

$$\therefore \quad \text{from (1), } X = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}.$$

$$\Rightarrow \quad x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$$

MASTERJEE CONCEPTS

In general if r rows (or columns) become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of the given determinant.

Anvit Tawar (JEE 2009 AIR 9)

7.3 Some Important Results

The lines: $a_1x+b_1y+c_1=0$ ... (i)

$a_2x+b_2y+c_2=0$ ... (ii)

$a_3x+b_3y+c_3=0$ ... (iii)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

This is the condition for the consistency of three simultaneous linear equations in 2 variables.

(a) $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines if

$$abc+2fgh-af^2-bg^2-ch^2=0=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(b) Area of a triangle whose vertices are (x_r, y_r) ; r = 1, 2, 3 is : D = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. If D = 0 then the three points are collinear.

(c) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$.

(d) If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

E.g., $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

It should be noted that while applying operations on determinants at least one row (or column) must remain unchanged i.e.

Maximum number of simultaneous operations = order of determinant – 1

MASTERJEE CONCEPTS

Always expand a determinant along a row or a column with maximum zeros.

To find the value of the determinant, the following steps are taken.

Take any row (or column); the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the elements with a proper sign, given by $(-1)^{p+q}$ where p and q are the no. of row and the no. of column respectively.

Vaibhav Krishnan (JEE 2009 AIR 22)

8. DIFFERENTIATION AND INTEGRATION OF DETERMINANTS

Let $\Delta(x)=\begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are functions of x. Then,

$$\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix} \quad \text{Also, } \Delta'(x)=\begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged. If we write $\Delta(x)=[C_1 \quad C_2]$, where C_i denotes the i^{th} column, then $\Delta'(x)=[C_1' \quad C_2]+[C_1 \quad C_2']$, where C_i' denotes the column obtained by differentiating functions in the i^{th} column C_i. Also, if $\Delta(x)=\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, then $\Delta'(x)=\begin{bmatrix} R_1' \\ R_2 \end{bmatrix}+\begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$ Similarly, we can differentiate determinants of higher order.

Note: Differentiation can also be done column wise by taking one column at a time.

If f(x), g(x) and h(x) are functions of x and a, b, c, α, β and γ are constants such that

$\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}$, then the integration of $\Delta(x)$ is given by $\int \Delta(x)dx=\begin{vmatrix} \int f(x)dx & \int g(x)dx & \int h(x)dx \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}$

Illustration 22: If $\Delta(x)=\begin{vmatrix} \sin^2 x & \log\cos x & \log\tan x \\ n^2 & 2n-1 & 2n+1 \\ 1 & -2\log 2 & 0 \end{vmatrix}$, then evaluate $\int_0^{\pi/2} \Delta(x)\,dx$. **(JEE MAIN)**

Sol: By applying integration on variable elements of determinant we will solve the given problem.

We have, $\Delta(x)=\begin{vmatrix} \sin^2 x & \log\cos x & \log\tan x \\ n^2 & 2n-1 & 2n+1 \\ 1 & -2\log 2 & 0 \end{vmatrix}$; $\int_0^{\pi/2} \Delta(x)\,dx=\begin{vmatrix} \int_0^{\pi/2} \sin^2 x\,dx & \int_0^{\pi/2} \log\cos x\,dx & \int_0^{\pi/2} \log\tan x\,dx \\ n^2 & 2n-1 & 2n+1 \\ 1 & -2\log 2 & 0 \end{vmatrix}$

$$=\begin{vmatrix} \frac{\pi}{4} & -\frac{\pi}{2}\log 2 & 0 \\ n^2 & 2n-1 & 2n+1 \\ 1 & -2\log 2 & 0 \end{vmatrix}=\frac{\pi}{4}\begin{vmatrix} 1 & -2\log 2 & 0 \\ n^2 & 2n-1 & 2n+1 \\ 1 & -2\log 2 & 0 \end{vmatrix}=\frac{\pi}{4}\times 0=0$$

Illustration 23: If $f(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then show that $\frac{d^n}{dx^n}\{f(x)\}=0$ at x = 0. **(JEE ADVANCED)**

Sol: By applying integration on variable elements of the determinant we will solve the given problem.

We have, $f(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$; $\frac{d^n}{dx^n}\{f(x)\}=\begin{vmatrix} \frac{d^n}{dx^n}(x^n) & \frac{d^n}{dx^n}(\sin x) & \frac{d^n}{dx^n}(\cos x) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$

$$= \begin{vmatrix} n! & \sin\left(x+\frac{n\pi}{2}\right) & \cos\left(x+\frac{n\pi}{2}\right) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}; \qquad \left(\frac{d^n}{dx^n}\{f(x)\}\right)_{x=0} = \begin{vmatrix} n! & \sin\frac{n\pi}{2} & \cos+\frac{n\pi}{2} \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} = 0$$

PROBLEM-SOLVING TACTICS

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and in general

$\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of f(x).

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$, where a, b, c, l, m and n are constants.

$$\Rightarrow \int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

If the elements of more than one column or rows are functions of x then the integration can be done only after evaluation/expansion of the determinant.

