Limits, Continuity and Differentiability

Solved Examples

JEE Main/Boards

Example 1: $\lim_{x\to 0}\dfrac{(\cos x)^{1/2}-(\cos x)^{1/3}}{\sin^2 x}$ is

Sol: Use L' Hôpital's rule to solve this problem.

$$=\lim_{x\to 0}\frac{-\frac{1}{2}(\cos x)^{-1/2}\sin x+\frac{1}{3}(\cos x)^{-2/3}\sin x}{2\sin x\cos x}$$

(using L'Hôpital's rule)

$$=\lim_{x\to 0}\frac{-\frac{1}{2}(\cos x)^{-1/2}+\frac{1}{3}(\cos x)^{-2/3}}{2\cos x}$$

$$=-\frac{1}{2}\left[\frac{1}{2}-\frac{1}{3}\right]=-\frac{1}{12}$$

Example 2: $\lim_{x\to 1}\left[\left(\dfrac{4}{x^2-x^{-1}}-\dfrac{1-3x+x^2}{1-x^3}\right)^{-1}+3\dfrac{x^4-1}{x^3-x^{-1}}\right]$ is

Sol: Simply using algebra, we can solve this problem.

$$\lim_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+3\frac{x^4-1}{x^3-x^{-1}}\right]$$

$$=\lim_{x\to 1}\left[\left(\frac{4x}{x^3-1}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+\frac{3x(x^4-1)}{x^4-1}\right]$$

$$=\lim_{x\to 1}\left[\left(\frac{4x+1-3x+x^2}{x^3-1}\right)^{-1}+3x\right]$$

$$=\lim_{x\to 1}[x-1+3x]=3$$

Example 3: $\lim_{x\to\pi/3}\dfrac{2\sin(x-\pi/3)}{1-2\cos x}$ is

Sol: We can solve this problem using trigonometric formulae's.

$$\lim_{x\to\pi/3}\frac{2\sin(x-\pi/3)}{1-2\cos x}\quad\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to\pi/3}\frac{2\cos(x-\pi/3)}{2\sin x}$$

$$=\lim_{x\to\pi/3}\frac{2\cos(x-\pi/3)}{2\sin x}$$

$=\cos 0/\sin(\pi/3)=2/\sqrt{3}$

Example 4: If $f(x)=\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$ then find $\lim_{x\to\infty}f(x)$

Sol: Solve it by using 1^∞ form

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$$

$$=\lim_{x\to\infty}\left(\frac{1+(5/x)+(3/x^2)}{1+(1/x)+(2/x^2)}\right)^x\ \left\{1^\infty\text{form}\right\}=e^a\quad\ldots\quad\text{(i)}$$

$$a = \lim_{x\to\infty} x\left(\frac{1+(5/x)+(3/x^2)}{1+(1/x)+(2/x^2)}-1\right)$$

$$= \lim_{x\to\infty}\frac{4+(1/x)}{1+(1/x)+(2/x^2)} = 4$$

$$\Rightarrow \lim_{x\to\infty} f(x) = e^4$$

Example 5: If a, b are chosen from {1, 2, 3, 4, 5, 6, 7} randomly with replacement. The probability that

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = 7 \text{ is}$$

Sol: Similar to above example.

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} \left[1^\infty \text{form}\right] = e^a$$

Where,

$$a = \lim_{x\to 0}\frac{2}{x}\left(\frac{a^x+b^x}{2}-1\right) = \lim_{x\to 0}\left(\frac{a^x+b^x-2}{x}\right)\left\{\frac{0}{0}\text{form}\right\}$$

$$= \lim_{x\to 0}\left(a^x\log a + b^x \log b\right) = \log ab$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = e^{\log ab} = ab$$

Given $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{2/x} = 7$

Hence the favourable outcomes are (1, 7) and (7, 1). Therefore, the required probability is 2/49.

Example 6: Find $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$

Sol: Solve this by putting $x = \frac{\pi}{2}+h$

Put $x - \pi/2 = \theta$, so that

$$\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$$

$$= \lim_{\theta\to 0}\frac{\cot(\pi/2+\theta)-\cos(\pi/2+\theta)}{(-2\theta)^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\theta^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\frac{\sec^2\theta-\cos\theta}{3\theta^2}$$

$$= \frac{1}{24}\lim_{\theta\to 0}\frac{2\sec^2\theta\cdot\tan\theta+\sin\theta}{2\theta}$$

$$= \frac{1}{24}\left(\frac{2+1}{2}\right) = \frac{1}{16}$$

Example 7: Given function $g(x) = \sqrt{6-2x}$ and h(x) = $2x^2 - 3x + a$. Then

(i) Evaluate h(g(2)) (ii) If $f(x) = \begin{cases} g(x); x \le 1 \\ h(x); x > 1\end{cases}$. Find 'a' so that f is continuous.

Sol: Equate left hand limit and right hand limit of f(x).

(i) $g(2) = \sqrt{6-4} = \sqrt{2}$

$h(g(2)) = h(\sqrt{2}) = 4 - 3\sqrt{2} + a$

(ii) $f(x) = \begin{cases} g(x);\ x\le 1 \\ h(x);\ x>1\end{cases}$ $f(x) = \begin{cases}\sqrt{6-2x}, & x \le 1 \\ 2x^2-3x+a, & x>1\end{cases}$

$f(1)=2$ R.H.L. $|_{x=1} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x^2-3x+a)$

$= a - 1$ .... (i)

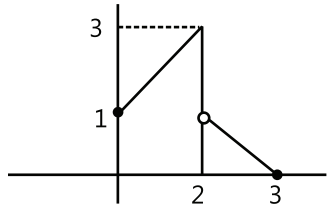
L.H.L. $|_{x=1} = \lim_{x\to 1^-}\sqrt{6-2x} = 2$

Since function is continuous

L.H.L. $|_{x=1}$ = R.H.L. $|_{x=1}$ = f(1) $\Rightarrow$ a = 3

Example 8: Let $f(x) = \begin{cases} 1+x\ ;\ 0\le x\le 2 \\ 3-x\ ;\ 2<x\le 3\end{cases}$.

Determine the form of g(x) = f[f(x)] & hence find the point of discontinuity of g (if any)



Graph of f(x)

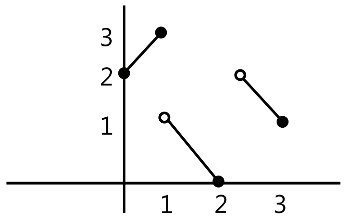
Sol: Sketch the graph of f(f(x)) to obtain the point of discontinuity.

$$f(x) = \begin{cases} 1+x\ ;\ 0\le x\le 2 \\ 3-x\ ;\ 2<x\le 3\end{cases}$$

$(fof)(x) = f(f(x))$

$$= \begin{cases} 1 + f(x) \;;\; 0 \le f(x) \le 2 \\ 3 - f(x) \;;\; 2 < f(x) \le 3 \end{cases}$$

Let f(x) = y



Graph of f(f(x))

$$f(f(x)) = \begin{cases} 1 + y & 0 \le y \le 2 \\ 3 - y & 2 < y \le 3 \end{cases}$$

$$= \begin{cases} 2 + x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \\ 4 - x & 2 < x \le 3 \end{cases}$$

Clearly from the graph we can see that at x=1 and 2 the function is discontinuous.

Example 9: Determine the value of a, b & c for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ; \quad x < 0 \\ c & ; \quad x = 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & ; \quad x > 0 \end{cases}$$

is continuous at x = 0

Sol: Here the function is continuous at x = 0, therefore by equating Left hand limit and Right hand limit of the given function we can obtain required values.

$f(0) = c$; LHS $\big|_{x=0} \lim\limits_{x \to 0^-} \dfrac{\sin(a+1)x + \sin x}{x}$

$= \lim\limits_{x \to 0^-} (\cos(a+1)x)(a+1) + \cos x = (\cos 0)\,(a + 1)$

$+ \cos 0 = a+1 +1 = a+2$

RHS $\big|_{x=0} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$

$$= \lim_{x \to 0^+} \frac{(1 + bx)^{\frac{1}{2}} - 1}{bx} = \lim_{x \to 0^+} \frac{1}{2}(1 + bx)^{-\frac{1}{2}} = \frac{1}{2}$$

Since the function is continuous

$$c = a + 2 = \frac{1}{2} \Rightarrow a = -\frac{3}{2}, c = \frac{1}{2} \text{ and } b \in R - \{0\}$$

Example 10: Find the locus of (a, b) for which

the function $f(x) = \begin{cases} ax - b & ; \quad x \le 1 \\ 3x & ; \quad 1 < x < 2 \\ bx^2 - a & ; \quad x \ge 2 \end{cases}$ is continuous

at x = 1 but discontinuous at x = 2

Sol: Solve this example by using given condition.

at x =1 function should be continuous

$\Rightarrow$ a – b = 3 .....(i)

at x = 2 the function should be discontinuous

$\Rightarrow \quad 6 \ne 4b - a$

$\Rightarrow \quad 6 \ne 4b - b - 3 \quad \Rightarrow \; (a, b) \ne (6, 3)$

Hence the locus of (a,b) is y=x-3 and x6

Example 11: Let f: R [0, ∞) be such that $\lim\limits_{x \to 5} f(x)$ exists and $\lim\limits_{x \to 5} \dfrac{f(x)^2 - 9}{\sqrt{|x - 5|}} = 0$.

Then $\lim\limits_{x \to 5} f(x)$ equals?

Sol: Here in the limit, denominator becomes zero but the limit has a finite value. So, numerator should also be zero (whether to apply L'Hôpital's rule or to factorize the common factor).Given,

$$\lim_{x \to 5} \frac{(f(x))^2 - 9}{\sqrt{|x - 5|}} = 0$$

Here in the limit, the denominator becomes zero but the limit has a finite value. So the numerator should also be zero (whether to apply L'Hôpital's rule or to factorize the common factor).

Hence, $\lim\limits_{x \to 5} (f(x))^2 - 9 = 0 \Rightarrow \lim\limits_{x \to 5} f(x) = \pm 3$

Since the range of f is [0, ∞), $\lim\limits_{x \to 5} f(x) = 3$

JEE Advanced/Boards

Example 1: Evaluate

$$\lim_{x \to 0} \frac{\sqrt[3]{8 + x} - \sqrt[3]{8 + x^2 - x^3}}{\sqrt[3]{8 + x} - \sqrt[3]{8 + x^2 + x^3}}$$

Sol: $\lim_{x\to 0}\dfrac{\sqrt[3]{8+x}-\sqrt[3]{8+x^2-x^3}}{\sqrt[3]{8+x}-\sqrt[3]{8+x^2+x^3}}\left[\dfrac{0}{0}\text{form}\right]$

$$= \lim_{x\to 0}\frac{\frac{1}{3}\left(\sqrt[3]{8+x}\right)^{-2}-\frac{1}{3}\left(\sqrt[3]{8+x^2-x^3}\right)^{-2}\left(2x-3x^2\right)}{\frac{1}{3}\left(\sqrt[3]{8+x}\right)^{-2}-\frac{1}{3}\left(\sqrt[3]{8+x^2+x^3}\right)^{-2}\left(2x+3x^2\right)}=1$$

Example 2: Evaluate $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

Sol: Multiply and divide to given limit by

$\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)$.

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$$

$$= \lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{1+(1/\sqrt{x})}}{\left\{\sqrt{1+\sqrt{(x+\sqrt{x})/x^2}}+1\right\}} = \frac{1}{1+1}=\frac{1}{2}$$

Example 3: Evaluate

$$\lim_{n\to\infty}\frac{1\cdot\sum_1^n r+2\cdot\sum_1^{n-1} r+3\cdot\sum_1^{n-2} r+.......n\cdot 1}{n^4}$$

Sol: By using summation formula of n numbers we can evaluate given limit.

consider $m\cdot\sum_1^{n-m+1} r$

$$= m\cdot\frac{(n-m+1)(n-m+2)}{2}$$

$$= \frac{m}{2}\{n^2-(2m-3)n+(m-1)(m-2)\}$$

$$= \frac{n^2}{2}\cdot m-\frac{n}{2}m(2m-3)+\frac{m}{2}(m^2-3m+2)$$

$$= \frac{n^2}{2}\cdot m-n\cdot m^2+\frac{3n}{2}\cdot m+\frac{m^3}{2}-\frac{3}{2}m^2+m$$

$$= \left(\frac{n^2}{2}+\frac{3n}{2}+1\right)m-\left(n+\frac{3}{2}\right)m^2+\frac{1}{2}m^3$$

$$\Rightarrow \sum_1^n\left\{m\sum_1^{n-m+1} r\right\}$$

$$= \left(\frac{n^2}{2}+\frac{3n}{2}+1\right)\sum_1^n m-\left(n+\frac{3}{2}\right)\sum_1^n m^2+\frac{1}{2}\sum_1^n m^3$$

$$= \frac{n^2+3n+2}{2}\cdot\frac{n(n+1)}{2}$$

$$-\frac{2n+3}{2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2(n+2)}{4}-\frac{n(n+1)(2n+1)(2n+3)}{12}+\frac{n^2(n+1)^2}{8}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1\cdot\sum_1^n r+2\cdot\sum_1^{n-1} r+3\cdot\sum_1^{n-2} r+.......n\cdot 1}{n^4}$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\cdot n^4\left\{\frac{\left((1/n)+1\right)^2\left((2/n)+1\right)}{4}-\frac{\left(1+(1/n)\right)\left(2+(1/n)\right)\left(2+(3/n)\right)}{12}+\frac{\left(1+(1/n)\right)^2}{8}\right\}$$

$$= \frac{1}{4}-\frac{4}{12}+\frac{1}{8}=\frac{1}{24}$$

Example 4: Evaluate $\lim_{x\to-\infty}\dfrac{x^4\sin(1/x)+x^2}{1+|x|^3}$

Sol: Simply by putting x = –y in given limit, we can evaluate this.

Let x = –y. Then $y\to\infty$ when $x\to-\infty$

$$\Rightarrow \lim_{x\to-\infty}\frac{x^4\sin(1/x)+x^2}{1+|x|^3}=\lim_{y\to\infty}\frac{y^4\sin(-1/y)+y^2}{1+|-y|^3}$$

$$= = \lim_{y\to\infty}\frac{-y^4\sin(1/y)+y^2}{1+y^3}\quad[\because \text{ y is positive}]$$

$$= \lim_{y\to\infty}\left\{\frac{\sin(1/y)}{(1/y)}\cdot\frac{-y^3}{1+y^3}+\frac{y^2}{1+y^3}\right\}$$

$$= \lim_{y\to\infty}\left\{\frac{\sin(1/y)}{(1/y)}\cdot\frac{-1}{(1/y^3)+1}+\frac{(1/y)}{\left((1/y^3)+1\right)}\right\}$$

$$= 1\cdot\frac{-1}{1}+0=-1$$

Example 5: Without using expansions or using L'Hôpital's rule, Prove that $\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=\frac{1}{6}$.

Sol: Consider $\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=p$, and after that replace θ by 3θ to prove above problem.

Let the limit be p. then $p=\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}$

$$=\lim_{\theta\to 0}\frac{3\theta-\sin 3\theta}{(3\theta)^3} \text{ (replacing } \theta \text{ by } 3\theta)$$

$$=\lim_{\theta\to 0}\left[\frac{3\theta-3\sin\theta+4\sin^3\theta)}{(3\theta)^3}\right]$$

$$=\lim_{\theta\to 0}\left\{\frac{\theta-\sin\theta}{9\theta^3}+\frac{4}{27}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3\right\}$$

$$=\frac{1}{9}\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}+\frac{4}{27}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3=\frac{1}{9}p+\frac{4}{27}\cdot 1^3$$

$$\Rightarrow p-\frac{1}{9}p=\frac{4}{27}\Rightarrow\frac{8p}{9}=\frac{4}{27}\Rightarrow p=\frac{1}{6}$$

Example 6:

$$f(x)=\begin{cases}\dfrac{a^x-1}{x^n\sin x}\left(\dfrac{b\sin x-\sin bx}{\cos x-\cos bx}\right)^n & x>0\\[2ex] \dfrac{a^x\sin bx-b^x\sin ax}{\tan bx-\tan ax} & x<0\end{cases}$$

is continuous at x = 0 (a, b > 0, b ≠ 1, a ≠ b). Obtain f(0) and a relation between a, b and n.

Sol: Here the given function is continuous at x = 0, therefore $\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(0-h)$.

$$\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{a^h-1}{\sin h\cdot h^n}\left[\frac{b\sin h-\sin bh}{\cos h-\cos bh}\right]^n$$

$$=\lim_{h\to 0}\frac{a^h-1}{h}\cdot\frac{h}{\sin h}\left[\left(\frac{b\sin h-\sin bh}{h}\right)\frac{1}{\cos h-\cos bh}\right]^n$$

$$=\left(\lim_{h\to 0}\left[\left(\frac{b\sin h-\sin bh}{h^3}\right)\frac{h^2}{\cos h-\cos bh}\right]^n\right)\ln a$$

$$=\left(\lim_{h\to 0}\left[\frac{b(\sin h-h)-(\sin bh-bh)}{h^3}\right]\cdot\left[b^2\left(\frac{1-\cos bh}{b^2h^2}\right)-\left(\frac{1-\cos h}{h^2}\right)\right]^{-1}\right)^n\ln a$$

$$=\left(\lim_{h\to 0}\left[b\left(\frac{\sin h-h}{h^3}\right)-\left(\frac{\sin bh-bh}{b^3h^3}\right)b^3\right]\cdot\left[\left(\frac{b^2}{2}-\frac{1}{2}\right)^{-1}\right]\right)^n\ln a$$

$$=\left(\lim_{h\to 0}\left[b\left(-\frac{1}{6}\right)-b^3\left(-\frac{1}{6}\right)\right]\cdot\frac{2}{(b^2-1)}\right)^n\ln a$$

$$=\left(\lim_{h\to 0}\frac{b(b^2-1)}{6}\cdot\frac{2}{(b^2-1)}\right)^n\ln a=\ln a\left(\frac{b}{3}\right)^n$$

hence $f(0^+)=\ln a\cdot\left(\frac{b}{3}\right)^n$

$$f(0^-)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\frac{a^{-h}\sin(-bh)-b^{-h}\sin(-ah)}{\tan(-bh)-\tan(-ah)}$$

$$=\lim_{h\to 0}\frac{a^h\sin ah-b^h\sin bh}{a^h\cdot b^h[\tan ah-\tan bh]}$$

$$=\lim_{h\to 0}\frac{\dfrac{ah\cdot a^h\sin(ah)}{ah}-\dfrac{b^h\sin(bh)}{bh}\cdot bh}{ah\cdot\dfrac{\tan(ah)}{ah}-bh\dfrac{\tan(bh)}{bh}}=1$$

If f(x) is to be continuous at x = 0, then f(0) = 1 and $\ln a\cdot\left(\frac{b}{3}\right)^n=1$

Example 7: Find a, b and c such that

$$\lim_{x\to 0}\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}=2$$

Sol: By using L' hôpital rule solve the given limit.

$$\text{Limit}=\lim_{x\to 0}\frac{a(e^x+xe^x)-b\cdot\left((1)/(1+x)\right)+c(e^{-x}-xe^{-x})}{2x\sin x+x^2\cos x}$$

{using L'Hôpital's rule} $=\frac{a-b+c}{0}$

But, the limit is given to be 2. So, the indeterminate form $\frac{0}{0}$ should continue.

So, a – b + c = 0 ....(i)

Then, limit =

$$\lim_{x\to 0}\frac{\left\{a(1+x)e^x+ae^x+(b/(1+x)^2)+c(-1)e^{-x}+c(1-x)e^{-x}\cdot(-1)\right\}}{2\sin x+2x\cos x+2x\cos x-x^2\sin x}$$

{using L' Hospital's rule}

$$= \lim_{x\to 0}\frac{a(2+x)e^{x}+(b/(1+x)^{2})-c(2-x)e^{-x}}{2\sin x+4x\cos x-x^{2}\sin x}$$

$$= \frac{2a+b-2c}{0}$$

But, the limit is given to be 2.

So, 2a + b – 2c = 0 ... (ii)

Then, limit =

$$\lim_{x\to 0}\frac{\left\{a(2+x)e^{x}+ae^{x}-\dfrac{2b}{(1+x)^{3}}+c(2-x)e^{-x}+ce^{-x}\right\}}{6\cos x-4x\sin x-2x\sin x-x^{2}\cos x}$$

$$=\frac{2a+a-2b+2c+c}{6}=2\,(\text{given})$$

∴ 3a – 2b + 3c = 12 ... (iii)

Solving (i),(ii) and (iii) we get a = 3, b = 12, c= 9

Example 8: Consider the function

$$g(x)=\begin{cases}\dfrac{1-a^{x}+xa^{x}\ln a}{a^{x}x^{2}} & ;\ x<0\\ \dfrac{2^{x}a^{x}-x\ln 2-x\ln a-1}{x^{2}} & ;\ x>0\end{cases}$$

find the value of 'a' & g (0) so that the function g(x) is continuous at x = 0

Sol: By obtaining left hand limit and right hand limit and equating them we can easily solve given limit.

$$\text{L.H.L.}\big|_{x=0}=\lim_{x\to 0^{-}}\left(\frac{1-a^{x}+xa^{x}\ln a}{a^{x}x^{2}}\right)$$

Put x = 0 – h.= $\lim_{h\to 0}\dfrac{1-a^{-h}-ha^{-h}\ln a}{a^{-h}h^{2}}$

$$=\lim_{h\to 0}\frac{a^{h}-1-h\ln a}{h^{2}}=\lim_{h\to 0}\left(\frac{a^{h}\ln a-0-\ln a}{2h}\right)$$

$$=\lim_{h\to 0}\frac{a^{h}(\ln a)^{2}-0}{2}=\frac{(\ln a)^{2}}{2}$$

$$\text{R.H.L.}\big|_{x=0}=\lim_{x\to 0^{+}}\left(\frac{2^{x}a^{x}-x\ln 2-x\ln a-1}{x^{2}}\right)$$

Put x = 0 + h = $\lim_{h\to 0^{+}}\left(\dfrac{2^{h}a^{h}-h\ln 2-h\ln a-1}{h^{2}}\right)$

$$=\lim_{h\to 0}\frac{(2a)^{h}\ln 2a-\ln 2a}{2h}$$

$$=\lim_{h\to 0}\frac{(2a)^{h}(\ln 2a)^{2}-0}{2h}=\frac{(\ln 2a)^{2}}{2}$$

Since the function is continuous

$g(0) = \text{L.H.L.}\big|_{x-0} = \text{R.H.L}\big|_{x-0}$

$$\frac{(\ln(2a))^{2}}{2}=\frac{(\ln a)^{2}}{2}\ \Rightarrow (\ln(2a)+\ln a)(\ln 2a-\ln a)=0$$

$(\ln(2a)\ \ln 2 = 0 \Rightarrow \ln(2a^{2}) = 0$

$$\Rightarrow 2a^{2}=1\Rightarrow a=\pm\frac{1}{\sqrt{2}}\Rightarrow a=\frac{1}{\sqrt{2}}$$

$$g(0)=\frac{(\ln a)^{2}}{2}=\frac{(\ln 2^{-1/2})^{2}}{2}=\frac{1}{8}(\ln 2)^{2}$$

Example 9: Evaluate: $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}}$

Sol: Here the given limit is in the form of $1^{\infty}= e^{a}$.

$\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}}$ is of the form $1^{\infty}= e^{a}$

Where $a=\lim_{x\to 0}\left(\dfrac{(\sin x)/x}{1-((\sin x)/x)}\right)\left(\dfrac{\sin x}{x}-1\right)=-1$

$$\therefore\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}}}=e^{-1}$$

Example 10: Let f(x) = $x^3 - x^2 - 3x - 1$, g(x) = (x+1)a and $h(x)=\dfrac{f(x)}{g(x)}$ where h is a rational function such that

(a) It is continuous everywhere except when x = –1

(b) $\lim_{x\to\infty}h(x)=\infty$ and

(c) $\lim_{x\to -1}h(x)=\dfrac{1}{2}$ .Find $\lim_{x\to 0}(3h(x)+f(x)-2g(x))$

Sol: Simply by following given condition we can solve above example.

$$h(x)=\frac{f(x)}{g(x)}=\frac{x^{3}-x^{2}-3x-1}{(x+1)a} \quad \text{... (i)}$$

Given that $\lim_{x\to -1} h(x) = \frac{1}{2} \Rightarrow \lim_{x\to -1} \frac{x^3 - x^2 - 3x - 1}{(x+1)a} = \frac{1}{2}$

$$\Rightarrow \lim_{x\to -1} \frac{(x^2 - 2x - 1)(x+1)}{(x+1)a} = \frac{1}{2}$$

$$\Rightarrow \frac{2}{a} = \frac{1}{2} \Rightarrow a = 4\ ;\ h(x) = \frac{x^3 - x^2 - 3x - 1}{(x+1)4}$$

$g(x) = 4(x + 1)$; $\lim_{x\to 0}(3h(x) + f(x) - 2g(x))$ ...(ii)

$$= \lim_{x\to 0}\left(\left(3\cdot\left(\frac{x^3 - x^2 - 3x - 1}{(x+1)4}\right)\right) + \left(x^3 - x^2 - 3x - 1\right) - 2\left(4\left(x+1\right)\right)\right)$$

$$= \frac{3(-1)}{4} - 1 - 8 = \frac{-3 - 4 - 32}{4} = \frac{-39}{4}$$

Example 11: Let

$$f(x) = \begin{cases} \dfrac{1 - \cos 4x}{x^2}, & x < 0. \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & x > 0 \end{cases}$$

If possible, find the value of a so that the function is continuous at x = 0.

Sol: f(x) will be continuous at x = 0 if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0).$$

f(x) will be continuous at x = 0 if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0) \quad \text{.... (i)}$$

Now, $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{\sqrt{0+h}}{\sqrt{16 + \sqrt{0+h}} - 4}$

$$= \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4} = \lim_{h\to 0} \frac{\sqrt{h}\left\{\sqrt{16 + \sqrt{h}} + 4\right\}}{16 + \sqrt{h} - 16}$$

$$= \lim_{h\to 0}\{\sqrt{16 + \sqrt{h}} + 4\} = 8$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1 - \cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0} \frac{1 - \cos 4h}{h^2} = \lim_{h\to 0} \frac{2\sin^2 2h}{h^2} = \lim_{h\to 0} 2\cdot\left(\frac{\sin 2h}{2h}\right)^2 \times 4 = 8$$

$\therefore$ if a = 8 functions will be continuous at x = 0.

Example 12: If $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n} + 1}$ examine the continuity of f(x) at x = 1.

Sol: In order to examine the continuity at x = 1, we are required to derive the definition of f(x) in the intervals x <1, x > 1 and at x = 1, i.e., on and around x=1.

In order to examine the continuity at x = 1, we are required to derive the definition of f(x) in the intervals x < 1, x > 1 and at x = 1, i.e., on and around x = 1.

Now, if 0 < x < 1,

$$f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n} + 1}$$

$$= \frac{\log(x+2) - 0\sin x}{0 + 1} = \log(x+2)$$

if x = 1, $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - 1\cdot\sin x}{1 + 1} = \frac{\log(x+2) - \sin x}{2}$

if x > 1, $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n} + 1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(x+2) - \sin x}{1 + \frac{1}{x^{2n}}} = -\sin x$$

Thus, we have

$$f(x) = \begin{cases} \log\left(x+2\right) & 0 < x < 1 \\ \dfrac{\log\left(x+2\right) - \sin x}{2} & x = 1 \\ -\sin x & x > 1 \end{cases}$$

$$f(1^+) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\{-\sin(1+h)\} = -\sin 1$$

$$f(1^-) = \lim_{h\to 0} f(1-h)\ \lim_{h\to 0} \log(1-h+2) = \log 3$$

Clearly $f(1^+) \neq f(1^-)$

So f(x) is not continuous at x=1

Example 13: Is f(x) differentiable at x = 0 if

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases} ?$$

Sol: Here if $f'(0^+)$ is equal to the $f'(0^-)$ then only the given function is differentiable otherwise not differentiable.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{(0+h)/(1+e^{1/(0+h)})-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+e^{1/h}} = \frac{1}{1+e^{\infty}} = 0$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{(0-h)/(1+e^{1/(0-h)})-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$\therefore f'(0^+) \neq f'(0^-)$

Hence f(x) is not differentiable at x = 0

Example 14: Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max\{f(t), 0 \le t \le x\}, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}.$$

Discuss the continuity and differentiability of g(x) in the interval [0, 2]

Sol: Similar to above example.

Here, $f(x) = x^3 - x^2 + x + 1$

$\therefore f'(x) = 3x^2 - 2x + 1$ or $f'(x) = 3\left\{x^2 - \frac{2}{3}x\right\} + 1$

$$= 3\left\{x^2 - \frac{2}{3}x + \frac{1}{9}\right\} + 1 - \frac{1}{3} = 3\left\{x - \frac{1}{3}\right\}^2 + \frac{2}{3} > 0 \text{ for all } x$$

$\therefore$ f(x) is an increasing function of x (monotonic function)

$\therefore$ in $0 \le x \le 1$, max $\{f(t), 0 \le t \le x\} = f(x)$

$\therefore g(x) = f(x) = x^3 - x^2 + x + 1, 0 \le x \le 1$

$3 - x, 1 < x \le 2$

As polynomial functions are continuous and differentiable everywhere, g(x) is also continuous and differentiable everywhere except at the turning point of definition x = 1

Now, $g(1^+) = \lim_{h\to 0} g(1+h) = \lim_{h\to 0} \{3-(1+h)\} = 2$

$$g(1^-) = \lim_{h\to 0} g(1-h)$$

$$= \lim_{h\to 0}\{(1-h)^3 - (1-h)^2 + (1+h) + 1\} = 2$$

$g(1) = 1^3 - 1^2 + 1 + 1 = 2$

$\therefore \quad g(1^+) = g(1^-) = g(1)$

$\therefore \quad$ g(x) is continuous at x = 1

$$\text{Next, } g'(1^+) = \lim_{h\to 0}\frac{g(1+h)-g(1)}{h} = \lim_{h\to 0}\frac{\{3-(1+h)\}-2}{h} = -1$$

$$g'(1^-) = \lim_{h\to 0}\frac{g(1-h)-g(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^3-(1-h)^2+(1-h)+1-2}{-h}$$

$$= \lim_{h\to 0}\frac{-h^3+2h^2-2h}{-h} = \lim_{h\to 0}(h^2-2h+2) = 2$$

$\therefore g'(1+0) \neq g'(1-0)$;

So g(x) is not differentiable at x = 1.

