27. 3D GEOMETRY

1. COORDINATE OF A POINT IN SPACE

Let P be a point in the space. If a perpendicular from that point is dropped to the xy-plane, then the algebraic length of this perpendicular is considered as z-coordinate. From the foot of the perpendicular, drop a perpendicular to x and y axes, and algebraic lengths of perpendicular are considered as y and x coordinates, respectively.

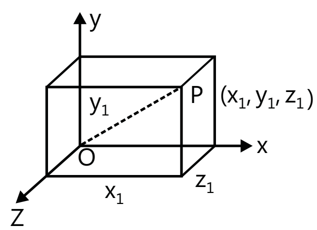


Figure 27.1

2. VECTOR REPRESENTATION OF A POINT IN SPACE

If (x, y, z) are the coordinates of a point P in space, then the position vector of the point P w.r.t. the same origin is $\vec{OP} = x\hat{i} + y\hat{i} + z\hat{k}$.

3. DISTANCE FORMULA

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two points, then the distance between them can be calculated by the following formula: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

3.1 Vector Method

If OA and OB are the position vectors of two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2), then $AB = |\vec{OB} - \vec{OA}|$

$\Rightarrow AB = |(x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)|$ $\quad \Rightarrow AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3.2 Distance of a Point from Coordinate Axes

Let PA, PB and PC be the distances of the point P(x, y, z) from the coordinates axes OX, OY and OZ, respectively. Then $PA = \sqrt{y^2 + z^2}$, $PB = \sqrt{z^2 + x^2}$, $PC = \sqrt{x^2 + y^2}$

Illustration 1: Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form a right-angled isosceles triangle.

(JEE MAIN)

Sol: By using distance formula we can find out length of sides formed by these points and if it satisfies Pythagoras theorem then these points form a right angled triangle.

Let A ≡ (0, 7, 10), B ≡ (–1, 6, 6), C ≡ (–4, 9, 6)$AB^2 = (0 + 1)^2 + (7 – 6)^2 + (10 – 6)^2 = 18$

∴ $AB = 3\sqrt{2}$ Similarly $BC = 3\sqrt{2}$ and AC = 6; Clearly $AB^2 + BC^2 = AC^2$ and AB = BC

Hence, ΔABC is isosceles right angled.

Illustration 2: Find the locus of a point which moves such that the sum of its distance from points A(0, 0, –α) and B(0, 0, α) is constant. **(JEE MAIN)**

Sol: Consider the point whose locus is required be P(x, y, z). As sum of its distance from point A and B is constant therefore PA + PB = constant = 2a.

Let P(x, y, z) be the variable point whose locus is required

Given that PA + PB = constant = 2a(say)

$$\therefore \quad \sqrt{(x-0)^2+(y-0)^2+(z+\alpha)^2}+\sqrt{(x-0)^2+(y-0)^2+(z-\alpha)^2}=2a$$

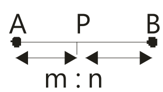
$$\Rightarrow \sqrt{x^2+y^2+(z+\alpha)^2}=2a-\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow x^2+y^2+z^2+\alpha^2+2z\alpha=4a^2+x^2+y^2+z^2+\alpha^2-2z\alpha-4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

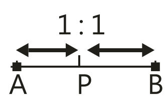
$$\Rightarrow 4z\alpha-4a^2=-4a\sqrt{x^2+y^2+(z-\alpha)^2} \Rightarrow \frac{z^2\alpha^2}{a^2}+a^2-2z\alpha=x^2+y^2+z^2+\alpha^2-2z\alpha \Rightarrow \frac{x^2+y^2}{a^2-\alpha^2}+\frac{z^2}{a^2}=1$$

4. SECTION FORMULA

If a point P divides the distance between the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio of m:n, then the coordinates of P are $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$



Note: Midpoint $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$



5. DIRECTION COSINES AND DIRECTION RATIOS

(a) Direction cosines: If α, β, γ are the angles which the line makes with the positive directions of the axes x, y and z coordinates, respectively, then cosα, cosβ, cosγ are called the direction cosines (d.c.s) of the line. The direction cosines are usually denoted by (*l*, m, n), where *l* = cosα, m = cosβ and n = cos γ.

(b) If ℓ, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

(c) Direction ratios: If the intercepts a, b, c are proportional to the direction cosines ℓ,

(d) m, n, then a, b, c are called the direction ratios (d.r.s).

(e) If ℓ, m, n are the direction cosines and a, b, c are the direction ratios of a vector, then

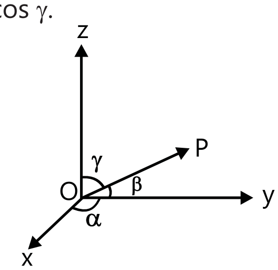


Figure 27.2

$$\ell=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},n=\frac{c}{\sqrt{a^2+b^2+c^2}} \quad \text{or}$$

$$\ell=\frac{-a}{\sqrt{a^2+b^2+c^2}},m=\frac{-b}{\sqrt{a^2+b^2+c^2}},n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$

(f) If OP = r, where O is the origin and *l*, m, n are the direction cosines of OP, then the coordinates of P are (*l*r, mr, nr) If direction cosines of the line AB are *l*, m, n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1), then the coordinates of B are $(x_1 + rl, y_1 + rm, z_1 + rn)$

Illustration 3: Let α, β, γ be the angles made with the coordinate axes. Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(JEE ADVANCED)

Sol: Here line makes angles α, β, γ with the co-ordinates axes, hence by using its direction cosine we can prove given equation.

Since a line makes angles α, β, γ with the coordinates axes, $\cos\alpha, \cos\beta, \cos\gamma$, are direction cosines.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1 \Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Illustration 4: Find the direction cosines l, m, n of a line using the following relations: $l + m + n = 0$ and $2mn + 2ml - nl = 0$.

(JEE ADVANCED)

Sol: By solving these two equations simultaneously, we will be get $l : m : n$.

Given, $\ell + m + n = 0$ ... (i) $\quad 2mn + 2m\,l - n\,l = 0$ ...(ii)

From equation (i), $n = -(l + m)$

Substituting $n = -(l + m)$ in equation (ii), we get,

$-2m(l + m) + 2m\,l + (l + m)\,l = 0 \quad \Rightarrow -2ml - 2m^2 + 2ml + l^2 + ml = 0$

$\Rightarrow l^2 + m\,l - 2m^2 = 0 \quad \Rightarrow \left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right) - 2 = 0 \Rightarrow \frac{l}{m} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

Case I: When $\frac{\ell}{m} = 1$: In this case $m = \ell$ From equation (1), $2\,l + n = 0$

$\Rightarrow n = -2\,l$

$\therefore \; l : m : n = 1 : 1 : -2 \qquad \therefore$ Direction ratios of the line are 1, 1, –2

$\therefore$ Direction cosines are $\pm\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm\frac{-2}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \text{ or } -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

Case II: When $\frac{\ell}{m} = -2$: In this case $\ell = -2m$

From equation (i), $-2m + m + n = 0 \quad$ Þ $\quad n = m \qquad \therefore \; \ell : m : n = -2m : m : m$

$\therefore$ Direction ratios of the line are –2, 1, 1

$\therefore$ Direction cosines are given by

$$\frac{-2}{\sqrt{(-2)^2 + 1^2 + 1^2}}, \frac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}}, \frac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \quad \text{or} \quad \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$$

6. ANGLE BETWEEN TWO LINE SEGMENTS

If a_1, b_1, c_1 and a_2, b_2, c_2, are the direction ratios of any two lines, respectively, then $a_1i + b_1j + c_1k$ and $a_2i + b_2j + c_2k$ are the two vectors parallel to the lines, and the angle between them is given by the following formula:

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(a) The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) The lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(c) Two parallel lines have same direction cosines, i.e. $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$

Illustration 5: Prove that the lines, whose direction cosines given by the relations $a^2 l + b^2m + c^2n = 0$ and $mn + nl + lm = 0$, are perpendicular if $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=0$ and parallel, if $a \mp b \pm c = 0$ **(JEE ADVANCED)**

Sol: Here if two lines are perpendicular then, $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ and if they are parallel then,

$\ell_1 = \ell_2,\ m_1 = m_2,\ n_1 = n_2$

Given that, $a^2 l + b^2m + c^2n = 0$ ... (i)

and $mn + n l + l m = 0$ ... (ii)

Eliminating m from equations (i) and (ii), we have $-\frac{1}{b^2}(a^2\ell + c^2n)n + n\ell - \frac{1}{b^2}(a^2\ell + c^2n)\ell = 0$

$\Rightarrow a^2n\ell + c^2n^2 - b^2n\ell + a^2\ell^2 + c^2\ell n = 0 \qquad \Rightarrow a^2\frac{\ell}{n} + c^2 - b^2\frac{\ell}{n} + a^2\frac{\ell^2}{n^2} + c^2\frac{\ell}{n} = 0$

$\Rightarrow a^2\left(\frac{\ell}{n}\right)^2 + \left(a^2 - b^2 + c^2\right)\left(\frac{\ell}{n}\right) + c^2 = 0$ ... (iii)

Let $\frac{\ell_1}{n_1}, \frac{\ell_2}{n_2}$ be the roots of the equation (iii).