FORMULAE SHEET

(a) Determinant of order 3 × 3 = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ b_3 & b_3 \end{vmatrix}$

(b) In the determinant D = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, minor of a_{12} is denoted as $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

(c) Cofactor of an element $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

(d) Properties of determinants:

(i) Reflection property: $\left|A_{i\times j}\right| = \left|A_{j\times i}\right|$

(ii) All-zero property: If all the elements of a row (or column) are zero, then the determinant is zero.

(iii) Proportionality (Repetition) Property: If all the elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

(iv) Switching Property: The interchange of any two rows (or columns) of the determinant changes its sign.

(v) Scalar Multiple Property: If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

(vi) Sum Property: $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

(vii) Property of Invariance: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$.

(viii) Triangle Property: $\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$

(e) Cramer's rule : if $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ and $a_3x+b_3y+c_3z=d_3$ then $x=\frac{\Delta_1}{\Delta}, y=\frac{\Delta_2}{\Delta}, z=\frac{\Delta_3}{\Delta}$ where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

And if $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then $x=\frac{\Delta_1}{\Delta}$ $y=\frac{\Delta_2}{\Delta}$.

Where $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

(f) **(i)** lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(ii) $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(iii) area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is : $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(iv) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(g) If $\Delta(x)=\begin{vmatrix} f_1(x) & g_1(x)\\ f_2(x) & g_2(x)\end{vmatrix}$ then $\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1'(x)\\ f_2(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1(x)\\ f_2'(x) & g_2'(x)\end{vmatrix}$ or $\begin{vmatrix} f_1'(x) & g_1(x)\\ f_2'(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x)\\ f_2(x) & g_2'(x)\end{vmatrix}$

(h) If $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ a & b & c\\ \alpha & \beta & \gamma\end{vmatrix}$ then $\int\Delta(x)dx=\begin{vmatrix} \int f(x)dx & \int g(x)dx & \int h(x)dx\\ a & b & c\\ \alpha & \beta & \gamma\end{vmatrix}$

Solved Examples

JEE Main/Boards

Example 1: Prove that

$\begin{vmatrix} pa & qb & rc\\ qc & ra & pb\\ rb & pc & qa\end{vmatrix}=pqr\begin{vmatrix} a & b & c\\ c & a & b\\ b & c & a\end{vmatrix}$. Use p + q + r = 0.

Sol: By using the expansion formula of determinants we can prove this.

L.H.S.= $\begin{vmatrix} pa & qb & rc\\ qc & ra & pb\\ rb & pc & qa\end{vmatrix}=$

$pa\begin{vmatrix} ra & pb\\ pc & qa\end{vmatrix}-qb\begin{vmatrix} qc & pb\\ rb & qa\end{vmatrix}+rc\begin{vmatrix} qc & ra\\ rb & pc\end{vmatrix}$

$=pa(a^2qr-p^2bc)-qb(q^2ac-prb^2)+rc(pqc^2-r^2ab)$

$=a^3pqr-p^3abc-q^3abc+b^3pqr-r^3abc$

$=pqr(a^3+b^3+c^3)-abc(p^3+q^3+r^3)$

$\because\ p+q+r=0$ ... (given)

$(p + q + r)^3 = 0$

$\Rightarrow p^3 + q^3 + r^3 - pqr = 0 \Rightarrow p^3 + q^3 + r^3 = 3pqr$

$\Rightarrow$ L.H.S. $=pqr(a^3+b^3+c^3)-abc(3pqr)$

$\Rightarrow$ L.H.S. $=pqr(a^3+b^3+c^3-3abc)$ .... (i)

R.H.S. = $pqr\left[a\begin{vmatrix} a & b\\ c & a\end{vmatrix}-b\begin{vmatrix} c & b\\ b & a\end{vmatrix}+c\begin{vmatrix} c & a\\ b & c\end{vmatrix}\right]$

$=pqr[a(a^2-bc)-b(ca-b^2)+c(c^2-ab)]$

$=pqr[a^3-abc-abc+b^3+c^3-abc]$

$\Rightarrow$ R.H.S. = $pqr(a^3+b^3+c^3-3abc)$ .... (ii)

From eq. (i) and (ii), we get

$\therefore$ L.H.S. = R.H.S.

Example 2: Prove that the determinant

$\begin{vmatrix} x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$ is independent of θ.

Sol: Simply by expanding the given determinant we can prove it.

We have, $\begin{vmatrix} x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$

$=x\begin{vmatrix} -x & 1\\ 1 & x\end{vmatrix}-\sin\theta\begin{vmatrix} -\sin\theta & 1\\ \cos\theta & x\end{vmatrix}+\cos\theta\begin{vmatrix} -\sin\theta & -x\\ \cos\theta & 1\end{vmatrix}$

$x(-x^2-1)-\sin\theta(-x\sin\theta-\cos\theta)+\cos\theta(-\sin\theta+x\cos\theta)$

$=-x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+x\cos^2\theta$

$=-x^3-x+x(\sin^2\theta+\cos^2\theta)=-x^3-x+x$

Thus, the determinant is independent of θ.

Example 3: Solve the equation $\begin{vmatrix} x+a & x & x\\ x & x+a & x\\ x & x & x+a\end{vmatrix}=0$, $a\neq 0$.

Sol: We can expand the above determinant by applying the invariance and scalar multiple properties, and hence we can easily solve this problem.