$\therefore$ g(x) is continuous in [0, 2] and differentiable in [0, 2] except at the point x =1

Example 15: If $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x) \cdot G(x)$ where $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} G(x)$ exists. Prove that f(x) is continuous at all $x \in R$.

Sol: Simply by following the given condition we can solve this example. $\lim_{x\to 0} g(x) = 0$

$$\Rightarrow \lim_{h\to 0} g(0+h) = \lim_{h\to 0} g(0-h) = 0$$

$$\Rightarrow \lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h) = 0 \quad \text{.... (i)}$$

$$\lim_{x\to 0} G(x) \text{ exists} \Rightarrow \lim_{h\to 0} G(0+h) = \lim_{h\to 0} G(0-h)$$

$$\Rightarrow \lim_{h\to 0} G(h) = \lim_{h\to 0} G(-h) = \text{finite} \quad \text{.... (ii)}$$

Now, $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x)\cdot f(h) = f(x)\lim_{h\to 0} f(h)$

$\{\because f(x + y) = f(x) \cdot f(y)\}$

$= f(x)\cdot\lim_{h\to 0}\{1 + g(h)G(h)\}$, using the given relation

$$= f(x)\cdot\left\{1 + \lim_{h\to 0} g(h)\cdot\lim_{h\to 0} G(h)\right\}$$

$= f(x) \cdot \{1 + 0 \cdot \text{finite}\}$, using (i) and (ii) $= f(x)$ Also,

$$\lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x)\cdot f(-h) = f(x)\cdot\lim_{h\to 0} f(-h)$$

$= f(x)\cdot\lim_{h\to 0}\{1 + g(-h)\cdot G(-h)\}$, using the given relation

$$= f(x)\cdot\left\{1 + \lim_{h\to 0} g(-h)\cdot\lim_{h\to 0} G(-h)\right\}$$

$= f(x)\cdot \{1 + 0 \cdot \text{finite}\}$, using (i) and (ii) $= f(x)$

$$\therefore \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h) = f(x)$$

$\therefore$ f(x) is continuous everywhere

JEE Main/Boards

Exercise 1

Limits

Q.1 $\lim\limits_{x\to -4} \dfrac{\sqrt{5+x}-1}{x^2+4x}$

Q.2 $\lim\limits_{x\to 2}\left[\left(\dfrac{x^3-4x}{x^3-8}\right)^{-1}\right]$

Q.3 $\lim\limits_{x\to 1}\dfrac{x^2-x\ell nx+\ell nx-1}{x-1}$

Q.4 $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

Q.5 $\lim\limits_{x\to 2}\left[\dfrac{1}{x(x-2)^2}-\dfrac{1}{x^2-3x+2}\right]$

Q.6 (a) $\lim\limits_{x\to 0}\tan^{-1}\dfrac{a}{x^2}$ where $a \in R$

(b) Plot the graph of the function

$f(x)=\lim\limits_{t\to 0}\left(\dfrac{2x}{\pi}\tan^{-1}\dfrac{x}{t^2}\right)$

Q.7 $\lim\limits_{x\to 1}\dfrac{\left(\sum\limits_{K=1}^{100}x^k\right)-100}{x-1}$

Q.8 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

Q.9 $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$

Q.10 $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos x^2}}{1-\cos x}$

Q.11 $\lim\limits_{x\to \pi/6}\dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

Q.12 $\lim\limits_{\theta\to \frac{\pi}{4}}\dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

Q.13 $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$

Q.14 If $\lim\limits_{x\to 0}\dfrac{a\sin x-\sin 2x}{\tan^3 x}$ is finite then find the value of 'a' and the limit

Q.15 $\lim\limits_{x\to \pi/4}\tan 2x\,\tan(\pi/4-x)$

Q.16 $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$

Q.17 $\lim\limits_{n\to\infty} n\cdot\cos\left(\dfrac{\pi}{4n}\right)\cdot\sin\left(\dfrac{\pi}{4n}\right)$

Q.18 Evaluate: $\lim\limits_{x\to 2}\dfrac{(\cos\alpha)^x+(\sin\alpha)^x-1}{x-2}$

Q.19 $\lim\limits_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

Q.20 $\lim\limits_{x\to I}\dfrac{\sin\{x\}}{\{x\}}$ where $\{x\}$ is the fractional part function & I is any integer

Q.21 $\lim\limits_{x\to 0}\left[\dfrac{n\sin x}{x}\right]+\lim\limits_{x\to 0}\left[\dfrac{n\tan x}{x}\right]$ where [*] denotes the greater function and $n \in I-0$

Q.22 ABC is an isosceles triangle inscribed in a circle of radius r. If AB = AC and h is the altitude from A to BC. Then in triangle ABC evaluate $\lim\limits_{h\to 0}\dfrac{\Delta}{P^3}$, where Δ is area of the triangle and P is the perimeter.

Q.23 $\lim\limits_{x\to 0}\dfrac{\ln(1+x+x^2)+\ln(1-x+x^2)}{x(e^x-1)}$

Q.24 $\lim\limits_{n\to\infty} n^2\left(a^{\frac{1}{n}}-a^{\frac{1}{n+1}}\right)$, $a>0$

Q.25 $\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^x=4$ then find c

Continuity

Q.1 Find all possible values of a and b so that f (x) is continuous for all $x \in R$ if

$$f(x) = \begin{cases} |ax+3|, & x \le -1 \\ |3x+a|, & -1 < x \le 0 \\ \dfrac{b\sin 2x}{x} - 2b, & 0 < x < \pi \\ \cos^2 x - 3, & x \ge \pi \end{cases}$$

Q.2 The function

$$f(x) = \begin{cases} \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5}}, & 0 < x < \dfrac{\pi}{2} \\ b+2, & x = \dfrac{\pi}{2} \\ \left(1+|\cos x|\right)^{\left(\frac{|a\tan x|}{b}\right)}, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$

Q.3 Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and

$$h(x) = \begin{cases} \dfrac{f(x)}{x-3}, & x \ne 3 \\ K, & x = 3 \end{cases} \text{ then}$$

(a) Find all zeros of f(x)

(b) Find the value of K that makes h continuous at x = 3.

(c) Using the value of K, determine whether h is an even function

Q.4 Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + ... + \dfrac{x^2}{(1+x^2)^{n-1}}$

and $y(x) = \lim\limits_{n\to\infty} y_n(x)$.

Discuss the continuity of $y_n(x)$ ($n \in N$) and y(x) at x = 0.

Q.5 Find the number of points of discontinuity of the function f(x) = [5x] + {3x} in [0, 5] where [y] denote greater integer function and {y} denote fractional part of y.

Q.6 Examine the continuity of $f(x) = \lim\limits_{n\to\infty} \dfrac{x}{(2\sin x)^{2n} + 1}$ for $x \in R$.

Q.7 Let $f(x) = \begin{cases} \dfrac{\ln\cos x}{\sqrt[4]{1+x^2} - 1}, & x > 0 \\ \dfrac{e^{\sin 4x} - 1}{\ln(1+\tan 2x)}, & x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0? If yes, then what is the value of f(0), if not then indicate the nature of discontinuity.

Q.8 If $f(x+y) = f(x).f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x).G(x)$ where $\lim\limits_{x\to 0} g(x) = 0$ and $\lim\limits_{x\to 0} G(x)$ exists, prove that f (x) is continuous at all $x \in R$.

Q.9 Find the number of ordered pair(s) (a, b) for which the function $f(x) = \text{sgn}((x^2 - ax + 1)(bx^2 - 2bx + 1))$ is discontinuous at exactly one point (where a, b are integers).

[Note: sgn(x) denotes signum function of x.]

Q.10 Let the equation $x^3 + 2x^2 + px + q = 0$ and $x^3 + x^2 + px + r = 0$ have two roots in common and the third root of each equation are represented by α and β respectively.

$$\text{If } f(x) = \begin{cases} e^{x\log_{\tan x}|\alpha+\beta|}, & -1 < x < 0 \\ a, & x = 0 \\ b\dfrac{\ell n(e^{x^2} + \alpha\beta\sqrt{x}),}{\tan\sqrt{x}} & 0 < x < 1 \end{cases}$$

is continuous at x = 0, then find the value of 2(a + b).

Q.11 A function f: $R \to R$ is defined as

$f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c\cdot e}$ where f is continuous on R.

Find the values of a, b and c.

Q.12 Let $f(x) = \dfrac{1-\cos 4x}{x^2}$, $x < 0$, a, x = 0

$\dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, x > 0$. If possible, find the value of a so that the function may be continuous at x = 0.

Q.13 Let $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n-1} + ax^3 + bx^2}{x^{2n} + 1}$.

If f(x) is continuous for all $x \in R$ find the bisector of angle between the lines $2x + y - 6 = 0$ and $2x - 4y + 7 = 0$ which contains the point (a, b).

Q.14 $f(x) = x^4, x^2 < 1$

$x, x^2 \geq 1$

Discuss the existence of limit at x = 1, -1.

Q.15 Let

$$f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x}, & \frac{-\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{|x|}}}{a\,e^{\frac{2}{x}} + b e^{\frac{3}{|x|}}}, & 0 < x < \frac{\pi}{2} \end{cases}$$

If f(x) is continuous at x = 0, then find the value of $(a^2 + b^2)$.

Differentiability

Q.1 Find the derivative of $\cos(x^2+1)$ w.r.t. x using the first principle.

Q.2 Find the derivative of $\tan\sqrt{x}$ w.r.t. x using first principle.

Q.3 Differentiae $e^{\sin x} + (\tan x)^x$ w.r.t.x

Q.4 Find the derivative of $\sin x^2$ w.r.t.x using first principle.

Q.5 Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t.x

Q.6 Differentiate w.r.t. x: $\tan^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

Q.7 If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, then find $\frac{dy}{dx}$

Q.8 If $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$

Examine the continuity of f (x) at x = 1.

Q.9 Find from first principle, the derivative of $\sqrt{\cos x}$ w.r.t. x.

Q.10 Differentiate $\sqrt{\tan x}$ w.r.t. x from first principle

Q.11 Check differentiability of $f(x)=|x|$ at x=0.

Q.12 If f(x) is differentiable at x = a and f′(a)=¼, then find $\lim_{h\to 0}\frac{f(a+2h^2)-f(a-2h^2)}{h^2}$.

Q.13 Let f be a real valued continuous function on R and satisfying $f(-x) - f(x) = 0 \ \forall\ x \in R$. If f(–5) = 5, f (–2) = 4, f(3) = –2 and f(0) = 0 then find the minimum number of zero's of the equation f(x) = 0.

Q.14 (a) If g: [a, b] onto [a,b] is continuous show that there is some c ∈ [a,b] such that g(c) = c.

(b) Let f be continuous on the interval [0, 1] to R such that f(0) = f(1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

Exercise 2

Limits

Single Correct Choice Type

Q.1 C is a point on the circumference of a circle & D is the foot of the perpendicular from C on a fixed diameter AB. Then the limit of $\frac{CD^2}{DB}$ as C tends to B along the circumference

(A) Does not exist

(B) Equal to one

(C) Is equal to the length AB

(D) None

Q.2 $\lim_{x\to 0}\left(1 + \log^2_{\cos\frac{x}{2}} \cos x\right)$

(A) Is equal to 4 (B) Is equal to 25

(C) Is equal to 289 (D) Is non existent

Q.3 Let α & β be the roots of the equation, $ax^2 + bx + c = 0$ where 1 < α < β, then

$\lim_{x\to m}\frac{|ax^2+bx+c|}{ax^2+bx+c} = 1$, when:

(A) a > 0 & m > 1 (B) a < 0 & m < 1

(C) a < 0 & α < m < b (D) $\frac{|a|}{a} = 1$ & m > α

Q.4 $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1}{(1+x)^{\frac{1}{3}}-1}$ is

(A) 1 (B) 0 (C) 3/2 (D) ∞

Q.5 $\lim_{x\to 0}(\log x - x)$

(A) Equals ∞ (B) Equals e

(C) Equals $-\infty$ (D) Does not exist

Q.6 $\lim_{x\to 0} x\tan\frac{1}{x}$

(A) Equals 0 (B) Equals 1 (C) Equals ∞

(D) Does not exist

Q.7 The limit of $\sqrt{x}\left(\sqrt{x+4}-\sqrt{x}\right)$ as $x\to\infty$

(A) Does not exist

(B) Exists and equals 0

(C) Exists and equals 1/2

(D) Exists and equals 2

Q.8 $\lim_{x\to 0^+}\frac{\ell n(\sin 2x)}{\ell n(\sin x)}$ is equal to -

(A) 0 (B) 1 (C) 2 (D) 4

Q.9 Centre of circle is the limit of point of intersection of t lines 3x + 5 y = 1 and (2 + c) x + 5c^2 y = 1 as c tends to 1. If it passes through (2, 0) its radius is -

(A) $\frac{\sqrt{1601}}{25}$ (B) $\frac{41}{25}$ (C) $\frac{1601}{\sqrt{25}}$ (D) $\sqrt{\frac{1601}{25}}$

Q.10 $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sum_{k=1}^{r}k}{\sum_{k=1}^{r}k^3}$ is equal to

(A) 1 (B) 3 (C) 4 (D) 2

Q.11 $\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{m}\sqrt[n]{(i-1)^n+i^n}}{m^2}\right)\right)$ is equal to

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5

Q.12 Let $f(x)=x\sin\left(\frac{1}{x}\right)$, then

(A) f(0) is not defined but $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$

(B) f(0) = 0 = $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$

(C) f(0) is defined but $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$ does not exist

(D) Both f(0) and $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$ are not defined

Continuity

Single Correct Choice Type

Q.1 Is f(x) differentiale at x = 0 if f(x) is defined as follows:

$f(x)=\frac{x}{1+e^{1/x}}$, $x \neq 0$

0, x= 0

(A) 0 (B) 1

(C) 2 (D) Not defferentiable

Q.2 Given f(x) = b ([x]^2 + [x]) + 1 for $x \geq -1$ = sin (π(x + a)) for x < –1

Where [x] denotes the integral part of x, then for what values of a, b the function is continuous at x = – 1 ?

(A) a = 2n + (3/2); $b \in R$; $n \in I$

(B) a = 4n + 2; $b \in R$; $n \in I$

(C) a = 4n + (3/2); $b \in R^+$; $n \in I$

(D) a = 4n + 1; $b \in R^+$ $n \in I$

Q.3 Given $f(x)=\frac{[|x|]e^{x^2}[x+|x|]}{e^{\frac{1}{x^2}}-1\,\text{sgn}(\sin x)}$ for $x \neq 0 = 0$

for x = 0 where {x} is the fractional part function; [x] is the step up function and sgn (x) is the signum function of x then, f(x) -

(A) Is continuous at x = 0

(B) Is discontinuous at x = 0

(C) Has a removable discontinuity at x = 0

(D) Has an irremovable discontinuity at x = 0

Q.4 f(x) has an isolated point discontinuity at x = a, then,

(A) $\frac{1}{f(x)}$ necessarily has an isolated point discontinuity at x = a

(B) $\frac{1}{f(x)}$ can be continuous at x = a

(C) $\frac{1}{f(x)}$ will have non-removable discontinuity at x = a

(D) $\frac{1}{f(x)}$ may have missing point discontinuity at x = a

Q.5 The number of points at which the function, f(x) = | x – 0.5 | + | x – 1| + tanx does not have a derivative in the interval (0, 2) is -

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 Which of the following functions defined below are NOT differentiable at the indicated point ?

(A) $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ -x^2 & \text{if } 0 \le x \le 1 \end{cases}$ at x = 0

(B) $g(x) = \begin{cases} x & \text{if } -1 \le x < 0 \\ \tan x & \text{if } 0 \le x \le 1 \end{cases}$ at x = 0

(C) $h(x) = \begin{cases} \sin 2x & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$ at x = 0

(D) $k(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 < x \le 2 \end{cases}$ at x = 1

Q.7 If $f(x) = \begin{cases} e^x & \text{for } x < 1 \\ a - bx & \text{for } x \ge 1 \end{cases}$ is

differentiable for x ∈ R, then:

(A) a = 1, b = e – 1 (B) a = 0, b = e

(C) a = 0, b = – e (D) a = e, b =1

Q.8 $f(x) = \begin{cases} x^2 + 2x + 3 & x \le 2 \\ \frac{a}{\pi}\sin(\pi x) + b & x > 2 \end{cases}$

If f(x) is derivable ∀ x ∈ R, then

(A) 2a + bπ = 7

(B) b + 2π = 3

(C) 2a + bπ = 13

(D) None of these

Q.9 The function f(x) is defined as follows

$f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ x^3 - x + 1 & \text{if } x > 1 \end{cases}$ then f(x) is

(A) Derivable and cont. at x = 0

(B) Derivable at x = 1 but not continuous at x = 1

(C) Neither derivable nor cont. at x = 1

(D) Not derivable at x = 0 but continuous at x = 1

Q.10 A function f defined as f(x) = x [x] for – 1 ≤ x ≤ 3 where [x] defines the greatest integer ≤ x is

(A) Continuous at all points in the domain of f but non-derivable at a finite number of points

(B) Discontinuous at all points & hence non-derivable at all points in the domain of f

(C) Discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) Discontinuous & also non-derivable at a finite number of points of f.

Q.11 Let $f(x) = \frac{|x|}{\sin x}$ for x ≠ 0 & f(0) = 1 then

(A) f(x) is continuous & differentiable at x = 0

(B) f(x) is continuous & not differentiable at x = 0

(C) f(x) is discontinuous & not differentiable at x = 0

(D) none of these

Q.12 Let f(x) = x^3 and g(x) = |x|, Then at x = 0, the composite functions

(A) gof is derivable but fog is not

(B) fog is derivable but gof is not

(C) gof and fog both are derivable

(D) neither gof nor fog is derivable

Q.13 [x] denotes the greatest integer less than or equal to x. If f(x) = [x] [sin π x] in (–1, 1) then f(x) is -

(A) Continuous at x = 0

(B) Continuous in (–1, 0) ∪ (0, 1)

(C) Differentiable in (–1, 1)

(D) None

Differentiability

Single Correct Choice Type

Q.1 Which of the following function is not differentiable at x = 0?

(A) x |x| (B) x^3

(C) e^{-x} (D) x + |x|

Q.2 Which of the following is differentiable function?

(A) x^2 sin 1/x (B) x |x|

(C) cosh x (D) All of the above

Q.3 The function f(x) = sin |x| is

(A) Continuous for all x

(B) Continuous only at certain points

(C) Differentiable at all points

(D) None of these

Q.4 If f(x) = |x – 3| , then f is

(A) Discontinuous at x = 2

(B) Not differentiable at x =2

(C) Differentiable at x = 3

(D) Continuous but not differentiable at x = 3

Q.5 If $f(x)=\frac{|x-1|}{x-1}$, $x \neq 1$, and f(1) = 1, then the correct statements

(A) Discontinuous at x = 1

(B) Continuous at x = 1

(C) Differentiable at x = 1

(D) Discontinuous for x > 1

Q.6 If $f(x)=\begin{cases} x+1, & x>1 \\ 0, & x=1 \\ 7-3x, & x<1 \end{cases}$, then f′(0) equals

(A) 1 (B) 2 (C) 0 (D) –3

Q.7 The function f(x) = |x| + |x –1| is not differentiable at

(A) x = 0, 1 (B) x = 0, –1

(C) x = –1, 1 (D) x = 1, 2

Q.8 If $f(x)=\begin{cases} e^{1/x}, & x \neq 0 \\ 0, & x=0 \end{cases}$, then which one is correct?

(A) f(x) is differentiable at x = 0

(B) f(x) is discontinuous at x = 0

(C) f(x) is continuous no where

(D) None of these

Q.9 Function [x] is not differentiable at

(A) Every rational number

(B) Every integer

(C) Origin

(D) Every where

Q.10 If $f(x)=\begin{cases} |x-3| & , \text{when} x \geq 1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & , \text{when} x \geq 1 \end{cases}$,

then correct statement is -

(A) f is discontinuous at x = 1

(B) f is discontinuous at x = 3

(C) f is differentiable at x = 1

(D) f is differentiable x = 3

Q.11 Function $f(x)=\frac{|x|}{x}$ is -

(A) Continuous everywhere

(B) Differentiable everywhere

(C) Differentiable everywhere except at x = 0

(D) None of these

Q.12 Let f(x) = |x – a| +|x–b|, then -

(A) f(x) is continuous for all $x \in R$

(B) f(x) is differential for $\forall$ $x \in R$

(C) f(x) is continuous except at x = a and b

(D) None of these

Q.13 Function f(x) = |x – 1| + |x–2| is differentiable in [0, 3], except at -

(A) x = 0 and x = 3 (B) x = 1

(C) x = 2 (D) x = 1 and x = 2

Q.14 If $f(x)=\begin{cases} 1, & \text{when } x>0 \\ 1+\sin x, & \text{when } 0 \leq x \leq \pi/2 \end{cases}$

then at x = 0, f ′ (x) equals -

(A) 1 (B) 0 (C) ¥ (D) Does not exist

Q.15 If $f(x) = \begin{cases} \dfrac{x}{1+6^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

then the function f(x) is differentiable for -

(A) $x \in R^+$ (B) $x \in R$

(C) $x \in R_0$ (D) None of these

Q.16 If $f(x) = \begin{cases} x^\alpha \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at x = 0, then -

(A) $\alpha > 0$ (B) $\alpha > 1$ (C) $\alpha \geq 1$ (D) $\alpha \geq 0$

Q.17 The function f(x) = x – | x | is not differentiable at -

(A) x = 1 (B) x = – 1 (C) x = 0 (D) No where

Previous Years' Questions

Q.1 For a real number y, let [y] denotes the greatest integer less than or equal to y. Then, the function

$f(x) = \dfrac{\tan[\pi(x-\pi)]}{1+[x]^2}$ is ***(1981)***

(A) Discontinuous at some x

(B) Continuous at all x, but the derivative f ' (x) does not exist for some x

(C) f'(x) exists for all x, but the derivative f '' (x) does not exist for some x

(D) f '(x) exists for all x

Q.2 There exists a function f(x) satisfying f(0) = 1. f' (0) = –1, f (x) > 0 for all x and ***(1982)***

(A) f "(x) < 0 for all x

(B) –1 < f " (x) < 0 for all x

(C) –2 ≤ f '' (x) ≤ – 1 for all x

(D) f '' (x) < – 2 for all x

Q.3 If $G(x) = -\sqrt{25-x^2}$, then $\lim\limits_{x\to 1} \dfrac{G(x)-G(1)}{x-1}$ has the value ***(1983)***

(A) $\dfrac{1}{\sqrt{24}}$ (B) $\dfrac{1}{5}$ (C) $-\sqrt{24}$ (D) None of these

Q.4 The function $f(x) = \dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0 so that it is continuous at x = 0, is ***(1983)***

(A) a – b (B) a + b

(C) log a + log b (D) None of these

Q.5 If f(a) = 2, f ' (a) = 1, g(a) = –1, g' (a) = 2, then the value of $\lim\limits_{x\to a} \dfrac{g(x)f(a)-g(a)f(x)}{x-a}$ is ***(1983)***

(A) –5 (B) $\dfrac{1}{5}$

(C) 5 (D) None of these

Q.6 $\lim\limits_{n\to\infty} \left(\dfrac{1}{1-n^2} + \dfrac{2}{1-n^2} + ... + \dfrac{n}{1-n^2} \right)$ is equal to ***(1984)***

(A) 0 (B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{2}$ (D) None of these

Q.7 If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

where [x] denotes the greatest integer less than or equal to x, ***(1985)***

(A) 1 (B) 0

(C) –1 (D) None of these

Q.8 If $f(x) = x\left(\sqrt{x} + \sqrt{(x+1)}\right)$, then ***(1985)***

(A) f(x) is continuous but not differentiable at x = 0

(B) f (x) is differentiable at x = 0

(C) f (x) is not differentiable at x = 0

(D) None of the above

Q.9 The set of all points, where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable, is ***(1987)***

(A) $(-\infty, \infty)$ (B) $[0, \infty)$

(C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

Q.10 If $y^2 = P(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ equals ***(1988)***

(A) $P'''(x) + P'(x)$ (B) $P''(x) \cdot P'''(x)$

(C) $P(x)\, P'''(x)$ (D) A constant

Q.11 If $f(x) = \frac{1}{2}x - 1$, then on the interval [0, p] ***(1989)***

(A) tan [f(x)] and 1/f(x) are both continuous

(B) tan [f(x)] and 1/f(x) are both discontinuous

(C) tan [f(x)] and $f^{-1}(x)$ are both continuous

(D) tan [f(x)] and 1/f is not continuous

Q.12 The value of $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x}$ ***(1989)***

(A) 1 (B) –1 (C) 0 (D) None of these

Q.13 The function $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)\pi$, [×] denotes the greatest integer function, is discontinuous at ***(1993)***

(A) All x (B) All integer points

(C) No x (D) x which is not an integer

Q.14 Let [×] denotes the greatest integer function and $f(x) = [\tan^2 x]$, then ***(1993)***

(A) $\lim_{x\to 0} f(x)$ does not exist

(B) f(x) is continuous at x = 0

(C) f(x) is not differentiable at x =0

(D) f ′ (0) = 1

Q.15 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & o \\ p & p^2 & p^3 \end{vmatrix}$,

where p is constant. Then $\frac{d^3}{dx^3}$ f(x) at x = 0 is ***(1997)***

(A) p (B) $p + p^2$

(C) $p + p^3$ (D) Independent of p

Q.16 Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$.

Then which one of the following is true? ***(2008)***

(A) f is neither differentiable at x =0 nor at x = 1

(B) f is differentiable at $x = 0$ and at $x = 1$

(C) f is differentiable at $x = 0$ but not at $x = 1$

(D) f is differentiable at $x = 1$ but not at $x = 0$

Q.17 Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then y′(1) equals ***(2009)***

(A) -1 (B) 1 (C) log 2 (D) –log 2

Q.18 Let $f: R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$.

Statement-I: $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement-II: $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in R$

(A) Statement-I is true, statement-II is true; statement-II is not the correct explanation for statement-I

(B) Statement-I is true, statement-II is false

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true; statement-II is the correct explanation for statement-I

Q.19 Let f: $R \to R$ be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty}\frac{f(2x)}{f(x)} =$. ***(2010)***

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) 3 (D) 1

Q.20 Let $f:(-1,1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$ Let $g(x) = [f(2f(x)+2)]^2$

Then $g'(0) =$ ***(2010)***

(A)-4 (B) 0 (C) -2 (D) 4

Q.21 $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ ***(2011)***

(A) Equals $\sqrt{2}$ (B) Equals $-\sqrt{2}$

(C) Equals $\frac{1}{\sqrt{2}}$ (D) Does not exist

Q.22 The value of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q & x=0\\ \dfrac{\sqrt{x+x^2}}{x^{3/2}} & x>0\end{cases}$$

is continuous for all x in R, is ***(2011)***

(A) $p=\frac{5}{2},q=\frac{1}{2}$ (B) $p=-\frac{3}{2},q=\frac{1}{2}$

(C) $p=-\frac{1}{2},q=\frac{3}{2}$ (D) $p=-\frac{1}{2},q=-\frac{3}{2}$

Q.23 $\lim\limits_{x\to0}\dfrac{(1-\cos2x)(3+\cos x)}{x\tan4x}$ is equal to ***(2013)***

(A) $-\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

Q.24 If $y=\sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at $x=1$ is equal to: ***(2013)***

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\sqrt{2}$

Q.25 $\lim\limits_{x\to0}\dfrac{\sin(\pi\cos^2x)}{x^2}$ is equal to ***(2014)***

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

Q.26 If f and g are differentiable functions in [0,1] satisfying $f(0)=2=g(1),g(0)=0$ and $f(1)=6$, then for some $c\in[0,1]$ ***(2014)***

(A) $f'(0)=2=g(1),g(0)=0$ (B) $f'(c)=2g'(c)$

(C) $2f'(c)=g'(c)$ (D) $2f'(c)=3g'(c)$

Q.27 $\lim\limits_{x\to0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to ***(2015)***

(A) 4 (B) 3 (C) 2 (D) $\frac{1}{2}$

Q.28 If the function $g(x)=\begin{cases}k\sqrt{x+1}\ , & 0\le x\le3\\ k\sqrt{x+1}\ , & 3<x\le5\end{cases}$

is differentiable, then the value of k + m is; ***(2015)***

(A) 2 (B) $\frac{16}{5}$ (C) $\frac{10}{3}$ (D) 4

Q.29 The normal to the curve, $x^2+2xy-3y^2=0$ at (1, 1) ***(2015)***

(A) Does not meet the curve again

(B) Meets the curve again in the second quadrant

(C) Meets the curve again in the third quadrant.

(D) Meets the curve again in the fourth quadrant.

Q.30 Let $p=\lim\limits_{x\to0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to: ***(2016)***

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) 2

Q.31 $\lim\limits_{x\to\infty}\left(\dfrac{(n+1)(n+2)\ldots3n}{n^{2n}}\right)^{1/n}$ is equal to: ***(2016)***

(A) $\frac{27}{e^2}$ (B) $\frac{9}{e^2}$ (C) $3\log3-2$ (D) $\frac{18}{e^4}$

JEE Advanced/Boards

Exercise 1

Limits

Q.1 Find $\lim_{x\to 1}\dfrac{\left[\sum_{K=1}^{100} x^k\right]-100}{x-1}$

Q.2 Find the sum of an infinite geometric series whose first term is the limit of the function $f(x)=\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$ as $x \to \pi/4$ and whose common ratio is the limit of the function $g(x)=\dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$ as $x \neq 1, \in R$.

