14. LIMITS, CONTINUITY AND DIFFERENTIABILITY

1. LIMITS

1.1 Intuitive Idea of Limits

(a) Suppose we are travelling from Kashmiri gate to Connaught Place by metro, which will reach Connaught place at 10 a.m. As the time gets closer and closer to 10 a.m. the distance of the train from Connaught place gets closer and closer to zero (here we assume that the train is not delayed). If we consider the time as an independent variable denoted by t and the distance remaining as a function of time, say f(t), then we say that f(t) approaches zero as t approaches zero. We can say that the limit of f(t) is zero as t approaches zero.

(b) Let a regular polygon be described in a circle of given radius. We notice the following points from the geometry.

(i) The area of the polygon cannot be greater than the area of the circle however large the number of sides may be.

(ii) As the number of sides of the polygon increases indefinitely, the area of the polygon continuously approaches the area of the circle.

(iii) Ultimately the difference between the area of the circle and the area of the polygon can be made as small as we please by sufficiently increasing the number of sides of the polygon.

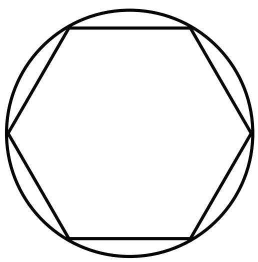

Figure 14.1

We can say that the limit of the area of the polygon inscribed in a circle is the area of the circle as the number of sides increases indefinitely.

1.2 Meaning of $x \to a$

Let x be a variable and a be constant. Since x is a variable; we can change its value at our pleasure. It can be changed in such a way that its value comes nearer and nearer to a. Then we say that x approaches a and it is denoted by $x \to a$:

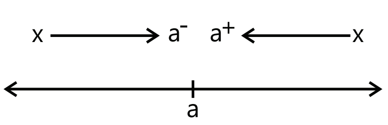

Figure 14.2

We know that $|x - a|$ is the distance between x and a on the real number line and $0 < |x - a|$ if $x \neq a$, "x tends to a" means

(a) $x \neq a$, i.e. $0 < |x - a|$,

(b) x takes up values nearer and nearer to a, i.e. the distance $|x - a|$ between x and a becomes smaller and smaller. One may ask "how much smaller"? The answer is, as much as we please. It may be less than 0.1 or 0.00001 or 0.0000001 and so on. In fact, we may choose any positive number δ. However small it may be, $|x - a|$ will always be less than δ. The above discussion leads up to the following definition of $x \to a$.

Let x be a variable and a be a constant.

Definition: Given a number $\delta > 0$ however small, if x takes up values, such that $0 < |x - a| < \delta$. Then x is said to tend to a, and is symbolically written as $x \to a$

Note. If x approaches a from values less than a, i.e. from the left side of a, we write $x \to a^-$. If x approaches a from values greater than a, i.e. from the right side of a, we write $x \to a^+$.

But $x \to a$ means both $x \to a^-$ and $x \to a^+$. So x approaches or tends to a means x approaches a from both sides right and left.

Neighbourhood of point a: The set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a. Neighbourhood of a $= (a - \delta, a + \delta)$; $x \in (a - \delta, a + \delta)$

1.3 Limit of a Function

Let us take some examples to find the limit of various functions:

Illustration 1: Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$. We investigate the behaviour of f(x) at the point x = 2 and near the point x =2. **(JEE MAIN)**

Sol: Here as $f(2) = 0$, therefore try to evaluate the value of f(x) when x is very near to 2.

$f(2) = \dfrac{4-4}{2-2} = \dfrac{0}{0}$, which is meaningless. Thus f(x) is not defined at x = 2.

Now we try to evaluate the value of f(x) when x is very near to 2 for some values of x less than 2 and then for x greater than 2.

$$f(1.9) = \frac{(1.9)^2 - 4}{1.9 - 2} = \frac{-0.39}{-0.1} = 3.9 \qquad f(2.1) = \frac{(2.1)^2 - 4}{2.1 - 2} = \frac{0.41}{0.1} = 4.1$$

$$f(1.99) = \frac{(1.99)^2 - 4}{1.99 - 2} = \frac{-0.0399}{-0.01} = 3.99 \qquad f(2.01) = \frac{(2.01)^2 - 4}{2.01 - 2} = \frac{0.0401}{0.01} = 4.01$$

$$f(1.999) = \frac{(1.999)^2 - 4}{1.999 - 2} = \frac{-0.003999}{-0.001} = 3.999 \qquad f(2.001) = \frac{(2.001)^2 - 4}{2.001 - 2} = \frac{0.004001}{0.001} = 4.001$$

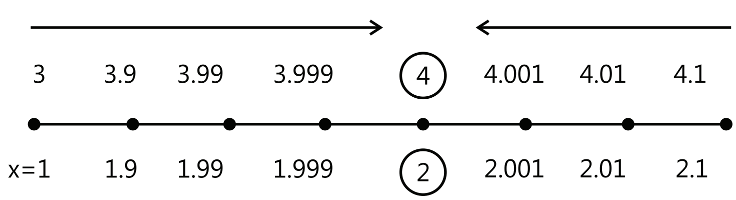

Figure 14.3

It is clear that as x gets nearer and nearer to 2 from either side, f(x) gets closer and closer to 4 from either side.

When x approaches 2 from the left hand side the function f(x) tends to a definite number 4. Thus we say that as x tends to 2 the left hand limit of the function f exists and equals to the definite number 4. Similarly, as x approaches 2 from the right hand side, the function f(x) tends to a definite number, 4.

Again we say that as x approaches 2 from the right hand side of 2, the right hand limit of f exists and equals to 4.

Illustration 2: Discuss the limit of the function $f(x)=\begin{cases}-1, \text{if } x<0\\ 1, \text{ if } x>0\end{cases}$ at x = 0 **(JEE MAIN)**

Sol: Sketch its graph when x is very near to 0 for range of x from – 1 to 1.

We have, $f(x)=\begin{cases}-1,\ x<0\\ +1,\ x>0\end{cases}$

Let us sketch its graph

x	-1	-0.5	-0.1	-0.01	-0.001	0.001	0.01	0.1	0.5	1
f(x)	-1	-1	-1	-1	-1	1	1	1	1	1

(i) As x approaches zero from the left of zero, f(x) remains at –1. And we say that the left hand limit of f exists at x = 0 and equals to –1. $\lim\limits_{x\to 0^-} f(x)=-1$

(ii) As x approaches zero from the right of zero, f(x) remains at 1. So we say that the right hand limit of f at x = 0 exists and equals to +1. $\lim\limits_{x\to 0^+} f(x)=+1$

(iii) Left hand limit of f(x) (at x = 0) ≠ Right hand limit of f(x) {at x = 0}.

So the $\lim\limits_{x\to 0} f(x)$ does not exist.

Illustration 3: Discuss the limit of the function $f(x)=\frac{1}{x}$ at x = 0 and its graph **(JEE MAIN)**

Sol: Same as above illustration.

We have, $f(x)=\frac{1}{x}$ Let us draw the graph of the given function $f(x)=\frac{1}{x}$.

x	-1	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1	1
f(x)	-1	-10	-100	-1000	-10000	10000	1000	100	10	1

(i) As x approaches zero from the left of zero the graph never approaches a finite number so we say that the left hand limit of f at x = 0 does not exist i.e. $\lim\limits_{x\to 0^-} f(x)$ does not exist

(ii) As x approaches zero from the right of zero, the graph again does not approach a finite number. Again we say that the right hand limit of *f* at x = 0 does not exist. $\lim\limits_{x\to 0^+} f(x)$ does not exist

(iii) At x=0 left hand limit of f ≠ right hand limit of f

Hence, the limit of f(x) does not exist.

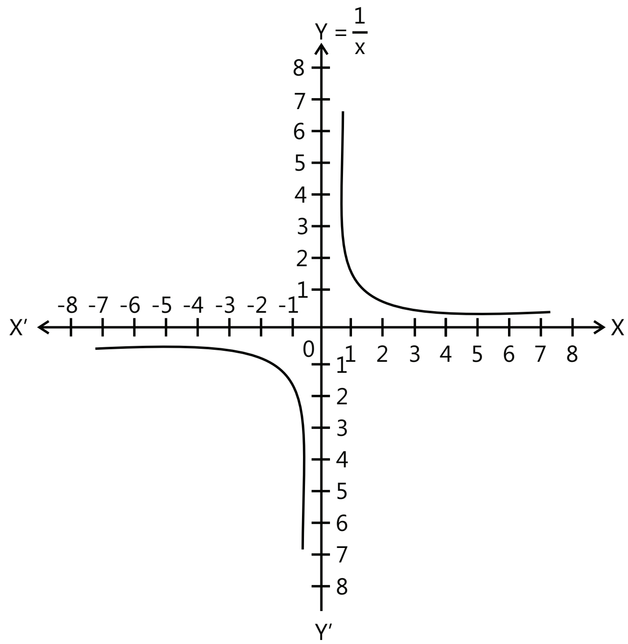

Figure 14.4

MASTERJEE CONCEPTS

(a) The fact that the limit of f(x) exists at x = a means that the graph of f(x) approaches the same value from both sides of x = a.

(b) The fact that f(x) is continuous at x = a means that there is no break in the graph as x moves from a^- to a^+.

Vaibhav Krishnan (JEE 2009, AIR 22)

1.4 Different Cases of Limits

Right hand limit is the limit of the function as x approaches a from the positive side.

Left hand limit is the limit of the function as x approaches a from the negative side.

Note: A function will have a limiting value only if its right hand limit equals its left hand limit.

Illustration 4: Discuss the limits of f(x) = |x| at x = 0 and draw its graph. **(JEE MAIN)**

Sol: We have f(x) = |x|, therefore f(x) is equals to x for x > 0 and – x for x < 0.

Increasing x → ← Decreasing x

x	-3	-2	-1	0	1	2	3
f(x)	3	2	1	0	1	2	3

Decreasing f(x) → Limit ← Decreasing f(x)

We have f(x) = |x| Þ $f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

Let us draw its graph.