$\therefore$ Product of roots $\frac{\ell_1}{n_1}\cdot\frac{\ell_2}{n_2} = \frac{c^2}{a^2} \qquad \Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{n_1n_2}{1/c^2} \Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{m_1m_2}{1/b^2} = \frac{n_1n_2}{1/c^2}$ [By symmetry]

$\Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{m_1m_2}{1/b^2} = \frac{n_1n_2}{1/c^2} = \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{1/a^2 + 1/b^2 + 1/c^2}$

For perpendicular lines $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 0$ Two lines are parallel if $\ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

$\Rightarrow \frac{\ell_1}{n_1} = \frac{\ell_2}{n_2}$ $\Rightarrow$ roots of equation (iii) are equal $\Rightarrow (a^2 - b^2 + c^2)^2 - 4a^2c^2 = 0 \Rightarrow a^2 - b^2 + c^2 = \pm 2ac$

$\Rightarrow a^2 + c^2 \pm 2ac = b^2 \Rightarrow (a \pm c)^2 = b^2 \qquad \Rightarrow (a \pm c) = \pm b \qquad \Rightarrow a \mp b \pm c = 0$

Note: In the above result, the two signs are independent of each other. So, the total cases would be $(a+b+c=0, a+b-c=0, a-b+c=0, a-b-c=0)$.

7. PROJECTION OF A LINE SEGMENT ON A LINE

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of P and Q, respectively, then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.

Vector form: Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $= \frac{\vec{a}\vec{b}}{|\vec{b}|}$. In the above case, we replace $2\sqrt{6}$ with $\overrightarrow{PQ}$ as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and $\Rightarrow (x-1)^2 = 11$ with $\ell\hat{i} + m\hat{j} + n\hat{k}$.

where $\ell|r|, m|r| \& n|r|$ are the projections of r in the coordinate axes OX, OY and OZ, respectively.

$r = |r|(\ell\hat{i} + m\hat{j} + n\hat{k})$

Illustration 6: Find the projection of the line joining the coordinates (1, 2, 3) and (–1, 4, 2) on line having direction ratios 2, 3, –6. **(JEE MAIN)**

Sol: Here projection of line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios 2, 3, –6 is given by $\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

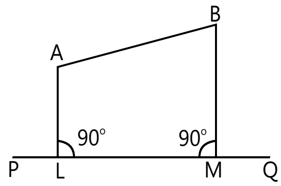


Figure 27.3

Let $A \equiv (1,2,3), B \equiv (-1,4,2)$. Direction ratios of the given line PQ are 2, 3, –6

$\therefore$ Direction cosines of PQ are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$

Projection of AB on PQ = $\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{2}{7}(-1-1) + \frac{2}{7}(4-2) - \frac{6}{7}(2-3) = \frac{-4+6+6}{7} = \frac{8}{7}$$

8. PLANE

If a line joining any two points on a surface entirely lies on it or if a line joining any two points on a surface is perpendicular to some fixed straight line, then the surface is called a plane. This fixed line is called the normal to the plane.

8.1 Equation of a Plane

(a) **Normal form:** The equation of a plane is given by lx + my + nz = p, where ℓ, m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(b) **General form:** The equation of a plane is given by ax + by + cz + d = 0, where a, b, c are the direction ratios of the normal to the plane.

(c) The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are the direction ratios of the normal to the plane.

(d) **Plane through three points:** The equation of a plane through three noncollinear points is given by

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \equiv \begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$

(e) **Intercept form:** The equation of a plane cutting the intercepts a, b, c on the axes is given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(f) **Vector form:** The equation of a plane passing through a point having a position vector $\vec{a}$ and unit vector normal to plane is $(\vec{r} - \vec{a}).\hat{n} = 0 \Rightarrow \vec{r}.\hat{n} = \vec{a}.\hat{n}$

(g) The equation of any plane parallel to the given plane ax + by + cz + d = 0 is given by ax + by + cz + λ = 0 (same direction ratios), where λ is any scalar.

(h) The equation of a plane passing through a given point $\vec{a}$ and parallel to two vectors $\vec{b}$ and $\vec{c}$ is given by $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ where $\vec{r}$ is a position vector of any point on the plane.

8.2 Plane Parallel to a Given Plane

The general equation of the plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + k = 0, where k is any scalar.

Distance between two parallel planes ax + by + cz + d_1 = 0 and ax + by + cz + d_2 = 0 is given by $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

Illustration 7: Find the distance between the planes 2x – y + 2z = 4 and 6x – 3y + 6z = 2. **(JEE MAIN)**

Sol: Here if two planes are parallel then the distance between them is equal to $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

Given planes are $2x - y + 2z - 4 = 0$ ... (i)

and $6x - 3y + 6z - 2 = 0$ ... (ii)

We find that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. Hence, planes (i) and (ii) are parallel

Plane (ii) may be written as $2x - y + 2z - 2/3 = 0$ ... (iii)

$\therefore$ Required distance between the planes $= \frac{|4-(2/3)|}{\sqrt{2^2+(-1)^2+2^3}} = \frac{10}{3.3} = \frac{10}{9}$

8.3 Plane Passing Through the Line of Intersection of Planes

Let π_1 and π_2 be the two planes represented by equations $\vec{r}\cdot\hat{n}_1 = d_1$ and $\vec{r}\cdot\hat{n}_2 = d_2$, respectively. The position vector of any point on the line of intersection must satisfy both equations.

If $\vec{t}$ is the position vector of a point on the line, then $\vec{t}\cdot\hat{n}_1 = d_1$ and $\vec{t}\cdot\hat{n}_2 = d_2$

Therefore, for all real values of λ, we have $\vec{t}\cdot(\hat{n}_1 + \lambda\hat{n}_2) = d_1 + \lambda d_2$

Because $\vec{t}$ is arbitrary, it satisfies for any point on the line. Hence, the equation $\vec{r}\cdot(\hat{n}_1 + \lambda\hat{n}_2) = d_1 + \lambda d_2$ represents a plane π_3 which is such that if any vector X satisfies the equations of both the planes π_1 and π_2, it also satisfies the equation of plane π_3.

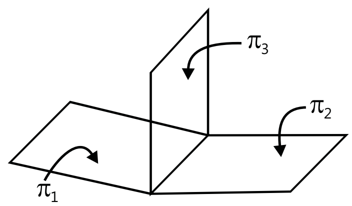


Figure 27.4

8.4 Cartesian Form

In a Cartesian system, let $n_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$, $n_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$ and $r = x\hat{i} + y\hat{j} + z\hat{k}$.

On substituting above values in vector equation we get,

$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$ or $(A_1 x + B_1 y + C_1 z - d_1) + \ell(A_2 x + B_2 y + C_2 z - d_2) = 0$

Illustration 8: Show that the points (0, –1, 0), (2, 1, –1), (1, 1, 1), (3, 3, 0) are coplanar. **(JEE MAIN)**

Sol: Equation of any plane passing through (x_1, y_1, z_1) is given by $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$, by using this formula we can obtain respective equation of plane.

Let A ≡ (0, –1, 0), B ≡ (2, 1, –1), C ≡ (1, 1, 1) and D ≡ (3, 3, 0)

Equation of a plane through A(0, –1, 0) is $a(x - 0) + b(y + 1) + c(z - 0) = 0$

$\Rightarrow ax + by + cz + b = 0$ ...(i)

If plane (i) passes through B(2, 1, –1) and C(1, 1, 1)

Then $2a + 2b - c = 0$ ...(2) and $a + 2b + c = 0$ ...(iii)

From equations (ii) and (iii), we have $\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2}$ or $\frac{a}{4} = \frac{b}{-3} = \frac{c}{2} = k$ (say)

Substituting values of a, b, c in equation (i), equation of the required plane is $4kx - 3k(y + 1) + 2kz = 0$

$\Rightarrow 4x - 3y + 2z - 3 = 0$ ...(iv)

Thus, point D(3, 3, 0) lies on plane (iv).