Q.3 $\lim_{x\to 1}\left(\dfrac{p}{1-x^p}-\dfrac{q}{1-x^q}\right) p,q \in N$

Q.4 $\lim_{x\to\infty}(x-\ln\cosh x)$ where cosh $t=\dfrac{e^{+t}+e^{-t}}{2}$

Q.5 $\lim_{x\to 0}\dfrac{\sin^4(3\sqrt{x})}{1-\sqrt{\cos x}}$

Q.6 (a) $\lim_{x\to \frac{1}{\sqrt{2}}}\dfrac{\cos^{-1}(2x\sqrt{1-x^2})}{x-\frac{1}{\sqrt{2}}}$

(b) $\lim_{x\to\frac{\pi}{4}}\dfrac{1-\sqrt{\sin 2x}}{\pi-4x}$

(c) $\lim_{x\to -7}\dfrac{[x]^2+15[x]+56}{\sin(x+7)\sin(x+8)}$

Where [ ] denotes the greatest integer function

Q.7 Find $\lim_{x\to\frac{3\pi}{4}}\dfrac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$

Q.8 Find $\lim_{x\to 0}\dfrac{8}{x^8}\left[1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right]$

Q.9 Find

$$\lim_{h\to 0}\frac{\sin\left((\pi/3)+4h\right)-4\sin\left((\pi/3)+3h\right)+6\sin\left((\pi/3)+2h\right)-4\sin\left((\pi/3)+h\right)+\sin(\pi/3)}{h^4}$$

Q.10 Find $\lim_{x\to\infty}x^2\left(\dfrac{\sqrt{x+2}}{x}-\dfrac{\sqrt[3]{x+3}}{x}\right)$

Q.11 Find $\lim_{x\to-\infty}\dfrac{(3x^4+2x^2)\sin(1/x)+|x|^3+5}{|x|^3+|x|^2+|x|+1}$

Q.12 If $\ell=\lim_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\dfrac{\pi}{r+1}-r\sin\dfrac{\pi}{r}\right)$

then find $\{\ell\}$. (where { } denotes the fractional part function)

Q.13 Find a & b if:

(i) $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$

(ii) $\lim_{x\to-\infty}\left(\sqrt{x^2-x+1}-ax-b\right)=0$

Q.14 $\lim_{x\to 0}[\ln(1+\sin^2 x)\cdot\cot(\ln^2(1+x))]$

Q.15 $\lim_{x\to 1}\dfrac{(\ln(1+x)-\ln 2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}]\cdot\sin(x-1)}$

Q.16 If $\lim_{x\to 0}\dfrac{e^{x^2}-3^{3x}}{\sin\left(\dfrac{x^2}{2}\right)-\sin x}=\ln K$ (where K∈N) find K.

Q.17 If $\lim_{x\to 3}\left(\dfrac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right)^{\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6}}$

can be expressed in the form $\dfrac{a\sqrt{b}}{c}$ where a, b, c ∈ N, then find the least value of $(a^2+b^2+c^2)$.

Q.18 If the $\lim_{x\to 0}\dfrac{1}{x^3}\left(\dfrac{1}{\sqrt{1+x}}-\dfrac{1+ax}{1+bx}\right)$

exists and has the value equal to ℓ, then find the value of $\dfrac{1}{a}-\dfrac{2}{\ell}+\dfrac{3}{b}$.

Q.19 Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

(i) $a_n + b_n + c_n = 2n + 1$

(ii) $a_nb_n + b_nc_n + c_na_n = 2n - 1$

(iii) $a_nb_nc_n = -1$

(iv) $a_n < b_n < c_n$

Then find the value of $\lim\limits_{n\to\infty}(na_n)$

Q.20 Let $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = x^2 + x - 2$

If $\lim\limits_{x\to1}\dfrac{f(x)}{g(x)} = 1$ and $\lim\limits_{x\to-2}\dfrac{f(x)}{g(x)} = 4$, then find the value of $\dfrac{c^2+d^2}{a^2+b^2}$

Differentiability

Q.1 Discuss the continuity & differentiability of the functions, $f(x) = \sin x + \sin|x|$, $x \in R$.

Q.2 If the function f(x) defined as

$$f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$$

is continuous but not derivable at x = 0, then find the range of n.

Q.3 Let $g(y) = \lim\limits_{x\to y}\dfrac{\tan x - \tan y}{1-\dfrac{x}{y}+\left(1-\dfrac{x}{y}\right)\cdot\tan x\tan y}$

and $f(x) = x^2$. If $h(x) = \text{Min.}(f(x), g(x))$, find the number of points where h(x) is non-derivable-

Q.4 Let f(0) = 0 and f ′ (0) = 1. For a positive integer k, show that

$$\lim_{x\to0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+....f\left(\frac{x}{k}\right)\right) =$$

$$1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{k}$$

Q.5 Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$;

$x \ne 0$, f(0) = 0, test the continuity & differentiability at x = 0 .

Q.6 If $f(x) = |x-1|\cdot([x]-[-x])$, then find $f'(1^+)$ & $f'(1^-)$. where [x] denotes greatest integer function.

Q.7 If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ \dfrac{-1}{|x|} & \text{if } |x| \ge 1 \end{cases}$

is derivable at x = 1. Find the values of a & b.

Q.8 Let $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$

and $g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 & \text{if } x \ge 0 \end{cases}$

If m, n and p are respectively the number of points where the functions f, g and gof are not derivable, find the value of (m + n + p)

Q.9 Let f(x) be defined in the interval [–2, 2] such that

$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$ & $g(x) = f(|x|) + |f(x)|$.

Test the differentiability of g(x) in (–2, 2).

Q.10 Examine for continuity & differentiability the points x = 1 & x = 2, the function f defined by

$f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)[x], & 2 \le x \le 3 \end{cases}$ where [x] = greatest integer less than or equal to x.

Q.11 Discuss the continuity & the derivability in [0, 2] of

$$f(x) = \begin{cases} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{cases}$$

where [ ] denotes greatest integer function

Q.12 Let f(x) = [3 + 4 sin x] (where [ ] denotes the greatest integer function). If sum of all the values of 'x' in [π, 2p] where f(x) fails to be differentiable, is $\dfrac{k\pi}{2}$, then find the value of k.

Q.13 The function

$$f(x) = \begin{cases} ax(x-1)+b & \text{when } x < 1 \\ x-1 & \text{when } 1 \le x \le 3 \\ px^2+qx+2 & \text{when } x > 3 \end{cases}$$

Find the values of the constants a, b, p, q so that -

(i) f(x) is continuous for all x

(ii) f′ (1) does not exist

(iii) f′ (x) is continuous at x = 3

Q.14 Let a_1 and a_2 be two value of a for which

$$f(x) = \begin{cases} x.\dfrac{\ln(1+x)+\ln(1-x)}{\sec x - \cos x}, & x \in (-1,0) \\ (a^2-3a+1)x + x^2, & x \in [0,\infty) \end{cases}$$

is differentiable at x = 0, then find the value of $(a_1^2 + a_2^2)$

Exercise 2

Limits

Single Correct Choice Type

Q.1 $\lim\limits_{n\to\infty} \dfrac{1^2 n + 2^2(n-1) + 3^2(n-2) + \ldots\ldots\ldots + n^2 \cdot 1}{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3}$ is equal to -

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{6}$

Q.2 If $\ell = \lim\limits_{x\to\infty}\left[\sqrt{x^2+2x} - x\right]$ & $m = \lim\limits_{x\to\infty}\left\{\sqrt{x^2-2x} + x\right\}$,

where [ℓ] & { ℓ } represent integral and fractional part respectively then ℓ + m is equal to -

(A) 0 (B) 1 (C) 2 (D) 3

Q.3 The limit of $x^3\sqrt{x^2+x^4+1} - x\sqrt{2}$ as $x\to\infty$

(A) Exists and equals $\frac{1}{2\sqrt{2}}$

(B) Exists and equals $\frac{1}{4\sqrt{2}}$

(C) Does not exist

(D) Exists and equals $\frac{3}{4\sqrt{2}}$

Q.4 Consider the function

$f(x) = \tan^{-1}\left(2\tan\frac{x}{2}\right)$ where $-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

($\lim\limits_{x\to\pi-0}$ means limit from the left at π and $\lim\limits_{x\to\pi+0}$ means limit from the right). Then

(A) $\lim\limits_{x\to\pi-0} f(x) = \frac{\pi}{2}$, $\lim\limits_{x\to\pi+0} f(x) = \frac{\pi}{2}$

(B) $\lim\limits_{x\to\pi-0} f(x) = -\frac{\pi}{2}$, $\lim\limits_{x\to\pi+0} f(x) = \frac{\pi}{2}$

(C) $\lim\limits_{x\to\pi} f(x) = \frac{\pi}{2}$

(D) $\lim\limits_{x\to\pi} f(x) = -\frac{\pi}{2}$

Q.5 If x_1, x_2, x_3............. x_n is a sequence of positive numbers such that

$x_n = x_{n-1} + x_{n-2}$. If $\lim\limits_{n\to\infty}\dfrac{x_n}{x_{n-1}}$ exists then it is equal to -

(A) $(\sqrt{5}+1)$ (B) $\sqrt{5}-1$ (C) $\frac{\sqrt{5}-1}{2}$ (D) $\frac{\sqrt{5}+1}{2}$

Q.6 If α and β be the roots of $ax^2 + bx + c = 0$, then

$\lim\limits_{x\to\alpha}(1 + ax^2 + bx + c)^{\frac{1}{(x-\alpha)}}$ is

(A) ln |a ($\alpha - \beta$)| (B) $e^{a(\alpha-\beta)}$

(C) $e^{a(\beta-\alpha)}$ (D) None of these

Multiple Correct Choice Type

Q.7 Identify the correct statement

(A) $\lim\limits_{x\to1}\dfrac{1-|\cos(x-1)|}{(x-1)^2} = \dfrac{1}{2}$ (B) $\lim\limits_{x\to0^+}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}} = 1$

(C) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x} = 1$ (D) $\lim\limits_{x\to\infty}\dfrac{\tan x}{x} = 0$

Q.8 If $\ell = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x}}$, $m = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x^2}}$

and $n = \lim\limits_{x\to0^+}(\cos x)^{\frac{1}{x^3}}$, then -

(A) ℓ =1 (B) $m^{-2} = e$

(C) ℓ and n are rational (D) $m^2 = \ln$

Q.9 Which of the following limit(s) is are finite

(A) $\lim\limits_{x\to0^+}(\cot x)^{\frac{\sin x - x}{x^4}}$ (B) $\lim\limits_{x\to0^+}(\cot x)^{\frac{x}{\sin x - x}}$

(C) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$ (D) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x} - x\right)$

Q.10 Let $f(x) = \dfrac{x2^x - x}{1-\cos x}$ & $g(x) = 2^x\sin\left(\dfrac{\ln 2}{2^x}\right)$ then

(A) $\lim\limits_{x\to0} f(x) = \ln 2$ (B) $\lim\limits_{x\to\infty} g(x) = \ln 4$

(C) $\lim\limits_{x\to0} f(x) = \ln 4$ (D) $\lim\limits_{x\to\infty} g(x) = \ln 2$

Q.11 Which of the following limits is unity ?

(A) $\lim\limits_{t\to 0}\dfrac{\sin(\tan t)}{\sin t}$ (B) $\lim\limits_{x\to \pi/2}\dfrac{\sin(\cos x)}{\cos x}$

(C) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (D) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2}}{x}$

Q.12 Which of the following limits vanishes?

(A) $\lim\limits_{x\to\infty} x^{\frac{1}{4}}\sin\dfrac{1}{\sqrt{x}}$ (B) $\lim\limits_{x\to\pi/2}(1-\sin x)\cdot\tan x$

(C) $\lim\limits_{x\to\infty}\dfrac{2x^2+3}{x^2+x-5}\cdot\text{sgn}(x)$ (D) $\lim\limits_{x\to 3^+}\dfrac{[x]^2-9}{x^2-9}$

where [ ] denotes greatest integer function.

Q.13 Which of the following limits vanishes?

(A) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\tan x}\right)$ (B) $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+1}{2x^2-1}\right)^{\frac{x^3}{1-x}}$

(C) $\lim\limits_{x\to\frac{\pi+}{4}}\left[\tan\left(x+\dfrac{\pi}{8}\right)\right]^{\tan 2x}$ (D) $\lim\limits_{x\to 1}\dfrac{x^4-2x^2+1}{x^3-1}$

Continuity

Single Correct Choice Type

Q.1 If $f(x)=\begin{cases} x/2-1 & , 0\le x<1 \\ 1/2 & , 1\le x<2\end{cases}$,

g(x) = (2x + 1) (x – k) + 3, 0 ≤ x < ∞, then g [f(x)], will be continuous at x = 1 if k is equal to -

(A) 1/2 (B) 1/6 (C) 11/6 (D) 13/6

Q.2 If function $f(x)=\dfrac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$, is continuous function, then f(0) is equal to-

(A) 2 (B) 1/4 (C) 1/6 (D) 1/3

Q.3 If function $f(x)=\dfrac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/3}}(x\ne 0)$

is continuous at x = 0, then f(0) is equal to

(A) 2 (B) 4 (C) 6 (D) 2/3

Q.4 If

$$f(x)=\begin{cases}\dfrac{\log(1+2ax)-\log(1-bx)}{x} & ,x\ne 0\\ k & ,x=0\end{cases}$$

is continuous at x = 0, then k is equal to -

(A) 2a + b (B) 2a – b (C) b – 2a (D) a + b

Q.5 If $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & , x\ne 0\\ 1 & , x=0\end{cases}$

then at x = 0, f(x) is

(A) Continuous (B) Left continuous

(C) Right continuous (D) None of these

Q.6 If the function

$$f(x)=\begin{cases}1+\sin\dfrac{\pi}{2}x & , -\infty<x\le 1\\ ax+b & , 1<x<3\\ 6\tan\dfrac{\pi x}{12} & , 3\le x<6\end{cases}$$

is continuous in the interval (–∞, 6), then the value of a and b are respectively

(A) 0, 2 (B) 1, 1 (C) 2, 0 (D) 2, 1

Q.7 If

$$f(x)=\begin{cases}\dfrac{\sin[x]}{[x]+1} & , x>0\\ \dfrac{\cos\pi[x]/2}{[x]} & , x<0\\ K & , x=0\end{cases}$$

is a continuous function at x = 0, then the value of K ([·] denotes greatest integer function) is -

(A) 0 (B) 1 (C) –1 (D) None of these

Q.8 If $f(x)=\begin{cases}x, & \text{if } x \text{ rational}\\ -x, & \text{if } x \text{ irrational}\end{cases}$, then $\lim\limits_{x\to 0} f(x)$, is -

(A) 0 (B) 1

(C) –1 (D) Indeterminate

Q.9 Function f(x) = $[x]^2 - [x^2]$, where [x] greatest integer ≤ x is discontinuous at -

(A) All integers

(B) All integers except 0 & 1

(C) At x = 1 only

(D) All integers except 1

Q.10 Function $f(x)=\begin{cases} x+2 & , \ 1\le x<2 \\ 4 & , \ x=2 \\ 3x-2 & , \ x>2 \end{cases}$ is continuous -

(A) Only at x = 2 (B) For $x \le 2$

(C) For $x \ge 2$ (D) None of these

Q.11 If function $f(x)=\begin{cases} 5x-4 & , 0<x\le 1 \\ 4x^2+3bx & , 1<x<2 \end{cases}$

is continuous at every point of its domain, then b is equal to -

(A) 0 (B) 1 (C) –1 (D) 13/13

Q.12 If function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$

is continuous at every point of its domain, then f(0) is equal to -

(A) 1/3 (B) –1/3 (C) 2/3 (D) 2

Q.13 If

$$f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & , \ x<0 \\ a & , \ x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & , \ x>0 \end{cases}$$

then correct statement is -

(A) f(x) is discontinuous at x = 0 for any value of a

(B) If f(x) is continuous at x = 0 when a = 8

(C) f(x) is continuous at x = 0 when a = 0

(D) None of these

Q.14 Function f(x) = 1 + |sin x| is -

(A) Continuous at all points

(B) Discontinuous at all points

(C) Continuous only at x = 0

(D) None of these

Q.15 The sum of two discontinuous functions

(A) Is always discontinuous

(B) May be continuous

(C) Is always continuous

(D) None of these

Q.16 If $f(x)=\begin{cases} x^{\alpha}\cos 1/x & , \ x\ne 0 \\ 0 & , \ x=0 \end{cases}$

is continuous at x = 0, then

(A) $\alpha < 0$ (B) $\alpha > 0$

(C) $\alpha = 0$ (D) $\alpha \ge 0$

Q.17 If $f(x)=\begin{cases} \dfrac{1-\sin x}{\pi-2x} & , x\ne \pi/2 \\ k & , x=\pi/2 \end{cases}$

is continuous at $x=\dfrac{\pi}{2}$, then k is equal to -

(A) 0 (B) 1 (C) –1 (D) 1/2

Q.18 If $f(x)=\begin{cases} x\sin 1/x & , x\ne 0 \\ k & , x=0 \end{cases}$

is continuous at x = 0, then the value of k will be -

(A) 1 (B) –1

(C) 0 (D) None of these

Q.19 Function $f(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ is discontinuous at -

(A) Every x (B) No x

(C) Every integral point (D) Every non-integral point

Q.20 If

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x & , \ 0<x<\pi/4 \\ 2x\cot x+b & , \ \pi/4\le x\le \pi/2 \\ a\cos 2x-b\sin x & , \ \pi/2<x\le \pi \end{cases}$$

is continuous at x = π/4, then a –b is equal to -

(A) π/2 (B) 0 (C) 1/4 (D) π/4

Q.21 At origin, the function $f(x)=|x|+\dfrac{|x|}{x}$ is-

(A) Continuous

(B) Discontinuous because |x| is discontinuous there

(C) Discontinuous because $\dfrac{|x|}{x}$ is discontinuous there

(D) Discontinuous because |x| and $\dfrac{|x|}{x}$ both are discontinuous there

Q.22 If $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, then

(A) f(x) is discontinuous at x = 0

(B) f(x) is continuous at x = 0

(C) f(x) is continuous at x = 0 if $f(0) = a^2$

(D) Alternative (A) and (C)

Q.23 If $f(x) = \dfrac{1 - \sin x + \cos x}{1 + \sin x + \cos x}$,

is continuous at x = π, then f(π) is equal to -

(A) –1 (B) 2 (C) 1/4 (D) p

Q.24 If $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

is continuous function, then k is equal to -

(A) 1 (B) 3 (C) 1/3 (D) 0

Q.25 If $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

is continuous at x = 0, then k is equal to -

(A) ¼ (B) –1/2 (C) 0 (D) 1/2

Q.26 Consider $f(x) = \begin{bmatrix} x[x]^2 \log_{(1+x)} 2 & \text{for } -1 < x < 0 \\ \dfrac{\ln\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{bmatrix}$

where [*] & {*} are the greatest integer function & fractional part function respectively, then -

(A) f(0) = ln2 ⇒ f is continuous at x = 0

(B) f(0) = 2 ⇒ f is continuous at x = 0

(C) $f(x) = e^2$ ⇒ f is continuous at x = 0

(D) f has an irremovable discontinuity at x = 0

Q.27 Let f(x) = [2+3 sin x] (where [] denotes the greatest integer function) x ∈ (0, π). Then number of points at which f(x) is discontinuous is -

(A) 0 (B) 4 (C) 5 (D) Infinite

Q.28 y = f(x) is a continuous function such that its graph passes through (a, 0). Then

$\lim\limits_{x \to a} \dfrac{\log_e(1 + 3f(x))}{2f(x)}$ is -

(A) 1 (B) 0 (C) 3/2 (D) 2/3

Q.29 'f' is a continuous function on the real line. Given that $x^2 + (f(x)–2)\,x - \sqrt{3}$.

$f(x) + 2\sqrt{3} - 3 = 0$ then the value of f $\sqrt{3}$ j.

(A) Cannot be determined (B) Is $2(1- \sqrt{3})$

(C) Is zero (D) Is $\dfrac{2(\sqrt{3}-2)}{3}$

Q.30 Let $f(x) = \begin{bmatrix} x^2 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{bmatrix}$ then

(A) f(x) is discontinuous for all x

(B) Discontinuous for all x except at x = 0

(C) Discontinuous for all except at x = 1 or –1

(D) None of these

Q.31 Consider $f(x) = \lim\limits_{n \to \infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for x ? 0, x ≠ 1 f(1) = 0 then -

(A) f is continuous at x = 1

(B) f has a finite discontinuity at x = 1

(C) f has an infinite or oscillatory discontinuity at x = 1

(D) f has a removable type of discontinuity at x = 1

Q.32 If $f(x) = a\,|\sin x| + be^{|x|} + c\,|x|^3$ and if f(x) is differentiable at x = 0 then -

(A) b = 0, c = 0, a is any real

(B) a = 0, b = 0, c is any real

(C) c = 0, a = 0 b is any real

(D) None of these

Q.33 The number of points in (1, 3) where $f(x) = a^{[x^2]}$, a > 1 and [x] denote the greatest integer function is not differentiable is -

(A) 1 (B) 3 (C) 5 (D) 7

Q.34 A function f defined as f(x) = x [x] for $-1 \le x \le 3$ where [x] defines the greatest integer $\le$ x is:

(A) continuous at all points in the domain of f but non-derivable at a finite number of points

(B) discontinuous at all points & hence non-derivable at all points in the domain of f.

(C) discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) discontinuous & also non-derivable at a finite number of points of f

Q.35 If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$

where {x} denotes the fractional part function, then:

(A) 'f ' is continuous & differentiable at x = 0

(B) 'f ' is continuous but not differentiable at x = 0

(C) 'f ' is continuous & differentiable at x =2

(D) None of these

Multiple Correct Choice Type

Q.36 If $f(x) = \begin{cases} \dfrac{x\ln(\cos x)}{\ln(1+x^2)} & x \ne 0 \\ 0 & x = 0 \end{cases}$ then

(A) f is continuous at x = 0

(B) f is continuous x = 0 but not differentiable at x = 0

(C) f is differentiable at x = 0

(D) f is not continuous at x = 0

Differentiability

Single Correct Choice Type

Q.1 If $f(x) = \begin{cases} e^x, & x \le 0 \\ |1-x|, & x > 0 \end{cases}$ then f(x) is -

(A) Continuous at x = 0

(B) Differentiable at x = 0

(C) Differentiable at x = 1

(D) Differentiable both at x = 0 and 1

Q.2 Which of the following function is not differentiable at x = 1

(A) $\sin^{-1} x$ (B) tan x (C) a^x (D) cos h x

Previous Years' Questions

Q.1 If x + | y | = 2y, then y as a function of x is ***(1984)***

(A) Defined for all real x

(B) Continuous at x = 0

(C) Differentiable for all x

(D) Such that $\dfrac{dy}{dx} = \dfrac{1}{3}$ for x < 0

Q.2 The function f(x) = 1+ |sin x| is ***(1986)***

(A) Continuous no where

(B) Continuous every where

(C) Differentiable at x = 0

(D) Not differentiable at infinite number of points

Q.3 Let [x] denote the greatest integer less than or equal to x. If f(x) = [x sin px], then f(x) is ***(1986)***

(A) Continuous at x = 0 (B) Continuous in (–1, 0)

(C) Differentiable at x = 1 (D) Differentiable in (–1, 1)

Q.4 The function

$f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is ***(1988)***

(A) Continuous at x = 1 (B) Differentiable at x = 1

(C) Discontinuous at x = 1 (D) Differentiable at x = 3

Q.5 The following functions are continuous on $(0, \pi)$ ***(1991, 2M)***

(A) tan x

(B) $\int_0^x t\sin\frac{1}{t}\,dt$

(C) $\begin{cases} 1, & 0 \le x \le \dfrac{3\pi}{4} \\ 2\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$

(D) $\begin{cases} x\sin x, & 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

Match the Columns

Match the conditions/expression is column I with statement in column II.

Q.6 ***(1992)***

	Column-I		Column-II
(A)	sin (p[x])	(p)	differentiable every where
(B)	sin (π(x-[x]))	(q)	nowhere differentiable
		(r)	not differentiable at 1 and –1

Q.7 In the following [x], denotes the greatest integer less than or equal to x. ***(2007)***

	Column-I		Column-II
(A)	x|x|	(p)	continuous in (–1, 1)
(B)	$\sqrt{\|x\|}$	(q)	differentiable in (–1, 1)
(C)	x + [x]	(r)	strictly increasing(–1, 1)
(D)	|x –1| + |x +1|	(s)	not differentiable at least at one point in (–1, 1)

Analytical and Descriptive Questions

Q.8 Evaluate the following limit

$\lim_{x\to1}\left(\frac{x-1}{2x^2-7x+5}\right)$ ***(1978)***

Q.9 Evaluate $\lim_{x\to1}(1-x)\tan\left(\frac{\pi x}{2}\right)$ ***(1978)***

Q.10 Differentiate from first principle $\sin(x^2+1)$ ***(1978, 3M)***

Q.11 If f(x) = x $\tan^{-1}x$, find f '(1) from first first principle ***(1978)***

Q.12 Find f ' (1) if

$$f(x)=\begin{cases}\frac{x-1}{2x^2-7x+5}, & \text{when } x\neq1\\ -\frac{1}{3}, & \text{when } x=1\end{cases}$$ ***(1979, 3M)***

Q.13 Evaluate $\lim_{x\to0}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$ ***(1979)***

Q.14 Evaluate $\lim_{h\to0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$ ***(1980)***

Q.15 Given $y=\frac{5x}{3\sqrt{(1-x)^2}}+\cos^2(2x+1)$,

find $\frac{dy}{dx}$ ***(1980)***

Q.16 Let f: $R\to R$ be a continuous function defined by

$f(x)=\frac{1}{e^x+2e^{-x}}$.

Statement-I: $f(c)=\frac{1}{3}$, for some $c\in R$. Statement-I:

$o\prec f(x)\leq\frac{1}{2\sqrt{2}}$, for all $x\in R$ Statement-I:

$o\prec f(x)\leq\frac{1}{2\sqrt{2}}$, for all $x\in R$.

(A) Statement-I is true, statement-II is true; statement-II is not the correct explanation for statement-I

(B) Statement-I is true, statement-II is false

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true; statement-II is the correct explanation for statement-I ***(2010)***

Q.17 Let $g(x)=\log(f(x))$ where $f(x)$ is a twice differentiable positive function on $(0,\infty)$ such that $f(x+1)=xf(x)$. Then, for $N=1,2,3,$.

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$ ***(2008)***

Paragraph 1:

Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ defined by

$$f(x)=\frac{x^2-ax+1}{x^2+ax+1},0<a<2.$$

Q.18 Which of the following is true?

(A) $(2+a)^2 f'(1)+(2-a)^2 f'(-1)=0$

(B) $(2+a)^2 f'(1)-(2-a)^2 f'(-1)=0$

(C) $f'(1)f'(-1)=(2-a)^2$

(D) $f'(1)f'(-1)=-(2-a)^2$ ***(2008)***

Q.19 Let

$$g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)};0<x<2,m$$

and n are integers, $m\neq 0,n>0$, and let p be the left hand derivative of $|x-1|$ at x = 1.

If $\lim_{x\to 1^+} g(x)=p$, then ***(2008)***

(A) $n=1,m=1$ (B) $n=1,m=-1$

(C) $n=1,m=2$ (D) $n>1,m=n$

Q.20 Let f and g be real valued functions defined on interval $(-1,1)$ such that g″(x) is continuous $g(0)\neq 0,g'(0)=0,g''(0)\neq 0$ and $f(x)=g(x)\sin x$

And

Statement-II: $f(0)=g(0)$

(A) Statement-I is True, statement-II is True; statement-II is a correct explanation for statement-I

(B) Statement-I is True, statement-II is True; statement-II is NOT a correct explanation for statement-I

(C) Statement-I is True, statement-II is False

(D) Statement-I is False, statement-II is True ***(2008)***

Paragraph 1: Consider the functions defined implicitly by the equation $y^3-3y+x=0$ on various intervals in the real line. If $x\in(-\infty,-2)\cup(2,\infty)$, the equation implicitly defines a unique real valued differentiable function y = f(x). if $x\in(-2,2)$, the equation implicitly defines a unique real valued differentiable function y = g(x) satisfying g(0) = 0.

Q.21 If $f(-10\sqrt{2})=2\sqrt{2}$, then $f(-10\sqrt{2})=$ ***(2008)***

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

Q.22 For function $f(x)=x\cos\frac{1}{x},x\geq 1$

(A) For atleast one x in interval $[1,\infty),f(x+2)-f(x)>2$

(B) $\lim_{x\to\infty} f'(x)=1$

(C) For all x in the interval $[1,\infty),f(x+2)-f(x)>2$

(D) $f(x)$ is strictly decreasing in the interval $[1,\infty)$

(2009)

Q.23 Let p(x) be a polynomial of degree 4 having extremum at x = 1,2 and

$\lim_{x\to\infty} f\left(1+\frac{p(x)}{x^2}\right)=2$. Then the value of p(2) is ***(2009)***

Q.24 Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4},a>0$. If finite, then

(2009)

(A) (1) $a=2$ (B) $a=1$ (C) $L=\frac{1}{64}$ (D) $L=\frac{1}{32}$

Q.25 Let f be a real-valued function defined on the interval $(0,\infty)$ by $f(x)=\ln x+\int_0^x\sqrt{1+\sin t}\,dt$.

Then which of the following statement (s) is (are) true?