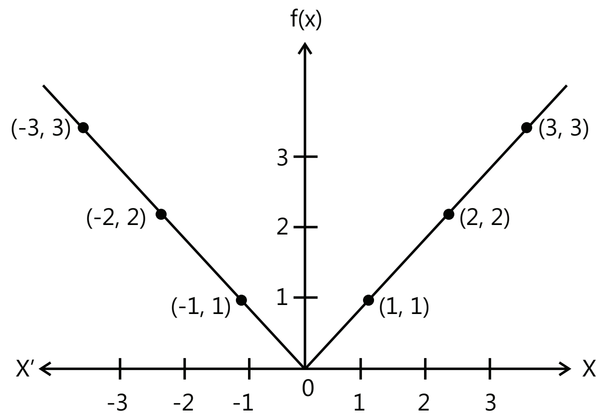

Figure 14.5

(i) As x approaches zero from left of zero, f(x) = 0. And we say that left hand limit of f(x) exists, and is equal to zero.

$$\lim_{x\to 0^-} f(x) = 0$$

(ii) As x approaches zero from the right of zero f(x) is equal to zero. So we say that the right hand limit of f at x = 0 exists, and is equal to zero. $\lim_{x\to 0^+} f(x) = 0$ Here $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$

Hence we say that the limit of *f* (x) at x = 0 exists and equals to zero.

Illustration 5: Evaluate: $\lim_{x\to 3} x + 3$ **(JEE MAIN)**

Sol: Simply taking value of x very near to 3 we can obtain value of the function at these points.

Let us compute the value of function f(x) for x very near to 3. Some of the points near and to the left of 3 are 2.9, 2.99, 2.999.

Values of the function are given in the table below. Similarly, some of the numbers near and right of 3 are 3.001,3.01,3.1. Value of the function at these points are also given in the table.

Increasing x → ← Decreasing x

x	2.9	2.99	2.999	3	3.001	3.01	3.1
f(x)	5.9	5.99	5.999	6	6.001	6.01	6.1

Increasing f(x) → Limit ← Decreasing f(x)

From the table we deduce that the value of f(x) at x = 3 should be greater than 5.999 and less than 6.001.

It is reasonable to assume that the value of function f(x) at x = 3 from the left of 3 is 5.999.

$$\lim_{x\to 3^-} f(x) \approx 5.999 \qquad \text{... (i)}$$

Similarly, when x approaches x = 3 from the right, f(x) should be 6.001

$$\lim_{x\to 3^+} f(x) \approx 6.001 \qquad \text{... (ii)}$$

From (i) and (ii), we conclude that the limit is equal to 6.

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3} f(x) = 6$

Illustration 6: Evaluate: $\lim\limits_{x\to 2} \dfrac{3x^2 - x - 10}{x^2 - 4}$ **(JEE MAIN)**

Sol: Similar to above problem.

Let us compute the value of function f(x) for x very near to 2. Some of the points near and to the left of 2 are 1.9, 1.99, 1.999.

Values of the function are given in the table below. Similarly, some of the numbers near and to the right of 2 are 2.001, 2.01, 2.1. Values of the function at these points are also given in the table.

Increasing x → ← Decreasing x

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	2.743	2.749	2.7499	2.75	2.750	2.7506	2.756

Increasing f(x) → Limit ← Decreasing f(x)

From the table we deduce that the value of f(x) at x = 2 should be greater than 2.7499 and less than 2.750.

It is reasonable to assume that the value of function f(x) at x = 3 from the left of 2 is 2.7499.

$$\therefore \quad \lim_{x\to 2^-} f(x) \approx 2.7499 \qquad \text{.... (i)}$$

Similarly, when x-approaches x = 2 from the right *f* (x) should be 2.750. $\lim\limits_{x\to 2^+} f(x) \approx 2.750$ ....(ii)

From (i) and (ii), we conclude that the limit is equal to 2.75. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2} f(x) = 2.75$

MASTERJEE CONCEPTS

- If $\mathop{\text{Lim}}\limits_{x\to \bar{a}} f\left(a-b\right) = \mathop{\text{Lim}}\limits_{x\to a^+} + \left(a+b\right)$, then we can say that both the left hand limit and right hand limits exist at x = a, and irrespective of the value of the function at a, i.e f(a), the function does not have a limit at x = a, that is $\mathop{\text{Lim}}\limits_{x\to a} f\left(x\right)$ does not exist
- If both the left hand limit and right hand limit of f(x) at x = a exist and at least one of them is not equal to f(a), then the limit of f at x = a does not exist.

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1.5 Working Rule for Evaluation of Left and Right Hand Limits

Right hand limit of f(x), when $x \to a = \lim\limits_{x\to a^+} f(x)$

Step I. Put x = a + h and replace a^+ by a.

Step II. Simplify $\lim\limits_{h\to 0} f(a+h)$.

Step III. The value obtained in step 2 is the right hand limit of f(x) at x = a.

Similarly for evaluating the left hand limit put x = a – h.

Evaluate the left-hand and right-hand limits of the following function at x = 1.

$$f(x)=\begin{cases} 5x-4, & \text{if } 0<x\le 1 \\ 4x^2-3x, & \text{if } 1<x<2 \end{cases}$$

Illustration 7: Does $\lim\limits_{x\to 1} f(x)$ exist? **(JEE MAIN)**

Sol: By taking left hand limit and right hand limit we can conclude that the given limit exist or not.

Left hand limit $= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (5x-4)$ [$\because$ f(x) = 5x – 4, if 0 < x ≤ 1]

$= \lim\limits_{h\to 0}[5(1-h)-4] = \lim\limits_{h\to 0}\left[5-5h-4\right] = 5-4 = 1$ [Put x = 1 – h]

Right hand limit $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (4x^2-3x)$ [$\because$ f(x) = $4x^2$ – 3x, if 1 < x < 2]

$= \lim\limits_{h\to 0}[4(1+h)^2-3(1+h)]$ [Put x = 1 + h]

$= \lim\limits_{h\to 0}[4+8h+4h^2-3-3h] = \lim\limits_{h\to 0}[1+5h+4h^2] = 1$

$\therefore$ At x = 1, Left hand limit = Right hand limit $\Rightarrow \lim\limits_{x\to 1} f(x)$ exists and it is equal to 1.

Illustration 8: Evaluate the left hand and right-hand limits of the following function at x = 1 **(JEE MAIN)**

$$f(x)=\begin{cases} 1+x^2, & \text{if } 0\le x\le 1 \\ 2-x, & \text{if } x>1 \end{cases} \qquad \text{does } \lim_{x\to 1} f(x) \text{ exist ?}$$

Sol: By putting x = 1 – h and 1 + h, we can conclude left hand limit and right hand limit respectively. If both are equal then the given limit exist otherwise not exist.

Left hand limit $= \lim\limits_{x\to 1^-} f(x)$

$= \lim\limits_{x\to 1^-} (1+x^2)$ [$\because$ f(x) = 1 + x^2, if 0 ≤ x ≤ 1]

$= \lim\limits_{h\to 0}[1+(1-h)^2] = 1+1 = 2$ [Put x = 1 – h]

Right hand limit $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2-x)$ [$\because$ f(x) = 2 – x, if x > 1]

$= \lim\limits_{h\to 0}[2-(1+h)] = 2-1 = 1$ (Put x = 1 + h)

Therefore, At x = 1, Left hand limit ≠ Right hand limit $\Rightarrow \lim\limits_{x\to 1} f(x)$ does not exist.

Illustration 9: Evaluate the right hand limit of the function $f(x)=\begin{cases} \dfrac{|x-6|}{x-6}, & x\ne 6 \\ 0, & x=6 \end{cases}$ at x = 6 **(JEE MAIN)**

Sol: Here Right hand limit of the given function f(x) at x = 6 is $\lim\limits_{h\to 0} f(6+h)$.

Right hand limit of f(x) at x = 6 = $\lim\limits_{x\to 6^+} f(x)$, = $\lim\limits_{h\to 0} f(6+h)$,

$= \lim\limits_{h\to 0}\dfrac{|6+h-6|}{6+h-6}, = \lim\limits_{h\to 0}\dfrac{|h|}{h} = \lim\limits_{h\to 0}\dfrac{h}{h}, = \lim\limits_{h\to 0} 1 = 1$

Illustration 10: Evaluate the left hand limit of the function: $f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$ at x = 4 **(JEE MAIN)**

Sol: Here Left hand limit of the given function f(x) at x = 4 is $\lim\limits_{h\to 0} f(4-h)$.

Left hand limit of f(x) at x = 4 = $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{h\to 0} f(4-h) = \lim\limits_{h\to 0}\dfrac{|4-h-4|}{4-h-4} = \lim\limits_{h\to 0}\dfrac{|-h|}{-h}$

$= \lim\limits_{h\to 0}\dfrac{h}{-h} = \lim\limits_{h\to 0} -1 = -1$

Illustration 11: Evaluate the left hand and right hand limits of the following function at x = 2: **(JEE MAIN)**

$f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ x+5, & \text{if } x>2\end{cases}$ Does $\lim\limits_{x\to 2} f(x)$ exist?

Sol: Similar to illustration 7.

Left hand limit = $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(2x+3)$ ($\because$ f(x) = 2x + 3, if $x \le 2$)

$= \lim\limits_{h\to 0}[2(2-h)+3]$ (Put x = 2 – h) = 4 + 3 = 7

Right hand limit = $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} f(x+5)$ ($\because$ f(x) = x + 5, if x > 2)

$= \lim\limits_{h\to 0}(2+h+5) = 7$ (put x = 2 + h)

Therefore, the left hand limit = right hand limit [at x = 2] $\Rightarrow \lim\limits_{x\to 2} f(x)$ exists and it is equal to 7.

Illustration 12: For what integers m and n does $\lim\limits_{x\to 0} f(x)$ exist, if $f(x)=\begin{cases}mx^2+n, & x<0\\ nx+m, & 0\le x\le 1\\ nx^2+m, & x>1\end{cases}$ **(JEE ADVANCED)**

Sol: As the given limit exist, therefore its Left hand limit must be equal to its Right hand limit.

Limit at x = 0 ; $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0-h}(mx^2+n) = \lim\limits_{h\to 0}\left[m(0-h)^2+n\right] = n$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to (0+h)}(nx+m) = \lim\limits_{h\to 0}[n(0+h)+m] = m$

Limit exists if $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) \Rightarrow n = m$

Illustration 13: Suppose $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$ and if $\lim\limits_{x\to 1} f(x) = f(1)$. What are possible values of a and b? **(JEE Advanced)**

Sol: Here $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} f(x)$.