Because the points on the plane passes through A, B, C, the points A, B, C and D are coplanar.

Illustration 9: Find the equation of the plane upon which the length of normal from the origin is 10 and direction ratios of this normal are 3, 2, 6. **(JEE ADVANCED)**

Sol: Let p be the length of perpendicular from the origin to the plane and ℓ, m, n be the direction cosines of this normal. The equation is given by

$\ell x + my + nz = p$ ... (i)

From the data provided, p = 10 and the direction ratios of the normal to the plane are 3, 2, 6.

$\therefore$ Direction cosines of normal to the required plane are $\ell = \frac{3}{7}, m = \frac{2}{7}, n = \frac{6}{7}$

Substituting values of ℓ, m, n, p in equation (i), equation of the required plane is $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$

$\Rightarrow$ 3x + 2y + 6z = 70

Illustration 10: A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$ A plane through P and perpendicular to OP meets the coordinate axes in A, B and C. If the planes through A, B and C parallel to the planes x = 0, y = 0, z = 0 intersect in Q, find the locus of Q. **(JEE ADVANCED)**

Sol: Similar to above problem.

Given plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ... (i)

Let P ≡ (h, k, ℓ), Then, $\frac{h}{a} + \frac{k}{b} + \frac{\ell}{c} = 1$ ... (ii)

(OP) = $\sqrt{h^2 + k^2 + \ell^2}$

Direction cosines of OP are $\frac{h}{\sqrt{h^2 + k^2 + \ell^2}}, \frac{k}{\sqrt{h^2 + k^2 + \ell^2}}, \frac{\ell}{\sqrt{h^2 + k^2 + \ell^2}}$

$\therefore$ Equation of the plane through P and normal to OP is

$$\frac{h}{\sqrt{h^2 + k^2 + \ell^2}}x + \frac{k}{\sqrt{h^2 + k^2 + \ell^2}}y + \frac{\ell}{\sqrt{h^2 + k^2 + \ell^2}} = \sqrt{h^2 + k^2 + \ell^2}$$

$$\Rightarrow hx + ky + \ell z = (h^2 + k^2 + \ell^2),\ A \equiv \left(\frac{h^2 + k^2 + \ell^2}{h}, 0, 0\right), B = \left(0, \frac{h^2 + k^2 + \ell^2}{k}, 0\right), C \equiv \left(0, 0\frac{h^2 + k^2 + \ell^2}{\ell}\right)$$

$$\Rightarrow A = \left(\frac{h^2 + k^2 + \ell^2}{h}, 0, 0\right), B = \left(0, \frac{h^2 + k^2 + \ell^2}{k}, 0\right), C = \left(0, 0\frac{h^2 + k^2 + \ell^2}{\ell}\right)$$

Let $Q \equiv (\alpha, \beta, \gamma)$, then $\alpha = \frac{h^2 + k^2 + \ell^2}{h}, \beta = \frac{h^2 + k^2 + \ell^2}{k}, \gamma = \frac{h^2 + k^2 + \ell^2}{\ell}$ ... (iii)

Now $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{h^2 + k^2 + \ell^2}{(h^2 + k^2 + \ell^2)^2} = \frac{1}{(h^2 + k^2 + \ell^2)}$ ... (iv)

From equation (iii), $h = \frac{h^2 + k^2 + \ell^2}{\alpha}$

$\therefore \frac{h}{a} = \frac{h^2 + k^2 + \ell^2}{a\alpha}$ Similarly $\frac{k}{b} = \frac{h^2 + k^2 + \ell^2}{b\beta}$ and $\frac{\ell}{c} = \frac{h^2 + k^2 + \ell^2}{c\gamma}$

$\therefore \frac{h^2 + k^2 + \ell^2}{a\alpha} + \frac{h^2 + k^2 + \ell^2}{b\beta} + \frac{h^2 + k^2 + \ell^2}{c\gamma} = \frac{h}{a} + \frac{k}{b} + \frac{\ell}{c} = 1$ [from equation (ii)] or, $\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+\ell^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ [from equation (iv)]

$\therefore$ Required locus of Q(α, β, γ) is $\frac{1}{ax}+\frac{1}{by}=\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$.

8.5 Plane and a Point

(a) A plane divides the three-dimensional space into two equal segments. Two points A(x_1 y_1 z_1) and B(x_2 y_2 z_3) lie on the same sides of the plane ax + by + cz + d = 0 if the two expressions $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_3 + d$ are of same sign, and lie on the opposite sides of plane if both of these expressions are of opposite sign.

(b) Perpendicular distance of the point (x′, y′, z′) from the plane ax + by + cz + d = 0 is given by

$$\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}.$$

(c) The length of the perpendicular from the point having a position vector $\vec{a}$ to the plane $\vec{r}.\vec{n}=d$ is given by

$$P=\frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$$

(d) The coordinates of the foot of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ are given by $\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(e) If P′(x′, y′, z′) is the image of a point P(x_1, y_1, z_1) w.r.t. the plane ax + by + cz + d = 0, then

$$\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

MASTERJEE CONCEPTS

The distance between two parallel planes ax + by + cx + d = 0 and ax + by + cx + d′ = 0 is given by

$$\frac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$$

If a variable point P moves so that $PA^2 - PB^2 = K$, where K is a constant and A and B are the two points, then the locus of P is a plane.

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 11: Show that the points (1, 2, 3) and (2, –1, 4) lie on the opposite sides of the plane x + 4y + z – 3 = 0. **(JEE MAIN)**

Sol: Substitute given points in to the given equation of plane, if their values are in opposite sign then the points are on opposite sides of the plane.

Since 1 + 4 × 2 + 3 – 3 = 9 and 2 – 4 + 4 – 3 = –1 are of opposite sign, the points are on opposite sides of the plane.

8.6 Angle Between Two Planes

Let us consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d = 0. Angle between these planes is the angle between their normals. Let (a, b, c) and (a′, b′, c′) be the direction ratios of their normals of the two planes, respectively, and the angle θ between them is given by

$$\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

The planes are perpendicular if aa′ + bb′ + cc′ = 0 and the planes are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

In vector form, if θ is the angle between the planes $\vec{r}.\vec{n}=d_1$ and $\vec{r}.\vec{n}_2=d_2$, then $\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$. The planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2=0$ and the planes are parallel if $\vec{n}_1=\lambda\vec{n}_2$.

8.6.1 Angle Bisectors

(a) Equations of the planes bisecting the angle between the two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(b) Equation of bisector of the angle containing the origin is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$ [where d_1 and d_2 are positive]

(c) In order to find the bisector of acute/obtuse angle, both the constant terms should be positive. If

$a_1 a_2 + b_1 b_2 + c_1 c_2 > 0$ Þ then the origin lies in the obtuse angle

$a_1 a_2 + b_1 b_2 + c_1 c_2 < 0$ Þ then the origin lies in the acute angle

now apply step (ii) according to the question.

8.7 Family of Planes

(a) The equation of any plane passing through the line of intersection of nonparallel planes or through the given line is

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, i.e. $P_1 = 0$ and $P_2 = 0$

$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, i.e. $P_1+\lambda P_2=0$

(b) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1=d_1$ & $\vec{r}.\vec{n}_2=d_2$ is $r.(n_1+\lambda n_2)=d_1+\lambda d_2$, where λ is an arbitrary scalar.

Illustration 12: The plane x – y – z = 4 is rotated through 90° about its line of intersection with the plane x + y + 2z = 4. Find its equation in the new position. **(JEE MAIN)**

Sol: As the required plane passes through the line of intersection of given planes, therefore its equation may be taken as x + y + 2z – 4 + k(x – y – z – 4) = 0

$\Rightarrow$ (1 + k) x + (1 – k) y + (2 – k) z – 4 – 4k = 0 ... (iii)

Thus, planes (i) and (iii) are mutually perpendicular.

$\therefore$ (1 + k) – (1 – k) – (2 – k) = 0 $\Rightarrow$ 1 + k – 1 + k – 2 + k = 0 $\Rightarrow$ k = 2/3

Substituting k = 2/3 in equation (iii), we get, 5x + y + 4z = 20. This is the required equation of the plane in its new position.

Illustration 13: Find the equation of the plane through the point (1, 1, 1) and passing through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ **(JEE MAIN)**

Sol: Similar to above illustration.