(A) f″(x) exists for all $x\in(0,\infty)$

(B) f′(x) exists for all $f'(2^-)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

and f′ is continuous $(0,\infty)$, but not differentiable on $(0,\infty)$

(C) there exists $\alpha>1$ such that $|f'(x)|<|f(x)|$ for all $x\in(\alpha,\infty)$

(D) there exists $\beta>1$ such that $|f(x)|+|f'(x)|\leq\beta$ for all $x\in(0,\infty)$ ***(2010)***

Q.26 If $\lim_{x\to 0}\left[1+x\ln(1+b^2)\right]^{-1/x}=2b\sin^2\theta,b>0$

and $\theta\in(-\pi,\pi]$, then the value of θ is

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

(2011)

Q.27 If $f(x)=\begin{cases}-x-\dfrac{\pi}{2}, & x\le -\dfrac{\pi}{2}\\ -\cos x, & 0<x\le 1\\ \ln x, & x>1\end{cases}$ then, ***(2011)***

(A) f(x) is continuous at x = – π/2

(B) f(x) is not differentiable at x = 0

(C) f(x) is differentiable at x = 1

(D) f(x) is differentiable at x = -3/2

Q.28 Let y′(x) + y(x)g′(x) = g(x)g′(x), y(0) = 0, x ∈ R, where f′(x) denotes df(x) dx and g(x) is a given noncontact differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is ***(2011 (II))***

Q.29 Let f: R → R be a function such that f(x + y) = f(x) + f(y), ∀ x, y ∈ R. If f(x) is differentiable at x = 0, then ***(2011 (I))***

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous ∀ x ∈ R

(C) f′(x) is constant ∀ x ∈ R

(D) f(x) is differentiable except at finitely many points

Q.30 If $\lim_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right)=4$, then ***(2012 (I))***

(A) a = 1, b = 4 (B) a = 1, b = –4

(C) a = 2, b = – 3 (D) a = 2, b = 3

Q.31 Let $f(x)=\begin{cases}x^2\left|\cos\dfrac{\pi}{x}\right| & x\ne 0\\ 0 & x=0\end{cases}$,x ∈ then f is

(A) Differentiable both at x = 0 and at x = 2

(B) Differentiable at x = 0 but not differentiable at x = 2

(C) Not differentiable at x = 0 but differentiable at x = 2

(D) Differentiable neither at x = 0 nor at x = 2 ***(2012)***

Q.32 For a ∈ R (the set of all real numbers), a ≠ –1,

$$\lim_{n\to\infty}\frac{\left(1^a+2^a+....+n^a\right)}{(n+1)^{a-1}\left[(na+1)+(na+2)+....+(na+n)\right]}=\frac{1}{60}$$

Then a = ***(2013)***

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

Q.33 Which of the following is true for 0 < x < 1 ? ***(2013)***

(A) $0<f(x)<\infty$ (B) $-\frac{1}{2}<f(x)<\frac{1}{2}$

(C) $-\frac{1}{4}<f(x)<1$ (D) $-\infty<f(x)<0$

Q.34 Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f(x)<2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval ***(2013)***

(A) $(2-1,2e)$ (B) $(e-1,2e-1)$

(C) $\left(\frac{e-1}{2},e-1\right)$ (D) $\left(0,\frac{e-1}{2}\right)$

Q.35 Let $f:[0,2]\to R$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with f(0) = 1.

Let $F(x)=\int_0^{x^2} f(\sqrt{t})dt$ for $x\in[0,2]$. If $F'(x)$ for all $x\in(0,2)$ with f(0) = 1. ***(2014)***

(A) e^2-1 (B) e^2-1 (C) $e-1$ (D) e^4

Q.36 The value of $g\left(\frac{1}{2}\right)$ is ***(2014)***

(A) $(-1,0)\cup(0,2)$ (B) 2π

$f'(x)-3g'(x)=0$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

Q.37 The value of $g'\left(\frac{1}{2}\right)$ is ***(2014)***

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

Q.38 Let $f(x)=\dfrac{19x^2}{2+\sin^4 \pi x}$ for all $x\in R$ with $f\left(\dfrac{1}{2}\right)=0$. If $m\le\int_{1/2}^{1} f(x)dx \le M$, then the possible values of *m* and *M* are ***(2015)***

(A) $m=13, M=24$

(B) $m=\dfrac{1}{4}, M=\dfrac{1}{2}$

(C) $m=-11, M=0$

(D) $m=1, M=12$

Q.39 Let $f,g:[-1,2]\to R$ be continuous functions which are twice differentiable on the interval $(-1,2)$ Let the values of f and g at the points $-1, 0$ and 2 be as given in the following table:

In each of the intervals $(-1,0)$ and $(1,0)$ the function $(f-3g)^n$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x)-3g'(x)=0$ has exactly three solutions in $(-1,0)\cup(0,2)$

(B) $(-1,0)\cup(0,2)$ $f'(x)-3g'(x)=0$ has exactly one solution in $(-1,0)$

(C) $f'(x)-3g'(x)=0$ has exactly one solution in $(0,2)$

(D) $f'(x)-3g'(x)=0$ has exactly two solutions in $(-1,0)$ and exactly two solutions in (0, 2) ***(2015)***

Q.40 The correct statement (s) is (are)

(A) $f'(1)<0$

(B) $f(2)<0$

(C) $f'(x)\neq 0$ for any $x\in(1,3)$

(D) $f'(x)=0$ for some $x\in(1,3)$ ***(2015)***

Q.41 Let $g:R\to R$ be a differential function with $g(0)=0, g'(0)$ and $g(1)\neq f(x)=\begin{cases}\dfrac{x}{|x|}g(x) & x\neq 0\\ 0 & x=0\end{cases}$

And $h(x)=e^{|x|}$ $x\in R$ (f o h)(x) denote $f(h(x))$ denote $f(h(x))$ Then which of the following is (are) true?

(A) f is differentiable at $x=0$

(B) *h* is differentiable at $x=0$

(C) f o h is differentiable at $x=0$

(D) h o f is differentiable at $x=0$ ***(2015)***

Q.42 Let $a,b\in R$ and $R\to R$ be defined by $f(x)=a\cos\left(\left|x^3-x\right|\right)+b|x\sin\left(\left|x^3+x\right|\right)|$ Then f is

(A) Differentiable at $x=0$ if $a=0$ and $b=1$

(B) Differentiable at $x=1$ if $a=1$ and $b=0$

(C) NOT differentiable at $x=0$ if $a=1$ and $b=0$

(D) NOT differentiable at $x=1$ if $a=1$ and $b=1$ ***(2016)***

Q.43 Let
$$f(x)=\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)....\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right),$$
for all x > 0. Then ***(2016)***

(A) $f\left(\dfrac{1}{2}\right)\ge f(1)$

(B) $f\left(\dfrac{1}{3}\right)\le f\left(\dfrac{2}{3}\right)$

(C) $f'(2)\le 0$

(D) $\dfrac{f'(3)}{f(3)}\ge\dfrac{f'(2)}{f(2)}$

Q.44 Let $f:\left[-\dfrac{1}{2},2\right]\to R$ and $g:\left[-\dfrac{1}{2},2\right]\to R$ be functions defined by $f(x)=\left[x^2-3\right]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where [y] denotes the greatest integer less than or equal to y For $y\in\mathbb{R}$. Then ***(2016)***

(A) f is discontinuous exactly at three points in $\left[-\dfrac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\dfrac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\dfrac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\dfrac{1}{2},2\right)$

Q.45 let $f:(0,\infty)\to R$ be a differentiable function such that $f(x)=2-\frac{f(x)}{x}=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. Then

(A) $\lim_{x\to 0^+} x^2 f'\left(\frac{1}{x}\right)=1$ (B) $\lim_{x\to 0_+} x^2 f'\left(\frac{1}{x}\right)=2$

(C) $\lim_{x\to 0_+} x^2 f'(x)=2$

(D) $\left|f(x)\leq 2\right|$ for all $x\in(0,2)$ ***(2016)***

Q.46 Let f: R → R, g: R → R and $h:R\to R$ be differentiable functions such that

$f(x)=x^3+3x+2, g(f(x)=x)$ for all $x\in R$ ***(2016)***

(A) $g'(2)=\frac{1}{15}$ (B) $(1)=666$

(C) $h(0)=16$ (D) $h(g(3))=36$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Limits

Q.7 Q.13 Q.20 Q.22 Q.25

Continuity

Q.4 Q.9 Q.11 Q.13

Differentiability

Q.7 Q.8 Q.13 Q.14

Exercise 2

Limits

Q.1 Q.7 Q.9 Q.10

Continuity

Q.3 Q.4 Q.6 Q.9 Q.11 Q.14

Differentiability

Q.3 Q.7 Q.11 Q.15 Q.16

Previous Years' Questions

Q.1 Q.6 Q.10 Q.13 Q.15

JEE Advanced/Boards

Exercise 1

Limits

Q.6 Q.11 Q.17 Q.19 Q.20

Continuity

Q.4 Q.7 Q.12 Q.17 Q.19 Q22

Differentiability

Q.3 Q.6 Q.9 Q.11 Q.14

Exercise 2

Limits

Q.2 Q.4 Q.6 Q.8 Q.10

Continuity

Q.3 Q.6 Q.8 Q.10

Differentiability

Q.1

Previous Years' Questions

Q.1 Q.3 Q.7 Q.9 Q.14 Q.15 Q.16

Answer Key

JEE Main/Boards

Exercise 1

Limits

Q.1 $-\frac{1}{8}$

Q.2 $\frac{3}{2}$

Q.3 2

Q.4 $\frac{1}{2\sqrt{x}}$ if $x > 0$; ∞ if $x = 0$

Q.5 ∞

Q.6 (a) $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$ (b) $f(x) = |x|$

Q.7 5050

Q.8 $\frac{3}{2}$

Q.9 2

Q.10 $\sqrt{2}$

Q.11 –3

Q.12 $\frac{1}{16\sqrt{2}}$

Q.13 2

Q.14 a = 2; limit = 1

Q.15 0.5

Q.16 Does not exist

Q.17 $\frac{\pi}{4}$

Q.18 $\cos^2 \alpha/n \cos \alpha + \sin 2\alpha/n \sin a$

Q.19 $\sqrt{8}\, 2[\ell n3]^2$

Q.20 does not exist

Q.21 (2n–1)

Q.22 $\frac{1}{128r}$

Q.23 1

Q.24 ln a

Q.25 c = ln2

Continuity

Q.1 a = 0, b = 1

Q.2 a = 0, b = –1

Q.3 (a) –2, 2, 3 (b) K = 5 (c) even

Q.4 $y_n(x)$ is continuous at x = 0 for all n and y(x) is discontinuous at x = 0

Q.5 30

Q.7 $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define

Q.9 6

Q.10 9

Q.11 c = 1, a, b $\in$ R

Q.12 8

Q.13 6x – 2y – 5 =0

Q.14 Does not exist

Q.15 $e^2 + e^{-2}$

Differentiability

Q.1 $-2x \sin(x^2+1)$

Q.2 $-\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Q.3 $\frac{dy}{dx} = (\tan x)^x[\log(\tan x) + 2x\,\text{cosec}2x]$

Q.4 $2x \cos x^2$

Q.5 $\frac{1}{2}\left(\frac{1}{1+x^2}\right)$

Q.6 $\frac{1}{2}$

Q.7 $\frac{dy}{dx} = x^{\cos x}\left[-\sin x \cdot \log x + \frac{\cos x}{x}\right] + \cos x^{\sin x}\left[-\sin x \tan x + \cos x \cdot \log \cos x\right]$

Q.8 x = 1 **Q.9** $-\frac{\sin x}{2\sqrt{\cos x}}$ **Q.10** $\frac{1}{2\sqrt{\tan x}}\sec^2 x$

Q.11 Not differentiable **Q.12** 1 **Q.13** 5

Exercise 2

Single Correct Choice Type

Limits

Q.1 C **Q.2** C **Q.3** C **Q.4** C **Q.5** C **Q.6** D

Q.7 D **Q.8** B **Q.9** A **Q.10** D **Q.11** A **Q.12** A

Continuity

Q.1 D **Q.2** A **Q.3** A **Q.4** D **Q.5** C

Q.6 D **Q.7** C **Q.8** D **Q.9** D **Q.10** D **Q.11** A

Q.12 C **Q.13** B

Differentiability

Q.1 D **Q.2** D **Q.3** A **Q.4** D **Q.5** A **Q.6** D

Q.7 A **Q.8** B **Q.9** B **Q.10** C **Q.11** C **Q.12** A

Q.13 D **Q.14** D **Q.15** C **Q.16** B **Q.17** C

Previous Years' Questions

Q.1 D **Q.2** A **Q.3** A **Q.4** B **Q.5** C **Q.6** B

Q.7 D **Q.8** C **Q.9** A **Q.10** C **Q.11** B **Q.12** D

Q.13 C **Q.14** B **Q.15** D **Q.16** A **Q.17** A **Q.18** A

Q.19 D **Q.20** A **Q.21** D **Q.22** B **Q.23** D **Q.24** A

Q.25 B **Q.26** D **Q.27** C **Q.28** A **Q.29** D **Q.30** B

Q.31 A

JEE Advanced/Boards

Exercise 1

Limits

Q.1 5050

Q.2 $a = 2; r = \frac{1}{4}; S = \frac{8}{3};$

Q.3 $\frac{p-q}{2}$

Q.4 ln2

Q.5 324

Q.6 (a) Does not exist (b) does not exist (c) 0

Q.7 $-\frac{1}{3}$

Q.8 $\frac{1}{32}$

Q.9 $\frac{\sqrt{3}}{2}$

Q.10 1/2

Q.11 –2

Q.12 $\pi - 3$

Q.13 (i) a = 1, b = –1 (ii) a = 1, $b = -\frac{1}{2}$

Q.14 1

Q.15 $-\frac{9}{4}\ln\frac{4}{e}$

Q.16 27

Q.17 29

Q.18 72

Q.19 –1/2

Q.20 16

Differentiability

Q.1 f(x) is continuous but not derivable at x = 0

Q.2 $0 < n \le 1$

Q.3 2

Q.4 f is cont. but not diff. at x = 0

Q.5

Q.6 $f'(1^+) = 3, f'(1^-) = -1$

Q.7 a = 1/2, b = 3/2

Q.8 5

Q.9 Not derivable at x = 0 & x = 1

Q.10 Discontinuous & not derivable at x = 1, continuous but not derivable at x = 2

Q.11 f is conti. at x = 1, 3/2 & discount. at x = 2, f is not diff. at x = 1, 3/2, 2

Q.12 24

Q.13 $a \neq 1, b = 0, p = \frac{1}{3}$ and q = –1

Q.14 5

Exercise 2

Limits

Single Correct Choice Type

Q.1 A **Q.2** A **Q.3** B **Q.4** A **Q.5** D **Q.6** B

Multiple Correct Choice Type

Q.7 A, B, C **Q.8** A, B, C **Q.9** A, B, C, D **Q.10** C, D **Q.11** A, B, C **Q.12** A, B, D

Q.13 A, B, C, D

Continuity

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** A	**Q.5** C	**Q.6** C
Q.7 A	**Q.8** A	**Q.9** D	**Q.10** C	**Q.11** C	**Q.12** A
Q.13 B	**Q.14** A	**Q.15** B	**Q.16** B	**Q.17** A	**Q.18** C
Q.19 B	**Q.20** D	**Q.21** C	**Q.22** A	**Q.23** A	**Q.24** B
Q.25 D	**Q.26** D	**Q.27** C	**Q.28** C	**Q.29** B	**Q.30** C
Q.31 B	**Q.32** A	**Q.33** D	**Q.34** D	**Q.35** D	

Multiple Correct Choice Type

Q.36 A, C

Differentiability

Single Correct Choice Type

Q.1 A **Q.2** A

Previous Years' Questions

Q.1 A, B, D	**Q.2** B, D	**Q.3** A, B, D	**Q.4** A, B	**Q.5** B, C	**Q.6** A $\to$ p; B $\to$ r
Q.7 A $\to$ p, q, r; B $\to$ p,s; C $\to$ r, s; D $\to$ p, q			**Q.16** A	**Q.17** A	**Q.18** A
Q.19 C	**Q.20** B	**Q.21** B	**Q.22** B, C, D	**Q.24** A, C	**Q.25** B, C
Q.26 D	**Q.27** A, B, C, D	**Q.29** B, C	**Q.30** B	**Q.31** B	**Q.31** B, D
Q.32 B, D	**Q.33** D	**Q.34** D	**Q.35** B	**Q.36** A	**Q.37** D
Q.38 D	**Q.39** B, C	**Q.40** A, B, D	**Q.41** A, B	**Q.42** A, D	**Q.43** A, B
Q.44 B, C	**Q.45** B, C	**Q.46** A			

Analytical and Descriptive Questions

Q.8 $-\frac{1}{3}$ **Q.9** $\frac{2}{\pi}$ **Q.10** $2x\cos(x^2+1)$ **Q.11** $\frac{1}{2}+\frac{\pi}{4}$

Q.12 $f'(1)=-\frac{2}{9}$ **Q.13** 0 **Q.14** $a^2\cos a + 2a\sin a$

Q.15. $$\begin{cases} \dfrac{5}{3(1-x)^2}-2\sin(4x+2) & x\le 1 \\ -\dfrac{5}{3(x-1)^2}-2\sin(4x+2) & x>1 \end{cases}$$

Solutions

JEE Main/Boards

Exercise 1

Limits

Sol 1: $\lim\limits_{x\to-4}\dfrac{\sqrt{5+x}-1}{x^2+4x}=\lim\limits_{x\to-4}\dfrac{\left(\sqrt{5+x}-1\right)}{x\left(x+4\right)}\times\dfrac{\sqrt{5+x}+1}{\sqrt{5+x}+1}$

$$=\lim_{x\to-4}\frac{\left(5+x-1\right)}{x\left(x+4\right)\left(\sqrt{5+x}+1\right)}=\frac{1}{-4\times2}=-\frac{1}{8}$$

Sol 2: $\lim\limits_{x\to2}\left\{\dfrac{x\left(x-2\right)\left(x+2\right)}{\left(x-2\right)\left(x^2+2x+4\right)}\right\}^{-1}$

$$=\left\{\frac{2\times4}{12}\right\}^{-1}=\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$$

Sol 3: $\lim\limits_{x\to1}\dfrac{\left(x^2-1\right)-\left(x-1\right)\ln x}{\left(x-1\right)}=2$

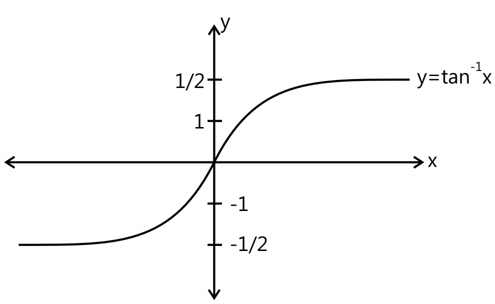
Sol 4: $\lim\limits_{x\to0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}=\lim\limits_{h\to0}\dfrac{x+h-x}{h\left\{\sqrt{+h}+\sqrt{x}\right\}}=\dfrac{1}{2\sqrt{x}}$

Sol 5: $\lim\limits_{x\to2}\left[\dfrac{1}{x\left(x-2\right)^2}-\dfrac{1}{x^{2-3x+2}}\right]$

$$=\lim_{x\to2}\frac{\left(x-1\right)-x\left(x-2\right)}{\left(x-2\right)^2\left(x-1\right).x}=\lim_{x\to2}\frac{-x^2+3x-1}{x\left(x-2\right)^2\left(x-1\right)}=\infty$$

Sol 6: (a) $\lim\limits_{x\to0}\tan^{-1}\dfrac{a}{x^2}$

Graph of $y=\tan^{-1}x$



So in $\lim\limits_{x\to0}\tan^{-1}\dfrac{a}{x^2}$

As $x\to0$ $\dfrac{a}{x^2}\to\infty\Rightarrow\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

Depends upon value of a.

(b) $f\left(x\right)=\lim\limits_{t\to0}\left(\dfrac{2x}{\pi}.\tan^{-1}\dfrac{x}{t^2}\right)$

$f\left(0\right)=-x=x$

$f\left(\infty\right)=\infty$

Sol 7: $\lim\limits_{x\to1}\dfrac{\sum\limits_{k=1}^{100}x^k-100}{x-1}=\lim\limits_{x\to1}\dfrac{\left(x+x^2+x^3+.....+x^{100}\right)}{\left(x-1\right)}$

$$=\lim_{x\to1}\frac{\left(x-1\right)\left\{1+\left(1+x\right)+\left(1+x+x^2\right)+......\right\}}{\left(x-1\right)}$$

$$=1+2+....+100=\frac{100\times101}{2}=5050$$

Sol 8: Use Binomial Expansion.

Sol 9:

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}\times\frac{1+\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(1-\tan x\right)\left(1+\sqrt{2}\sin x\right)}{1-2\sin^2x}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(1-\tan x\right)\left(1+\sqrt{2}\sin x\right)}{\left(\dfrac{1-\tan^2x}{1+\tan^2x}\right)}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(1+\tan^2x\right)\left(1+\sqrt{2}\sin x\right)}{\left(1+\tan x\right)}=2$$

Sol 10: Use $1-\cos2\theta=2\sin^2\theta$

Sol 11: $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{\left(2\sin x-1\right)\left(\sin x+1\right)}{\left(2\sin x-1\right)\left(\sin x-1\right)}=-3$

Sol 12:

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-(\cos\theta+\sin\theta)}{16\left(\theta-\frac{\pi}{4}\right)^2}=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)}{16\left(\theta-\frac{\pi}{4}\right)^2}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left\{2\sin^2\frac{\left(\theta-\frac{\pi}{4}\right)}{2}\right\}}{16.\left(\theta-\frac{\pi}{4}\right)^2}=\frac{1}{16\sqrt{2}}$$

Sol 13:

$$\lim_{x\to0}\frac{2\sin^2\frac{x}{2}+2\sin x-\sin^3x-x^2+3x^4}{\tan^3x-6\sin^2x+x-5x^3}$$

$$=\lim_{x\to0}\frac{\frac{\sin\frac{x}{2}}{x}.\sin\frac{x}{2}\times2+\frac{2\sin x}{x}-\frac{2\sin x}{x}-x+3x^3}{\frac{\tan^3x}{x}-\frac{6\sin^2x}{x}+1-5x^2}$$

$=2$

Sol 14: $\because \lim\limits_{x\to0}\frac{a\sin x-\sin 2x}{\tan^3 x}$ is finite

$\therefore \lim\limits_{x\to0} a\cos x-(\cos 2x).2=0$

$\to a-2=0$

$\to a=2$

Sol 15: $\lim\limits_{x\to\frac{\pi}{4}}\frac{2\tan x}{1-\tan^2x}\times\frac{1-\tan x}{1+\tan x}=\lim\limits_{x\to\frac{\pi}{4}}\frac{2\tan x}{(1+\tan x)^2}$

$=\frac{2}{(2)^2}=\frac{1}{2}$

Sol 16: Does not exist.

Sol 17:

$$\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$$

$$=\lim_{n\to\infty}\cos\left(\frac{\pi}{4n}\right)\frac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\times\frac{\pi}{4}=\frac{\pi}{4}$$

Sol 18: Use L'Hospitals Rule

Sol 19: $\lim\limits_{x\to0}\frac{9^x(3^x-1)-1(3^x-1)}{\sqrt{2}\left\{1-\cos\frac{x}{2}\right\}}=\lim\limits_{x\to0}\frac{(3^x-1)(9^x-1)}{2\sqrt{2}\sin^2\frac{x}{4}}$

$$=\lim_{x\to0}\frac{\frac{(3^x-1)}{x}\times\frac{9^x-1}{x}}{\frac{2\sqrt{2}\sin^2\frac{x}{4}}{\frac{x^2}{(4)^2}\times(4)^2}}=\frac{8}{\sqrt{2}}\log3\times\log9$$

$=8\sqrt{2}(\log3)^2$

Sol 20: Does not exist.

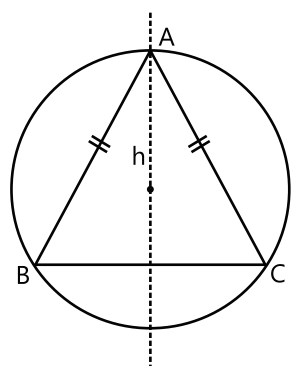
Hint: Use LHL and RHL.

Sol 21: $\lim\limits_{x\to0}\left[\frac{n\sin x}{x}\right]+\lim\limits_{x\to0}\left[\frac{n\tan x}{x}\right]$

$$=\lim_{x\to0}\left[\frac{nx\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right\}}{x}\right]+\lim_{x\to0}\left[\frac{nx\left\{1+\frac{x^2}{3}+\frac{2x^4}{15}\cdots\right\}}{x}\right]$$

$=n-1+n=2n-1$

Sol 22: From the diagram,



$h^2+\frac{a^2}{4}=b^2$

or, $h^2=b^2-\frac{a^2}{4}$

$$\lim_{h\to0}\frac{\Delta}{p^3}=\lim_{h\to0}\frac{\frac{ab^2}{4r}}{(a+2b)^3}$$

$$=\lim_{h\to\frac{a}{2}}\frac{a.\frac{a^2}{4}}{4r.8a^3}=\frac{1}{128r}$$

Sol 23: $\lim\limits_{x\to0}\frac{\log\left\{(1+x^2)^2-x^2\right\}}{x(e^x-1)}$

$$=\lim_{x\to0}\frac{\log\{1+x^2+x^4\}}{(x^2+x^4)}\times\frac{(x^2+x^4)}{x^2\left(\frac{e^x-1}{x}\right)}=1$$

Sol 24: $= \lim_{n\to\infty} n^2\left\{\left(a^{\frac{1}{n}}-1\right)-\left(a^{\frac{1}{n+1}}-1\right)\right\}$

$= \lim_{n\to\infty} n\left\{\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}\right\}-\frac{n^2}{(n+1)}\left\{\frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\right\}$

$= \lim_{n\to\infty} n\log a-\frac{n^2}{(n+1)}\log a=\log a\lim_{n\to\infty}\frac{n}{n+1}=\log a$

Sol 25: $\lim_{x\to\infty}\left(\frac{(x+c)}{x-c}\right)^x=4\Rightarrow\lim_{x\to\infty}\left(1+\frac{2c}{x-c}\right)^x=4$

Continuity

Sol 1: f(x) = $\begin{cases}|ax+3| & , \quad x\le -1\\ |3x+a| & , \quad -1<x\le 0\\ \frac{b\sin 2x}{x}-2b & , \quad 0<x<\pi\\ \cos^2 x-3 & , \quad x\ge\pi\end{cases}$

For x < –1, f(x) = |ax + 3| and it will be continuous.

At x = – 1,

Vf(x = –1) = |-a + 3| = |a – 3|

LHL = |a – 3|

RHL = $\lim_{x\to -1^+}|3x+a|$ = |a – 3|

At x = –1, function is continuous.