We have, $f(x)=\begin{cases}a+bx, & x<1\\ 4, & x=1\\ b-ax, & x>1\end{cases}$ Left hand $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}[a+bx] = \lim\limits_{h\to 0}[a+b(1-h)] = a+b$

Right hand $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} [b - ax] = \lim\limits_{h\to0}[b - a(1+h)] = b - a$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1} f(x) \Rightarrow a + b = b - a = 4 \Rightarrow \quad a = 0, b = 4$

Illustration 14: If $f(x) = \begin{cases} \dfrac{x-|x|}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$ show that $\lim\limits_{x\to0} f(x)$ does not exist **(JEE ADVANCED)**

Sol: Here if left hand limit is not equal to the right hand limit of the given function then the $\lim\limits_{x\to0} f(x)$ does not exist.

Left hand limit of f at x = 0 = $\lim\limits_{x\to0^-} f(x) = \lim\limits_{h\to0} f(0-h) = \lim\limits_{h\to0} \dfrac{(0-h)-|0-h|}{(0-h)}$ [Putting x = 0 – h]

$$= \lim_{h\to0} \frac{-h-h}{-h} = \lim_{h\to0} \frac{-2h}{-h} = \lim_{h\to0} 2 = 2$$

Right hand limit of f at x = 0 = $\lim\limits_{x\to0^+} f(x) = \lim\limits_{h\to0} f(0+h) = \lim\limits_{h\to0} \dfrac{(0+h)-|0+h|}{(0+h)}$ [Putting x = 0 + h]

$$= \lim_{h\to0} \frac{h-|h|}{h} = \lim_{h\to0} \frac{h-h}{h} = \lim_{h\to0} \frac{0}{h} = \lim_{h\to0} 0 = 0$$

Here, left hand limit of f (at x = 0) ≠ right hand limit of f(at x = 0). Therefore $\lim\limits_{h\to0} f(x)$ does not exist at x = 0

MASTERJEE CONCEPTS

If f(x) denotes the greatest integer function then $\lim\limits_{x\to0} f(x) = [0] = 0$ this representation is wrong.

The correct form is

$$\lim_{x\to0^+} f(x) = \lim_{h\to0} [0+h] = 0$$

$$\lim_{x\to0^-} f(x) = \lim_{h\to0} [0-h] = -1$$

Hence the limit doesn't exist. Remember that the limit must be applied only after complete simplification.

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1.6 Value of a Function at a Point and Limit at a Point

Case I: $\lim\limits_{x\to a} f(x)$ and f(a) both exist but are not equal.

Example: $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$; $\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to1}(x+1) = 2$, f(x) exists at x = 1

f(1) = 0, value of f also exists at x = 1. But $\lim\limits_{x\to1} f(x) \neq f(1)$.

Case II: $\lim\limits_{x\to a} f(x)$ and f(a) both exist and are equal.

Example: $f(x) = x^2$; $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(x^2) = 1$ limit exists, and $f(1) = (1)^2 = 1$; $\Rightarrow$ value of f also exists. $\Rightarrow$ $\lim\limits_{x\to 1} f(x) = f(1)$

1.7 Properties of Limits

Let f and g be two real functions with common domain D, then.

(a) $\lim\limits_{x\to a}(f+g)(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

(b) $\lim\limits_{x\to a}(f-g)(x) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

(c) $\lim\limits_{x\to a}(c\cdot f)(x) = c\lim\limits_{x\to a} f(x)$ [c is a constant]

(d) $\lim\limits_{x\to a}(fg)(x) = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$

(e) $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

(f) $\lim\limits_{x\to a}\left(f(x)\right)^n = \left(\lim\limits_{x\to a} f(x)\right)^n$

(g) $\lim\limits_{x\to a}\left(f(x)\right)^{g(x)} = \left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$

(h) If f(x)g(x) xa then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

MASTERJEE CONCEPTS

$\lim\limits_{x\to a} f(x) = \ell$, then $\lim\limits_{x\to a} |f(x)| = |\ell|$

The converse of this may not be true i.e.

$\lim\limits_{x\to a} |f(x)| = |\ell| \not\Rightarrow \lim\limits_{x\to a} f(x) = \ell$

$\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} g(x) = B$ then $\lim\limits_{x\to a} f(x)^{g(x)} = A^B$

Shivam Agarwal (JEE 2009, AIR 27)

1.8 Cancellation of Common Factor

Let $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$. If by substituting x = a, $\dfrac{f(x)}{g(x)}$ reduces to the form $\dfrac{0}{0}$, then (x–a) is a common factor of f(x) and g(x). So we first factorize f(x) and g(x) and then cancel out the common factor to evaluate the limit.

Working Rule:

To find out $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$, where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$.

Step I: Factorize f(x) and g(x).

Step II: Cancel the common factor (s).

Step 3: Use the substitution method to obtain the limit.

Important formulae for factorization

(a) $(a^2 - b^2) = (a - b)(a + b)$

(b) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(c) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(d) $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2)$

(e) If $f(\alpha) = 0$, then $x - \alpha$ is a factor of f(x)

Illustration 15: Evaluate $\lim\limits_{x\to1}\frac{x^3-1}{x-1}$. **(JEE MAIN)**

Sol: Factorize the numerator and denominator.

If we put x = 1, the expression $\frac{x^3-1}{x-1}$ assumes the indeterminate form $\frac{0}{0}$. Therefore (x – 1) is a common factor of $(x^3 - 1)$ and (x – 1). Factorising the numerator and denominator, we have,

$$\lim_{x\to1}\frac{x^3-1}{x-1}\ \left[\frac{0}{0}\text{Form}\right]$$

$$=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)}=\lim_{x\to1}(x^2+x+1)\ \text{[After cancelling (x–1)]}=1^2+1+1=3$$

Illustration 16: Evaluate $\lim\limits_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}$ **(JEE MAIN)**

Sol: Similar to above illustration, By factorizing numerator and denominator we can evaluate given limit.

If we put $x=\frac{1}{2}$, the expression $\frac{4x^2-1}{2x-1}$ assumes the indeterminate from $\frac{0}{0}$.

Therefore $\left(x-\frac{1}{2}\right)$ i.e. (2x – 1) is a common factor of $(4x^2 - 1)$ and (2x – 1). Factorising the numerator and denominator, we have,

$$\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}\quad\left[\frac{0}{0}\text{Form}\right]=\lim_{x\to\frac{1}{2}}\frac{(2x-1)(2x+1)}{(2x-1)}\ [\because a^2-b^2=(a-b)(a+b)]$$

$$=\lim_{x\to\frac{1}{2}}(2x+1)\ \text{[After cancelling (2x – 1)]}=2\times\frac{1}{2}+1=1+1=2$$

Illustration 17: Evaluate $\lim\limits_{x\to5}\frac{x^2-9x+20}{x^2-6x+5}$. **(JEE MAIN)**

Sol: Take (x – 5) common from numerator and denominator to solve this problem.

If we put x = 5, the expression $\frac{x^2-9x+20}{x^2-6x+5}$ assumes the indeterminate form $\frac{0}{0}$.

Therefore (x – 5) is a common factor of the numerator and denominator both. Factorising the numerator and denominator, we have $\left[\frac{0}{0}\text{Form}\right] = \lim\limits_{x\to 5}\frac{(x-4)\,(x-5)}{(x-1)\,(x-5)} = \lim\limits_{x\to 5}\frac{(x-4)}{(x-1)}$

[After cancelling (x – 5)] $= \frac{5-4}{5-1} = \frac{1}{4}$

Illustration 18: Evaluate $\lim\limits_{x\to 2}\frac{x^3-7x^2+14x-8}{x^2+2x-8}$ **(JEE MAIN)**

Sol: Same as above illustration.

When x = 2 the expression $\frac{x^3-7x^2+14x-8}{x^2+2x-8}$ assumes the form $\frac{8-28+28-8}{4+4-8} = \frac{0}{0}$

Therefore (x – 2) is a common factor of the numerator and denominator.

Factorising the numerator and denominator, we get

$$\lim_{x\to 2}\frac{(x-1)(x-2)(x-4)}{(x-2)(x+4)} = \lim_{x\to 2}\frac{(x-1)(x-4)}{(x+4)} = \frac{(2-1)(2-4)}{(2+4)} = \frac{1\times(-2)}{6} = \frac{-2}{6} = -\frac{1}{3}$$

Illustration 19: Evaluate $\lim\limits_{x\to 1}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$ **(JEE MAIN)**

Sol: Same as above illustration.

On putting x = 1 in $\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$, we get $\frac{0}{0}$.

It means (x – 1) is the common factor of the numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x\to 1}\frac{(x-1)(x^3-2x^2-2x-2)}{(x-1)(x^2-4x-1)} = \lim_{x\to 1}\frac{(x^3-2x^2-2x-2)}{x^2-4x-1} \quad \text{[Cancellation of (x – 1)]}$$

$$= \frac{1^3-2\times 1^2-2\times 1-2}{1^2-4\times 1-1} \quad \text{[Substitution method]}$$

$$= \frac{1-2-2-2}{1-4-1} = \frac{-5}{-4} = \frac{5}{4}$$

Note: When the degree of the polynomial is higher, then it is difficult to factorize. So, we apply L'Hôpital's rule

1.9 L'Hôpital's Rule

L'Hôpital's rule states that for functions f and g which are differentiable:

If $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$ has a finite value then $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$

Note: Most common indeterminate forms are $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty$ and ∞^0

MASTERJEE CONCEPTS

Evaluation of limits using L'Hôpital's rule is applicable only when $\frac{f(x)}{g(x)}$ becomes of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$ simplify the given expression till it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply the rule.

To apply L'Hôpital's rule differentiate the numerator and the denominator separately.

What you may overlook is the fact that the LH rule is applicable only when the modified limit (obtained by differentiating the numerator and denominator) also exists. Let's consider an example to illustrate this point. Consider the limit $\lim\limits_{x\to\infty}\frac{x^2+\sin x}{x^2}$. We see that the limit is of the indeterminate form $\frac{\infty}{\infty}$. Applying the LH rule two times in succession, we obtain: $\lim\limits_{x\to\infty}\frac{x^2+\sin x}{x^2}=\lim\limits_{x\to\infty}\frac{2x+\cos x}{2x}$ (again $\frac{\infty}{\infty}$ form)

$$=\lim_{x\to\infty}\frac{2-\sin x}{2}=1-\frac{1}{2}\lim_{x\to\infty}\sin x$$

Which does not exist. However, only a few moments of consideration are required to conclude that the limit must exist, because the numerator is x^2 + sinx, and since x tends to infinity, the term sinx can be ignored in comparison to x^2 (as sinx only ranges from -1 to 1); the denominator is x^2 and so the limit must be 1. Why did the LH rule go wrong?