Given planes are $x + y + z - 6 = 0$ ... (i)

and $2x + 3y + 4z + 5 = 0$ ... (ii)

Given point is P(1, 1, 1).

Equation of any plane passing through the line of intersection of the planes (i) and (ii) is

$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ ... (iii)

If plane (iii) passes through a point P, then $1 + 1 + 1 - 6 + k(2 + 3 + 4 + 5) = 0$

$\Rightarrow k = \frac{3}{14}$

From equation (i), the required plane is $20x + 23y + 26z - 69 = 0$

Illustration 14: If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$. **(JEE ADVANCED)**

Sol: Here the plane passing through the line of intersection of planes $x - cy - bz = 0$ and $cx - y + az = 0$ is same as the plane $bx + ay - z = 0$. Hence by using family of planes we can obtain required result.

Given planes are $x - cy - bz = 0$ ... (i)

$cx - y + az = 0$ ... (ii)

$bx + ay - z = 0$ ... (iii)

Equation of any plane passing through the line of intersection of the planes (i) and (ii) may be written as

$x - cy - bz + \lambda(cx - y + az) = 0 \quad \Rightarrow x(1 + lc) - y(c + \lambda) + z(-b + a\lambda) = 0$ ... (iv)

If planes (3) and (4) are the same, then equations (iii) and (iv) will be identical.

$$\therefore \underset{\text{(i)}}{\frac{1+c\lambda}{b}} = \underset{\text{(ii)}}{\frac{-(c+\lambda)}{a}} = \underset{\text{(iii)}}{\frac{-b+a\lambda}{-1}};$$

From equations (i) and (ii), $a + ac\lambda = -bc - b\lambda$

$\Rightarrow \lambda = -\frac{(a+bc)}{(ac+b)}$ ... (v)

From equations (ii) and (iii), $c + \lambda = -ab + a^2\lambda$

$\Rightarrow \lambda = -\frac{(ab+c)}{1-a^2}$ ... (vi)

From equations (v) and (vi), we have, $\frac{-(a+bc)}{ac+b} = \frac{-(ab+c)}{(1-a^2)}$

$\Rightarrow a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc \quad \Rightarrow 2a^2bc + ac^2 + ab^2 + a^3 - a = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

Illustration 15: Through a point P(h, k, ℓ), a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\frac{p^5}{2hk\ell}$. **(JEE ADVANCED)**

Sol: Here line OP is normal to the plane, therefore $\ell x + my\ nz = p$. where ℓ, m and n are direction cosines of given plane.

$OP = \sqrt{h^2 + k^2 + \ell^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is the normal to the plane, therefore, equation of the plane will be

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

$\Rightarrow$ $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$

$$\therefore\ A = \left(\frac{p^2}{h}, 0, 0\right), B = \left(0, \frac{p^2}{k}, 0\right), C = \left(0, 0, \frac{p^2}{\ell}\right)$$

Thus, area of $\Delta ABC = \frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{2}$

$$= \frac{\left|\left(\frac{p^2}{h}\hat{i} - \frac{p^2}{k}\hat{j}\right) \times \left(\frac{p^2}{h}\hat{i} - \frac{p^2}{\ell}\hat{k}\right)\right|}{2} = \frac{\left|\left(\frac{p^4}{h\ell}\hat{j} + \frac{p^4}{kh}\hat{k} + \frac{p^4}{k\ell}\hat{i}\right)\right|}{2}$$

$$= \frac{1}{2}\sqrt{p^8\left(\frac{1}{h^2\ell^2} + \frac{1}{h^2k^2} + \frac{1}{k^2\ell^2}\right)} = \frac{1}{2}\sqrt{\frac{P^8}{h^2\ell^2k^2}\left(\ell^2 + h^2 + k^2\right)} = \frac{p^5}{2hk\ell}$$

9. TETRAHEDRON

Volume of a tetrahedron given the coordinates of its vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ can be calculated by

$$V = \frac{1}{6}\left\|\begin{matrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & x_4 & 1 \end{matrix}\right\| = \frac{1}{6}\left\|\begin{matrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{matrix}\right\|$$

MASTERJEE CONCEPTS

Four points (x_r, y_r, z_r); r = 1, 2, 3, 4; will be coplanar if the volume of the tetrahedron with the points as vertices is zero. Therefore, the condition of coplanarity of the points

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$

Vaibhav Gupta (JEE 2009 AIR 54)

Centroid of a Tetrahedron

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ be the vertices of a tetrahedron.

The coordinate of its centroid (G) is given as $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}, \frac{\Sigma z_i}{4}\right)$

Illustration 16: If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular. **(JEE MAIN)**

Sol: If two lines are perpendicular then summation of product of their respective direction ratios is equals to zero.

Let OABC be the tetrahedron where O is the origin and coordinate of A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), respectively.

Let $OA \perp BC$ and $OB \perp CA$. We have to prove that $OC \perp BA$

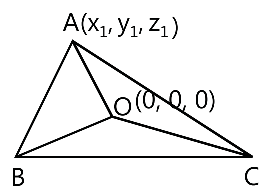


Figure 27.5

Direction ratios of OA are $x_1 - 0$, $y_1 - 0$, $z_1 - 0$ or x_1, y_1, z_1

Direction ratios of BC are $(x_3 - x_2)$, $(y_3 - y_2)$, $(z_3 - z_2)$

$OA \perp BC$

$\Rightarrow x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 + z_2) = 0$ ... (i)

Similarly, $OB \perp CA$

$\Rightarrow x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$ ... (ii)

On adding equations (1) and (2), we obtain the following equation:

$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ [$\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2)$, $(y_1 - y_2)$, $(z_1 - z_2)$]

10. LINE

10.1 Equation of a line

A straight line in space will be determined if it is the intersection of two given nonparallel planes and therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non-symmetrical form.

(a) The equation of a line passing through the point (x_1, y_1, z_1) with a, b, c as direction ratios is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetrical form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(b) Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar.

(c) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_3) is

(d) $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

(e) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(f) Reduction of Cartesian form of equation of a line to vector form and vice versa is as

(g) $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \Leftrightarrow \quad r = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$

MASTERJEE CONCEPTS

Straight lines parallel to coordinate axes

Straight lines	Equations
(i) Through origin	y = mx, z = nx
(ii) x-axis	y = 0, z = 0
(iii) y-axis	x = 0, z = 0
(iv) z-axis	x = 0, y = 0
(v) Parallel to x-axis	y = p, z = q
(vi) Parallel to y-axis	x = h, z = q
(vii) Parallel to z-axis	x = h, y = p

The number of lines which are equally inclined to the coordinate axes are 4.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 17: Find the equation of the line passing through the points (3, 4, –7) and (1, –1, 6) in vector form as well as in Cartesian form. **(JEE MAIN)**

Sol: Here line in vector form is given by $r = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$ and in Cartesian form is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Let $A \equiv (3, 4, -7)$, $B \equiv (1, -1, 6)$; Now, $\vec{a} = \overrightarrow{OA} = 3\hat{i} + 4\hat{j} - 7\hat{k}, \vec{b} = \overrightarrow{OB} = \hat{i} - \hat{j} + 6\hat{k}$

Equation (in vector form) of the line passing through A(a) and B($\vec{b}$) is $r = a + t(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = 3\vec{i} + 4\vec{j} - 7\vec{k} + t(-2\vec{i} - 5\vec{j} + 13\vec{k})$ ... (i)

Equation in Cartesian form is $\frac{x-3}{3-1} = \frac{y-4}{4+1} = \frac{z+7}{-7-6} \Rightarrow \frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$

Illustration 18: Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines. **(JEE MAIN)**

Sol: The given lines will intersect if any point on respective lines coincide for some value of λ and r.

Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ .... (i)

and $\frac{z-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ .... (ii)

Any point on line (1) is P(2r + 1, 3r + 2, 4r + 3) and any point on line (2) is Q(5λ + 4, 2λ + 1, λ)

Lines (i) and (ii) will intersect if P and Q coincide for some value of λ and r.