At x = 0,

LHL =VF(x = 0) = RHL

$\lim_{x\to 0^-}|3x+a|$ = |a| = $\lim_{x\to 0^+}\frac{b\sin 2x}{x}-2b$

LHL = f(0) = |a|

RHL = $\lim_{x\to 0^+}\frac{2.b\sin 2x}{2x}-2b$ = 2b – 2b = 0

$\therefore$|a| = 0 $\Rightarrow$ a = 0

At x = p

LHL = f(π) = RHL

$\lim_{x\to\pi^-}\frac{b\sin 2x}{x}-2b$ = $\cos^2\pi - 3$ = $\lim_{x\to\pi^+}\cos^2 x-3$

$\Rightarrow$–2b = –2 = – 2 $\Rightarrow$b = 1

$\therefore$ a = 0, b = 1

Sol 2: LHL = $\lim_{x\to\frac{\pi}{2}^-}\left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$

x = $\frac{\pi}{2}$ – h, as x $\to$ $\frac{\pi}{2}$; h $\to$ 0

$\therefore$LHL = $\lim_{h\to 0}\left(\frac{6}{5}\right)^{\frac{\tan 6\left(\frac{\pi}{2}-h\right)}{\tan 5\left(\frac{\pi}{2}-h\right)}}$ = $\lim_{h\to 0^+}\left(\frac{6}{5}\right)^{\frac{-\tan 6h}{\cot 5h}}$

= $\lim_{h\to 0^+}\left(\frac{6}{5}\right)^{-\tan 6h\,\tan 5h}$ = 1

$f\left(\frac{\pi}{2}\right)$ = b + 2 =LHL

$\Rightarrow$ b + 2 = 1

b = – 1

RHL = $\lim_{x\to\frac{\pi}{2}^+}(1+|\cos x|)^{\left(\frac{|a\tan x|}{b}\right)}$

x = $\frac{\pi}{2}$ + h, as x $\to$ $\frac{\pi}{2}^+$, h $\to$ 0^+

RHL = $\lim_{h\to 0^+}\left(1+\left|\cos\left(\frac{\pi}{2}+h\right)\right|\right)^{\frac{\left|a\tan\left(\frac{\pi}{2}+h\right)\right|}{b}}$

= $\lim_{h\to 0^+}(1+|\sin h|)^{\frac{|a\cot h|}{b}}$ = $\lim_{h\to 0^+}(1+|\sin h|)^{\frac{|a|\cot h}{b}}$

(as sin h > 0, for h $\to$ 0^+ and cot h > 0, for h $\to$ 0^+)

Now, this limit will be of the form equation for non zero values of a for a = 0,

RHL = $\lim_{h\to 0^+}(1+\sinh)^{o}$ = 1 = LHL = RHL = $f\left(\frac{\pi}{2}\right)$

Sol 3: (a) f(x) = $x^3 - 3x^2 - 4x + 12$

= (x – 3) $(x^2 - 4)$

f(x) = (x – 2) (x + 2) (x – 3)

zeros : 2, –2, 3

(b) h(x) = $\begin{cases}\frac{(x-2)(x+2)(x-3)}{x-3} & , \quad x\ne 3\\ k & ,, \quad x=3\end{cases}$

Atx = 3, for the continuity

h(x = 3) = LHL = RHL

LHL = 1. 5 = 5 = k

(c) We have,

$h(x) = \frac{(-x-2)(-x+2)(-x-3)}{(-x-3)}$, $x \neq -3$

= (x + 2) (x – 2) , x ≠ 3 = h(x)

hence even

At x = –3,

h(x) = –1x – 5 = 5 = h(3)

Hence, even function

Sol 4: $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + ... + \frac{x^2}{(1+x^2)^{n-1}}$

This is summation of a G P with n terms, a = x^2,

$r = \frac{1}{1+x^2}$

$\therefore y_n(x) = \frac{a(1-r^n)}{1-r} = \frac{x^2}{1-\frac{1}{1+x^2}}\left(1-\left(\frac{1}{1+x^2}\right)^n\right)$

$= \frac{x^2(1+x^2)}{1+x^2-1}\left[1-\left(\frac{1}{1+x^2}\right)^n\right]$

$y_n(x) = (1 + x^2)\left[1-\left(\frac{1}{1+x^2}\right)^n\right]$

For $y_n(x)$, at x = 0

$y_n(x) = 1\left(1-\left(\frac{1}{1}\right)^n\right) = 1$

LHL = RHL = 1

$\text{LHL} = \lim_{x\to 0}(1+x^2)\left(1-\left(\frac{1}{1+x^2}\right)^n\right) = 1$

Therefore, $y_n(x)$ is continuous

LHL = RHL, as it is an even function

$y(x) = \lim_{n\to\infty}(1+x^2)\left[1-\frac{1}{(1+x^2)^n}\right]$

$\lim_{x\to 0} y(x) = \lim_{x\to 0}\lim_{n\to\infty}(1+x^2)\left[1-\left(\frac{1}{1+x^2}\right)^n\right]$

At x = 0

y(x) will be indeterminate further LHL = RHL

$1-\left(\frac{1}{1+x^2}\right)^n$, as $n \to \infty$, approaches

$\therefore \lim_{x\to 0}\lim_{n\to\infty} y_n(x) = 1$

Thus Vf(x = 0) ≠ LHL, therefore discontinuous

Sol 5: f(x) = [5x] + {3x}

at x = 0, f(0) = 0

$\text{RHL} = \lim_{x\to 0}[5x]+\{3x\} = 0$

∴ continuous at x = 0

At $x = \frac{1}{5}$,

$f\left(\frac{1}{5}\right) = \left[5\times\frac{1}{5}\right] + \left\{3\times\frac{1}{5}\right\} = 1 + \frac{3}{5} = \frac{8}{5}$

$\text{LHL} = \lim_{x\to\frac{1}{5}^-}[5x]+\{3x\}$

$= 0 + \frac{3}{5} = \frac{3}{5}$

$\text{LHL} \neq \text{VF}\left(x = \frac{1}{5}\right)$

Therefore discontinuous at $x = \frac{1}{5}$

Similarly discontinuous at $\frac{1}{5}$, i. ∈ (1, 24) , i ≠ 5, 10, 15, 20

At $x = \frac{1}{3}$, $f\left(x = \frac{1}{3}\right) = \left[\frac{5}{3}\right] + \left\{3\times\frac{1}{3}\right\} = 0 + 0 = 0$

$\text{LHL} = \lim_{x\to\frac{1}{3}^-}[5x]+\{3x\} = 1 \neq \text{Vf}\left(x = \frac{1}{3}\right)$

Therefore, discontinuous

Similarly discontinuous at $x = \frac{1}{3}$, i ∈ (1, 14) ,

i ≠ 3, 6, 9, 12

Now, at Total 10 points

f(x = 1) = [5 × 1] + {3 × 1} = 5

$\text{LHL} = \lim_{x\to 1^-}[5x]+\{3x\} = 4 + 1 = 5$

$\text{RHL} = \lim_{x\to 1^+}[5x]+\{3x\} = 5 + 0 = 5$

∵ LHL = RHL = Vf(x = 1) , continuous at x = 1

similarly continuous at x= 2, 3, 4

∴ D is continuous at total 20 + 10 = 30 points

Sol 6: If $|2\sin x| < 1, f(x) = \lim_{n\to\infty}\frac{x}{(2\sin x)^{2n}+1} = \frac{x}{0+1} = x.$

If $|2\sin x| = 1, f(x) = \lim_{n\to\infty} \frac{x}{(2\sin x)^{2n}+1} = \frac{x}{1+1} = \frac{1}{2}x.$

If $|2\sin x| > 1, f(x) = \lim_{n\to\infty} \frac{x}{(2\sin x)^{2n}+1} = \frac{x}{\infty+1}0.$

But $|2\sin x| < 1 \Rightarrow |\sin x| < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < \sin x < \frac{1}{2} \Rightarrow n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$

$|2\sin x| = 1 \Rightarrow |\sin x| = \frac{1}{2} \Rightarrow \sin x = \pm\frac{1}{2}$

$\Rightarrow x = n\pi + (-1)^n . \left(\pm\frac{\pi}{6}\right) = n\pi \pm \frac{\pi}{6}$

$|2\sin x| > 1 \Rightarrow |\sin x| > \frac{1}{2}$

$\Rightarrow \sin x > \frac{1}{2}$ or $\sin x < -\frac{1}{2}$

$\Rightarrow n\pi + \frac{\pi}{6} < x < n\pi + \frac{5\pi}{6}.$

Thus, we have $f(x) = x, n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$

$\frac{1}{2}x, x = n\pi \pm \frac{\pi}{6}$

$0, n\pi + \frac{\pi}{6} < x < n\pi + \frac{5\pi}{6}$

As polynomial functions are continuous everywhere, only doubtful points are $x = n\pi \pm \frac{\pi}{6}$. Clearly,

$f\left(n\pi - \frac{\pi}{6} + 0\right) \neq f\left(n\pi - \frac{\pi}{6} - 0\right)$ because $n\pi - \frac{\pi}{6} \neq 0$

$f\left(n\pi + \frac{\pi}{6} + 0\right) \neq f\left(n\pi + \frac{\pi}{6} - 0\right)$ because $0 \neq n\pi + \frac{\pi}{6}$

$\therefore f(x)$ is not continuous at $x = n\pi \pm \frac{\pi}{6}$

$\therefore$ The function f (x) is continuous everywhere in R except the set of points $\left\{x \middle| x = n\pi \pm \frac{\pi}{6}, n \in Z\right\}$

Sol 7: f(x) = $\begin{cases} \frac{\ell n\cos x}{\sqrt[4]{1+x^2}-1}, & x > 0 \\ \frac{e^{\sin^4 x}-1}{\ell n(1+\tan 2x)}, & x < 0 \end{cases}$

LHL = $\lim_{x\to 0^-} \frac{e^{\sin^4 x}-1}{\ell n(1+\tan 2x)}$

LHL= $\lim_{x\to 0^-} \left(\frac{e^{\sin^4 x}-1}{\sin^4 x}\right).\left(\frac{1}{\frac{\sin(1+\tan 2x)}{\tan 2x}}\right).\frac{\sin^4 x}{\tan 2x}$

$= \lim_{x\to 0^-} \frac{\sin 4x}{4x} . \frac{4x}{2x} . \frac{1}{\frac{\tan 2x}{2x}} = 2$

RHL = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ell n\cos x}{\sqrt[4]{1+\pi^2}-1}$, $\frac{0}{0}$ form

Using L Hospital rule.

RHL = $\lim_{x\to 0^+} \frac{1}{\cos x} \frac{(-\sin x)}{\frac{1}{4}(1+x^2)^{-3/4}.2x}$

$= \lim_{x\to 0} \frac{-2}{\cos}\left(\frac{\sin x}{x}\right)(1+x^2)^{3/4} = -2\left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$

$\because$ LHL $\neq$ RHL, it is not possible to make the function continuous. This is jump discontinuity.

Sol 8: $\lim_{x\to 0} g(x), = 0 \Rightarrow \lim_{h\to 0} g(0+h) = \lim_{h\to 0} g(0-h) = 0$

$\Rightarrow \lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h) = 0$ .....(i)

$\lim_{x\to 0} G(x)$ exists $\Rightarrow \lim_{h\to 0} G(0+h) = \lim_{h\to 0} G(0-h)$

$\Rightarrow \lim_{h\to 0} G(h) = \lim_{h\to 0} G(-h) =$ finite ....(ii)

Now, $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x).f(h) = f(x)\lim_{h\to 0} f(h)$

$\{\because f(x+y) = f(x).f(y)\}$

$= f(x).\lim_{h\to 0}\{1+g(h)G(h)\},$

Using given relation

$= f(x).\left\{1+\lim_{h\to 0} g(h).\lim_{h\to 0} G(h)\right\}$

$= f(x).\{1+0.\text{ finite}\}$, using (1) and (2)

= f (x)

Also, $\lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x).f(-h) = f(x).\lim_{h\to 0} f(-h)$

$= f(x).\lim_{h\to 0}\{1+g(-h).G(-h)\}$, using given relation

$= f(x).\left\{1+\lim_{h\to 0} g(-h).\lim_{h\to 0} G(-h)\right\}$

$= f(x).\{1+0.\text{ finite}\}$, using (i) and (ii)

$= f(x)$

$\therefore \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x-h) = f(x).$

$\therefore f(x)$ is continuous everywhere.

Sol 9: $f(x) = \text{sgn}((x^2 - ax + 1)(bx^2 - 2bx + 1))$

This function will be discontinuous when $(x^2 - ax + 1)(bx^2 - 2bx + 1) = 0$

Therefore, for this to be discontinuous at exactly one point if

$(x^2 - ax + 1)(bx^2 - 2bx + 1)$ has exactly one root.

For $x^2 - ax + 1$, $D = a^2 - 4$

$D \le 0 \Rightarrow a^2 \le 4$

$\Rightarrow a = -2, -1, 0, 1, 2$ ($\because a \in z$)

for $bx^2 - 2bx + 1$, $D = 4(b^2 - b) \le 0$

$\Rightarrow b = 1, 0$ ($\because b \in z$)

At $b = 0$, $bx^2 - 2bx + 1 = 1$, which has no root pairs (a, b) for which exactly root,

(–1, 1) , (0, 1) , (1, 1) , (–2, 0) , (2, 1) , (2, 0)

x = 1, x = 1, x = 1, x = –1, x = 1,x = 1

Total 6 ordered pairs.

Sol 10: Let the common roots be λ_1 and λ_2.

Then, we have from $x^3 + 2x^2 + px + q = 0$

$\alpha\lambda_1 + \alpha\lambda_2 + \lambda_1\lambda_2 = p$

$\alpha(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = p$ ... (i)

from $x^3 + x^2 + px + r = 0$

$\beta(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = p$ ... (ii)

from (i) and (ii)

$\alpha(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = \beta(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$

$\alpha(\lambda_1 + \lambda_2) = \beta(\lambda_1 + \lambda_2)$

$\Rightarrow \lambda_1 + \lambda_2 = 0$ (for non zero α, β)

Now, from (a), $\alpha + \lambda_1 + \lambda_2 = -2 \Rightarrow \alpha = -2$

from (b), $\beta + \lambda_1 + \lambda_2 = -1$

$\beta = -1$

$\therefore |\alpha + \beta| = 3, \alpha\beta = 2$

Now, for f(x) at x = 0,

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{x\log_{1+x} 3} = \lim_{x\to 0^-} 3^{x\log_{1+x} e}$$

$$= \lim_{x\to 0^-} 3^{\frac{x}{\log(1+x)}} = 3 \left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$$

RHL

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} b\frac{\ell n(e^{x^2} + \alpha\beta\sqrt{x})}{\tan\sqrt{x}}$$

$$= b \lim_{x\to 0^+} b\frac{\ell n\left(e^{x^2}\left(1+\frac{\alpha\beta\sqrt{x}}{e^{x^2}}\right)\right)}{\tan\sqrt{x}}$$

$$= b\left[\lim_{x\to 0^+}\frac{x^2}{\tan\sqrt{x}} + \frac{\ell n\left(1+\frac{\alpha\sqrt{x}}{e^{x^2}}\right)}{\frac{\alpha\beta\sqrt{x}}{e^{x^2}}}.\frac{\alpha\beta\sqrt{x}}{\tan\sqrt{x}}.\frac{1}{e^{x^2}}\right]$$

$$= b\left[\lim_{x\to 0^+}\left(\frac{x^{3/2}}{\frac{\tan\sqrt{x}}{\sqrt{x}}}\right) + \frac{\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}\left(\frac{2}{\frac{\tan\sqrt{x}}{\sqrt{x}}}\right)\frac{1}{e^{x^2}}\right]$$

$= b(0/1 + 1.\ 2.\ 1) = 2b$

Now, for continuity

LHL= RHL = Vf(x = 0)

$\therefore 3 = 2b = a \quad \therefore a = 3, b = \frac{3}{2}$

Hence, $2(a + b) = 2\left(3+\frac{3}{2}\right) = 9$

Sol 11: $f(x) = \lim_{x\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$

For x = 0

$f(x = 0) = \lim_{n\to\infty}\frac{c+1}{1+c}$

For x < 0,

$$\text{LHL} = \lim_{x\to 0^-}\lim_{n\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + ce^{nx}}$$

as $x \to 0^-$, $e^{nx} \to 0$

$$\therefore \text{LHL} = \lim_{x\to 0^-}\lim_{n\to\infty}\frac{ax^2 + bx + c}{1+0} = c$$

$$\text{RHL} = \lim_{x\to 0^+}\lim_{n\to\infty}\frac{ax^2 + bx + c + e^{nx}}{1 + ce^{nx}}$$

$$= \lim_{x\to 0^+}\lim_{n\to\infty}\frac{\frac{ax^2}{e^{nx}}+\frac{bx}{e^{nx}}+\frac{c}{e^{nx}}+1}{\frac{1}{e^{nx}}+c} = \frac{1}{c}$$

For continuity, LHL = RHL ⇒ $\frac{1}{c} = c \Rightarrow c = \pm 1$

But, for c = –1,

f(x = 0) = indeterminate therefore, c = 1

Sol 12: f (x) will be continuous at x = 0 if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0) \quad \text{... (i)}$$

Now, $\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4}$

$$= \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}\left\{\sqrt{16+\sqrt{h}}+4\right\}}{16+\sqrt{h}-16}$$

$$= \lim_{h\to 0}\left\{\sqrt{16+\sqrt{h}}+4\right\} = 8$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1-\cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 4h}{h^2} = \lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$$

$$= \lim_{h\to 0} 2.\left(\frac{\sin 2h}{2h}\right)^2 \times 4 = 8$$

$f(0) = a$ ∴ by (1), 8 = 8 = a; a = 8.

Sol 13: We have to check continuity at x = ±1 only as function will be continuous for other points.

At x = 1,

$$\text{LHL} = \lim_{x\to 1^-}\frac{x^{2n-1}+ax^3+bx^2}{x^{2n}+1}$$

as $x \to 1^-$, $x^{2n-1} \to 0$, $x^{2n} \to 0$

$$\text{LHL} = \lim_{x\to 1^-}\frac{ax^3+bx^2}{1} = a + b$$

$$\text{Vf}(x = 1) = \frac{1+a+b}{1+1}$$

for continuity, LHL = Vf(x = 1)

$$\therefore a + b = \frac{1+a+b}{2} \Rightarrow a + b = 1 \quad \text{... (i)}$$

At x = 1

$$\text{RHL} = \lim_{x\to -1^+}\frac{x^{2n-1}+ax^3+bx^2}{x^{2n}+1}$$

as $x \to -1^+$, x^{2n-1} and $x^{2n} \to 0$

$$\therefore \text{RHL} = \lim_{x\to -1^+}\frac{ax^3+b^2}{1} = -a + b$$

$$f(x = -1) = \frac{-1-a+b}{1+1}$$

for continuity RHL = Vf(x = –1)

$$\therefore -a + b = \frac{-1-a+b}{2} \Rightarrow -2a + 2b - 1 - a + b$$

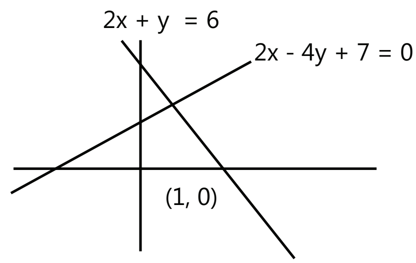
$$\Rightarrow a - b = 1 \quad \text{... (ii)}$$

Solving (i) and (ii) , a = 1, b = 0

∴ (a, b) = (1, 0)

Lines –2x – y + 6 = 0

2x – 4y + 7 = 0



We see that origin is on same side as (1, 0) .

∴ equation of angle bisector

$$\frac{2x-y+6}{\sqrt{5}} = \frac{2x-4y+7}{2\sqrt{5}}$$

–4x – 2y + 12 = 2x – 4y + 7

for 6x – 2y – 5 = 0

Sol 14: We have $x^2 < 1 \Rightarrow (x+1)(x-1) < 0$.

From the sign-scheme, $x \le -1$ or $x \ge 1$.

Thus, the function is $f(x) = x, x \le 1$

$x^4, -1 < x < 1$

$x, x \ge 1$

Now, $\lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\{1+h\} = 1$

$$\lim_{x\to 1-0} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^4 = 1$$

∴ $\lim_{x\to 1} f(x)$ exists and it is equal to 1.

Next, $\lim\limits_{x\to 1+0} f(x)=\lim\limits_{h\to 0} f(-1+h)=\lim\limits_{h\to 0}(-1+h)^4=1$

$\lim\limits_{x\to 1-0} f(x)=\lim\limits_{h\to 0} f(-1-h)=\lim\limits_{h\to 0}\{-1-h\}=-1$

$\therefore \lim\limits_{x\to -1} f(x)$ does not exist.

Sol 15: At x = 0

LHL = $\lim\limits_{x\to 0^-}(\sin x+\cos x)^{\operatorname{cosec} x}$

$=\lim\limits_{x\to 0^-}(1+(\sin x+\cos x-1))^{\operatorname{cosec} x}$

$=e^{\lim\limits_{x\to 0}\operatorname{cosec} x(\sin x+\cos x-1)} = e^{1+\lim\limits_{x\to 0}\frac{\cos x-1}{\sin x}}$

$=e^{1+\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}} = e^{1-\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}}} = e$

RHL = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{|x|}}}{ae^{\frac{2}{x}}+be^{\frac{3}{|x|}}}$

$=\lim\limits_{x\to 0^+}\frac{e^{-\frac{1}{x}}+e^{-\frac{1}{x}}+1}{ae^{\frac{-1}{x}}+b} = \frac{1}{b}$

for continuity, e = a = $\frac{1}{b}$

$\therefore$ a = e, b = e^{-1} $a^2+b^2=e^2+e^{-2}$

Differentiability

Sol 1: $f(x)=\cos\left(x^2+1\right)$

$f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\frac{\cos\left((x+h)^2+1\right)-\cos\left(x^2+1\right)}{h}$

$=\frac{\cos\left[x^2+h^2+2xh+1\right]-\cos\left(x^2+1\right)}{h}$

$=\frac{\cos\left[x^2+1+h^2+2xh\right]-\cos\left(x^2+1\right)}{h}$

Apply $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\Rightarrow \lim\limits_{h\to 0}$
$\frac{\cos\left(x^2+1\right).\cos\left(h^2+2xh\right)-\sin\left(x^2+1\right).\sin\left(h^2+2xh\right)-\cos\left(x^2+1\right)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\cos\left(x^2+1\right)\left[\cos\left(h^2+2xh\right)-1\right]-\sin\left(x^2+1\right).\sin\left(h^2+2xh\right)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\cos\left(x^2+1\right).\left(\cos\left(h^2+2hx\right)-1\right)}{h(h+2x)}\times$

$(h+2x)-\frac{\sin\left(x^2+1\right).\sin\left(h^2+2xh\right)}{h(h+2x)}\times(h+2x)$

$\Rightarrow \lim\limits_{h\to 0}\cos\left(x^2+1\right)(h+2x).\frac{\left[\cos\left(h^2+2hx\right)-1\right]}{h^2+2hx}-$

$\sin\left(x^2+1\right),(h+2x).\frac{\sin\left(h^2+2xh\right)}{h^2+2hx}$

$\because \lim\limits_{x\to 0}\frac{\cos x-1}{x}=0$ and $\because \lim\limits_{x\to 0}\frac{\sin x}{x}=1$

$\cos\left(x^2+1\right)(2x)(0)-\sin\left(x^2+1\right).(2x)$

$\Rightarrow -2x.\sin\left(x^2+1\right)$

Sol 2: $f(x)=\tan\sqrt{x}$

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A+\tan B}$

$f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\tan\sqrt{x+h}-\tan\sqrt{x}}{h}$

$\lim\limits_{h\to 0}\frac{\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right).\tan\left(\sqrt{x+h}-\sqrt{x}\right)}{h\left(\sqrt{x+h}-\sqrt{x}\right)}\times\left(\sqrt{x+h}-\sqrt{x}\right)$

$\Rightarrow \lim\limits_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\lim\limits_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$

$\Rightarrow \lim\limits_{h\to 0}\left(1+\tan\sqrt{x+h}.\tan\sqrt{x}\right)\lim\limits_{h\to 0}-\frac{1}{2}\frac{\frac{1}{\sqrt{x+h}}}{1}$

$$\Rightarrow \left(1+\tan^2\sqrt{x}\right)\left(-\frac{1}{2\sqrt{x}}\right) \Rightarrow -\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Sol 3: $e^{\sin x} + (\tan x)^x = f(x)$

$$\frac{df(x)}{dx} = \frac{de^{\sin x}}{dx} + \frac{d(\tan x)^x}{dx}$$

↓↓

III

Part-I:

$$\frac{de^{\sin x}}{dx} = \frac{de^{\sin x}}{d\sin x}.\frac{d\sin x}{dx} = \cos x\, e^{\sin x}$$

Part-II:

$$\frac{d(\tan x)^x}{dx}$$

$\mu = (\tan x)^x$

$\log \mu = x \log (\tan x)$

$$\therefore \frac{1}{\mu}\frac{d\mu}{dx} = \log(\tan x) + \frac{x}{\tan x}\times \sec^2 x$$

$$\therefore \frac{dy}{dx} = (\tan x)^x[\log(\tan x) + 2x\,\mathrm{cosec}2x]$$

Sol 4: $f(x) = \sin x^2$

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)^2 - \sin x^2}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{x^2+h^2+2xh+x^2}{2}\right)\sin\left(\frac{h^2+2xh}{2}\right)}{h}$$

$$= 2\cos x^2 \lim_{h\to 0}\frac{\sin\left(\frac{h^2}{2}+xh\right)}{\left(\frac{h^2}{2}+xh\right)}\times\frac{\left(\frac{h^2}{2}+xh\right)}{h} = 2x\cos x^2$$

Sol 5: $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$f'(x) = \frac{d\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)}{d\left(\frac{\sqrt{1+x^2}-1}{x}\right)}\times\frac{d\left(\frac{\sqrt{1+x^2}-1}{x}\right)}{dx}$$

$$= \frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x}\right)^2}\left[\frac{\frac{2x^2}{2\sqrt{1+x^2}}-(\sqrt{1+x^2}-1)}{x^2}\right]$$

$$= \frac{x^2}{2\sqrt{1+x^2}(\sqrt{1+x^2}-1)}\,\frac{(x^2-1-x^2+\sqrt{1+x^2})}{x^2\sqrt{1+x^2}}$$

$$= \frac{1}{2(1+x^2)}$$

Alternative

Put $x = \tan\theta$

Sol 6: $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = f(x)$

$$\Rightarrow \tan^{-1}\left(\frac{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}+\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}-\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right)$$

$$\Rightarrow \tan^{-1}x\tan\frac{x}{2} = \frac{x}{2} = f(x)$$

$$\therefore f'(x) = \frac{d(x/2)}{dx} = \frac{1}{2}$$

Sol 7: $y = (x)^{\cos x} + (\cos x)^{\sin x}$

$y = u(x) + v(x)$ where $u(x) = (x)^{\cos x}$

$\log u = \cos x \log x$

$$\frac{1}{u}\frac{du}{dx} = \frac{\cos x}{x} - \sin x\log x$$

$$\therefore \frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x\log x\right)$$

$v(x) = (\cos x)^{\sin x}$

$\log v = \sin x \log \cos x$

$$\frac{1}{v}\frac{dv}{dx} = \frac{\sin x}{\cos x}(-\sin x) + \cos x\log\cos x$$

$$\therefore \frac{dv}{dx} = (\cos x)^{\sin x}\left(\cos x\log\cos x - \frac{\sin^2 x}{\cos x}\right)$$

$$\therefore \frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right)$$

$$+ (\cos x)^{\sin x}\left(\cos x \log \cos x - \frac{\sin^2 x}{\cos x}\right)$$

$$= \frac{1}{2\sqrt{\tan x}} \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h(\cos(x+h)\cos(x))}$$

$$= \frac{1}{2\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \lim_{h\to 0} \frac{\sin(x+h-x)}{h}$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}} \lim_{h\to 0} \frac{\sinh}{h} = \frac{s\sec^2 x}{2\sqrt{\tan x}}$$

Sol 8: In order to examine the continuity at x = 1, we are required to derive the definition of f (x) in the intervals x < 1, x > 1 and at x = 1, i.e., on and around x = 1.

Now, if $0<x<1$,

$$f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$$

$$= \frac{\log(x+2) - 0.\sin x}{0+1} = \log(x+2)$$

If $x=1$, $f(x) = \lim_{n\to\infty} \frac{\log(x+2) - 1.\sin x}{1+1} = \frac{\log(x+2) - \sin x}{2}$

If $x>1, f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(x+2) - \sin x}{1+\frac{1}{x^{2n}}} = -\sin x$$

Thus, we have

$$f(x) = \begin{cases} \log(x+2) & ; \quad 0<x<1 \\ \frac{1}{2}\{\log(x+2) - \sin x\} & ; \quad x=1 \\ -\sin x & ; \quad x>1 \end{cases}$$

$$\therefore f(1+0) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} f\{-\sin(1+h)\} = -\sin 1$$

$$f(1-0) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \log(1-h+2) = \log 3$$

$$\therefore f(1+0) \neq f(1-0)$$

So f (x) is not continuous at x = 1.

Sol 9: $f(x) = \sqrt{\cos x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\sqrt{\cos(h+x)} - \sqrt{\cos x}}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\cos(x+h) + \cos x}{h\left(\sqrt{\cos(x+h)} + \sqrt{\cos x}\right)} \quad (\because \text{Rationalise})$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{2\sqrt{\cos x}} . \frac{\cos x(\cosh - 1) - \sin x.\sinh}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{2\sqrt{\cos x}} . \left[\frac{\cos x(\cosh - 1)}{h} - \frac{\sin x.\sinh}{h}\right]$$

$$\Rightarrow -\frac{\sin x}{2\sqrt{\cos x}}$$

Sol 10: Here $f(x) = \sqrt{\tan x}$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} . \frac{\sqrt{\tan(x+h)} + \sqrt{\tan x}}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\tan(x+h) - \tan x}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\frac{\sin(x+h).\cos x - \sin x.\cos(x+h)}{\cos(x+h).\cos x}}{h\left(\sqrt{\tan(x+h)} + \sqrt{\tan x}\right)}\right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sin(x+h-x)}{h\cos(x+h).\cos x\left(\sqrt{\tan(x+h)}+\sqrt{\tan x}\right)} \right]$$

$$\Rightarrow f'(x) = \lim_{(h\to 0)} \left[\frac{\sin h}{h\cos(x+h).\cos x\left(\sqrt{\tan(x+h)}+\sqrt{\tan x}\right)} \right]$$

$$\Rightarrow f'(x) = \frac{1}{\cos(x+0).\cos x\left(\sqrt{\tan(x+0)}+\sqrt{\tan x}\right)} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

$$\left[\because \lim_{(h\to 0)} \frac{\sinh}{h} = 1\right]$$

Sol 11: $f(x) = \begin{cases} -x & : \quad x<0 \\ 0 & : \quad x=0 \\ x & : \quad x>0 \end{cases}$

L.H.D. $f'(0^-) = -1$

R.H.D. $f'(0^+) = 1$

Sol 12: $f'(a) = \frac{1}{4}$

$$\lim_{h\to 0} \frac{f(a+2h^2) - f(a-2h^2)}{h^2}$$

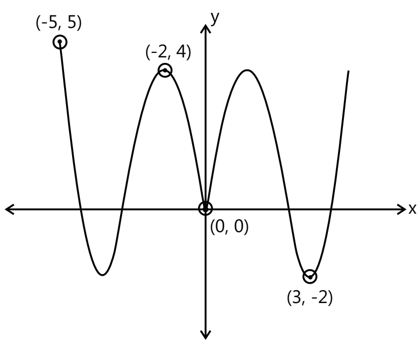
$$\lim_{h\to 0} \frac{f(a+2h^2)}{h^2} - \lim_{h\to 0} \frac{f(a-2h^2)}{h^2}$$

$$\Rightarrow \lim_{h\to 0} \frac{f'(a+2h^2).4h}{2h} - \lim_{h\to 0} \frac{f'(a-2h^2).(-4h)}{2h}$$

$$\Rightarrow \lim_{h\to 0} 2.f'(a+2h^2) + \lim_{h\to 0} 2.f'(a-2h^2)$$

$$\Rightarrow 2.f'(a) + 2.f'(a) \quad \Rightarrow 2\times\frac{1}{4} + 2\times\frac{1}{4} \quad \Rightarrow 1$$

Sol 13:



Sol 14: (a) Proof. We consider the function g (x) = f (x) – x . g is continuous on [a, b], and $g(a) = f(a) - a \geq 0$ because the range of f is [a, b]. By the same reason, $g(b) = f(b) - b \leq 0$. Now by the intermediate value theorem, there exists $c \in [a, b]$ such that $g(c) = 0$ i.e., $f(c) = c$.

(b) Let f be continuous on the interval [0, 1] to $\mathbb{R}$ and such that f (0) = f (1). Prove that there exists a point $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

Proof. (Paige Cudworth) Define

$h(x) = f(x) - f\left(x + \frac{1}{2}\right)$. Then $h(0) = f(0) - f\left(\frac{1}{2}\right)$ and $h\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0)$. So $h(0) = -h\left(\frac{1}{2}\right)$.

Case 1: $h(0) > 0 > h\left(\frac{1}{2}\right)$. Then by the Location of Roots Theorem, there exists $c \in \left(0, \frac{1}{2}\right)$ such that $h(c) = 0$. So

$$0 = f(c) = f\left(c + \frac{1}{2}\right) \Rightarrow f(c) = f\left(c + \frac{1}{2}\right)$$

Case 2: $h(0) < 0 < h\left(\frac{1}{2}\right)$. Similar to case 1.

