Limits, Continuity and Differentiability

PROBLEM SOLVING TACTICS

Above we have discussed L'Hôpital's rule by an example. Let us consider the same example again

$$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2}$$

These kind of problems where oscillating functions are involved and $x\to\infty$ are solved using the sandwich theorem.

It states that Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have:

$g(x)\le f(x)\le h(x)$

and also suppose that: $\lim_{x\to a} g(x)=\lim_{x\to a} h(x)=L$

Then $\lim_{x\to a} f(x)=L$

Now we know that $-1\le \sin x\le 1 \Rightarrow \frac{-1}{x^2}\le\frac{\sin x}{x^2}\le\frac{1}{x^2}\Rightarrow\frac{x^2-1}{x^2}\le\frac{x^2+\sin x}{x^2}\le\frac{x^2+1}{x^2}$

$$\lim_{x\to\infty}\frac{x^2+1}{x^2}=\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)=1;\ \lim_{x\to\infty}\frac{x^2-1}{x^2}=\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)=1\ \Rightarrow\ \lim_{x\to\infty}\frac{x^2+\sin x}{x^2}=1$$

While examining the continuity and differentiability of a function f(x) at a point x = a, if you start with the differentiability and find that f(x) is differentiable then you can conclude that the function is also continuous. But if you find f(x) is not differentiable at x = a, you will also have to check the continuity separately. Instead, if you start with the continuity and find that the function is not continuous then you can conclude that the function is also non-differentiable. But if you find f(x) is continuous, you will also have to check the differentiability separately.

FORMULAE SHEET

	Let f and g be two real functions with a common domain D, then.
(i)	$\lim_{x\to a}(f+g)(x)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$
(ii)	$\lim_{x\to a}(f-g)(x)=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$
(iii)	$\lim_{x\to a}(c\cdot f)(x)=c\lim_{x\to a}f(x)$ [c is a constant]
(iv)	$\lim_{x\to a}(fg)(x)=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$
(v)	$\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$
(vi)	$\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a}f(x)\right)^n$
(vii)	If f(x) < = g(x), then $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$
(viii)	$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$
(ix)	$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$
(x)	$\lim_{x\to 0}\frac{\sin x}{x}=1$
(xi)	$\lim_{x\to 0}\frac{e^x-1}{x}=1$
(xii)	$\lim_{x\to 0}\left(1+ax\right)^{1/x}=e^a=\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$
(xiii)	if $\lim_{x\to a}f(x)=1$ and $\lim_{x\to a}g(x)=\infty$ then $\lim_{x\to a}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to a}g(x)\left[f(x)-1\right]}$ sin x < x < tanx
Expansions of Some Functions	
1.	$(1+x)^n=\left\{1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....\right\}$ [| x | < 1]

2.	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \frac{x^n}{3!} + \ldots\ldots$
3.	$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \ldots\ldots\ldots$
4.	$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$
5.	$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$
6.	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$
7.	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$
8.	$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$
9.	$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots\ldots\ldots$
10.	$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \ldots.(-1 < x < 1)$
11.	$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ (-1 < x < 1)$
L'Hôpital's rule	
For functions f and g which are differentiable: if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ has a finite value then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$	
Common Indeterminate Forms	
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \mathrm{x} \infty, \infty - \infty, 0^0, 1^\infty$ and ∞^0	
Differentiation	
$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$	
Continuity	
Function f(x) is continuous at x = a if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$	

Differentiability
f(x) is said to be derivable or differentiable at x = a if $f'(a^+) = f'(a^-)$ = finite quantity
Rolle's theorem
If a function f defined on the closed interval [a, b] is continuous on [a, b] and derivable on (a, b) and f(a) = f(b), then there exists at least one real number c between a and b (a < c < b), such that f ' (c) = 0
Mean Value Theorem
If a function f defined on the closed interval [a, b], is continuous on [a, b] and derivable on (a, b), then there exists at least one real number c between a and b (a < c < b), such that $f'(c) = \frac{f(b) - f(a)}{b - a}$