Ravi Vooda (JEE 2009, AIR 71)

1.10 Theorems

1. Let n be any positive integer. Then. $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

First Proof: Putting x = a + h, we get $\frac{x^n-a^n}{x-a}=\frac{(a+h)^n-a^n}{a+h-a}=\frac{1}{h}\left[(a+h)^n-a^n\right]$

$$=\frac{1}{h}\left[(a^n+{}^nC_1a^{n-1}h+....+h^n)-a^n\right]\text{[Using the Binomial theorem]}$$

$$=\frac{1}{h}\left[{}^nC_1a^{n-1}h+{}^nC_2a^{n-2}h^2+....+h^n\right]={}^nC_1a^{n-1}+{}^nC_2a^{n-2}h+....+h^{n-1}$$

$$\therefore\quad \lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\left[{}^nC_1a^{n-1}+{}^nC_2a^{n-2}h+....h^{n-1}\right]=na^{n-1}\qquad(\text{as } {}^nC_1=n)$$

Second Proof: We know that,

$x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+...a^{n-2}x+a^{n-1})$ Therefore, $\frac{x^n-a^n}{(x-a)}=x^{n-1}+ax^{n-2}+....a^{n-2}x+a^{n-1}$

$$\Rightarrow\lim_{x\to a}\frac{x^n-a^n}{(x-a)}=a^{n-1}+a(a^{n-2})+....a^{n-2}a+a^{n-1}=a^{n-1}+a^{n-1}+....a^{n-1}+a^{n-1}\text{ (n terms)}=n\,a^{n-1}$$

Illustration 20: Evaluate: $\lim\limits_{x\to a}\frac{x^m-a^m}{x^n-a^n}$ **(JEE MAIN)**

Sol: By using L' Hospital rule, we can solve this problem.

We have, $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$

$$\left[\frac{0}{0}\text{ form}\right]=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\cdot\frac{x-a}{x^n-a^n}\right\}=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\div\frac{x^n-a^n}{x-a}\right\}=\lim_{x\to a}\left\{\frac{x^m-a^m}{x-a}\right\}\div\lim_{x\to a}\left\{\frac{x^n-a^n}{x-a}\right\}$$

$$=ma^{m-1}\div na^{n-1}=\frac{ma^{m-1}}{na^{n-1}}=\frac{m}{n}a^{m-1-n+1}=\frac{m}{n}a^{m-n}$$

Illustration 21: Evaluate: $\lim\limits_{x\to 2}\dfrac{x^5-32}{x^3-8}$ **(JEE MAIN)**

Sol: Dividing numerator and denominator by (x – 2) we can evaluate given limit.

When x = 2, the expression $\dfrac{x^5-32}{x^3-8}$ assumes the indeterminate form $\dfrac{0}{0}$.

Now $$\lim_{x\to 2}\frac{x^5-32}{x^3-8}=\lim_{x\to 2}\frac{x^5-2^5}{x^3-2^3}=\lim_{x\to 2}\frac{(x^5-2^5)/(x-2)}{(x^3-2^3)/(x-2)}=\lim_{x\to 2}\left\{\frac{x^5-2^5}{x-2}\right\}\div\lim_{x\to 2}\left\{\frac{x^3-2^3}{x-2}\right\}$$

$$=5\times 2^{5-1}\div 3\times 2^{3-1}\qquad\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=5\times 2^4\div 3\times 2^2=\frac{5\times 2^4}{3\times 2^2}=\frac{5}{3}\times 2^2=\frac{20}{3}$$

Illustration 22: Find all possible values of n, if $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$, $n\in N$ **(JEE MAIN)**

Sol: Simply using L'Hospital rule we can obtain value of n.

We have $$\lim_{x\to 2}\frac{x^n-2^n}{x-2}=80\ \Rightarrow\ n\cdot 2^{n-1}=80\qquad\left[\therefore \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$\Rightarrow\quad n\cdot 2^{n-1}=5\times 2^{5-1}\Rightarrow\quad n=5$$

Illustration 23: $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$ **(JEE ADVANCED)**

Sol: Put x + 2 = y and a + 2 = b and after that solve this by using L'hospital rule.

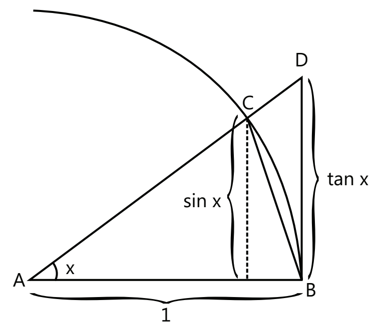
Putting x + 2 = y and a + 2 = b, we get

$$=\lim_{y\to b}\frac{y^{\frac{5}{2}}-b^{\frac{5}{2}}}{y-b}=\frac{5}{2}b^{\frac{5}{2}-1}=\frac{5}{2}b^{\frac{3}{2}}\qquad\left[\because \lim_{y\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=\frac{5}{2}(a+2)^{\frac{3}{2}}$$

Illustration 24: Prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Sol: Proof by geometry



Draw a circle of radius unity and with its centre at O. Let $\angle AOB$ = x radians

Join AB. Draw AC $\perp$ OA at A. Produce OB to meet AC at C. Draw BD $\perp$ OA

From the figure

Area of DOAB < Area of sector OAB < Area of DOAC

$$\Rightarrow \frac{1}{2}(OA)(BD) < \left(\frac{x}{2\pi}\right)\pi(OA)^2 < \frac{1}{2}(OA)\cdot(AC)$$

$$\Rightarrow \frac{1}{2}(OA)(OB)\sin x < \frac{1}{2}(OA)^2 x < \frac{1}{2}(OA)\cdot(OA)\tan x \qquad \left[\because \frac{BD}{OB} = \sin x, \tan x = \frac{AC}{OA}\right]$$

$$\Rightarrow \frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x \qquad [\because OA = OB]$$

$$\Rightarrow \sin x < x < \tan x \qquad \Rightarrow \sin x < x < \frac{\sin x}{\cos x}$$

$$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad \text{(Dividing by sin x)}$$

$$\Rightarrow \frac{x}{\sin x} \text{ lies between 1 and } \frac{1}{\cos x}$$

When $x \to 0$, cos x = 1

$\therefore$ When $x \to 0$, $\frac{x}{\sin x}$ lies between 1 and 1

$$\therefore \lim_{x\to 0}\frac{x}{\sin x} = 1 \text{ or } \lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{Proved}$$

Proof by algebra

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(\frac{x-(x^3/3!)+(x^5/5!)-....}{x}\right) = \lim_{x\to 0}\frac{x\left(1-(x^2/3!)+(x^4/5!).....\right)}{x} = \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}.....\right) = 1$$

Illustration 25: Evaluate: $\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$ **(JEE MAIN)**

Sol: As we know $\lim_{x\to 0}\frac{\sin\theta}{\theta} = 1$, Therefore we can reduce given equation by multiplying and dividing by 25.

We have, $\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$ **=** $\lim_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{\sin 5x}{x} = \lim_{x\to 0}5\left(\frac{\sin 5x}{5x}\right)\times\lim_{x\to 0}5\left(\frac{\sin 5x}{5x}\right)$

= $5(1)\times 5(1) = 25$

Illustration 26: Evaluate: $\lim_{x\to 0}\frac{(a+x)^2\sin(a+x)-a^2\sin a}{x}$ **(JEE ADVANCED)**

Sol: By using algebra we can reduce given limit in the form of $\lim_{x\to 0}f(x)+\lim_{x\to 0}g(x)+\lim_{x\to 0}h(x)$, and then by solving we will get the result.

$$\lim_{x\to 0}\frac{(a+x)^2\sin(a+x)-a^2\sin a}{x} = \lim_{x\to 0}\frac{(a^2+2ax+x^2)\sin(a+x)-a^2\sin a}{x}$$

$$= \lim_{x\to 0} \frac{a^2 \sin(a+x) + (2ax + x^2)\sin(a+x) - a^2 \sin a}{x}$$

$$= \lim_{x\to 0} \frac{a^2\left(\sin(a+x) - \sin a\right)}{x} + \lim_{x\to 0} \frac{2ax\sin(a+x)}{x} + \lim_{x\to 0} \frac{x^2 \sin(a+x)}{x}$$

$$= a^2 \lim_{x\to 0} \frac{2\cos\left(a + (x/2)\right)\sin(x/2)}{x} + \lim_{x\to 0} 2a\sin(a+x) + \lim_{x\to 0} x\sin(a+x)$$

$$= a^2 \lim_{x\to 0} \cos\left(a + \frac{x}{2}\right) \lim_{x\to 0} \frac{\sin(x/2)}{(x/2)} + \lim_{x\to 0} 2a\sin(a+x) + \lim_{x\to 0} x\sin(a+x)$$

$$= a^2 \cos(a+0) + 2a\sin(a+0) + 0\sin(a+0)$$

$$= a^2 \cos a + 2a \sin a$$

Illustration 27: Evaluate: $\lim_{x\to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$ **(JEE ADVANCED)**

Sol: By using trigonometric formulae, we can evaluate this problem.

We have, $\lim_{x\to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x} = \lim_{x\to 0} \frac{\left((\cos 2x)/(\sin 2x)\right) - \left(1/(\sin 2x)\right)}{x} = \lim_{x\to 0} \frac{\cos 2x - 1}{x \cdot \sin 2x}$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{x \cdot 2\sin x \cdot \cos x} = -\lim_{x\to 0} \frac{\sin x}{x \cdot \cos x} = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = -1 \cdot \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \cos x} = -1$$

Illustration 28: Evaluate the following limits: $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$ **(JEE ADVANCED)**

Sol: Same as above problem.

We have, $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\left((\sin x)/(\cos x)\right) - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{x^3 \cos x}$

$$= \lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3 \cdot \cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{(1-\cos x)}{x^2} \cdot \frac{1}{\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{2\sin^2(x/2)}{x^2} \cdot \lim_{x\to 0} \cdot \frac{1}{\cos x}$$

$$= 1 \cdot \lim_{x\to 0} \frac{2}{4}\left(\frac{\sin(x/2)}{(x/2)}\right)^2 \cdot \lim_{x\to 0} \frac{1}{\cos x} = (1)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \frac{1}{2}$$

Illustration 29: Evaluate: $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ **(JEE ADVANCED)**

Sol: Reduce given equation in the form of $\frac{\sin\theta}{\theta}$ by using trigonometric formula.