Case 3: $h(0) = 0 = h\left(\frac{1}{2}\right)$. Then $0 = f(0) = f\left(\frac{1}{2}\right)$, so $f(0) = f\left(0 + \frac{1}{2}\right)$.

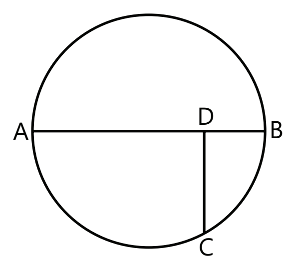
Exercise 2

Limits

Single Correct Choice Type

Sol 1: (C)

Hint : $CD^2 = AD \times DB$



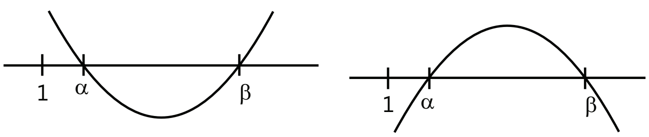
Sol 2: (C)

$$\lim_{x\to 0}\left(1+\log^2_{\cos\frac{x}{2}}\cos x\right)^2$$

$$\lim_{x\to 0}\log_{\cos\frac{x}{2}}\cos x=\lim_{x\to 0}\frac{\log(\cos x)}{\log\left(\cos\frac{x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{\frac{1}{2}\frac{-\sin x/2}{\cos x/2}}=\lim_{x\to 0}\frac{4\cos^2\frac{x}{2}}{\cos x}=4$$

Sol 3: (C)



Sol 4: (C) Use $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

Sol 5: (C) $\lim\limits_{x\to 0}\ \log x-x=\infty$

Sol 6: (D)

$$\lim_{x\to 0}\ x.\left\{\frac{\frac{1}{x}}{1}+\frac{\frac{1}{x}3}{3}+\frac{2\frac{1}{x}5}{15}+......\right\}$$

$$=\lim_{x\to 0}\ x\left\{\frac{1}{x}+\frac{1}{3x^3}+\frac{2}{15x^5}+......\right\}$$

$$=\lim_{x\to 0}\ 1+\frac{1}{3x^2}+\frac{2}{15x^5}.....$$

Hence, (D).

Sol 7: (D)

$$\lim_{x\to\infty}\ \sqrt{x}\frac{(\sqrt{x+4}-\sqrt{x})(\sqrt{x+4}+\sqrt{x})}{(\sqrt{x+4}+\sqrt{x})}$$

$$=\lim_{x\to\infty}\ \frac{\sqrt{x}(4)}{\sqrt{x+4}+\sqrt{x}}=\frac{4}{2}=2$$

Sol 8: (B)

$$\lim_{x\to 0^+}\frac{\ln(\sin 2x)}{\ln(\sin x)}=\lim_{h\to 0}\frac{\ln(\sin 2h)}{\ln(\sinh)}$$

Use L'Hospitals rule

Sol 9: (A)

On solving for x and y we get

$$x=\lim_{c\to 1}\frac{1-C^2}{2+C-3C^2}\lim_{c\to 1}\frac{(1-c)(1+c)}{(2+3c)(1-c)}=\frac{2}{5}$$

$$\therefore y=-\frac{1}{25}$$

Now, radius can be found by distance formula.

Sol 10: (D) $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{\sum_{k=1}^{r}k}{\sum_{k=1}^{r}k^3}\Rightarrow\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{r\frac{(r+1)}{2}}{\frac{r^4}{4}+\frac{r^3}{2}+\frac{r^2}{4}}$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{r}{2}(r+1)}{\frac{r^4+2r^3+r^2}{4}}\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r^2+2r}{r^4+2r^3+r^2}$$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r(r+1)}{r^2(r^2+2r+1)}\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r(r+1)}{r^2(r+1)^2}$$

$$\Rightarrow\lim_{n\to\infty}\sum_{r=1}^{n}\frac{2}{r(r+1)}\Rightarrow\lim_{n\to\infty}2.\sum_{r=1}^{n}\frac{1}{r(r+1)}$$

$$\Rightarrow\lim_{n\to\infty}2.\frac{n}{n+1}\Rightarrow\lim_{n\to\infty}2.\frac{1}{1+\frac{1}{n}}\Rightarrow 2$$

Sol 11: (A) Take i^n common and use binomial theorem.

Sol 12: (A) $\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=\lim\limits_{x\to 0}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}\left(\because\left|\sin\frac{1}{x}\le 1\right|\right)=0$

But f (0) is not defined

Hence, (A)

Continuity

Single Correct Choice Type

Sol 1: $f'(0+0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{0+h}{1+e^{\frac{1}{0+h}}}-0}{h}$

$$=\lim_{h\to 0}\frac{1}{1+e^{1/h}}=\frac{1}{1+e^{\infty}}=0$$

$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{\frac{0-h}{1+e^{\frac{1}{0-h}}}-0}{-h}$$

$$= \lim_{-h\to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$\therefore f'(0+0) \neq f'(0-0)$.

Hence, f(x) is not differentiable at x = 0.

Sol 2: (A) At x = –1,

LHL = $\lim_{x\to -1} f(x) = \lim_{x\to -1^-} \sin(\pi(x+a))$

= $\sin(\pi(-1 + a)) = \sin(\pi - \pi a) = -\sin \pi a$

RHL = $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} b([x]^2+[x])+1$

= b(1 – 1) + 1 = 1

f(x = –1) = –b(1 – 1) + 1 = 1

For continuity, at

LHL = Vf(x = – 1) = RHL

$\Rightarrow -\sin \pi a = 1$

$\pi a = \left(2n+\frac{3}{2}\right)\pi$

$a = 2n + \frac{3}{2}$, $n \in I$, and $b \in R$

Sol 3: (A) $f(x) = \begin{cases} \frac{[|x|]e^{x^2}[x+|x|]}{e^{1/x^2}-1\,\text{sgn}(\sin x)} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$

LHL = $\lim_{x\to 0^-} f(x)$

Let x = 0 – h, as $x \to 0^-$, $h \to 0^+$

$\therefore$ LHL = $\lim_{h\to 0^+} \frac{[|-h|]e^{(-h)^2}[-h+|-h|]}{e^{\frac{1}{(-h)^2}}-[x\,\text{sgn}(\sin(-h))]}$

for $h \to 0^+$

Now, |-h| = h, [h] = 0,

sin(-h) = –sin h < 0 $\forall$ sgn |sin(-h)) | = –1

$\therefore$ LHL = $\lim_{h\to 0^+} \frac{0.[-h+h]}{e^{1/h^2}+1} = 0$

RHL = $\lim_{x\to 0^+} \frac{[|x|]e^{x^2}.[x+|x|]}{e^{\frac{1}{x^2}}-|x\,\text{sgn}(\sin x)} = \lim_{x\to 0^+} \frac{0.e^{x^2}.[x+x]}{e^{\frac{1}{x^2}}-1} = 0$

LHL = Vf(x = 0) = RHL

f(x) is continuous at x = 0

Sol 4: (D) Consider a function

$f(x) = \begin{cases} x-3, & x \neq 3 \\ 6, & x = 3 \end{cases}$

This function has a isolated point continuity at x = 3.

Now, $g(x) = \frac{1}{f(x)} = \begin{cases} \frac{1}{x-3}, & x \neq 3 \\ \frac{1}{6}, & x = 3 \end{cases}$

Here, the function is not defined at x = 3. Therefore, g(x) has missing point discontinuity.

Sol 5: (C) f(x) = |x– 0. 5| + |x – 1| + tanx

We know |x – a| is not differentiable at x = a

$\therefore$ x = 0. 5 and x = 1 are two points of discontinues further in (0, 2) , tan x is not differentiable at $x = \frac{\pi}{2}$.

$\therefore$ Total three points at which function is not derivable.

Sol 6: (D) (A) for f(x) ,at x = 0,

LHD = $\left.\frac{d}{dx}x^2\right|_{x=0} = 2x = 0$

RHD = $\frac{d}{dx}\left.|-x^2|\right|_{x=0} = -2x = 0$

$\because$ LHD = RHD, f(x) is differentiable

(B) for g(x) , at x = 0

LHD = $\left.\frac{d}{dx}(x)\right|_{x=0} = 1$

RHD = $\left.\frac{d}{dx}(\tan x)\right|_{x=0} = \sec^2 x = 1$

$\because$ LHD = RHD, g(x) is differentiable

(C) for h(x) at x = 0

LHD = $\left.\frac{d}{dx}(\sin 2x)\right|_{x=0} = 2\cos^2 x = 2$

RHD = $\left.\frac{d}{dx}(2x)\right|_{x=0} = 2$

∵ LHD = RHD, g(x) is differentiable

(D) for k(x) , at x = 1

LHD = $\frac{d}{dx}(x)_{x=1} = 1$

RHD = $\frac{d}{dx}(2-x)_{x=1} = -1$

∵ LHD ≠ RHD, k(x) is not differentiable at x = 1

Sol 7: (C) It should also be continuous for continuity.

LHL = RHL= Vf(x = 1)

$\lim_{x\to1^-} e^x = \lim_{x\to1^+} a-bx = a-b \Rightarrow e^1 = a-b$ . . . (i)

for differentiability

LHD = RHD

$\frac{d}{dx}(e^x)\Big|_{x=1} = \frac{d}{dx}(a-bx)\Big|_{x=1} \Rightarrow e = -b$

Putting in (i) , a = 0

Sol 8: (D) f(x) must also be continuous at x = 2

∴ LHL = RHL = Vf(x = 2)

∴ $\lim_{x\to2^-} x^2+2x+3 = \lim_{x\to2^+} \frac{a}{\pi}\sin(\pi x)+b = 4+4+3$

$\Rightarrow \frac{a}{\pi}\sin 2\pi + b = 11 \Rightarrow b - 11$

for derivability,

LHD = RHD

$\frac{d}{dx}(x^2+2x+3)\Big|_{x=2} = \frac{d}{dx}\left(\frac{a}{\pi}\sin(\pi x)+b\right)\Big|_{x=2}$

$(2x+2)|_{x=2} = (a\cos(\pi x))_{x=2} \Rightarrow 6 = a$

∴ b = 11, a = 6

∴ 2a + bπ = 12 + 11π ≠ 7

b + 2π = 11 + 2π ≠ 3

2a + bπ = 12 + 11π ≠ 13

∴ No correct option.

Sol 9: (D) $f(x) = \begin{cases} x, & x<0 \\ x^2, & 0\le x\le 1 \\ x^3-x+1, & x>0 \end{cases}$

At x = 0

LHL = $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} x = 0$

RHL = $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x^2 = 0$

Vf(x = 0) = 0^2 = 0

∵ LHL = RHL = Vf(x = 0) , continuous at x = 0

for derivability LHD = $\frac{d}{dx}(x)\Big|_{x=0} = 1$

RHD = $\frac{d}{dx}(x^2)\Big|_{x=0} = 2x = 0$

∵ LHD ≠ RHD

f(x) is not derivable at x = 0

At x = 1,

LHL = $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1$

RHL = $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^3-x+1) = 1$

f(x = 1) = 1

LHL = Vf(x = 1) = RHL, function is continuous at x = 1

Now, LHD = $\frac{d}{dx}(x^2)\Big|_{x=1} = 2x|_{x=1}$

RHD = $\frac{d}{dx}(x^3-x+1)\Big|_{x=1} = (3x^2-1)_{x=1} = 2$

∵ LHD = RHD,

f(x) is derivable at x = 1

Sol 10: (D) Discontinuous & also non-derivable at a finite number of points of f.

Sol 11: (A) $f(x) = \begin{cases} \frac{|x|}{\sin x}, & x\ne 0 \\ 1, & x=0 \end{cases}$

At x = 0,

LHL = $\lim_{x\to0^-} \frac{|x|}{\sin x}$

Let x = 0 – h, as x → 0^+, h → 0^+

∴ LHL = $\lim_{h\to0^+} \frac{|-h|}{-\sinh} = -\lim_{h\to0^+} \frac{|h|}{\sinh} = -1$

We see that LHL ≠ Vf(x = 0), therefore, f(x) is not continuous and hence not differentiable at x = 0

Sol 12: (C) f(x) = x^3, g(x) = |x|

fog(x) = $|x|^3$

gof = $|x|^3$

for gof, at x = 0,

gof (x) is continuous , gof(x = 0) = 0

$$LHD = \lim_{h\to 0^+} \frac{gof(0-h)-gof(0)}{-h}$$

$$= \lim_{h\to 0^+} \frac{|(-h)^3|-0}{-h} = \lim_{h\to 0^+} \frac{|h^3|-0}{-h}$$

As $\lim_{h\to 0^+} \frac{h^3}{-h} = 0$

$$RHD = \lim_{h\to 0^+} \frac{gof(0+h)-gof(0)}{h} = \lim_{h\to 0^+} \frac{|h^3|-0}{-h} = 0$$

Since LHD = RHD,

gof(x) is differentiable at x = 0

for fog(x) , at x = 0

$$LHD = \lim_{h\to 0^+} \frac{fog(0-h)-fog(0)}{-h} = \lim_{h\to 0^+} \frac{|-h|^3-0}{-h}$$

$$= \lim_{h\to 0^+} \frac{|h|^3}{-h} = \lim_{h\to 0^+} -h^2 = 0$$

$$RHD = \lim_{h\to 0^+} \frac{fog(0+h)-fog(0)}{h} = \lim_{h\to 0^+} \frac{|h^3|-0}{h}$$

$$= \lim_{h\to 0^+} h^2 = 0$$

Since, LHD = RHD,

fog(x) is differentiable at x = 0

Sol 13: (B) Function can be discontinuous at x = 0

At x = 0

$$LHL = \lim_{x\to 0^-} [x][\sin \pi x]$$

As $x \to 0^-$, $[x] \to -1$, $[\sin \pi x] \to -1$

$\therefore$ LHL = –1 × –1 = 1

$$RHL = \lim_{x\to 0^+} [x][\sin \pi x]$$

as $x \to 0^+$, $[x] \to 0$

$\therefore$ RHL = 0

$\because$ LHL $\neq$ RHL, f(x) is discontinuous at x = 0.

It can also be discontinuous at $x = \frac{1}{2}$

At $x = \frac{1}{2}$

$$LHL = \lim_{x\to \frac{1}{2}^-} [x][\sin \pi x] = 0$$

$$RHL = \lim_{x\to \frac{1}{2}^+} [x][\sin \pi x] = 0$$

continuous at $x = \frac{1}{2}$ ($\because$ LHL = RHL)

At $x = \frac{-1}{2}$,

$$LHL = \lim_{x\to \frac{1}{2}^-} [x][\sin \pi x]$$

as $x \to \frac{-1}{2}$, [x] = –1,

$\sin \pi x \to -1^-$,

$\therefore$ $[\sin \pi x] = -1$

$\therefore$ LHL = – 1 × –1 = 1

$$RHL = \lim_{x\to \frac{-1}{2}^+} [x][\sin \pi x]$$

as $x \to \frac{1}{2}^+$, [x] = –1, $\sin \pi x \to -1\uparrow$

$[\sin \pi x] = -1$

$\therefore$ RHL = –1 × –1 = 1

$\because$ LHL = RHL, continuous at $x = \frac{-1}{2}$

$\therefore$ f(x) is continuous in $x \in (-1, 0) \cup (0, 1)$

Differentiability

Single Correct Choice Type

Sol 1: (D) Considering the case (A), $x|x| = \begin{Bmatrix} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{Bmatrix}$

this is continuous at x = 0

$\because$ RHL = LHL = 0

(B) x^3 is polynomial function which is continuous for $x \in R$.

(C) e^{-x} is exponential function which is continuous for $x \in R$

(D) $f(x) = \begin{Bmatrix} 0 & x \leq 0 \\ 2x & x > 0 \end{Bmatrix}$

$$f'(0^+) = \lim_{h\to 0} \frac{(2(\alpha+h))-0}{h} = \lim_{h\to 0} \frac{2(0+h)}{2} = 2$$

$f'(0^-) = \lim_{h\to 0}\frac{2(0)}{-h} = 0$

$\because$ LHD $\neq$ RHD

$\therefore$ Function is not differentiable

Sol 2: (D) Considering the case A, $f(0^+)$

$$= \lim_{h\to 0}\frac{(x+h)^2\sin\left(\frac{1}{x+h}\right)-0}{h}$$

$$= \lim_{h\to 0}\frac{(x^2+2xh+h^2)\sin\left(\frac{1}{x+h}\right)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

$$f'(0^-) = \lim_{h\to 0}\frac{(-h)^2\sin\left(\frac{1}{-h}\right)}{-h} = \lim_{h\to 0}\ +h\sin\left(\frac{1}{h}\right) = 0$$

$\because$ LHD = RHD

$\therefore$ function is differentiable

(B) x | x |

As function changes the definition at x = 0. So we check differentiability at x = 0.

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$f(0^+) = \lim_{h\to 0}\left[\frac{+(0+h)^2-0}{h}\right] = 0$$

$$f(0^-) = \lim_{h\to 0}\frac{-(0-h)^2-0}{-h} = \lim_{h\to 0} h = 0$$

$\because$ LHD = RHD

so function is differentiable.

(C) $f(x) = \cosh x = \frac{e^x+e^{-x}}{2}$

$$f(0^+) = \lim_{h\to 0}\frac{e^{+h}+e^{-h}-2}{-2h} = \lim_{h\to 0}\frac{e^h-1}{-2h}+\left[\frac{e^{-h}-1}{(-h)}\right]-\frac{1}{2} = 0$$

$$f(0^-) = \lim_{h\to 0}\frac{e^{-h}+e^{h}+2}{-2h} = \lim_{h\to 0}\left(\frac{e^{-h}-1}{-h}\right)+\left(-\frac{1}{2}\right)\frac{e^h-1}{h} = 0$$

$\because$ LHD = RHD

So, given function is differentiable

So, all the three functions are differentiable.

Sol 3: (A) $f(x) = \sin|x| = \begin{cases} \sin x & x \geq 0 \\ -\sin x & x < 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(x)$$

$\therefore$ Function is continuous at x = 0

other than x = 0 the function is a trigonometric sin function which is continuous for $x \in R$.

$$f'(0^-) = \lim_{x\to 0}\frac{-\sin(-h)-0}{-h} = -1$$

$$f'(0^+) = \lim_{x\to 0}\frac{\sin(h)-0}{h} = 1$$

$\because$ LHD $\neq$ RHD

$\therefore$ Function is not derivable at x = 0

Sol 4: (D) $f(x) = |x-3| = \begin{cases} x-3 & x \geq 3 \\ 3-x & x < 3 \end{cases}$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3-x = 0$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-3 = 0$$

$\therefore$ Function is continuous at x = 3

$$f'(3^-) = \lim_{h\to 0}\frac{(3-(3-h))}{-h} = -1$$

$$f'(3^+) = \lim_{h\to 0}\frac{(3+h)-3}{h} = 1$$

$\because$ LHD $\neq$ RHD

Function is continuous at x = 3 and not derivable at x = 3.

Sol 5: (A) We have, $f(x) = \frac{|x-1|}{x-1}, x \neq 1, f(1) = 1$

$$f(x) = \begin{cases} 1 & x > 1 \\ -1 & x < 1 \\ 1 & x = 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = -1 \neq \lim_{x\to 1^+} f(x) = 1$$

$\therefore$ Function is discontinuous at x = 1

If function is discontinuous at x = a then it is also not differentiable at x = a.

Sol 6: (D) $f'(0) = \lim_{h\to 0}\frac{[7-3(0+h)]-7}{h}$

$$= \lim_{h\to 0}\frac{7-3h-7}{h} = -3$$

Sol 7: (A) $f(x) = |x| + |x-1| = \begin{cases} 2x-1 & x>1 \\ 1 & 0<x\le 1 \\ 1-2x & x\le 0 \end{cases}$

$f'(1^-) = \lim_{h\to 0}\frac{1-1}{-h} = 0$

$f'(1^+) = \lim_{h\to 0}\frac{2(1+h)-1-1}{h} = 2$

$f'(0^-) = \lim_{h\to 0}\frac{[1-2(0-h)]-1}{h} = 2$

$f'(0^+) = \lim_{h\to 0}\frac{1-1}{+h} = 0$

∴ Function is not differentiable at x = 0, 1.

Sol 8: (B) $f(x) = \begin{bmatrix} e^{1/x}, & x\ne 0 \\ 0 & x=0 \end{bmatrix}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{\frac{1}{x+h}} - e^{\frac{1}{x}}}{h} = e^{\frac{1}{x}}\lim_{x\to 0^+}\frac{e^{\frac{-h}{x(x+h)-1}}}{h}$

$= e^{\frac{1}{x}}\lim_{x\to 0^+}\frac{e^{\frac{-h}{x(x+h)}}-1}{\frac{-h}{x(x+h)}}\times\frac{1}{x(x+h)}(-1) = \frac{-e^{\frac{1}{x}}}{x^2}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{\frac{1}{x+h}} - e^{\frac{1}{x}}}{h} = \frac{-e^{\frac{1}{x}}}{x^2}$

LHL = RHL $\ne$ f(0) = 0

∴ Function is discontinuous at x = 0

Sol 9: (B) [x] = x – {x} = f(x)

$\left.\begin{aligned} \lim_{x\to a^+}[x] &= a \\ \lim_{x\to a^-}[x] &= a-1 \end{aligned}\right] a\in I$

∴ Function is discontinuous at every integer.

Sol 10: (C) $f(x) = \begin{bmatrix} |x-3| & x<1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & x\ge 1 \end{bmatrix}$

$f'(1^-) = \lim_{h\to 0}\frac{3-(1-h)-2}{-h} = -1$

$f'(1^+) = \lim_{h\to 0}\left[\frac{\left(\frac{(1+h)^2}{4}-\frac{3(1+h)}{2}+\frac{13}{4}\right)-\left(\frac{1}{4}-\frac{3}{2}+\frac{13}{4}\right)}{h}\right]$

$= \lim_{h\to 0}\frac{\frac{(1+h)^2}{4}-\frac{3(1+h)}{2}-\frac{1}{4}+\frac{3}{2}}{-h} = \lim_{h\to 0}\frac{1}{4}\frac{h(2+h)}{h}-\frac{3}{2} = -1$

∴ LHD = RHD

So function is differentiable at x = 1

Sol 11: (C) $f(x) = \begin{cases} 1 & x\ge 0 \\ -1 & x<0 \end{cases}$

$\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$

∴ Function is discontinuous and not differentiable at x = 0

Sol 12: (A) $f(x) = \begin{cases} 2x-a-b & x\ge b \\ b-a & a<x<b \\ a+b-2x & x\le a \end{cases}$ let $b > a$

$\lim_{x\to b^-}(2a-a-b) = b-a = \lim_{x\to b^+}(b-a)$

∴ Function is discontinuous at x = b

$\lim_{x\to a^-}(a+b-2a) = b-a$

$\lim_{x\to a^+}(b-a) = b-a$

∴function is continuous at x = a

$f'(b^+) = \lim_{h\to 0}\frac{(2(b+h)-a-b)-(b-a)}{h} = +2$

$f'(b^-) = \lim_{h\to 0}\frac{(b-a)-(b-a)}{-h} = 0$

$f'(a^+) = 0$

$f'(a^-) = \lim_{h\to 0}\frac{[a+b-2(a-h)]-(b-a)}{-h} = -2$

∴ Function is not differentiable at x = b, a

Sol 13: (D) $f(x) = \begin{bmatrix} 2x-3 & x\ge 2 \\ 1 & 1<x<2 \\ 3-2x & x\le 1 \end{bmatrix}$

f(x) is not differentiable at x = 1 & 2

$f'(1^-) = -2$ and $f'(1^+) = 0$ and $f'(2^-) = 0 f'(2^+) = 2$

Sol 14: (D) $f'(0^-) = \lim_{h\to 0}\frac{[1+\sin(-h)-1]}{-h} = 1$

$f'(0^+) = 0$

given function is not differentiable.

Sol 15: (C) $f'(0^-) = \lim_{h\to 0}\dfrac{\dfrac{-h}{1+6^{\frac{-1}{h}}}}{-h} = \lim_{h\to 0} + \dfrac{1}{1+\dfrac{1}{6^{\frac{1}{h}}}} = +1$

$f'(0^+) = \lim_{h\to 0}\dfrac{\dfrac{h}{1+6^{\frac{1}{h}}}}{h} = \lim_{h\to 0}\dfrac{1}{1+6^{\frac{1}{h}}} = 0$

$\therefore$ function is not differentiable at x = 0

Sol 16: (B) $f'(0^-) = \lim_{h\to 0}\dfrac{(-h)^{\alpha}\sin\left(\dfrac{1}{-h}\right)}{(-h)}$

$f'(0^+) = \lim_{h\to 0}\dfrac{(h)^{\alpha}\sin\left(\dfrac{1}{h}\right)}{(h)}$

$\therefore$ for function to be differentiable

$f'(0^-) = f'(0^+)$

$\Rightarrow -(-h)^{\alpha-1} = (h)^{\alpha-1}$

$\Rightarrow (-1)^{\alpha}(h)^{\alpha-1} = (h)^{\alpha-1}$

$\therefore \alpha$ = even $\Rightarrow \alpha > 1$

Sol 17: (C) $f(x) = \begin{cases} 0 & x \ge 0 \\ 2x & x < 0 \end{cases}$

$f'(0^-) = \lim_{h\to 0}\dfrac{2(-h)-0}{-h} = 2$

$f'(0^+) = 0$

$\therefore$ Function is not differentiable at x = 0.

for $x \ge 0$; f(x) = 0 which is derivable

for x < 0; f(x) = 2x which is a polynomial function

So, it is differentiable at x < 0

Previous Years' Questions

Sol 1: (D) Here $f(x) = \dfrac{\tan\pi[(x-\pi)]}{1+[x]^2}$

Since, we know $\pi[(x-\pi)]$ = nx and tan nπ = 0

$\because 1 + [x]^2 \ne 0$

$\therefore$ f(x) = 0 for all x

Thus, f(x) is a constant function

$\therefore$ f ′ (x) exists for all x.

Sol 2: (A) Since, f(x) is continuous and differentiable where f(0) = 1 thus f(x) is decreasing for x > 0 and concave down.

$\Rightarrow f''(x) < 0$

Therefore, (A) is answer.

Sol 3: (A) Given, $G(x) = -\sqrt{25-x^2}$

$\therefore \lim_{x\to 1}\dfrac{G(x)-G(1)}{x-1} = \lim_{x\to 1}\dfrac{G'(x)-0}{1-0}$

(using L' Hospital's rule)

$= G'(1) = \dfrac{1}{\sqrt{24}}$

$\left[\because G(x) = -\sqrt{25-x^2} \Rightarrow G'(x) = \dfrac{2x}{2\sqrt{25-x^2}}\right]$

Sol 4: (B) For f(x) to be continuous, we must have $f(0) = \lim_{x\to 0} f(x)$

$= \lim_{x\to 0}\dfrac{\log(1+ax)-\log(1-bx)}{x}$

$= \lim_{x\to 0}\dfrac{a\log(1+ax)}{ax} + \dfrac{b\log(1-bx)}{-bx}$ = a · 1 + b · 1

{using $\lim_{x\to 0}\dfrac{\log(1+x)}{x} = 1$)

= a+b $\therefore$ f(0) = (a + b)

Sol 5: Given f(a) = 2, f ′ (a) = 1,

g(a) = –1 g′ (a) = 2

$\therefore \lim_{x\to a}\dfrac{g(x)f(a)-g(a)f(x)}{x-a} = \lim_{x\to a}\dfrac{g'(x)f(a)-g(a)f'(x)}{1-0}$

(using L' Hospital's rule)

= g ′ (a) f(a) – g (a) f ′ (a) = 2(2) – (–1) (1) = 5

Sol 6: (B) $\lim_{x\to\infty}\left(\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\ldots+\dfrac{n}{1-n^2}\right)$

$= \lim_{x\to\infty}\dfrac{1+2+3+\ldots.+n}{(1-n^2)}$

$= \lim_{x\to\infty}\dfrac{n(n+1)}{2(1-n)(1+n)} = \lim_{x\to\infty}\dfrac{\pi}{2(1-n)} = -\dfrac{1}{2}$

Sol 7: (D) since, $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x] \ne 0 \\ 0, & [x] = 0 \end{cases}$

$$\Rightarrow f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & x\in R-[0,1)\\ 0, & 0\le x<1\end{cases}$$

At x = 0

RHL = $\lim_{x\to 0^+} 0 = 0$ and

LHL = $\lim_{x\to 0^-}\dfrac{\sin[x]}{[x]}=\lim_{h\to 0}\dfrac{\sin[0-h]}{[0-h]} = \lim_{h\to 0}\dfrac{\sin(-1)}{-1}=\sin 1$

Since , RHL ≠ LHL

∴ Limit does not exist

Sol 8: (C) Given f(x) = $f(x)=x(\sqrt{x}+\sqrt{x+1})$

⇒ f(x) would exists when x ≥ 0 and x + 1 ≥ 0

⇒ f(x) would exist when x ≥ 0

∴ f(x) is not continuous as x = 0, because LHL does not exist.

Hence, option (C) is correct

Sol 9: (A) Given $f(x)=\dfrac{x}{1+|x|}=\begin{cases}\dfrac{x}{1+x}, & x\ge 0\\ \dfrac{x}{1-x}, & x<0\end{cases}$

$$\therefore\ f'(x)=\begin{cases}\dfrac{(1+x)1\cdot -x\cdot 1}{(1+x)^2}, & x\ge 0\\ \dfrac{(1-x)\cdot 1-x(-1)}{(1-x)^2}, & x<0\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}\dfrac{1}{(1+x)^2}, & x\ge 0\\ \dfrac{1}{(1-x)^2}, & x<0\end{cases}$$

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∴ RHD at x = 0 $\lim_{x\to 0}\dfrac{1}{(1+x)^2}=1$

and LHD at x = 0

$\Rightarrow \lim_{x\to 0}\dfrac{1}{(1-x)^2}=1$

Hence f(x) is differentiable for all x.