$\Rightarrow 2r + 1 = 5\lambda + 4$ $\Rightarrow 2r - 5\lambda = 3$ ... (iii)

$\Rightarrow 3r + 2 = 2\lambda + 1$ $\Rightarrow 3r - 2\lambda = -1$ ... (iv)

$\Rightarrow 4r + 3 = \lambda$ $\Rightarrow 4r - \lambda = -3$ ... (v)

Solving equations (iii) and (iv), we get r = –1, λ = –1; these obtained values of r and λ clearly satisfy equation (v)

$\Rightarrow$ P ≡ (–1, –1, –1). Hence, lines (i) and (ii) intersect at (–1, –1, –1)

Illustration 19: Find the angle between the lines x – 3y – 4 = 0, 4y – z + 5 = 0 and x + 3y – 11 = 0, 2y – z + 6 = 0. **(JEE MAIN)**

Sol: $\frac{x-4}{3}=\frac{y-0}{1}=\frac{z-5}{4}$ ... (i)

$\frac{x-11}{-3}=\frac{y-0}{1}=\frac{z-6}{2}$ ... (ii)

a = 3, b = 1, c = 4 $\quad\therefore\quad a^1=-3, b^1=1, c^1=2$

$aa^1+bb^1+cc^1=-9+1+8=0 \Rightarrow \cos\theta=0 \qquad \theta=90$

Illustration 20: Find the equation of the line drawn through point (1, 0, 2) to meet at right angle with the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$. **(JEE ADVANCED)**

Sol: If two lines are perpendicular then summation of product of their direction ratios are equal to zero. Hence by obtaining direction ratio of these line, we will be get the result.

Given line is $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ ... (i)

Let P ≡ (1, 0, 2); coordinates of any point on line (i) may be written as Q≡ (3r – 1, –2r + 2, –r –1).

Direction ratios of PQ are 3r – 2, –2r + 2, –r –3

Direction ratios of the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ are 3, –2, –1

Since PQ ⊥ to the line $\Rightarrow$ 3(3r – 2) – 2(–2r + 2) – 1 (–r – 3) = 0

$\Rightarrow$ 9r – 6 + 4r – 4 + r + 3 = 0 $\Rightarrow$ 14r = 7 $\Rightarrow$ r = $\frac{1}{2}$ $\quad\therefore$ Direction ratios of PQ are $-\frac{1}{2},1,-\frac{7}{2}$

Hence, equation of the line PQ is $\frac{x-1}{-1}=\frac{y-0}{2}=\frac{z-2}{-7}$

Illustration 21: Find the equation of the line of intersection of the planes 4x + 4y – 5z = 12, 8x + 12y – 13z = 32 in the symmetric form **(JEE ADVANCED)**

Sol: Consider the line of intersection meet the xy-plane at P(α, β, 0), therefore obtain the value of α and β and direction ratios of line of intersection to solve the problem.

Given planes are 4x + 4y – 5z – 12 = 0 ... (i)

and 8x + 12y – 13z – 32 = 0 ... (ii)

Let ℓ, m, n be the direction ratios of the line of intersection. Then,

$4\ell+4m-5n=0$ and $8\ell+12m-13n=0 \Rightarrow \frac{\ell}{-52+60}=\frac{m}{-40+52}=\frac{n}{48-32} \Rightarrow \frac{\ell}{2}=\frac{m}{3}=\frac{n}{4}$

Direction ratios of the line of intersection are 2, 3, 4.

Let the line of intersection meet the xy-plane at P(α, β, 0)

Then P lies on planes (i) and (i) $\Rightarrow$ 4α + 4β – 12 = 0 $\Rightarrow$ α + β – 3 = 0

and 8α + 12β – 32 = 0 ... (v)

or 2α + 3β – 8 = 0 ... (vi)

Solving equations (v) and (vi), we get $\frac{\alpha}{-8+9}=\frac{\beta}{-6+8}=\frac{1}{3-2} \Rightarrow \alpha=1, \beta=2$

Hence, equation of the line of intersection in symmetrical form is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-0}{4}$

10.1 Coplanar Lines

Coplanar lines are lines that entirely lie on the same plane.

(i) If $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{x-\beta'}{m'}=\frac{z-\gamma'}{n'}$, are the lines, then the condition for intersection/coplanarity is $\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0$ and the plane containing the aforementioned lines is $\begin{vmatrix}x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0.$

(ii) Condition of coplanarity if both lines are in general form.

Let the lines be ax + by + cz + d = 0 = a′x + b′y + c′z + d′ and ax + by + gz + δ = 0 = α'x+ β'y + γ'z + δ' = 0

$$\text{If } \Delta = \begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix}=0\text{, then they are coplanar.}$$

10.2 Skew Lines

Skew lines are two lines that do not intersect and are not parallel.

$$\text{If } \Delta = \begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}\neq 0,\text{ then the lines are skew.}$$

Shortest distance

Let the equation of the lines be $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{x-\beta'}{m'}=\frac{z-y'}{n'}$

$$\text{S.D.} = \frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(n\ell'-n'\ell)+(\gamma-\ell')(\ell m'-\ell' m)}{\sqrt{\sum(mn'-m'n)^2}} = \begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}\Bigg/\sqrt{\sum(nm'-m'n)^2}$$

Vector form

For lines $\vec{a}_1+\lambda\vec{b}_1$ and $\vec{a}_2+\lambda\vec{b}_2$ to be skew, the following condition should be satisfied: $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_1-\vec{a}_2)\neq 0$

MASTERJEE CONCEPTS

Shortest distance between two skew lines is perpendicular to both the lines.

Anvit Tawar (JEE 2009 AIR 9)

10.3 Intersecting Lines

Two or more lines that intersect at a point are called intersecting lines, and their shortest distance between the two lines is

zero, i.e. $\left|\dfrac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0 \Rightarrow (\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_1 - \vec{a}_2) = 0 \Rightarrow [\vec{b}_1 \vec{b}_2 (\vec{a}_2 - \vec{a}_1)] = 0 \Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & a_2 & c_2 \end{vmatrix} = 0$

10.4 Parallel Lines

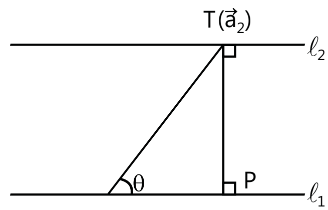


Figure 27.6

Parallel lines are lines that never intersect, and are coplanar.

Let $\left|-\dfrac{8}{3}, \dfrac{1}{3}, \dfrac{16}{3}\right|$ and $\left|\dfrac{8}{3}, \dfrac{1}{3}, \dfrac{16}{3}\right|$ be the two parallel lines.

Let the lines be given by $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ ... (i)

$\vec{r} = \vec{a}_2 + \mu\vec{b}$ ... (ii)

where $\vec{a}_1$ is the position vector of a point S on $\dfrac{x}{2} = \dfrac{y-1}{3} = \dfrac{z}{5}$ and $\vec{a}_2$ is the position vector of a point T on ℓ_2. As ℓ_1 and ℓ_2 are coplanar, if the foot of the perpendicular from T on the line ℓ_1 is P, then the distance between the lines ℓ_1 and ℓ_2 = |TP|. Let θ be the angle between the vectors $\overrightarrow{ST}$ and $\vec{b}$.

Then $\vec{b} \times \overrightarrow{ST} = (|\vec{b}||\overrightarrow{ST}|\sin\theta)\hat{n}$ ... (iii)

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines $a\ell + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d \neq 0$. But $\overrightarrow{ST} = \vec{a}_2 - \vec{a}_1$

Therefore, from equation (iii), we get $\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|PT\hat{n}$ (as PT = ST sin θ)

$\Rightarrow |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|PT \cdot 1 (\text{as} |\hat{n}| = 1)$ Hence, the distance between the given parallel lines is $d = |\overrightarrow{PT}| = \left|\dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$.

10.5 Angular Bisector

If $a(x-\alpha) + b(y-\beta)$, m_1, n_1 and $+c(z-\gamma) = 0$, , m_2, n_2 are the direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the

(a) Internal bisectors of the angle between the lines are $\dfrac{\ell_1 + \ell_2}{2\cos(\theta/2)}, \dfrac{m_1 + m_2}{2\cos(\theta/2)}$ and $\dfrac{n_1 + n_2}{2\cos(\theta/2)}$.

(b) External bisectors of the angle between the lines are $\dfrac{\ell_1 - \ell_2}{2\sin(\theta/2)}, \dfrac{m_1 - m_2}{2\sin(\theta/2)}$ and $\dfrac{n_1 - n_2}{2\sin(\theta/2)}$.

10.6 Reduction to Symmetric Form

Let the line in nonsymmetrical form be represented as $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$.

To find the equation of the line in symmetrical form, (i) its direction ratios and (ii) coordinate of any point on it must be known.