We have, $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2(x/2)}{x^2} = \frac{2}{4} \lim_{x\to 0} \left(\frac{\sin(x/2)}{(x/2)}\right)^2 = \frac{1}{2}$

Illustration 30: Prove that: $|\sin x| \le |x|$, holds for all x. **(JEE ADVANCED)**

Sol: We know that sin x < x < tanx therefore by applying the cases, $0 \le x \le 1 < \frac{\pi}{2}$, $-1 \le x \le 0$ and $|x| \ge 1$, we can prove this illustration.

We know that sin x < x < tanx

If $0 \le x \le 1 < \frac{\pi}{2} \Rightarrow |\sin x| = \sin x \le x \le |x|$

If $-1 \le x \le 0 \Rightarrow |\sin x| = -\sin x \Rightarrow \sin(-x) \le -x \le |x|$

If neither of the two cases hold, i.e. if $|x| \ge 1$ then $|\sin x| \le 1 \le |x| \Rightarrow |\sin x| \le |x|$

1.11 Trigonometric Limits

Now, we will learn to evaluate the trigonometric limits when the variable tends to a non-zero number.

Woking Rule:

Let the variable tend to a. ($x \to a$).

Step 1: Replace x by a + h, where $h \to 0$.

Step 2: Now the problem is transformed in h where $h \to 0$. Use the method already discussed in the previous exercise.

Illustration 31: Evaluate: $\lim\limits_{x\to\pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$ **(JEE MAIN)**

Sol: By replacing x by $\pi + h$.

We have, $\lim\limits_{x\to\pi} \frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}\lim\limits_{x\to\pi}\frac{\sin(\pi-x)}{(\pi-x)} = \frac{1}{\pi}\lim\limits_{h\to 0}\frac{\sinh}{h} = \frac{1}{\pi}$ $\left[\begin{array}{l}\because x \to \pi \\ \Rightarrow \pi - x \to 0\end{array}\right]$

Illustration 32: Evaluate: $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$ **(JEE MAIN)**

Sol: Simply replacing x by $\pi + h$ and using trigonometric limit we can solve above problem.

$\lim\limits_{x\to\pi}\frac{1+\cos x}{\tan^2 x} = \lim\limits_{h\to 0}\frac{1+\cos(\pi+h)}{\tan^2(\pi+h)}$ [Putting x = +h]

$= \lim\limits_{h\to 0}\frac{1-\cosh}{\tan^2 h}$ [$\because$ tan (+ h) = tan h] $= \lim\limits_{h\to 0}\frac{2\sin^2(h/2)}{\sin^2 h}\cos^2 h$

$= \lim\limits_{h\to 0}\frac{2\sin^2(h/2)}{\left(4\sin^2(h/2)\cos^2(h/2)\right)}\cos^2 h = \lim\limits_{h\to 0}\frac{\cos^2 h}{2\cos^2(h/2)} = \frac{1}{2}\frac{\times 1}{\times(1)} = \frac{1}{2}$

Illustration 33: Evaluate: $\lim\limits_{x\to a}\frac{\sin x - \sin a}{\sqrt{x}-\sqrt{a}}$ **(JEE ADVANCED)**

Sol: Same as above illustration replace x by a +h.

We have, $\lim\limits_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}} = \lim\limits_{h\to 0}\frac{\sin(a+h)-\sin a}{\sqrt{a+h}-\sqrt{a}}$ (Putting x = a + h)

$= \lim\limits_{h\to 0}\frac{2\cos\left(a+(h/2)\right)\sin(h/2)}{(a+h)-a}(\sqrt{a+h}+\sqrt{a}) \quad = \lim\limits_{h\to 0} 2\cos\left(a+\frac{h}{2}\right)\frac{\sin(h/2)}{h}\ [\sqrt{a+h}+\sqrt{a}]$

$= 2\cos a \lim\limits_{\frac{h}{2}\to 0}\left(\frac{1}{2}\frac{\sin(h/2)}{(h/2)}\right)\lim\limits_{h\to 0}\left(\sqrt{a+h}+\sqrt{a}\right) \quad \left(\because h \to 0 \Rightarrow \frac{h}{2} \to 0\right)$

$= 2\cos a\left[\frac{1}{2}\right](1)[\sqrt{a+0}+\sqrt{a}] = 2\sqrt{a}\cos a$

Illustration 34: Evaluate: $\lim_{x\to\frac{\pi}{2}} \frac{1+\cos(2x)}{(\pi-2x)^2}$ **(JEE ADVANCED)**

Sol: Replace x by $\frac{\pi}{2}+h$ and then by using trigonometric formulae's we can evaluate above problem.

We have, $\lim_{x\to\frac{\pi}{2}} \frac{1+\cos(2x)}{(\pi-2x)^2}$ $\lim_{x\to\frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{h\to 0} \frac{1+\cos 2\left((\pi/2)+h\right)}{\left[\pi-2\left((\pi/2)+h\right)\right]^2}$ $\left(\frac{0}{0}\text{form}\right)$ $\left[\text{Put}\, x=\frac{\pi}{2}+h\right]$

$= \lim_{h\to 0} \frac{1+\cos(\pi+2h)}{(\pi-\pi-2h)^2} = \lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$ $\left(\frac{0}{0}\text{form}\right)$ $= \lim_{h\to 0}\frac{2\sin^2 h}{4h^2} = \frac{1}{2}\lim_{h\to 0}\left(\frac{\sinh}{h}\right)^2 = \frac{1}{2}(1)^2 = \frac{1}{2}$

Illustration 35: Evaluate: $\lim_{x\to a}\frac{\cos x-\cos a}{\cot x-\cot a}$ **(JEE ADVANCED)**

Sol: By using trigonometric limit method we can solve this.

We have, $\lim_{x\to a}\frac{\cos x-\cos a}{\cot x-\cot a} = \lim_{h\to 0}\frac{\cos(a+h)-\cos a}{\cot(a+h)-\cot a} = \lim_{h\to 0}\frac{\cos(a+h)-\cos a}{\left((\cos(a+h)/(\sin(a+h)\right)-\left((\cos a)/(\sin a)\right)}$ [Put x = a+ h]

$= \lim_{h\to 0}\frac{2\sin\left((a-a-h)/2\right)\sin\left((a+a+h)/2\right)}{\cos(a+h)\sin a-\cos a\,\sin(a+h)}\sin(a+h)\sin a$

$= \lim_{h\to 0}\frac{2\sin\left(-(h/2)\right)\sin\left((2a+h)/2\right)}{\sin(a-(a+h))}\sin(a+h)\sin a$

$= \lim_{h\to 0}\frac{-2\sin\frac{h}{2}\sin\left((2a+h)/2\right)}{-\sinh}\sin(a+h)\sin a = \lim_{h\to 0}\frac{2\sin(h/2)\sin\left((2a+h)/2\right)}{2\sin(h/2)\cos(h/2)}\sin(a+h)\sin a$

$= \lim_{h\to 0}\frac{\sin\left((2a+h)/2\right)}{\cos(h/2)}\sin(a+h)\sin a = \frac{\sin\left((2a+0)/2\right)}{(1)}\sin(a+0)\sin a$

$= \sin a \cdot \sin a \cdot \sin a = (\sin a)^3 = \sin^3 a$

Illustration 36: Evaluate: $\lim_{x\to\frac{\pi}{6}}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$ **(JEE ADVANCED)**

Sol: Replace x by $\frac{\pi}{6}+h$.

We have, $\lim_{x\to\frac{\pi}{6}}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2} = \lim_{h\to 0}\frac{2-\sqrt{3}\cos\left((\pi/6)+h\right)-\sin\left((\pi/6)+h\right)}{\left[6\left((\pi/6)+h\right)-\pi\right]^2}$ $\left[\text{Put } x=\frac{\pi}{6}+h\right]$

$= \lim_{h\to 0}\frac{2-\sqrt{3}\left(\cos(\pi/6)\cosh-\sin(\pi/6)\sinh\right)-\left[\sin(\pi/6)\cosh+\cos(\pi/6)\sinh\right]}{\left[\pi+6h-\pi\right]^2}$

$= \lim_{h\to 0}\frac{2-\sqrt{3}\left((\sqrt{3}/2)\cosh-(1/2)\sinh\right)-\left((1/2)\cosh+(\sqrt{3}/2)\sinh\right)}{36h^2}$

$= \lim_{h\to 0}\frac{2-(3/2)\cosh+(\sqrt{3}/2)\sinh-(1/2)\cosh-(\sqrt{3}/2)\sinh}{36h^2}$

$$= \lim_{h\to 0} = \frac{2-2\cos h}{36h^2} = \lim_{h\to 0}\frac{1-\cos h}{18h^2} = \lim_{h\to 0}\frac{2\sin^2(h/2)}{18h^2} = \frac{1}{9}\lim_{h\to 0}\left(\frac{\sin(h/2)}{h}\right)^2$$

$$= \frac{1}{9x4}\left(\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}\right)^2 = \frac{1}{36}$$

Illustration 37: Evaluate: $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-(\pi/4)}$ **(JEE ADVANCED)**

Sol: Replace x by $\frac{\pi}{4}+h.$

We have, $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-(\pi/4)}$ $\begin{bmatrix}\text{put } x=(\pi/4)+h\\ \text{As } x\to(\pi/4)\Rightarrow h\to 0\end{bmatrix}$

$$= \lim_{h\to 0}\frac{\sin\big((\pi/4)+h\big)-\cos\big((\pi/4)+h\big)}{(\pi/4)+h-(\pi/4)} \quad \begin{bmatrix}\cos\big((\pi/4)+h\big)\\ =\sin\big((\pi/4)+h+(\pi/2)\big)\\ =\sin\big(h+(3\pi/4)\big)\end{bmatrix}$$

$$= \lim_{h\to 0}\frac{\sin\big((\pi/4)+h\big)-\sin\big(h+(3\pi/4)\big)}{h} = \lim_{h\to 0}\frac{2\cos\big(h+(\pi/2)\big)\sin\big(-(\pi/4)\big)}{h}$$

$$= \lim_{h\to 0}\frac{(-\sin h)\left[\sin\big(-(\pi/4)\big)\right]}{h} \qquad = \lim_{h\to 0}2\sin\frac{\pi}{4}\left(\frac{\sin h}{h}\right) = 2\sin\frac{\pi}{4}(1) = 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

Note: All of the above illustrations can be solved using L'Hôpital's rule.