Sol 10: (C) Since, $y^2 = P(x)$

On, differentiating both sides, we get $2yy_1 = P'(x)$,

Again, differentiating, we get

$2yy_2 + 2y_1^2 = P''(x)$

$\Rightarrow 2y^3y_2 + 2y^2y_1^2 = y^2P''(x)$

$\Rightarrow 2y^3y_2 = y^2P''(x) - 2(yy_1)^2$

$\Rightarrow 2y^3y_2 = P(x)\cdot P''(x)-\dfrac{\{P'(x)\}^2}{2}$

Again, differentiating, we get

$2\dfrac{d}{dx}(y^3y_2) = P'(x)\Rightarrow P''(x) + P(x)\cdot P'''(x)$

$-\dfrac{2P'(x)\cdot P''(x)}{2}$

$\Rightarrow 2\dfrac{d}{dx}(y^3y_2)=P(x)\cdot P'''(x)$

$\Rightarrow 2\dfrac{d}{dx}\left(y^3\cdot\dfrac{d^2y}{dx^2}\right)=P(x)\cdot P'''(x)$

Sol 11: (B) Given $f(x)=\dfrac{1}{2}x-1$ for $0\le x\le p$

$$\therefore\ [f(x)]=\begin{cases}-1, & 0\le x<2\\ 0 & 2\le x\le\pi\end{cases}$$

$$\tan[f(x)]\ \begin{cases}\tan(-1), & 0\le x<2\\ \tan 0, & 2\le x\le\pi\end{cases}$$

$\Rightarrow \lim_{x\to 2^-}\tan[f(x)]=-\tan 1$

and $\lim_{x\to 2^+}\tan[f(x)]=0$

So, tan f(x) is not continuous at x = 2

Now, $f(x)=\dfrac{1}{2}x-1$

$\Rightarrow f(x)=\dfrac{x-2}{2}\ \Rightarrow\ \dfrac{1}{f(x)}=\dfrac{2}{x-2}$

Clearly, 1/f(x) is not continuous at x = 2

So, tan[f(x)] and $\tan\left[\dfrac{1}{f(x)}\right]$ are both discontinuous at x = 2

Sol 12: (D)

$$\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x}=\lim_{x\to 0}\frac{1}{\sqrt{2}}\cdot\frac{|\sin x|}{x}$$

Atx = 0

RHL $\lim_{h\to 0}\dfrac{1}{\sqrt{2}}\dfrac{\sinh}{h}=\dfrac{1}{\sqrt{2}}$ and LHL = $\lim_{h\to 0}\dfrac{1}{\sqrt{2}}\dfrac{\sinh}{-h}=-\dfrac{1}{\sqrt{2}}$

Here, RHL ≠ LHL

∴ Limit does not exist.

Sol 13: (C) Here, $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$

$$f(x)=\begin{cases}-\cos\left(\frac{2x-1}{2}\right)\pi & ; -1\le x<0\\ 0 & ; 0\le x<1\\ \cos\left(\frac{2x-1}{2}\right)\pi & ; 1\le x<2\\ 2\cos\left(\frac{2x-1}{2}\right)\pi & ; 2\le x<3\end{cases}$$

which shows

RHL = LHL at x = n $\in$ Integer as

if x = 1

$\Rightarrow \lim\limits_{x\to1^+}\cos\left(\frac{2x-1}{2}\right)\pi=0$ and $\lim\limits_{x\to1^-}0=0$

Also, f(1) = 0

$\therefore$ continuous at x = 1

Similarly, when x = 2

$\Rightarrow \lim\limits_{x\to2^+}f(x)=\lim\limits_{x\to2^-}f(x)=0$

Thus, function is discontinuous at no x

Hence, (c) is the correct answer

Sol 14: (B) Given, f(x) = $[\tan^2 x]$

Now, –45° < x < 45°

$\Rightarrow$ tan (–45°) < tan x < tan (45°)

$\Rightarrow$ –tan 45° < tan x < tan (45°)

$\Rightarrow$ – 1 < tan x < 1

$\Rightarrow 0<\tan^2 x<1 \quad \Rightarrow [\tan^2 x]=0$

i.e f(x) is zero for all value of x from

x = –45° to 45°. Thus f(x) exists when x$\Rightarrow$0 and also it is continuous at x = 0.

Also, f(x) is differentiable at x = 0

and has a value of zero.

Sol 15: (D) Given, $f(x)=\begin{vmatrix}x^3 & \sin x & \cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$

On differentiating w.r.t.x, we get

$$f(x)=\begin{vmatrix}3x^2 & \cos x & -\sin x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}+\begin{vmatrix}x^3 & \sin x & \cos x\\ 0 & 0 & 0\\ p & p^2 & p^3\end{vmatrix}$$

$$+\begin{vmatrix}x^3 & \sin x & \cos x\\ 6 & -1 & 0\\ 0 & 0 & 0\end{vmatrix}$$

$$\Rightarrow f'(x)=\begin{vmatrix}3x^2 & \cos x & -\sin x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$$

$$\Rightarrow f''(x)=\begin{vmatrix}6x & -\sin x & -\cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}+0+0$$

$$\text{and } f'''(x)=\begin{vmatrix}6 & -\cos x & -\sin x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}+0+0$$

$$\therefore f'''(0)=\begin{vmatrix}6 & -1 & 0\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=0$$

Sol 16: (A) $f'(1)=\lim\limits_{h\to0}\frac{f(1+h)-f(1)}{h}$

$\Rightarrow f'(1)=\lim\limits_{h\to0}\sin\left(\frac{1}{h}\right)$

$\therefore$ f is not differentiable at $x=1$

Similarly, $f'(0)=\lim\limits_{h\to0}\frac{f(h)-f(0)}{h}$

$$\Rightarrow f'(0)=\lim_{h\to0}\frac{(h-1)\sin\left(\frac{1}{h-1}\right)-\sin(1)}{h}$$

$\Rightarrow$ f is also not differentiable at $x=0$

Sol 17: (A)

$x^{2x}-2x^x\cot y-1=0$ ... (i)

Now x = 1,

$1-2\cot y-1=0\Rightarrow \cot y=0\Rightarrow y=\frac{\pi}{2}$

Now differentiating eq. (i) w.r.t. ‘x’

$$2x^{2x}(1+\log x)-2\left[x^x\left(-\operatorname{cosec}^2 y\right)\frac{dy}{dx}+\cot y\, x^x(1+\log x)\right]=0$$

Now at $\left(1,\frac{\pi}{2}\right)$

$$2\left(1+\log 1\right)-2\left(1\left(-1\right)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}+0\right)=0$$

$$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{\left(t,\frac{\pi}{2}\right)}=0\Rightarrow\left(\frac{dy}{dx}\right)_{\left(t,\frac{\pi}{2}\right)}=-1$$

Sol 18: (A) $f\left(x\right)=\frac{1}{e^{x}+2e^{-x}}=\frac{e^{x}}{e^{2x}+2}$

$$f\left(x\right)=\frac{\left(e^{2x}+2\right)e^{x}-2e^{2x}.e^{x}}{\left(e^{2x+2}\right)^{2}}$$

$$f'\left(x\right)=0 \qquad \Rightarrow e^{2x}+2=2e^{2x}$$

$$e^{2x}=2 \qquad \Rightarrow e^{2x}+2=2e^{2x}$$

Maximum $f\left(x\right)=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$

$0\prec f\left(x\right)\le\frac{1}{2\sqrt{2}}$ $\Rightarrow$ for some $c\in R$

$$f\left(c\right)=\frac{1}{3}$$

Sol 19: (D) f(x) is a positive increasing function

$$\Rightarrow 0\prec f\left(x\right)\prec f\left(2x\right)\prec f\left(3x\right)$$

$$\Rightarrow 0\prec 1\prec\frac{f\left(2x\right)}{f\left(x\right)}\prec\frac{f\left(3x\right)}{f\left(x\right)}$$

$$\Rightarrow \lim_{x\to\infty}1\le\lim_{x\to\infty}\frac{f\left(2x\right)}{f\left(x\right)}\le\lim_{x\to\infty}\frac{f\left(3x\right)}{f\left(x\right)}$$

By sandwich theorem,

$$\Rightarrow \lim_{x\to\infty}\frac{f\left(2x\right)}{f\left(x\right)}=1$$

Sol 20: (A)

$$g'\left(x\right)=2\left(f\left(2f\left(x\right)+2\right)\right)\left(\frac{d}{dx}\left(f\left(2f\left(x\right)+2\right)\right)\right)$$

$$=2f\left(2f\left(x\right)+2\right)f'\left(2f\left(x\right)+2\right).\left(2f'\left(x\right)\right)$$

$$\Rightarrow g'\left(0\right)=2f\left(2f\left(0\right)+2\right).f'\left(2f\left(0\right)+2\right).2\left(f'\left(0\right)=4f\left(0\right)f'\left(0\right)\right)$$

$$=4\left(-1\right)\left(1\right)=-4$$

Sol 21: (D) $\lim_{x\to2}\frac{\sqrt{2\sin^{2}\left(x-2\right)}}{x-2}\Rightarrow\lim_{x\to2}\frac{\sqrt{2}\left|\sin\left(x-2\right)\right|}{x-2}$

R.H.L $=\sqrt{2}$, L.H.L $= =-\sqrt{2}$

Limit does not exist.

Sol 22: (B) $\lim_{x\to0}\frac{\sin\left(p+1\right)+\sin x}{x}=q=\lim_{x\to0}\frac{\sqrt{x+x^{2}}-\sqrt{x}}{x^{3/2}}$

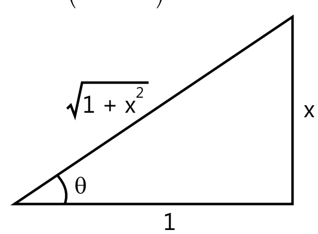
$$\lim_{x\to0}\left(p+1\right)\cos\left(p+1\right)x+\cos x=q=\frac{1}{2}$$

$$\Rightarrow p+1+1=\frac{1}{2}\Rightarrow p=-\frac{3}{2};q=\frac{1}{2}$$

Sol 23: (D) $I=\lim_{x\to0}\frac{\left(1-\cos 2x\right)}{x^{2}}\frac{\left(3+\cos x\right)}{1}.\frac{x}{\tan 4x}$

$$= I=\lim_{x\to0}\frac{2\sin^{2}x}{x^{2}}\frac{3+\cos x}{1}.\frac{x}{\tan 4x} = 2.4.\frac{1}{4}=2$$

Sol 24: (A) $y=\sec\left(\tan^{-1}x\right)$



Let $\tan^{-1}x=\theta$

$$y=\sec\theta$$

$$y=\sqrt{1+x^{2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+x^{2}}}.2x$$

at $x=1$

$$\frac{dy}{dx}=\frac{1}{\sqrt{2}}$$

Sol 25: (B) $\lim_{x\to0}\frac{\sin\left(\pi\cos^{2}x\right)}{x^{2}}$

$$=\lim_{x\to0}\frac{\sin\left(\pi(1-\sin^{2}x\right)}{x^{2}}=\lim_{x\to0}\sin\frac{\left(\pi-\pi\sin^{2}x\right)}{x^{2}}$$

$$=\lim_{x\to0}\sin\frac{\left(\pi\sin^{2}x\right)}{x^{2}} \quad \left[\because\sin\left(\pi-\theta\right)=\sin\theta\right]$$

$$= \lim_{x\to 0} \sin\frac{\left(\pi\sin^2 x\right)}{\left(\pi\sin^2 x\right)}\times\frac{\pi\sin^2 x}{x^2} = \lim_{x\to 0} 1\times\pi\left(\frac{\sin x}{x}\right)^2 = \pi$$

Sol 26: (D) Using, mean value theorem

$$f,(c) = \frac{f(1)-f(0)}{1-0} = 4$$

$$f'(c) = \frac{g(1)-g(0)}{1-0} = 2$$

So, $\boxed{f'(c) = g'(c)}$

Sol 27: (C)

$$\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}\frac{(3+\cos x)}{\frac{\sin 4x}{4x}}\frac{\cos x}{4} = 2$$

Sol 28: (A)

$$g(x) = \begin{cases} k\sqrt{x+1} & 0\le x\le 3 \\ k\sqrt{x+1} & 3 < x \le 5\end{cases}$$

$$L(g'(3)) = \lim_{x\to 3^-}\frac{k\sqrt{x+1}-2k}{x-3} = \lim_{x+3^-}\left\{\frac{(x+1-4)}{(x-3)\left(\sqrt{x+1}+2\right)}\right\} = \frac{k}{4}$$

$$R(g'(3)) = \lim_{x\to 3^+}\frac{mx+2-2k}{x-3}$$

Since this limit exists

$3m+2-2k = 0 \Rightarrow 2k = 3m+2$ ....(i)

So $R(g'(3)) = m$ by L-Hospital rule

Since g(x) is differentiable k = 4m ...(ii)

Solving (i) & (ii)

$m = \frac{2}{5}, k = \frac{8}{5} \quad \Rightarrow k+m = 2$ ....(ii)

Sol 29: (D) $x^2+2xy-3y^2 = 0$

$\Rightarrow 2x+2xy'+2y-6yy' = 0$

$\Rightarrow y' = \frac{x+y}{3y-x}$

At $x=1$, $y=1$ we have $\frac{dy}{dx} = 1$

Equation of normal at (1, 1) is y – 1 = - (x - 1)

$\Rightarrow x+y = 2$

Solving with curve, $x^2+2x(2-x)-3(2-x)^2 = 0$

$\Rightarrow x = 1,3$

$\Rightarrow$ P (1, 1) and Q (3, -1)

So normal meets curve again at (3, -1) in fourth quadrant.

Sol 30: (B) $p = \lim_{x\to 0^+}\left(1+\tan^2\sqrt{x}^{\frac{1}{2x}}\right)$ then log p =

$$p = e^{\lim_{x\to 0^+}\left(1+\tan^2\sqrt{x}-1\right)\frac{1}{2x}} = e^{\lim_{x\to 0^+}\frac{\left(\tan\sqrt{x}\right)^2}{2\left(\sqrt{x}\right)^2} = e^{\frac{1}{2}}}$$

$$\log p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

Sol 31: (A) $p = \lim_{x\to\infty}\left(\frac{(n+1)(n+2)...(n+2n)}{n^{2n}}\right)$

$$\log p = \frac{1}{n}\left(\lim_{x\to\infty}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)\right)$$

$$\log p = \int_0^2 \log(1+x)dx$$

$$\log p = \left(x\log(1+x)\right)_0^2 - \int_0^2\frac{x}{1+x}dx$$

$$\log p = 2\log 3 - \int_0^2\left(1-\frac{1}{1+x}\right)dx$$

$$\log p = 2\log 3 - \left(x-\log(1+x)\right)_0^2$$

$$\log p = 2\log 3 - (2-\log 3)$$

$$\log p = 3\log 3 - 2 = \log\frac{27}{e^2}$$

$$p = \frac{27}{e^2}$$

JEE Advanced/Boards

Exercise 1

Limits

Sol 1: Use L'Hospitals rule

Sol 2:

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{-\sec^2 x}{-\sqrt{2}.\cos x}=\frac{(\sqrt{2})^3}{\sqrt{2}}=2$$

$\therefore a = 2$

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}=\lim_{x\to 1}\frac{-\frac{1}{2\sqrt{x}}}{2.\cos^{-1}x.\left(\frac{-1}{\sqrt{1-x^2}}\right)}$$

$$=\lim_{x\to 1}\frac{\sqrt{1-x^2}}{\sqrt{x}.\cos^{-1}x}=\lim_{x\to 1}\frac{\frac{1}{2\sqrt{1-x^2}}.(-2x)}{\sqrt{x}.\left(\frac{-1}{\sqrt{1-x^2}}\right)+\frac{\cos^{-1}x}{2\sqrt{x}}}$$

$$=\lim_{x\to 1}\frac{-x.2\sqrt{x}}{\sqrt{1-x^2}.\cos^{-1}x-2x}=\frac{1}{4}$$

$\therefore\ r=\frac{1}{4}$

Now, use the formula for sum of infinite terms in a GP.

Sol 3: $\lim_{x\to 1}\left\{\frac{p}{1-x^p}-\frac{q}{1-x^9}\right\}=\lim_{x\to 1}\frac{p(1-x^9)-q(1-x^p)}{(1-x^P)(1-x^9)}$

$$=\lim_{x\to 1}\frac{(P-q)+(9x^p-px^q)}{1-x^p-x^q+x^{P+q}}$$

$$=\lim_{x\to 1}\frac{(q.Px^{P-1}-P.qx^{q-1})}{-P.x^{P-1}-qx^{q-1}+(P+q)x^{P+q-1}}$$

$$=\lim_{x\to 1}\frac{Pq(x^{p-1}-x^{q-1})}{P.x^{P-1}(x^9-1)+9x^{9-1}(x^P-1)}$$

$$=\lim_{x\to 1}\frac{Pq.x^{q-1}(x^{P-q}-1)}{P.x^{P-1}(x^9-1)+q.x^{q-1}(x^P-1)}$$

$$=\lim_{x\to 1}\frac{Pq.x^{9-1}\frac{\left(x^{P-q}-1\right)}{(x-1)}}{P.x^{P-1}\left(\frac{x^q-1}{x-1}\right)+q.x^{q-1}\left(\frac{x^P-1}{x-1}\right)}$$

$$=\frac{pq(p-q)}{pq+pq}=\frac{p-q}{2}$$

Sol 4: $\lim_{x\to\infty}x-\ln\cosh x=\lim_{x\to\infty}x-\ln\left(\frac{e^x+e^{-x}}{2}\right)$

$$=\lim_{x\to\infty}x-\ln\left\{e^x\left(\frac{1+e^{-2x}}{2}\right)\right\}$$

$$=\lim_{x\to\infty}x-\ln e^x-\ln\left(\frac{1+e^{-2x}}{2}\right)=-\ln\left(\frac{1}{2}\right)=\ln 2$$

Sol 5: $\lim_{x\to 0}\frac{\sin^4 3\sqrt{x}}{1-\sqrt{\cos x}}=\lim_{x\to 0}\left\{\frac{\sin 3\sqrt{x}}{3\sqrt{x}}\right\}^4\frac{3^4x^2(1+\sqrt{\cos x})}{1-\cos x}$

$$=\lim_{x\to 0}\frac{3^4.x^2}{2.\sin^2\frac{x}{2}}\times(1+\sqrt{\cos x})\ =3^4\times 2^2=324$$

Sol 6: (a) $\lim_{x\to\frac{1}{\sqrt{2}}}\frac{\cos^{-1}(2x\sqrt{1-x^2})}{x-\frac{1}{\sqrt{2}}}$

Use L'hospitals rule.

(b) $\lim_{x\to\frac{\pi}{4}}\frac{1-\sqrt{\sin 2x}}{\pi-4x}=\lim_{x\to\frac{\pi}{4}}\frac{\frac{-1}{2\sqrt{\sin 2x}}.\cos 2x\times 2}{-4}=0$

(c) $\text{RHL}=\lim_{h\to 0}\frac{4a-105+56}{\sinh\sin(1+h)}=0$

$$\text{LHL}=\lim_{h\to\infty}\frac{64-120+56}{-\sinh\sin(1-h)}=0$$

Hence. Limit = 0

Sol 7: Using L'Hospitals rule

$$\lim_{x\to\frac{3\pi}{4}}\frac{\frac{1}{3(\tan x)^{2/3}}.\sec^2 x}{-2.2\cos x(-\sin x)}$$

$$=\frac{(-\sqrt{2})^2}{3.(-1)^2(-2).2.\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}=-\frac{1}{3}$$

Sol 8:

$$\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}.\cos\frac{x^2}{4}\right]$$

$$=\lim_{x\to 0}\frac{8\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)}{x^8}$$

Now, use $1-\cos 2\theta=2\sin^2\theta$ to get the answer.

Sol 9:

$$\lim_{h\to 0}\frac{2\sin\left(\frac{\pi}{3}+2h\right)\cos 2h-8\sin\left(\frac{\pi}{3}+2h\right)\cosh+6\sin\left(\frac{\pi}{3}+2h\right)}{h^4}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{\pi}{3}+2h\right)\{\cos 2h - 4\cosh + 3\}}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right)\{\cos^2 h - 2\cosh + 1\}}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right).(1-\cosh)^2}{h^4}$$

$$= \lim_{h\to 0} \frac{4\sin\left(\frac{\pi}{3}+2h\right).4.\sin^4 \frac{4}{2}}{\left(\frac{h}{2}\right)^4 .2^4} = \frac{\sqrt{3}}{2}$$

Sol 10:

$$\lim_{x\to\infty} x^2 \left\{\sqrt{\frac{x+2}{x}} - \sqrt[3]{\frac{x+3}{x}}\right\}$$

$$= \lim_{x\to\infty} x^2 \left\{\left(1+\frac{2}{x}\right)^{\frac{1}{2}} - \left(1+\frac{3}{x}\right)^{\frac{1}{3}}\right\}$$

$$= \lim_{x\to\infty} x^2 \left[\left\{1+\frac{1}{2}.\frac{2}{x}+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{2}{x}\right)^2}{21}+.....\right\}\right.$$

$$\left.-\left\{1+\frac{1}{3}\left(\frac{3}{x}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{3}{x}\right)^2}{21}+.....\right\}\right] = \frac{1}{2}$$

Sol 11: $$\lim_{x\to-\infty} \frac{x(3x^2+2).\frac{\sin\frac{1}{x}}{\frac{1}{x}} + |x|^3 + 5}{|x|^3+|x|^2+|x|+1}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{2}{x^2}+\frac{|x|^3}{x^3}+5}{\frac{|x|^3}{x^3}+\frac{|x|^2}{x^3}+\frac{|x|}{x^3}+\frac{1}{x^3}} = \frac{3-1}{-1} = -2$$

Sol 12:

$$I = \lim_{n\to\infty} (n+1)\sin\frac{\pi}{(n+1)} - 2$$

$$= \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n+1}\right)}{\left(\frac{\pi}{n+1}\right)} \times \pi - 2 = \pi - 2$$

$$\therefore \{I\} = \{\pi - 2\} = \pi - 3$$

Sol 13:

(i) $$\lim_{x\to\infty} \frac{x^2+1-ax^2-ax-bx-b}{x+1} = 0$$

$\because$ limit is o

$\therefore$ The numerator should not have terms containing x^2 and x

$\Rightarrow$ $a = 1$ and $b = -1$

(ii) $$\lim_{x\to-\infty} \sqrt{x^2-x+1} - ax - b = 0$$

$$\Rightarrow \lim_{x\to-\infty} \frac{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} - a}{\frac{1}{x}} = b$$

Use L'Hospitals Rule

$$a = 1, \ b = \frac{-1}{2}$$

Sol 14:

$$\lim_{x\to 0} \frac{\ln 1 + \sin^2 x}{\tan(\ln^2 1 + x)} = \lim_{x\to 0} \frac{\frac{\ln(1+\sin^2 x)}{\sin^2 x}\frac{\sin^2 x}{x^2}}{\frac{\tan(\ln^2(1+x))}{\ln^2(1+x)}\left(\frac{\ln(1+x)}{x}\right)^2} = 1$$

Sol 15:

$$\lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)(3.4^{x-1}-3x)}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)\sin(x-1)}$$

$$= \lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)3.\left[\frac{(4^{x-1}-1)}{x-1}+\frac{(1-x)}{x-1}\right]}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)\frac{\sin(x-1)}{x-1}}$$

$$= \lim_{x\to 1} \frac{(\ln(1+x)-\ln 2)3(\ln 4-1)}{\left((7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}\right)}$$

$$= \lim_{x\to 1} \frac{\left(\frac{1}{1+x}\right)3(\ln 4-1)}{\frac{1}{3}(7+x)^{\frac{-2}{3}} - \frac{3}{2}(1+3x)^{\frac{-1}{2}}} = \frac{\left(\frac{1}{2}\right)3(\ln 4-1)}{\frac{1}{3}\left(\frac{1}{4}\right)-\frac{3}{2}\left(\frac{1}{2}\right)}$$

$$= \frac{-9}{4}\ln\frac{4}{e}$$

Sol 16:

$$\lim_{x\to 0} \frac{e^{x^2}-3^{3x}}{\sin\frac{x^2}{2}-\sin x} = \lim_{x\to 0} \frac{\left(\frac{e^{x^2}-1}{x^2}\right)x^2 - \left(\frac{27^x-1}{x}\right)x}{\left(\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\right)\frac{x^2}{2} - \left(\frac{\sin x}{x}\right)x}$$

$$= \lim_{x\to 0} \frac{x^2 - x\ln 27}{\frac{x^2}{2}-x}$$

$$\lim_{x\to 0} \frac{e^{x^2}-3^{3x}}{\sin\left(\frac{x^2}{2}\right)-\sin x}$$

Apply L.H. Rules

$$\lim_{x\to 0} \frac{e^{x^2}.2x - 3^{3x}.(\ln 3).3}{\cos\left(\frac{x^2}{2}\right).x - \cos x} \Rightarrow \frac{0-1.3.\ln 3}{0-1}$$

$\Rightarrow 3\ln 3 \quad \Rightarrow \ln 3^3 \quad \Rightarrow k = 27$

Sol 17: $\lim_{x\to 3} \left(\frac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right)^{\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6}}$

$$= \lim_{x\to 3}\left[\frac{2x+3-x^2}{x+1-(x^2-2x+1)}.\frac{\sqrt{x+1}+x-1}{\sqrt{2x+3}+x}\right]^{\frac{(x^2-2x+1)-(x^2-5)}{(x^2-5x+6)((x-1)+\sqrt{x^2-5})}}$$

$$= \lim_{x\to 3}\left(\left(\frac{x+1}{x}\right)\frac{\sqrt{x+1}+x-1}{\sqrt{2x+3}+x}\right)^{\frac{-2}{(x-2)((x-1)+\sqrt{x^2-5})}}$$

$$= \left[\frac{4}{3}\left(\frac{2+3-1}{3+3}\right)\right]^{\frac{-2}{(1)(2+2)}} = \left(\frac{8}{9}\right)^{\frac{-1}{2}} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}$$

$$= \frac{3}{4}\sqrt{2} = \frac{a\sqrt{6}}{c}$$

$$\therefore \left(a^2+b^2+c^2\right)_{\text{least}} = 3^2+2^2+4^2 = 29$$

Sol 18:

$$\lim_{x\to 0}\frac{1}{x^3}\left[\frac{1}{\sqrt{1+x}} - \frac{1+ax}{1+bx}\right]$$

$$= \lim_{x\to 0}\frac{1}{x^3}\left[(1+x)^{\frac{-1}{2}} - (1+ax)(1+bx)^{-1}\right]$$

$$= \lim_{x\to 0}\frac{1}{x^3}\left[\left(1-\frac{1}{2}x - \frac{\frac{1}{2}\left(\frac{-1}{2}-1\right)}{2}x^2 - \frac{\frac{1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\right) - (1+ax)\left(1-bx+b^2x^2-b^3x^3\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x^3}\left[\left(1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3\right) - \left[1+(a-b)x+b(b-a)x^2-b^2(b-a)x^3\right]\right]$$

$$= \lim_{x\to 0}\frac{1}{x^3}\left[-x\left(a-b+\frac{1}{2}\right)+x^2\left(\frac{3}{8}-b^2+ab\right) + x^3\left(-\frac{5}{16}-b^3+ab^2\right)\right]$$

$$\Rightarrow a-b = \frac{-1}{2} \quad ; \quad \frac{3}{8}-b(b-a) = 0$$

$$\Rightarrow b = \frac{+3}{4}, \quad a = +\frac{1}{4}$$

$$l = \frac{-5}{16}-b^3+ab^2 = \frac{-5}{16}+\frac{9}{16}\left[\frac{1}{2}\right] = \frac{-1}{32}$$

$$\frac{1}{a}-\frac{2}{l}+\frac{3}{b} = 4-2.(-32)+3.\left(\frac{+4}{3}\right) = 72$$

Sol 19: From the given condition we can write that a_n, b_n, c_n are roots of a cubic equation

$$f(n) = x^3-(2n+1)x^2+(2n-1)x+1$$

Clearly 1 is a root to this equation

$\therefore f(x) \equiv (x-1)(x^2-2nx-1)$

$\because a_n < b_n < c_n$

$$a_n = n-\sqrt{n^2+1},\ b_n = 1,\ c_n = n+\sqrt{n^2+1}$$

$$\therefore \lim_{x\to\infty} na_n = \lim_{x\to\infty} n(n-\sqrt{n^2+1})$$

$$= \lim_{x\to\infty} n\frac{-1}{n+\sqrt{n^2+1}} = \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{n_2}}} = \frac{-1}{2}$$

Sol 20: $\lim\limits_{x\to 1}\dfrac{f(x)}{a(x)}=1$ & $\lim\limits_{x\to -2}\dfrac{f(x)}{a(x)}=4$

$\Rightarrow$ $(x-1)$ & $(x+2)$ are factors of f(x)

$\Rightarrow \lim\limits_{x\to 1}\dfrac{a(x-1)(x+2)(x-k)}{(x-1)(x+2)}=1$

$\Rightarrow$ $a(1-k)=$ ...(1)

$\lim\limits_{x\to -2}\dfrac{a(x-1)(x+2)(x-k)}{(x-1)(x+2)}=4$

$\Rightarrow a(-2-k)=4$ ....(2)

$\Rightarrow k=2,\ a=-1$

$\Rightarrow f(x)=-(x-1)(x+2)(x-2)=-x^3+x^2+4x-4$

$\therefore \dfrac{c^2+d^2}{a^2+b^2}=\dfrac{4^2+4^2}{1^2+1^2}=16$

Differentiability

Sol 1: $f(x)=\begin{Bmatrix}2\sin x & x\ge 0\\ 0 & x<0\end{Bmatrix}$

$\lim\limits_{x\to 0^+} f(x)=0;\quad \lim\limits_{x\to 0^-} f(x)=0$

$\therefore$ Function is continuous at x = 0

$f'(0^-)=0$

$f'(0^+)=\lim\limits_{h\to 0}\dfrac{2\sin(0+h)}{h}=2$

$\because f'(0^-)\ne f'(0^+)$

$\therefore$ Function is not differentiable at x = 0

Sol 2: $f(x)=\begin{bmatrix}-\dfrac{x^2}{2} & x\le 0\\ x^n\sin\dfrac{1}{x} & x>0\end{bmatrix}$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} x^n\sin\dfrac{1}{x}=0$ for n > 0

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\left(\dfrac{-x^2}{2}\right)=0$

$f'(0^-)=\lim\limits_{h\to 0}\dfrac{-\dfrac{(0-h)}{2}-0}{-h}=\lim\limits_{h\to 0}\dfrac{h}{2}=0$

$f'(0^+)=\lim\limits_{h\to 0}\dfrac{h^n\sin\dfrac{1}{h}}{h}=\lim\limits_{h\to 0} h^{n-1}\sin\dfrac{1}{h}=0$

for function to be not differentiable

$n-1\le 0 \therefore n\le 1$

Also $n>0$ (for continuous)

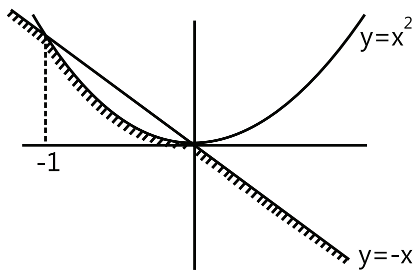
$\therefore 0<n\le 1$

Sol.3 $g(y)=\lim\limits_{x\to y}\left(\dfrac{\tan x-\tan y}{1+\tan x\tan y}\right)\dfrac{1}{\left(1-\dfrac{x}{y}\right)}$

$=\lim\limits_{x\to y} -y\dfrac{\tan(x-y)}{(x-y)}=-x; f(x)=x^2$

$h(x)=\min.(g(y), f(x))$

$x^2=-x; x=0,-1$



$\therefore$ h(x) is not derivable at x = –1, 0

$h(x)=\begin{Bmatrix}-x & x\le -1\\ x^2 & -1<x<0\\ -x & x\ge 0\end{Bmatrix}$

$f'(-1^+)=\lim\limits_{h\to 0}\dfrac{-(-1+h)-1}{+h}=-1$

$f'(-1^-)=\lim\limits_{h\to 0}\dfrac{(-1-h)^2-1}{-h}=2$

$f'(0^+)=-1; f'(0^-)=2$

Sol 4: $f(0)=0,\ f'(0)=1$

$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=1=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}\Rightarrow\lim\limits_{h\to 0}\dfrac{f(h)}{h}=1$

$\therefore \lim\limits_{h\to 0}\dfrac{f(h/2)}{(h/2)}\times\dfrac{1}{2}=\dfrac{1}{2}$

$\therefore \lim\limits_{x\to 0}\left[\dfrac{f(x)}{x}+\dfrac{f(x/2)}{x/2}.\dfrac{1}{2}\ \dfrac{f(x/3)}{x/3}\times\dfrac{1}{3}...\dfrac{f(x/k)}{(x/k)}.\dfrac{1}{k}\right]$

$=1+\dfrac{1}{2}+\dfrac{1}{3}\ ....\ .\dfrac{1}{k}$ hence proved.