(a) **Direction ratios:** Let ℓ, m, n be the direction ratios of the line. Since the line lies on both planes, it must be perpendicular to the normal of both planes. So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by using the method of cross-multiplication, i.e.

$$\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

Alternate method

The vector $\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i(b_1c_2 - b_2c_1) + j(c_1a_2 - c_2a_1) + k(a_1b_2 - a_2b_1)$ will be parallel to the line of intersection of the two given planes. Hence, $\ell : m : n = (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1)$.

(b) Coordinate of any point on the line: Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be nonzero. Let $a_1b_2 - a_2b_1 \neq 0$, so that the line cannot be parallel to xy-plane, and will intersect it. Let it intersect xy-plane at the point $(x_1, y_1, 0)$. These $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$. Solving these, we get a point on the line. Thus, we get the following equation:

$$\frac{x-x_1}{b_1c_2-b_2c_1} = \frac{y-y_1}{c_1a_2-c_2a_1} = \frac{z-0}{a_1b_2-a_2b_1} \text{ or } \frac{x-(b_1d_2-b_2d_1/a_1b_2-a_2b_1)}{b_1c_2-b_2c_1} = \frac{y-(d_1a_2-d_2a_1/a_1b_2-a_2b_1)}{c_1a_2-c_2a_1} = \frac{z-0}{a_1b_2-a_2b_1}$$

Note: If $\ell \neq 0$, take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on xz-plane as $(x_1, 0, z_1)$.

Alternate method:

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then put z = 0 in both equations and solve the equation $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ or put y = 0 and solve the equation $a_1x + c_1z + d_1 = 0$ and $a_2x + c_2z + d_2 = 0$.

10.7 Point and Line

Foot Length and Equation of Perpendicular from a Point to a Line

Cartesian form: Let equation of the line be $\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n} = r\,(\text{say})$ ...(i)

and $A(\alpha, \beta, \gamma)$ be the point. Any point on line (i) is P(lr + a, mr + b, nr + c). If it is the foot of the perpendicular from point A on the line, then AP is perpendicular to the line.

$\Rightarrow \ell(\ell r + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$, i.e. $r = [(\alpha - a)\ell + (\beta - b)m + (\gamma - c)n]/\, l^2 + m^2 + n^2$

Using this value of r, we get the foot of the perpendicular from point A on the given line. Because the foot of the perpendicular P is known, the length of the perpendicular $AP = \sqrt{(\ell r + a - \alpha^2) + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$ is given by the equation of perpendicular as $\frac{x-\alpha}{\ell r + a - \alpha} = \frac{y-\beta}{mr + b - \beta} = \frac{z-\gamma}{nr + c - \gamma}$

Illustration 22: Find the coordinates of the foot of the perpendicular drawn from the point A(1, 2, 1) to the line joining B(1, 4, 6) and C(5, 4, 4). **(JEE ADVANCED)**

Sol: Using section formula we will get co-ordinates of the foot D, and as AD is perpendicular to BC therefore $\overrightarrow{AD}.\overrightarrow{BC} = 0$.

Let D be the foot of the perpendicular drawn from A on BC,

and let D divide BC in the ratio k:1. Then, the coordinates

of D are $\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$ ...(i)

Now, $\overrightarrow{AD}$ = Position vector of D – Position vector of A

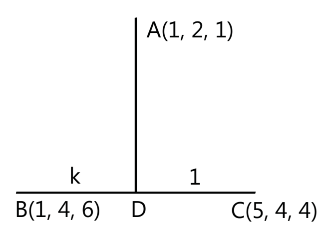


Figure 27.7

$$= \left(\frac{5k+1}{k+1}-1\right)\hat{i}+\left(\frac{4k+4}{k+1}-2\right)\hat{j}+\left(\frac{4K+6}{K+1}-1\right)\hat{K} = \left(\frac{4k}{k+1}\right)\hat{i}+\left(\frac{2k+2}{k+1}\right)\hat{j}+\left(\frac{3k+5}{k+1}\right)\hat{k}$$

and $\overrightarrow{BC}$ = Position vector of C – Position vector of B $= (5\hat{i}+4\hat{j}+4\hat{k})-(\hat{i}+4\hat{j}+6\hat{k}) = 4\hat{i}+0\hat{j}-2\hat{k}$

since $\overrightarrow{AD}\perp\overrightarrow{BC} \Rightarrow \overrightarrow{AD}.\overrightarrow{BC}=0 \Rightarrow \left[\left(\frac{4k}{k+1}\right)\hat{i}+\left(\frac{2k+2}{k+1}\right)\hat{j}+\left(\frac{3k+5}{k+1}\right)\hat{k}\right].(4\hat{i}+0\hat{j}-2\hat{k})=0$

$\Rightarrow 4\left(\frac{4k}{k+1}\right)+0\left(\frac{2k+2}{k+1}\right)-2\left(\frac{3k+5}{k+1}\right)=0 \quad \Rightarrow \frac{16k}{k+1}+0-2\frac{(3k+5)}{k+1}=0$

$\Rightarrow \quad 16k-6k-10=0 \qquad \Rightarrow \quad k = 1$

Substituting k = 1 in equation (i), we obtain the coordinates of D as (3, 4, 5).

10.8 Vector Form

Equation of a line passing through a point having position vector $\bar{\alpha}$ and perpendicular to the lines $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ a and $\vec{r}_1 = \vec{a}_2 + \lambda\vec{b}_2$ is parallel to $\vec{b}_1 \times \vec{b}_2$. So the vector equation of such line is $\vec{r} = \vec{\alpha} + \lambda(\vec{b}_1 \times \vec{b}_2)$. The equation of the perpendicular passing through $\vec{\alpha}$ is $\vec{r} = \vec{\alpha} + \mu\left((\vec{a}-\vec{\alpha}-\left(\frac{(\vec{a}-\vec{\alpha})\vec{b}}{|\vec{b}|^2}\right)b.\right)$

10.9 Image w.r.t. the Line

Let $L \equiv \frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ be the given line.

Let (x′, y′, z′) be the image of the point $P(x_1, y_1, z_1)$ w.r.t the line L. Then

(i) $a(x_1-x')+b(y_1-y')+c(z_1-z')=0$

(ii) $\frac{\frac{x_1-x'}{2}-x_2}{a} = \frac{\frac{y_1-y'}{2}-y_2}{b} = \frac{\frac{z_1-z'}{2}-z_2}{c} = \lambda$

From (ii), the value of x′, y′, z′ in terms of λ can be obtained as $x' = 2a\lambda + 2x_2 - x_1$, $y' =$ ɩ $- 2y_2 - y_1$, $z' = 2c\lambda + 2z_2 - z_1$

On substituting values of x′, y′, z′ in (i), we get λ and on re-substituting value of λ, we get (x′ y′ z′).

Illustration 23: Find the length of the perpendicular from P(2, –3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ **(JEE MAIN)**

Sol: Here Co-ordinates of any point on given line may be taken as Q ≡ (2r – 1, 3r + 3, –r – 2), therefore by using distance formula we can obtain required length.

Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ ...(i) P = (2, –3, 1)

Coordinates of any point on line (i) may be written as Q ≡ (2r – 1, 3r + 3, –r – 2)

Direction ratios of PQ are 2r – 3, 3r + 6, –r – 3.

Direction ratios of AB are 2, 3, –1.

Since PQ ⊥ AB $\Rightarrow$ 2(2r – 3) + 3(3r + 6) – 1(–r – 3) = 0 $\Rightarrow r = \frac{-15}{14}$

$\Rightarrow Q = \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right) \Rightarrow PQ = \frac{\sqrt{531}}{14}$ units

10.10 Plane Passing Through a Given Point and Line

Let the plane pass through the given point A($\vec{a}$) and line $\vec{r} = \vec{b} + \lambda\vec{c}$. For any position of point R (r) on the plane, vectors $\overrightarrow{AB}, \overrightarrow{RA}$ and $\vec{c}$ are coplanar.

Then $\left[\vec{r}-\vec{a} \quad \vec{b}-\vec{a} \quad \vec{c}\right] = 0$, which is the required equation of the plane.

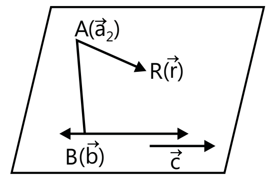


Figure 27.8

Angle between a plane and a line:

Angle between a line and a plane is complementary to the angle made by the line with the normal of plane. Hence, if θ is the angle between the line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane

ax + by + cz + d = 0, then $\sin\theta = \left(\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(\ell^2+m^2+n^2)}}\right)$

Vector form:

If θ is the angle between the line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$

Line and plane are perpendicular if $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, i.e. $\vec{b}\times\vec{n} = 0$.