1.12 Infinite Functions

Now, we will discuss the evaluation of the limits of two functions

(a) Exponential function (b) Logarithmic function

(a) Exponential function

Consider the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+.....+\frac{1}{n!}+.....\infty$

This infinite series is denoted by e. The domain of a function f(x) = e^x, x $\in$ R is R and the range is the set of positive real numbers

(b) Logarithmic function

Let e^y = x then it can be written as $\log_e x = y$

The domain of f(x) = $\log_e x$ is $R^{\oplus}$ and the range is R.

The graph of the logarithmic of a function is in the adjoining figure.

Some Important Functions

(i) $(1+x)^n = \left\{1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....\right\}$ $[|x| < 1]$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{3!} + ...$

(iii) $a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...$

(iv) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad (|x|<1)$

(v) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(ix) $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + ...$

(x) $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + ...(-1 < x < 1)$

(xi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} ..... \quad (-1 < x < 1)$

MASTERJEE CONCEPTS

Using these expansions is helpful where the limits are in the indeterminate form. But selecting the number of terms to use in the expansion varies with problems.

Vaibhav Gupta (JEE 2009, AIR 54)

Theorem 1: Prove that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof: We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \Rightarrow e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\Rightarrow \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ... \Rightarrow \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ...\right) = 1$$

Hence proved.

Theorem 2: Prove that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Proof: Let $\frac{\log_e(1+x)}{x} = y$. then,

$$\Rightarrow \log_e(1 + x) = xy \Rightarrow 1 + x = e^{xy} \Rightarrow e^{xy} - 1 = x \Rightarrow \frac{e^{xy} - 1}{x} = 1 \Rightarrow \frac{e^{xy} - 1}{xy} \cdot y = 1$$

Now taking limit, when $x \to 0$

$$\lim_{xy\to 0}\frac{e^{xy}-1}{xy}\cdot\lim_{x\to 0} y = 1 \qquad [\text{since } x \to 0 \Rightarrow xy \to 0]$$

$$\Rightarrow 1\cdot\lim_{x\to 0} y = 1 \quad \left[\because \lim_{xy\to 0}\frac{e^{xy}-1}{xy} = 1\right]$$

$$\Rightarrow \quad \lim_{x\to 0} y = 1 \;\Rightarrow \quad \lim_{xy\to 0}\frac{\log_e(1+x)}{x} = 1$$

Hence proved

Note: If no base of log is mentioned, then it is taken for granted that the base is e.

$\therefore$ log a is same as $\log_e a$

Illustration 38: Evaluate: $\lim_{x\to 0}\frac{e^x - x - 1}{x}$ **(JEE MAIN)**

Sol: As we already prove that $\lim_{x\to 0}\frac{e^x-1}{x} = 1$, Therefore by writing given limit as $\lim_{x\to 0}\frac{e^x-1}{x} - 1$ we can easily solve above problem.

$$\lim_{x\to 0}\frac{e^x - x - 1}{x} = \lim_{x\to 0}\frac{e^x-1}{x} - 1 = 1 - 1 = 0$$

Illustration 39: Evaluate: $\lim_{x\to 0}\frac{a^x - b^x}{x}$ **(JEE MAIN)**

Sol: Add and subtract 1 with numerator.

$$\lim_{x\to 0}\frac{a^x - b^x}{x} = \lim_{x\to 0}\frac{(a^x-1)-(b^x-1)}{x}$$

$$= \lim_{x\to 0}\left(\frac{a^x-1}{x} - \frac{b^x-1}{x}\right) = \lim_{x\to 0}\frac{a^x-1}{x} - \lim_{x\to 0}\frac{b^x-1}{x} = \log a - \log b = \log\frac{a}{b}$$

Illustration 40: Evaluate: $\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$ **(JEE ADVANCED)**

Sol: Reduce given limit in the form of $\lim_{x\to 0}\frac{e^x-1}{x}$.

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{x} = \lim_{x\to 0}\frac{e^x - \frac{1}{e^x}}{x} = \lim_{x\to 0}\frac{e^{2x}-1}{x}\lim_{x\to 0}\frac{1}{e^x} = 2\lim_{x\to 0}\frac{e^{2x}-1}{2x}\frac{1}{e^0} = 2(1) = 2$$

Illustration 41: Evaluate $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$ **(JEE ADVANCED)**

Sol: By multiplying and dividing by $\sin x$.

$$\lim_{x\to 0}\frac{e^{\sin x}-1}{x} = \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\frac{\sin x}{x}$$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\lim_{x\to 0}\frac{\sin x}{x} = \lim_{\sin x\to 0}\frac{e^{\sin x}-1}{\sin x}\times 1 = 1$$

MASTERJEE CONCEPTS

$\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$ then

$\lim\limits_{x\to a}\left[f(x)\right]^{g(x)} = e^{\lim\limits_{x\to a} g(x)\left[f(x)-1\right]}$

$\lim\limits_{n\to\infty} a^n$ = 0, if |a| < 1 Does not exist if |a| ≥ 1.

A common mistake made by students pertains to indeterminate limits. Consider a function f(x) = g(x) h(x). We are given that $\lim\limits_{x\to 0} g(x) = 0$. What is the value of $\lim\limits_{x\to 0} f(x)$? Many students would say that it is 0. However, the actual answer depends on $\lim\limits_{x\to 0} h(x) = 0$. If it is not finite, then the limit of f(x) is indeterminate. The point we are trying to make is that in calculating the limit of a function which is the product of two or more functions, if one of the function tends to 0, then that does not make it necessary for the entire limit to be 0 as well. Similar, remarks hold for other indeterminate forms. For example, if f(x) = $g(x)^{h(x)}$, and g(x) → 1 as x → a, it does not necessarily imply that f(x) → 1 as x → a, because if $\lim\limits_{x\to a} h(x)$ is not finite, then the limit on f(x) is indeterminate.

Akshat Kharaya (JEE 2009, AIR 235)

2. CONTINUITY

2.1 Introduction

The word 'continuous' means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be continuous. A function which is not continuous is called a discontinuous function.

Ex.

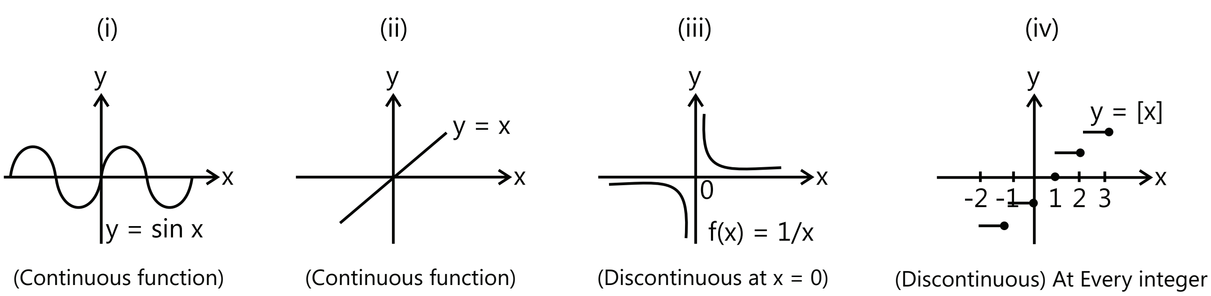


Figure 14.6

Following are examples of some continuous functions:

(i) f(x) = x (Identity function)

(ii) f(x) = c (Constant function)

(iii) f(x) = $a_0x^n + a_1x^{n-1} + \ldots\ldots + a^n$ (Polynomial function)

(iv) f(x) = sin x, cos x (Trigonometric function)

(v) f(x) = a^x, e^x, e^{-x} (Exponential function)

(vi) $f(x) = \log x$ (Logarithmic function)

(vii) $f(x) = \sinh x, \cosh x, \tanh x$ (Hyperbolic function)

(viii) $f(x) = |x|, x + |x|, x - |x|, x\,|x|$ (Absolute value functions)

Following are example of some discontinuous functions:

No.	Functions	Points of discontinuity
(i)	$[x]$	Every Integer
(ii)	$x - [x]$	Every Integer
(iii)	$\frac{1}{x}$	$x = 0$
(iv)	$\tan x, \sec x$	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}.$ ....
(v)	$\cot x, \operatorname{cosec} x$	$x = 0, \pm \pi, \pm 2\pi$....
(vi)	$\sin\frac{1}{x}, \cos\frac{1}{x}$	$x = 0$
(vii)	$e^{1/x}$	$x = 0$
(viii)	$\coth x, \operatorname{cosech} x$	$x = 0$

2.2 Continuity of a Function at a Point

A function f(x) is said to be continuous at a point x = a, if

(a) (i) f(a) exists

(b) (ii) $\lim\limits_{x\to a} f(x)$ exists and is finite

So $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x)$

(c) (iii) $\lim\limits_{x\to a} f(x) = f(a)$ or

Function f(x) is continuous at x = a

If $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$

i.e. If right hand limit at 'a' = left hand limit at 'a' = value of the function at 'a'.

If $\lim\limits_{x\to a} f(x)$ does not exist or $\lim\limits_{x\to a} f(x) \neq f(a)$, then f(x) is said to be discontinuous at x = a

2.3 Continuity from Left and Right

Function f(x) is said to be

(i) Left continuous at x = a if, $\lim\limits_{x\to a^-} f(x) = f(a)$ i.e. $f(a^-) = f(a)$

(ii) Right continuous at x = a if, $\lim\limits_{x\to a^+} f(x) = f(a)$ i.e. $f(a^+) = f(a)$

Thus, a function f(x) is continuous at a point x = a, if it is left continuous as well as right continuous at x = a.

Illustration 42: $f(x)=\begin{cases} x+a\sqrt{2}\sin x & 0\le x<\pi/4 \\ 2x\cot x+b & \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x & \frac{\pi}{2}<x\le\pi \end{cases}$ is continuous in [0, p]. Find a and b. **(JEE MAIN)**

Sol: By checking left continuous and right continuous for x = $\frac{\pi}{4}$ and x = $\frac{\pi}{2}$ we can obtain value of a and b.

$$f\left(\frac{\pi}{4}\right)=\frac{\pi}{2}+b;\ f\left(\frac{\pi}{4}^{-}\right)=\frac{\pi}{4}+a \Rightarrow \quad \frac{\pi}{4}+a=\frac{\pi}{2}+b \Rightarrow a-b=\frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right)=b\ ;\ f\left(\frac{\pi}{2}^{+}\right)=-a-b \Rightarrow a-b=b \Rightarrow 2b=-a \Rightarrow \quad a=\frac{\pi}{6}, b=-\frac{\pi}{12}$$

Illustration 43: Examine the continuity of the function $f(x)=\begin{cases} 2x^2+2, & x\le 2 \\ 2x, & x>0 \end{cases}$, at the point x = 2 **(JEE MAIN)**

Sol: By obtaining Left hand limit and right hand limit we will get to know that given function is continuous or not.