Line and plane are parallel if aℓ + bm + cn = 0, i.e. $\vec{b}\cdot\vec{n} = 0$.

MASTERJEE CONCEPTS

Condition for a Line to Lie on a Plane

(i) **Cartesian form:** Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie on a plane ax + by + cz + d = 0 if, $ax_1 + by_1 + cz_1 + d = 0$ and aℓ + bm + cn = 0

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\vec{b} +$ would lie on the plane $\vec{r}\cdot\hat{n} = d$ if $\vec{b}\cdot\hat{n} = 0$ & $\vec{a}\cdot\hat{n} = d$.

The number of lines which are equally inclined to the coordinate axes is 4.

If ℓ, m, n are the d.c.s of a line, then the maximum value of lmn $= \frac{1}{3\sqrt{3}}$.

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Illustration 24: Find the shortest distance and the vector equation of the lines of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$. **(JEE ADVANCED)**

Sol: Consider LM is the shortest distance between given lines therefore LM is perpendicular to these lines, hence by obtaining their direction ratios and using perpendicular formula we will get the result.

Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\,(3\hat{i} - \hat{j} + \hat{k})$ ... (i)

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\,(-3\hat{i} + 2\hat{j} + 4\hat{k})$ ... (ii)

Equations of lines (i) and (ii) in Cartesian form are

$$AB: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda \quad \text{and} \quad CD: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$$

Let $L \equiv (3\lambda+3, -\lambda+8, \lambda+3)$ and $M \equiv (-3\mu-3,\ 2\mu-7,\ 4\mu+6)$

Direction ratios of LM are $3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$ since LM $\perp$ AB

$\Rightarrow 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or, $11\lambda + 7\mu = 0$ ... (v)

Again $LM \perp CD$

$\therefore -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or, $-7\lambda - 29\mu = 0$ ... (vi)

Solving equations (v) and (vi), we get $\lambda = 0, \mu = 0$ $\Rightarrow$ L ≡ (3, 8, 3), M ≡ (–3, –7, 6)

Hence, the shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\vec{i} + 8\vec{j} + 3\vec{k} + t\,(-6\vec{i} + 15\vec{j} - 3\vec{k})$.

11. SPHERE

11.1 General Equation

The general equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where $(-u, -v, -w)$ is the center and $\sqrt{u^2 + u^2 + w^2 - d}$ is the radius of the sphere.

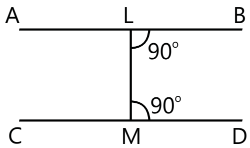

Figure 27.9

11.2 Diametric Form

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the extremities of a diameter of a sphere, then its equation is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

11.3 Plane and Sphere

If the perpendicular distance of the plane from the center of the sphere is equal to the radius of the sphere, then the plane touches the sphere. The plane $lx + my + nz = p$ touches the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if $(u\ell + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

11.4 Intersection of Straight Line and a Sphere

Let the equations of the sphere and the straight line be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (i)

and $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = r$, (say) ... (ii)

Any point on the line (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point lies on the sphere (i), then we have

$$(\alpha + \ell r)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + \ell r) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$$

$$\Rightarrow r^2[\ell^2 + m^2 + n^2] + 2r[\ell(u+\alpha) + m(v+\beta) + n(w+\gamma)] + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad \text{... (iii)}$$

This is a quadratic equation in r and thus two values of r are obtained. Therefore, the line (ii) intersects the sphere (i) at two points which may be real, coincident and imaginary, according to roots of (iii).

If ℓ, m, n are the actual d.c.s of the line, then $l^2 + m^2 + n^2 = 1$ and then the equation (iii) can be simplified.

11.5 Orthogonality of Two Spheres

Let the equation of the two spheres be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (i)

and $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ ... (ii)

If the sphere (i) and (ii) cut orthogonally, then $2uu' + 2vv' + 2ww' = d + d'$, which is the required condition. If the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cut orthogonally, then $d = a^2$.

Two spheres of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\dfrac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.

Illustration 25: A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$ **(JEE ADVANCED)**

Sol: Consider $P(\alpha, \beta, \gamma)$ be the foot of perpendicular from origin to, therefore by getting the direction ratios of OP we will get the required result.

Let the equation of the variable plane be

$\ell x + my + nz + d = 0$ ... (i)

Plane passes through the fixed point (a, b, c)

$\therefore\ \ell a + mb + nc + d = 0$ ... (ii)

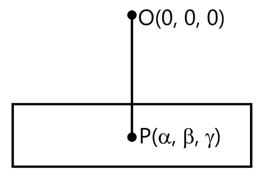

Figure 27.10

Let $P(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the origin to plane (i)

Direction ratios of OP are $\alpha - 0, \beta - 0, \gamma - 0$, i.e. α, β, γ

From equation (i), it is clear that the direction ratios of the normal to the plane, i.e. OP are ℓ, m, n, and α, β, γ are the direction ratio of the same line OP $\therefore\ \dfrac{\alpha}{\ell} = \dfrac{\beta}{m} = \dfrac{\gamma}{n} = \dfrac{1}{k}$(say) ; $\quad \ell = k\alpha, m = k\beta, n = k\gamma$ ... (iii)

Substituting values of ℓ, m, n from equation (iii) in equation (ii), we get, $ka\alpha + kb\beta + kc\gamma + d = 0$ ... (iv)

Since α, β, γ lies on plane (i) $\quad \therefore\ \ell\alpha + m\beta + n\gamma + d = 0$ ... (v)

Substituting values of ℓ, m, n from equation (iii) in equation (v), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ... (vi)

[substituting value of d from equation (iv) in equation (vi)] or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of the perpendicular from the point $P(\alpha, \beta, \gamma)$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$

Illustration 26: Find the equation of the sphere if it touches the plane $\vec{r}\cdot(2\vec{i} - 2\vec{j} - \vec{k}) = 0$ and the position vector of its center is $3\hat{i} + 6\hat{j} - 4\hat{k}$.
(JEE ADVANCED)

Sol: Here equation of the required sphere is $|\vec{r} - \vec{c}| = a$ where a is the radius of the sphere.

Given plane is $\vec{r}\cdot(2\vec{i} - 2\vec{j} - \vec{k}) = 0$ ... (i)

Let H be the center of the sphere, then $\vec{OH} = 3\vec{i} + 6\vec{j} - 4\vec{k} = c$(say)

Radius of the sphere = length of perpendicular from H to plane (i)

$$= \frac{|c\cdot(2i - 2j - k)|}{|2\vec{i} - 2\vec{j} - \vec{k}|} = \frac{|(3i + 6j - 4k)\cdot(2i - 2j - k)|}{(2\vec{i} - 2\vec{j} - \vec{k})} = \frac{|6 - 12 + 4|}{3} = \frac{2}{3} = a \text{ (say)}$$

Equation of the required sphere is $|\vec{r} - \vec{c}| = a$

$$\Rightarrow\ |x\vec{i} + y\vec{j} + z\vec{k} - (3\vec{i} + 6\vec{j} - 4\vec{k})| = \frac{2}{3} \text{ or } |(x-3)\vec{i} + (y-6)\vec{j} + (z+4)\vec{k}|^2 = \frac{4}{9} \Rightarrow (x-3)^2 + (y-6)^2 + (z+4)^2$$

$= 4/9$ or $9(x^2 + y^2 + z^2 - 6x - 12y + 8z + 61) = 4 \Rightarrow 9x^2 + 9y^2 + 9z^2 - 54x - 108y + 72z + 545 = 0$

Illustration 27: Find the equation of the sphere with the points (1, 2, 2) and (2, 3, 4) as the extremities of a diameter. Find the coordinates of its center. **(JEE MAIN)**

Sol: Equation of the sphere having (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

Let $A \equiv (1, 2, 2)$, $B \equiv (2, 3, 4)$

Equation of the sphere having (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Here $x_1 = 1, x_2 = 2, y_1 = 2, y_2 = 3, z_1 = 2, z_2 = 4$

$\therefore$ Required equation of the sphere is $(x - 1)(x - 2) + (y - 2)(y - 3) + (z - 2)(z - 4) = 0$ or

$x^2 + y^2 + z^2 - 3x - 5y - 6z + 16 = 0$

Center of the sphere is the midpoint of AB

$\therefore$ Center is $\left(\frac{3}{2}, \frac{5}{2}, 3\right)$.

Illustration 28: Find the equation of the sphere passing through the points (3, 0, 0), (0, –1, 0), (0, 0, –2) and whose center lies on the plane $3x + 2y + 4z = 1$ **(JEE ADVANCED)**

Sol: Consider the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As the sphere passes through these given points hence these points will satisfy equation of sphere. ... (i)

Let $A \equiv (3, 0, 0)$, $B \equiv (0, -1, 0)$, $C \equiv (0, 0, -2)$ since sphere (i) passes through A, B and C.

$\therefore 9 + 6u + d = 0$ ... (ii)

$1 - 2v + d = 0$ ... (iii)

$4 - 4w + d = 0$ ... (iv)

Since the center $(-u, -v, -w)$ of the sphere lies on plane $3x + 2y + 4z = 1 \therefore -3u - 2v - 4w = 1$ ....(v)

(ii) – (iii) $\Rightarrow 6u + 2v = -8$ ... (vi)

(iii) – (iv) $\Rightarrow -2v + 4w = 3$ ... (vii)

From (vi), $u = \frac{-2v - 8}{6}$ ... (viii)

From (vii), $4w = 3 + wv$

Substituting the values of u, v and w in equation (v), we get $\frac{2v + 8}{2} + 2v - 3 - 2v = 1$

$\Rightarrow 2v + 8 - 4v - 6 - 4v = 2 \Rightarrow v = 0$

From equation (viii), $u = \frac{0 - 8}{6} = -\frac{4}{3}$; From equation (ix), $4w = 3$ $\therefore w = 3/4$

From equation (iii), $d = 2v - 1 = 0 - 1 = -1$ From equation (i), the equation of the required sphere is

$x^2 + y^2 + z^2 - \frac{0 - 8}{6} - \frac{8}{3}x + \frac{3}{2}z - 1 = 0$ or $6x^2 + 6y^2 + 6z^2 - 16x + 9z - 6 = 0$

FORMULAE SHEET

(a) Distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(b) Coordinates of the point dividing the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m:n are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

(c) If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are vertices of a triangle, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

(d) If $A(x_1, y_1, z_1)$ and B (x_2, y_2, z_2) are the two points, the point which divides the line segment AB in ratio λ:1 is

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$$

(e) If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the two points on the line with x_2-x_1, y_2-y_1, z_2-z_1 as direction ratios, then their d.c.s are

$$\pm\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \pm\frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \pm\frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}$$

(f) If ℓ, m, n are d.c.s of a line, then $l^2 + m^2 + n^2 = 1$. Thus, if a line makes angles α, β, γ with axes, then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ and $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

(g) If a, b, c are the d.r.s of a line, then the d.c.s of the line are $\pm\frac{a}{\sqrt{\Sigma a^2}}, \pm\frac{b}{\sqrt{\Sigma a^2}}, \pm\frac{c}{\sqrt{\Sigma a^2}}$

(h) If p(x, y, z) is a point in space such that $\overrightarrow{OP}=\vec{r}$ has d.c.s ℓ, m, n, then

(a) $\ell|\vec{r}|, m|\vec{r}|, n|\vec{r}|$ are the projections on x-axis, y-axis and z-axis, respectively.

(b) $x=\ell|\vec{r}|, y=m|\vec{r}|, z=n|\vec{r}|$

(c) $\vec{r}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})$ and $\hat{r}=\ell\hat{i}+m\hat{j}+n\hat{k}$

Moreover, if a, b, c are d.r.s of a vector $\vec{r}$, then $\vec{r}=\frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$.

(i) Length of projection of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line with d.c.s ℓ, m, n is $|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$

(j) If θ is the angle between two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 then

$$\cos\theta=\pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$

(k) Two lines are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ and two lines are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

(l) Cartesian equations of a line passing through (x_1, y_1, z_1) and having direction ratios a, b, c are

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=t$$

(m) Vector equation of a line passing through the point $A(\vec{a})$ and parallel to vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ for scalar λ.

(n) Cartesian equation of a line passing through two points having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

(o) Vector equation of a line passing through two points having position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{\alpha} + \lambda(\vec{b} - \vec{a})$

(p) Distance between the parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $\dfrac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$

(q) Shortest distance (S.D.) between two lines with equations; $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is

$\dfrac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$. If θ is the angle between the lines, then $\cos\theta = \dfrac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

(r) The length of perpendicular from the point (α, β, γ) to the line $\dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ (ℓ, m, n being d.cs)

is given by $\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2 - [\ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)]^2}$

(s) If $\vec{a}$ and $\vec{b}$ are the unit vectors along the sides of an angle, then $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are the vectors, respectively, along the internal and external bisector of the angle. In fact, the bisectors of the angles between the lines,

$\vec{r} = x\vec{a}$ and $\vec{r} = y\vec{b}$ are given by $\vec{r} = \lambda\left(\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}\right); \lambda \in R$

(t) Equation of plane passing through the point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

(u) Equation of plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(v) Equation of a plane making intercepts a, b, c on axes is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

(w) Vector equation of a plane through the point $\vec{a}$ and perpendicular to the unit vector $\hat{n}$ is $(\vec{r} - \vec{a}).\hat{n} = 0$

(x) If θ is the angle between the two planes $\vec{r}.\hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$, then $\cos\theta = \dfrac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|}$

(y) Equation of a plane containing the line $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ and passing through the point (x_2, y_2, z_2) not on the line is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a & b & c \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \end{vmatrix} = 0$

(z) Equation of a plane through the line $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and parallel to the line $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

(aa) If θ is the angle between the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$, then

$$\sin\theta=\frac{|aA+bB+cC|}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}$$

(ab) Length of perpendicular from (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

(ac) The equation of a sphere with center at the origin and radius 'a' is $|\vec{r}|=a$ or $x^2+y^2+z^2=a^2$

(ad) Equation of a sphere with center (α,β,γ) and radius 'a' is $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2$

(ae) Vector equation of the sphere with center $\vec{c}$ and radius 'a' is $|\vec{r}-\vec{c}|=a$ or $(\vec{r}-\vec{c})\cdot(\vec{r}-\vec{c})=a^2$

(af) General equation of sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ whose center is $(-u,-v,-w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$

(ag) Equation of a sphere concentric with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $x^2+y^2+z^2+2ux+2vy+2wz+\lambda=0$, where λ is a real number.

Solved Examples

JEE Main/Boards

Example 1: Find the coordinates of the point which divides the join of P(2, –1, 4) and Q(4, 3, 2) in the ratio 2 : 3 (i) internally (ii) externally

Sol: By using section formula we can obtain the result.

Let R(x, y, z) be the required point

(i) $x=\frac{2\times4+3\times2}{2+3}=\frac{14}{5}$; $y=\frac{2\times3+3\times(-1)}{2+3}=\frac{3}{5}$

$z=\frac{2\times2+3\times4}{2+3}=\frac{16}{5}$

So, the required point is $R\left(\frac{14}{5},\frac{3}{5},\frac{16}{5}\right)$

(ii) $x=\frac{2\times4-3\times2}{2-3}=-2$; $y=\frac{2\times3-3\times(-1)}{2-3}=-9$

$z=\frac{2\times2-3\times4}{2-3}=8$

Therefore, the required point is R(–2. –9, 8)

Example 2: Find the points on X-axis which are at a distance of $2\sqrt{6}$ units from the point (1, –2, 3)

Sol: Consider required point is P(x, 0, 0), therefore by using distance formula we can obtain the result.

Let P(x, 0, 0) be a point on X-axis such that distance of P from the point (1, –2, 3) is $2\sqrt{6}$

$\Rightarrow\sqrt{(1-x)^2+(-2-0)^2+(3-0)^2}=2\sqrt{6}$

$\Rightarrow(x-1)^2+4+9=24$ $\qquad\Rightarrow(x-1)^2=11$

$\Rightarrow x-1=\pm\sqrt{11}$ $\qquad\Rightarrow x=1\pm\sqrt{11}$

Example 3: If a line makes angles α,β,γ with OX, OY, OZ, respectively, prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Sol: Same as illustration 2.

Let ℓ,m,n be the d.c.'s of the given line, then

$\ell=\cos\alpha, m=\cos\beta, n=\cos\gamma$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow(1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$

$\Rightarrow\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Example 4: Projections of a line segment on the axes are 12, 4 and 3 respectively. Find the length and direction cosines of the line segment.

Sol: Let ℓ,m,n be the direction cosines and r be the length of the given segment, then $\ell r, mr, nr$ are the projections of the segment on the axes.