$f(2) = 2^2 + 2 = 6$ ... (i)

L.H.L. $f(2^-)=\lim\limits_{h\to 0}((2-h)^2+1)=5$ ... (ii)

R.H.L. $f(2^+)=\lim\limits_{h\to 0}2(2+h)=4$ ... (iii)

$\because$ $f(2-0) \neq f(2+0) \neq f(2)$

$\therefore$ f(x) is not continuous at x = 2

Illustration 44: If $f(x)=\begin{cases} x+\lambda, & x<3 \\ 4, & x=3 \\ 3x-5, & x>3 \end{cases}$, is continuous at x = 3, then the value of λ is **(JEE MAIN)**

Sol: As given function is continuous at x = 3, hence its left hand limit will be equal to its right hand limit f(x) is continuous at x = 3

$$\therefore \quad f(3)=\lim_{x\to 3^-}f(x)=\lim_{x\to 3^+}f(x) \Rightarrow \quad 4=\lim_{h\to 0}((3-h)+\lambda)=3+\lambda \Rightarrow \lambda=1$$

2.4 Continuity of a Function in an Interval

(a) A function f(x) is said to be continuous in an open interval (a, b), if it is continuous at every point in (a, b). For example, function y = sin x, y = cos x, y = e^x are continuous in $(-\infty, \infty)$

(b) A function f(x) is said to be continuous in the closed interval [a, b], if it is:

(i) Continuous at every point of the open interval (a, b)

(ii) Right continuous at x = a, i.e. $RHL|_{x=a}$ = f(a)

(iii) Left continuous at x = b, i.e. $LHL|_{x=b}$ = f(b)

2.5 Reasons of Discontinuity

(a) Limit does not exist i.e. $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\lim_{x\to a} f(x) \neq f(a)$

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1, 2 and 3.

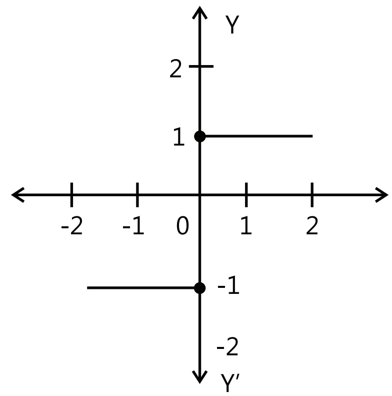


Figure 14.6

3 DIFFERENTIABILITY

3.1 Meaning of a Derivative

The instantaneous rate of change of a function with respect to the dependent variable is called the derivative. Let 'f' be a given function of one variable and let Dx denote a number (positive or negative) to be added to the number x. Let Df denote the corresponding change of 'f ', then

$$Df = f(x + Dx) - f(x) \Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $\frac{\Delta f}{\Delta x}$ approaches a finite value as Dx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is denoted by symbols such as

$$f'(x), \frac{df}{dx}, \frac{d}{dx}(f(x)) \text{ or if } y = f(x) \text{ by } \frac{dy}{dx} \text{ or } y' \Rightarrow \frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The process of finding derivative of a function is called differentiation. We also use the phrase differentiate f(x) with respect to x which means to find f '(x).

MASTERJEE CONCEPTS

The fact that f(x) is differentiable at x = a means that the graph is smooth as x moves from a^- to a to a^+.

Derivative of an even function is always an odd function.

Anvit Tawar (JEE 2009, AIR 9)

3.2 Existence of Derivative at x = a

(a) Right hand derivative:

The right hand derivative of f(x) at x = a, denoted by f ' (a^+) is defined as:

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$, provided the limit exists & is finite. (h>0)

(b) Left hand derivative:

The left hand derivative of f(x) at x = a, denoted by f ' (a^-) is defined as:

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h) - f(a)}{-h}$$, provided the limit exists & is finite. (h > 0)

Hence f(x) is said to be derivable or differentiable at x = a if

$f'(a^+) = f'(a^-)$ = finite quantity and it is denoted by $f'(a)$;

Right hand and left hand derivative at x = a is also denoted by Rf′(a) and Lf′(a) respectively.

MASTERJEE CONCEPTS

- If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
- If a function is differentiable at a point, then it is also continuous at that point.
- If a function is continuous at point x = a, then nothing can be guaranteed about the differentiability of that function at that point.
- If a function f(x) is not differentiable at x = a, then it may or may not be continuous at x = a
- If a function f(x) is not continuous at x = a, then it is not differentiable at x = a
- If the left hand derivative and the right hand derivative of f(x) at x = a are finite (they may or may not be equal), then f(x) is continuous at x = a.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 45: $f(x)=\begin{cases} ax+b & x\le -1 \\ ax^3+x+2b & x>-1 \end{cases}$ is differentiable for xÎR find ‘a’ & ‘b **(JEE ADVANCED)**

Sol: By equating left hand limit to its right hand limit we can obtain ‘a’ & ‘b**.**

$$f'(-1^-)=\lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0}\frac{(a(-1-h)+b)-(b-a)}{-h}=\lim_{h\to 0}\frac{-ah}{-h}=a$$

$$f'(-1^+)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{(a(h-1)^3+(h-1)+2b)-(b-a)}{h}$$

$$=\lim_{h\to 0}\left(a\left(h^2-3h+3\right)+1+\frac{(-a-1+b+a)}{h}\right)=3a+1+\lim_{h\to 0}\left(\frac{b-1}{h}\right)$$

For $f'(-1^+)$ to exist b = 1

Given f(x) is differentiable $\Rightarrow \quad 3a+1=a\Rightarrow a=-1/2$

3.3 Derivative Formula (Theorem)

If a function f(x) is derivable at x =a, then f(x) is continuous at x = a. i.e. every differentiable function is continuous.

Proof: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists. Also f(a + h) – f(a) = $\frac{f(a+h)-f(a)}{h}h\,[h\ne 0]$

$$\therefore \lim_{h\to 0}[f(a+h)-f(a)]=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot h=f'(a).0=0$$

$$\Rightarrow \lim_{h\to 0}[f(a+h)-f(a)]=0 \Rightarrow \lim_{h\to 0}f(a+h)=f(a)\Rightarrow \text{ f(x) is continuous at } x = a$$

MASTERJEE CONCEPTS

(i) Converse of the theorem above is not true.

(ii) Every differentiable function is necessarily continuous but every continuous function is not necessarily differentiable i.e. Differentiability $\Rightarrow$ continuity

but continuity $\not\Rightarrow$ differentiability

(iii) All polynomial, trigonometric, logarithmic and exponential functions are continuous and differentiable in their domains.

(iv) If f(x) & g(x) are differentiable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x) g(x) will also be differentiable at x = a & if $g(a) \neq 0$, then the function f(x) / g(x) will also be differentiable at x = a

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 46: Show that f(x) = |x – 3|, "x ∈ R, is continuous but not differentiable at x = 3. **(JEE ADVANCED)**

Sol: If left hand limit and right hand limit of the function is equal then the function is continuous and if the left hand limit and right hand limit of the derivative of the function is equal then the function is differentiable otherwise not differentiable.

$f(x) = |(x-3)| \Rightarrow f(x) = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$; $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} -(x-3) = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (x-3) = 0$ and f(3) =3 – 3 = 0 ; $\therefore \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)$

So, f(x) is continuous at x = 3 $Lf'(3) = \lim\limits_{x\to 3^-} \dfrac{f(x)-f(3)}{x-3} = \lim\limits_{x\to 3^-} \dfrac{-(x-3)-0}{x-3} = -1$

$Rf'(3) = \lim\limits_{x\to 3^+} \dfrac{f(x)-f(3)}{x-3} = \lim\limits_{x\to 3^+} \dfrac{(x-3)-0}{x-3} = 1$; $Lf'(3) \neq Rf'(3)$

So, f(x) is not differentiable at x = 3.

3.4 Differentiability in an Interval

(a) f(x) is said to be derivable over an open interval (a, b), if it is derivable at each & every point of the interval (a, b).

(b) A function f(x) is differentiable in a closed interval [a, b] if it is:

(i) Differentiable at every point of interval (a, b)

(ii) Right derivative exists at x = a

(iii) Left derivative exists at x = b.

MASTERJEE CONCEPTS

If a function is said to be differentiable over an interval, it is differentiable at each and every point in the interval. If it is not so, even at a single point, then we cannot say that it is differentiable over the interval.

Rohit Kumar (JEE 2012, AIR 79)

3.5 Differentiable Functions and their Properties

A function is said to be a differentiable function, if it is differentiable at every point of its domain

(a) Examples of some differentiable functions:

(i) Every polynomial function

(ii) Exponential functions: a^x, e^x, e^{-x} ......

(iii) Logarithmic functions: $\log_a x$, $\log_e x$ ....

(iv) Trigonometric functions: sinx, cosx,

(v) Hyperbolic functions: sinhx, coshx, ...

(b) Examples of some non-differentiable functions:

(i) $|x|$ at x = 0

(ii) $x \pm |x|$ at x = 0

(iii) $[x]$, $x \pm [x]$ at every $n \in Z$

(iv) $x\sin\left(\frac{1}{x}\right)$, at x = 0

(v) $\cos\left(\frac{1}{x}\right)$, at x = 0

Illustration 46: Check the differentiability of the function $f(x) = \begin{cases} x+2, & x > 3 \\ 5, & x = 3 \\ 8-x, & x<3 \end{cases}$ at x=3 **(JEE MAIN)**

Sol: For the given function to be differentiable f′(3 + h) =f′(3 – h).

For function to be differentiable f′(3 + h) =f′(3 – h)

$$f'(3+h) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \Rightarrow \lim_{h\to 0}\frac{(3+h+2)-5}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$f'(3-h) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h} \Rightarrow \lim_{h\to 0}\frac{8-(3-h)-5}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$f'(3+h) \neq f'(3-h)$$

So function is not differentiable.

Illustration 47: Check the differentiability of the function $f(x) = \begin{cases} 1, & x < 0 \\ 1+\sin x, & 0 \le x \le \pi/2 \\ 2+(x-\pi/2)^2, & \pi/2 < x \le \pi \end{cases}$, at x = π /2

(JEE ADVANCED)

Sol: Similar to above problem, Given function is differentiable if $f'\left(\frac{\pi}{2}+h\right) = f'\left(\frac{\pi}{2}-h\right)$.

$$\text{RHL } f'\left(\frac{\pi}{2}^{+}\right) = \frac{f(\pi/2+h)-f(\pi/2)}{h} = \lim_{h\to 0}\frac{2+(\pi/2+h-\pi/2)^2-(1+\sin\pi/2)}{h}$$

$$= \lim_{h\to 0}\frac{2+h^2-2}{h} = \lim_{h\to 0} h = 0\,;\ \text{LHL } f'\left(\frac{\pi}{2}^{-}\right) = \frac{f\left((\pi/2)-h\right)-f\left((\pi/2)\right)}{-h}$$

$= \lim_{h\to 0}\frac{1+\sin\left((\pi/2)-h\right)-\left(1+\sin(\pi/2)\right)}{-h} = \lim_{h\to 0}\frac{1+\cosh-2}{-h} = \lim_{h\to 0}\frac{\cosh-1}{-h} = \lim_{h\to 0} \sinh = 0$

So, the function is differentiable at $x = \frac{\pi}{2}$

Illustration 48: Check the differentiability of the function $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0 **(JEE ADVANCED)**

Sol: Here for the function to be differentiable $f'(0^+) = f'(0^-)$.

For the function to be differentiable

$f'(0^+) = f'(0^-) = \frac{f'(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(1/h)-0}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$

Which does not exist.

$f'(0^-) = \frac{f'(0-h)-f(0)}{h} = \lim_{h\to 0}\frac{(-h)\sin\left(-(1/h)\right)-0}{-h} = \lim_{h\to 0}\sin\left(-\frac{1}{h}\right)$

Which does not exist. So the function is not differentiable at x = 0

Illustration 49: A differentiable function f satisfies f(x +y) = f(x) f(y) ∀ x, yÎR find f(x) **(JEE ADVANCED)**

Sol: As given f(x +y) = f(x) f(y), By using this obtain f′(x).

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} = \lim_{h\to 0} f(x)\left(\frac{(f(h)-1)}{h}\right)$

$\Rightarrow \quad f'(x) = f(x)f'(0)$; $x = 0 \Rightarrow f'(0) = \left(f'(0)\right)^2 \Rightarrow f'(0) = 0$ or $f'(0) = 1$

$f'(0) = 0 \Rightarrow f'(x) = 0 \Rightarrow f(x)$ is constant

$f'(0) = 1 \Rightarrow f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1 \Rightarrow \ln f(x) = x + c \Rightarrow f(x) = ae^x$

$\because f'(0) = 1 \Rightarrow f(x) = e^x$

MASTERJEE CONCEPTS

The sum, difference, product, quotient (denominator 0) and composite of two differentiable functions is also differentiable.

Chen Reddy Sandeep Reddy (JEE 2012, AIR 62)

3.6 Rolle's Theorem

If a function f defined on the closed interval [a, b], is:

(i) Continuous on [a, b],

(ii) Derivable on (a, b) and

(iii) f(a) = f(b), then there exists at least one real number c between a and b (a < c < b), such that f′ (c) = 0

Geometrical interpretation

Let, the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn.

The theorem states that between two points with equal ordinates on the continuous graph of f, there exists at least one point where the tangent is parallel to x-axis.

MASTERJEE CONCEPTS

We can use this theorem to check if an equation has more than one root.

Consider an equation f(x)=0, where function f is continuous and differentiable and has more than 2 roots, then f satisfies all three conditions of Rolle's theorem and so we can say that the derivative f'(x) must be zero somewhere.

If the sign of the function doesn't change then the function can't have more than one root.

B Rajiv Reddy (JEE 2012, AIR 11)

3.7 Mean Value Theorem

If a function 'f' defined on the closed interval [a, b] is

(i) Continuous on [a, b] and

(ii) Derivable on (a, b), then there exists at least one real number c between a and b (a < c < b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation

The theorem states that between two points A and B on the graph of 'f' there exists at least one point, where the tangent is parallel to the chord AB.

Illustration 50: Verify Rolle's Theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2] **(JEE MAIN)**

Sol: By checking that the given function satisfies all the condition of Rolle's Theorem as mentioned above, we can verify Rolle's Theorem for the given function.

Given that f(x) is a polynomial function. So, it is always continuous and differentiable.

$f(0) = 0,\ f(2) = 2^3 - 3.\ (2)^2 + 2(2) = 0 \Rightarrow \quad f(0) = f(2)$

Thus, all the conditions of Rolle's Theorem are satisfied.

So, there must exist some $c \in (0, 2)$ such that $f'(c) = 0$

$\Rightarrow\ f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$

Where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's Theorem is verified.

Illustration 51: Verify the mean value theorem for the function f(x) = x – 2 sin x, in the interval [–π, p].**(JEE MAIN)**

Sol: Similar to above, by checking that the given function satisfies all the conditions of mean value Theorem as mentioned above, we can verify mean value Theorem for the given function.

Since x and sin x are everywhere continuous and differentiable, therefore f(x) is continuous on [–π, p] and differentiable on (–π, π). Thus, both the conditions of mean value theorem are satisfied. So, there must exist at least one $c \in (-\pi, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$

Now, f(x) = x – 2 sin x

$$\Rightarrow 1 - 2\cos c = \frac{\pi - (-\pi)}{\pi - (-\pi)} \Rightarrow 1 - 2\cos c = 1$$

$$\Rightarrow \cos c = 0 \Rightarrow c = \pm \pi/2$$

Thus, $c = \pm (\pi / 2) \in (-\pi, \pi)$; Hence the mean value theorem is verified

Illustration 52: Find c of the mean value theorem for the function $f(x) = 3x^2 + 5x+7$ in the interval [1, 3].

(JEE MAIN)

Sol: By using mean value theorem.

Given $f(x) = 3x^2 + 5x + 7$ ... (i)

$\Rightarrow f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$; $f'(x) = 6x + 5$

Now, from the mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 6c + 5 = \frac{f(3) - f(1)}{3 - 1} = \frac{49 - 15}{2} = 17 \Rightarrow c = 2$$

PROBLEM SOLVING TACTICS

Above we have discussed L'Hôpital's rule by an example. Let us consider the same example again

$$\lim_{x \to \infty} \frac{x^2 + \sin x}{x^2}$$

These kind of problems where oscillating functions are involved and $x \to \infty$ are solved using the sandwich theorem.

It states that Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have:

$$g(x) \le f(x) \le h(x)$$

and also suppose that: $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$

Then $\lim\limits_{x \to a} f(x) = L$

Now we know that $-1 \le \sin x \le 1 \Rightarrow \dfrac{-1}{x^2} \le \dfrac{\sin x}{x^2} \le \dfrac{1}{x^2} \Rightarrow \dfrac{x^2 - 1}{x^2} \le \dfrac{x^2 + \sin x}{x^2} \le \dfrac{x^2 + 1}{x^2}$

$$\lim_{x \to \infty} \frac{x^2 + 1}{x^2} = \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right) = 1 ; \lim_{x \to \infty} \frac{x^2 - 1}{x^2} = \lim_{x \to \infty} \left(1 - \frac{1}{x^2}\right) = 1 \Rightarrow \lim_{x \to \infty} \frac{x^2 + \sin x}{x^2} = 1$$

While examining the continuity and differentiability of a function f(x) at a point x = a, if you start with the differentiability and find that f(x) is differentiable then you can conclude that the function is also continuous. But if you find f(x) is not differentiable at x = a, you will also have to check the continuity separately. Instead, if you start with the continuity and find that the function is not continuous then you can conclude that the function is also non-differentiable. But if you find f(x) is continuous, you will also have to check the differentiability separately.

FORMULAE SHEET

	Let f and g be two real functions with a common domain D, then.
(i)	$\lim_{x\to a}(f+g)(x)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$
(ii)	$\lim_{x\to a}(f-g)(x)=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$
(iii)	$\lim_{x\to a}(c\cdot f)(x)=c\lim_{x\to a}f(x)$ [c is a constant]
(iv)	$\lim_{x\to a}(fg)(x)=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$
(v)	$\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$
(vi)	$\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a}f(x)\right)^n$
(vii)	If f(x) < = g(x), then $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$
(viii)	$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$
(ix)	$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$
(x)	$\lim_{x\to 0}\frac{\sin x}{x}=1$
(xi)	$\lim_{x\to 0}\frac{e^x-1}{x}=1$
(xii)	$\lim_{x\to 0}\left(1+ax\right)^{1/x}=e^a=\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$
(xiii)	if $\lim_{x\to a}f(x)=1$ and $\lim_{x\to a}g(x)=\infty$ then $\lim_{x\to a}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to a}g(x)\left[f(x)-1\right]}$ sin x < x < tanx
Expansions of Some Functions	
1.	$(1+x)^n=\left\{1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....\right\}$ [| x | < 1]

2.	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \frac{x^n}{3!} + \ldots\ldots$
3.	$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \ldots\ldots\ldots$
4.	$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$
5.	$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$
6.	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$
7.	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$
8.	$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$
9.	$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots\ldots\ldots$
10.	$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \ldots.(-1 < x < 1)$
11.	$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5}\ldots..\ (-1 < x < 1)$

L'Hôpital's rule
For functions f and g which are differentiable: if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ has a finite value then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$
Common Indeterminate Forms
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty$ and ∞^0
Differentiation
$\frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$
Continuity
Function f(x) is continuous at x = a if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Differentiability
$f(x)$ is said to be derivable or differentiable at $x = a$ if $f'(a^+) = f'(a^-)$ = finite quantity
Rolle's theorem
If a function f defined on the closed interval [a, b] is continuous on [a, b] and derivable on (a, b) and $f(a) = f(b)$, then there exists at least one real number c between a and b ($a < c < b$), such that $f'(c) = 0$
Mean Value Theorem
If a function f defined on the closed interval [a, b], is continuous on [a, b] and derivable on (a, b), then there exists at least one real number c between a and b ($a < c < b$), such that $f'(c) = \frac{f(b) - f(a)}{b - a}$