Sol 5: $f(x)=\begin{Bmatrix}xe^{-\left\{\frac{2}{x}\right\}} & x>0\\ x & x<0\\ 0 & x=0\end{Bmatrix}$

$\lim_{x\to 0^-} f(x) = 0 \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} \frac{x}{e^{2/x}} = 0$

$\therefore$ Function is continuous at x = 0

$f(0^-) = \lim_{h\to 0}\frac{(0-h)-0}{-h} = 1$

$f(0^+) = \lim_{h\to 0}\frac{he^{-\frac{2}{h}}-0}{h} = \lim_{h\to 0}\frac{1}{e^{2/h}} = 0$

$\therefore$ Function is not differentiable at x = 0.

Sol 6: f(x) = | x – 1 | ([x] – [–x])

$$f(x) = \begin{bmatrix} (x-1)[1-(-2)] = 3(x-1) & x>1 \\ (1-x)[0-(-1)] = (1-x) & x<1 \\ 0 & x=1 \end{bmatrix}$$

$f'(1^+) = \lim_{h\to 0}\frac{3(1+h-1)-0}{h} = 3$

$f'(1^-) = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$

Sol 7: $f(1^+) = \lim_{h\to 0}\frac{\left(-\frac{1}{1+h}\right)+1}{h} = \lim_{h\to 0}\left(\frac{h}{1+h}\right)\frac{1}{h} = 1$

$f(1^-) = \lim_{h\to 0}\frac{[a(1-h)^2-b]-[a-b]}{-h} = \lim_{h\to 0}\left[\frac{ah^2-2ah}{-h}\right] = 2a$

$\therefore$ 2a = 1 (for f(1⁻) = f(1⁺) i. e. function to be derivable.

$\therefore a = \frac{1}{2}$

Also function should be continuous at x = 1.

$\therefore$ a – b = –1;b = 1 + a;b = $\frac{3}{2}$

Sol 8: $f(x) = \begin{bmatrix} x+1 & x<0 \\ 1-x & 0\le x<1 \\ x-1 & x\ge 1 \end{bmatrix}$

$f'(0^-) = 1$

$f'(0^+) = -1$

$f'(1^-) = -1$

$f'(1^+) = 1$

$\therefore$ f(x) is not derivable at x = 0, 1

$\therefore$ m = 2

$g(x) = \begin{bmatrix} x+1 & x<0 \\ (x-1)^2 & x\ge 0 \end{bmatrix}$

$g'(0^-) = 1$

$g'(0^+) = -2$

$\therefore$ g(x) is not differentiable at x = 0

$\therefore$ n = 1

$$g\,of = \begin{bmatrix} x+2 & x<0 \\ x^2 & 0\le x<1 \\ (x-2)^2 & x\ge 1 \end{bmatrix}$$

$f'(0^-) = 1$

$f'(0^+) = 0$

$f'(1^-) = 2$

$f'(1^+) = -2$

gof is not differentiable at x=0,1

$\therefore$ p = 2

m + n + p = 1 + 2 + 2 = 5

Sol 9: $f(x) = \begin{bmatrix} -1 & -2\le x\le 0 \\ x-1 & 0<x\le 2 \end{bmatrix}$

g(x) = f|x| + |f(x)|

$f(x) = \begin{bmatrix} -x-1 & -2\le x\le 0 \\ x-1 & 0<x\le 2 \end{bmatrix}$

$|f(x)| = \begin{bmatrix} 1 & -2\le x\le 0 \\ 1-x & 0<x\le 1 \\ x-1 & 1<x\le 2 \end{bmatrix}$

$\therefore g(x) = \begin{bmatrix} -x & -2\le x\le 0 \\ 0 & 0<x\le 1 \\ 2x-2 & 1<x\le 2 \end{bmatrix}$

$g'(0) = -1$

$g'(0^+) = 0$

$g'(1^-) = 0$

$g'(1^+) = 2$

$\therefore$ g(x) is not differentiable at x = 0, 1

Sol 10: $f(x) = \begin{bmatrix} 0 & 0\le x<1 \\ x & 1\le x<2 \\ 2(x-1) & 2\le x<3 \\ 3(x-1) & x=3 \end{bmatrix}$

$f'(1^-) = 0$

$f'(1^+) = 1$

$f'(2^-) = \lim_{h\to 0}\frac{(2-h)-2}{-h} = 1$

$f'(2^+) = \lim_{h\to 0}\frac{2(2+h-1)-2}{h} = 2$

$\therefore$ Function f(x) is not derivable at x = 1,2.

$\lim_{x\to 1^-} f(x) = 0$ $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

$\therefore$ Function is not continuous at x = 1

$\lim_{x\to 2^-} f(x) = 2$ and $\lim_{x\to 2^+} f(x) = 2$

$\therefore$ Function is continuous at x = 2.

Sol 11: $f(x) = \begin{bmatrix} 3-2x & 1\le x<3/2 \\ 2x-3 & 3/2\le x<2 \\ 2 & x=2 \\ \sin(\pi/2x) & x<1 \end{bmatrix}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sin\frac{\pi}{2}x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3-2x) = 1$

$\therefore$ Function is continuous at x = 1

$\lim_{x\to \frac{3}{2}^-} f(x) = 3 - 2\times\frac{3}{2} = 0$

$\lim_{x\to \frac{3}{2}^+} f(x) = 2\times\frac{3}{2} - 3 = 0$

Function is continuous at $x = \frac{3}{2}$

$\lim_{x\to 2^-} f(x) = 2\times 2 - 3 = 1;$ $\lim_{x\to 2^+} f(x) = 2$

$\therefore$ Function is not continuous at x = 2

$f'(1^-) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$

$f'(1^+) = -2; f'(2^-) = 2; f'(2^0) = 0$

$f'\left(\frac{3^-}{2}\right) = -2; f'\left(\frac{3^+}{2}\right) = 2$

$\therefore$ Function is not derivable at x = 1, $\frac{3}{2}$, 2.

Sol 12: $f(x) = \begin{bmatrix} -1 & -1\le \sin x<\frac{-3}{4} \\ 0 & \frac{-3}{4}\le \sin x<\frac{-1}{2} \\ 1 & \frac{-1}{2}\le \sin x<\frac{-1}{4} \\ 2 & \frac{-1}{4}\le \sin x<0 \\ 3 & \sin x=0 \end{bmatrix}$

$\therefore$ Function is discontinuous at sin x = $\frac{-3}{4}, \frac{-1}{2}, \frac{-1}{4}, 0$

and so is not differentiable at these points.

$\therefore x = \pi + \sin^{-1}\left(\frac{-3}{4}\right)$,

$2\pi - \sin^{-1}\left(\frac{-3}{4}\right), \pi + \sin^{-1}\left(\frac{-1}{2}\right)$,

$2\pi - \sin^{-1}\left(\frac{-1}{2}\right), \pi + \sin^{-1}\left(\frac{-1}{4}\right)$,

$2\pi - \sin^{-1}\left(\frac{-1}{4}\right)$

$\pi, 2\pi$

$\therefore$ Sum of all x = $12\pi = 24\frac{\pi}{2}$

Ans. = 24

Sol 13: $f(x) = \begin{bmatrix} ax(x-1)+b & x<1 \\ x-1 & 1\le x\le 3 \\ px^2+qx+2 & x>3 \end{bmatrix}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$a(1-1) + b = 1 - 1 \Rightarrow b = 0$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$3 - 1 = 9p + 3q + 2$

$3p + q = 0$

$f'(1^-) = 2ax - a\big|_{x=1} = a$

$f'(1^+) = 1$ $\quad\therefore a \ne 1$

$f'(x) = \begin{cases} 2px+q & x>3 \\ 1 & 1\le x\le 3 \end{cases}$

$\therefore 2p\times 3 + q = 1$

$6p + q = 1$

$\therefore p = \frac{1}{3}, q = -1, b = 0, a \neq 1$

Sol 14: $f'(0) = \lim_{h\to 0} \frac{(-h)\left[\frac{\ell n(1-h)+\ell n(1+h)}{\sec(-h)-\cos(-h)}\right]}{(-h)}$

$= \lim_{h\to 0} \frac{\ell n(1-h)(1+h)}{(1-\cos^2 h)}\cos h$

$= \lim_{h\to 0} \frac{-\ell n(1-h^2)}{(-h^2)} \times \frac{\cos h}{\frac{(1-\cos^2 h)}{h^2}} = -1$

$f'(0^+) = a^2 - 3a + 1$

$\therefore a^2 - 3a + 2 = 0$

$(a - 2)(a - 1) = 0$

$a = 1, 2$

$a_1 = 1, a_2 = 2$

$\therefore a_1^2 + a_2^2 = 1 + 4 = 5$

Continuity

Sol 1: (A) $f(x) = \begin{cases} \frac{x}{2} - 1 & , \quad 0 \le x < 1 \\ \frac{1}{2} & , \quad 1 \le x < 2 \end{cases}$

$g(x) = (2x + 1)(x - k) + 3, 0 \le x < \infty$

$g[f(x)] = (2f(x) + 1)(f(x) - k) + 3,$

$0 \le f(x) < \infty$

for $0 \le x < 1$, $f(x) = \frac{x}{2} - 1$

$g(f(x)) = \left[2\left(\frac{x}{2}-1\right)+1\right]\left[\frac{x}{2}-1-k\right] + 3$

$= (x - 1)\left(\frac{x}{2}-(1+k)\right) + 3, 0 \le x < 1$

For $1 \le x < 2$, $f(x) = \frac{1}{2}$

$g(f(x)) = \left(2.\frac{1}{2}+1\right)\left(\frac{1}{2}-k\right) + 3 = 2\left(\frac{1}{2}-k\right) + 3$

For continuity at x = 1, LHL = RHL

$(1 - 1)\left[\frac{1}{2}-(1+k)\right]+3 = 2\left(\frac{1}{2}-k\right) + 3$

$0 = \frac{1}{2} - k \Rightarrow k = \frac{1}{2}$

Sol 2: (C) We have,

$f(x) = \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$

This is $\frac{0}{0}$ form, using L'Hospital rule, limit becomes

$\lim_{x\to 0} \frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{3}(1+x)^{-\frac{2}{3}} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Similar for LHL and RHL

$\therefore f(0) = \frac{1}{6}$

Sol 3: (A) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}}$

this is $\frac{0}{0}$ form

using L'Hospital rule

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\frac{1}{3}(27-2x)^{-2/3}(-2)}{-\frac{3}{5}(243+5x)^{-4/5}(5)}$

$= \frac{\frac{1}{3}\times\frac{-1}{9}\times(-2)}{\frac{3}{5}\times\frac{1}{81}\times(-5)} = \frac{1}{27} \times \frac{81\times 5}{3} \times \frac{2}{5} = 2$

Similar for LHL and RHL

$\therefore$ For function to be continuous,

$f(0) = 2$

Sol 4: (A) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+2ax)-\log(1-bx)}{x}$

$= \lim_{x\to 0} \frac{\log(1+2ax).2a}{2ax} - \frac{\log(1+(-bx).(-b)}{(-bx)}$

$= 2a + b \left(\because \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right)$

Similar for LHL and RHL

$\therefore$ For function to be continuous

$f(0) = \lim_{x\to 0} f(x) = 2a + b$

Sol 5: (C) LHL $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$x = 0 - h$, as $x \to 0^-$, $h \to 0^+$

$\therefore$ LHL = $\lim_{h\to0^+}\frac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1} = \lim_{h\to0^+}\frac{1-e^{1/h}}{1+e^{1/h}} = \lim_{h\to0^+}\frac{e^{1/h}-1}{e^{1/h}+1}$

This is $\frac{\infty}{\infty}$ form, using L Hospital

LHL = $-\lim_{h\to0}\frac{\frac{-1}{h^2}e^{\frac{1}{h}}}{\frac{-1}{h^2}e^{\frac{1}{h}}} = -1$

RHL = $\lim_{x\to0^+}\frac{e^{1/x}-1}{e^{1/x}+1}$

RHL = 1(using result of LHL)

Since, RHL = f(0) , this is right continuous

Sol 6: (C) At x = 1,

f(x = 1) = 1 + $\sin\frac{\pi}{2}$ = 2

LHL = $\lim_{x\to1^-}f(x) = \lim_{x\to1^-}1+\sin\frac{\pi}{2}x$

RHL = $\lim_{x\to1^+}f(x)$ = $\lim_{x\to1^+}ax+b$ = a + b

for continuity at x = 1,

a + b = 2 ... (i)

At x = 3,

f(x = 1) = 6 $\tan\frac{\pi}{12}$. 63 = 6

LHL = $\lim_{x\to3^-}ax+b$ = 3a + b

RHL = $\lim_{x\to3^+}6\tan\frac{\pi x}{12}$ = 6

for continuity at x = 3, 3a + b = 6 ... (ii)

from (i) and (ii) , a = 2, b = 0

Sol 7: (A) LHL = $\lim_{x\to0^-}f(x)$ = 0

$\lim_{x\to0^-}\frac{\cos\pi\frac{[x]}{2}}{[x]}$

at x $\to$ 0, [x] = –1; $\therefore$ LHL = $\frac{\cos\left(-\frac{\pi}{2}\right)}{-1}$ = 0

RHL = $\lim_{x\to0^+}f(x) = \lim_{x\to0^+}\frac{\sin[x]}{[x]+1}$

at x $\to$ 0^+, [x] = 0

$\therefore$ RHL = $\frac{\sin[0]}{0+1}$ = 0

$\therefore$ for continuity,

f(0) = LHL = RHL = 0 = k.

Sol 8: (A) f(0) = x = 0

In neighbourhood of x,

for rational x,

LHL = $\lim_{x\to0^-}x$ = 0

RHL = $\lim_{x\to0^+}x$ = 0

For irrational x,

LHL = $\lim_{x\to0^-}(-x)$ = 0

RHL = $\lim_{x\to0^+}(-x)$ = 0

Since, all limits are 0, $\lim_{x\to0}f(x)$ = 0

Sol 9: (D) f(x) = $[x]^2 - [x^2]$

for x = 1,

f(x = 1) = 1 – 1 = 0

LHL = $\lim_{x\to1^-}f(x)$ = $\lim_{x\to1^-}[x]^2-[x^2]$ = 0 – 0 = 0

RHL = $\lim_{x\to1}f(x)$ = $\lim_{x\to1^+}[x^2]-[x^2]$ = 1 – 1 = 0

$\because$ RHL = LHL = f(1)

continuous at x = 1

At x = N, N $\in$ Z, N $\neq$ 1

f(N) = $[N]^2 - [N^2]$ = 0

RHL = $\lim_{x\to N^+}[x]^2-[x^2]$ = $N^2 - N^2$ = 0

LHL = $\lim_{x\to N^-}[x]^2-[x^2]$ = $(N-1)^2 - (N^2-1)$

(Q[$(N^-)^2$] = $N^2 - 1$

= $N^2 + 1 - 2N - N^2 + 1$ = 2(1 – N) $\neq$ 0, N for N $\neq$ 1

$\therefore$ f(x) is discontinuous for x = N

Sol 10: (C) At x = 2,

f(x = 2) = 4

LHL = $\lim_{x\to2^-}f(x)$ = $\lim_{x\to2^-}(x+2)$ = 4

RHL = $\lim_{x\to2^+}f(x)$ = $\lim_{x\to2^+}(3x-2)$ = 4

$\therefore$ LHL = Vf(x = 2) = RHL, continuous at x = 2.

For x > 2, continuous as 3x – 2 is continuous .

$\therefore$ f(x) is continuous for $x \geq 2$.

Sol 11: (C) It can be discontinuous only at x = 1.

At x = 1

f(x = 1) = 5(1) – 4 = 1

LHL = $\lim_{x\to1^-} f(x) = \lim_{x\to1} 5x - 4 = 1$

RHL = $\lim_{x\to1} f(x) = \lim_{x\to1^+} 4x^2 + 3bx = 4 + 3b$

for continuity

LHL= RHL = Vf(x = 1)

$\Rightarrow 4 + 3b = 1 \Rightarrow b = -1$

Sol 12: (A) Discontinuity can arise only at x = 0

Now, for $x \neq 0$

$$f(x) = \frac{x\left(2-\frac{\sin^{-1}x}{x}\right)}{x\left(2+\frac{\tan^{-1}x}{x}\right)} = \frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}}$$

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}} = \frac{2-\lim_{x\to0}\frac{\sin^{-1}x}{x}}{2+\lim_{x\to0}\frac{\tan^{-1}x}{x}} = \frac{2-1}{2+1} = \frac{1}{3}$$

$\therefore$ For continuity, f(0) = $\frac{1}{3}$

Sol 13: (B) LHL = $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\frac{1-\cos^4 x}{x^2}$

$$= \lim_{x\to0^-}\frac{2\sin^2 2x}{x^2} = 2\lim_{x\to0^-}\left(\frac{\sin 2x}{2x^2}\right)^2 . 4 = 8$$

$$\text{RHL} = \lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to0^+}\frac{\sqrt{x}\left[\sqrt{16+\sqrt{x}}+4\right]}{16+\sqrt{x}-16} = 8$$

$\therefore$ For continuity,

f(0) = a = LHL = RHL = 8

Sol 14: (A) f(x) = 1 + |sin x|

We know, modulus function is continuous for all $x \in R$ and sin x is also continuous.

$\therefore$ |sin x| is also continuous for all $x \in R$

$\therefore$ 1 + |sin x| is continuous for all $x \in R$.

Sol 15: (B) Let f(x) = [x] and g(x) = {x}

we see that both f(x) and g(x) are discontinuous .

But, h(x) = x, which is a continuous function.

Thus, sum of two discontinuous functions may be continuous.

Sol 16: (B) We have, $\lim_{x\to0} f(x) = \lim_{x\to0} x^\alpha \cos\frac{1}{x}$

Let α = n, n > 0

Then, $\lim_{x\to0} f(x) = \lim_{x\to0} x^n \cos\frac{1}{x} = 0.\lim_{x\to0}\cos\frac{1}{x} = 0$

$\therefore$ continuous

for α = 0

$\lim_{x\to0} f(x) = \lim_{x\to0}\cos\frac{1}{x}$,

which is indeterminate

for α = –n, n > 0

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{\cos\frac{1}{x}}{x^n}$$

This is indeterminate

$\therefore$ For f(x) to be continuous , $\alpha > 0$

Sol 17: (A) LHL = $\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\pi-2x}$

Let x = $\frac{\pi}{2}$ – h, as $x \to \frac{\pi}{2}$, $h \to 0^+$

$$\therefore \text{LHL} = \lim_{h\to0^+}\frac{1-\sin\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to0^+}\frac{1-\cos h}{2h}$$

$$= \lim_{h\to0^+}\frac{2\sin\frac{2h}{2}}{2h} = \lim_{h\to0^+}\frac{h}{4}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = 0$$

$$\text{RHL} = \lim_{x\to\frac{\pi}{2}^+}\frac{1-\sin x}{\pi-2x}$$

x = $\frac{\pi}{2}$ + h, as $x \to \frac{\pi}{2}^+$, $h \to 0^+$

RHL = $\lim\limits_{x\to 0^+} \dfrac{1-\sin\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$ = $\lim\limits_{h\to 0^+} \dfrac{1-\cosh}{2h}$

= $\lim\limits_{h\to 0^+} \dfrac{-h}{4}\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$ = 0

∴ For continuity,

$f\left(\frac{\pi}{2}\right)$ = k = LHL = RHL = 0

Sol 18: (C) LHL = $\lim\limits_{x\to 0^+} x\sin\dfrac{1}{x}$ = 0. $\lim\limits_{x\to 0^+} \sin\dfrac{1}{x}$ = 0

RHL = $\lim\limits_{x\to 0^-} x\sin\dfrac{1}{x}$ = x $\lim\limits_{x\to 0^-} \sin\dfrac{1}{x}$ = 0

∴ For continuity,

f(0) = k = LHL= RHL = 0

Sol 19: (B) f(x) can be discontinuous at x = N, N ∈ Z

At x = z, z is an integer

LHL = $\lim\limits_{x\to z^-} f(x)$ = $\lim\limits_{x\to z^-} [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ = (2 – 1) . 0 = 0

RHL = $\lim\limits_{x\to z^+} f(x)$ = $\lim\limits_{x\to z^+} [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ = z.0=0

f(x = 2) ′ = 0

∵ LHL = Vf(x = z) = RHL,

f(x) is continuous

Sol 20: (D) At = $\dfrac{\pi}{4}$

LHL = $\lim\limits_{x\to \frac{\pi}{4}^-} f(x)$ = $\lim\limits_{x\to \frac{\pi}{4}^-} x + a\sqrt{2}\sin x$

π = $\dfrac{\pi}{4}$ + $a\sqrt{2}.\dfrac{1}{\sqrt{2}}$

LHL = $\dfrac{\pi}{4}$ + a

$f\left(x=\dfrac{\pi}{4}\right)$ = 2 . $\dfrac{\pi}{4}\cot\dfrac{\pi}{4}$ + b = $\dfrac{\pi}{2}$ + b

RHL = $\lim\limits_{x\to \frac{\pi}{4}^+} f(x)$ = $\lim\limits_{x\to \frac{\pi}{4}^+} 2x\cot x$ + b = $\dfrac{\pi}{2}$ + b

Now, for continuity at x = $\dfrac{\pi}{4}$,

LHL = $Vf\left(x=\dfrac{\pi}{4}\right)$

∴ $\dfrac{\pi}{4}$ + a = $\dfrac{\pi}{2}$ + b ⇒ a – b = $\dfrac{\pi}{4}$

Sol 21: (C) $\dfrac{|x|}{x}$ = sgn(x), which is discontinuous at x = 0.

Hence, f(x) = |x| + $\dfrac{|x|}{x}$, will be discontinuous

↓ ↓

Continuous at discontinuous

x = 0x = 0

at x = 0

Sol 22: (A) We have, at x = 0

RHL = $\lim\limits_{x\to 0^+} \dfrac{\sin^2 ax}{x}$ = $\lim\limits_{x\to 0^+} a^2x\left(\dfrac{\sin ax}{ax}\right)^2$ = a^2. 0 . 1 = 0

and f(x = 0) = 1

Now, since RHL ≠ Vf(x = 0) it is a discontinuous function

Sol 23: (A) At x = π,

LHL = $\lim\limits_{x\to \pi^+} f(x)$ = $\lim\limits_{x\to \pi^-} \dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$

Let x = π – h, as x → π^-, h → 0^+

∴ LHL = $\lim\limits_{h\to 0^+} \dfrac{1-\sin(\pi-h)+\cos(\pi-h)}{1+\sin(\pi-h)+\cos(\pi-h)}$

= $\lim\limits_{h\to 0^+} \dfrac{1-\sinh+\cosh}{1+\sinh-\cosh}$

This is $\dfrac{0}{0}$ form, using L hospital rule.

LHL = $\lim\limits_{h\to 0^+} \dfrac{\cosh+\sinh}{\cosh+\sinh}$ = –1

RHL = $\lim\limits_{x\to \pi^+} f(x)$ = $\lim\limits_{x\to \pi^+} \dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$

Let π = π + h, as x → π^+, h → $0^{+,}$

∴RHL = $\lim\limits_{x\to \pi^+} \dfrac{1-\sin(\pi+h)+\cos(\pi+h)}{1+\sin(\pi+h)+\cos(\pi+h)}$

= $\lim\limits_{h\to 0^+} \dfrac{1-\sinh-\cosh}{1+\sinh-\sinh}$ = –1 (same expression as in LHL)

Now, for continuity,

f(π) = LHL = RHL = –1

Sol 24: (B) At = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 3x}{\sin x} = \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{\frac{\sin x}{x}}\right).\left(\frac{3x}{x}\right)$$

= 3 × 1 × 1 = 3

for continuity, $f(0) = \lim_{x\to 0} f(x) = 3 = k$

Sol 25: (D) At x = 0, $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}\times 1 = \frac{1}{2}$$

Now, for continuity,

$$f(x=0) = k = \lim_{x\to 0} f(x) = \frac{1}{2}$$

Sol 26: (D) At x = 0,

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x[x]^2\log_{(1+x)} 2$$

At $x \to 0$, $[x] = -1 \Rightarrow [x]^2 = 1$

Let x = 0 – h, as $x \to 0^-$, $h \to 0^+$

$$\text{LHL} = \lim_{h\to 0^+} -h.\log_{(1-h)} 2 = \lim_{h\to 0^+}\frac{-h}{\log(1-h)}.\log 2$$

$$= \log 2 \lim_{h\to 0^+}\frac{(-h)}{\log(1+(-h))}. = \log 2$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\ln(e^{x^2}+2\sqrt{\{x\}})}{\tan\sqrt{x}}$$

for $x \to 0^+$, {x} = p

$$\therefore \text{RHL} = \lim_{x\to 0^+}\frac{\ln(e^{x^2}+2\sqrt{x})}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+}\frac{\ln\left(e^{x^2}\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)\right)}{\tan\sqrt{x}} = \lim_{x\to 0^+}\frac{\ell n e^{x^2}+\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+}\frac{x^2}{\tan\sqrt{x}} + \frac{\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}.\frac{2\sqrt{x}}{e^{x^2}}.\frac{1}{\tan\sqrt{x}}$$

$$= \lim_{x\to 0^+} x^{3/2}\left(\frac{\sqrt{x}}{\tan\sqrt{x}}\right) + \lim_{x\to 0^+}\frac{\ell n\left(1+\frac{2\sqrt{x}}{e^{x^2}}\right)}{\frac{2\sqrt{x}}{e^{x^2}}}$$

$$\lim_{x\to 0^+}\frac{2\sqrt{x}}{\tan\sqrt{x}}.\lim_{x\to 0^+}\frac{1}{e^{x^2}} = 0.(1) + 1\times 2\times 1 = 2$$

LHL ≠ RHL, this is an irremovable discontinuity.

Sol 27: (C) f(x) = [2 + 3 sinx], $x \in [0, \pi]$

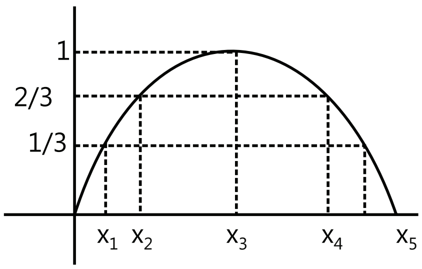
f(x) will be discontinuous at where $f(x) \in z$

we need to find x for which f(x) = z

Now, for $x \in (0, \pi)$, f(x) = z,

for sin x = 0, $\frac{1}{3}$, $\frac{2}{3}$, 1

Discontinuous for 5 values of x marked in graph of sin x for $x \in (0, \pi)$



Sol 28: (C) We have $\lim_{x\to a} f(x) = 0$, as graph of f(x) passes through (a, 0)

$$\text{Now, } L = \lim_{x\to a}\frac{\log_e(1+3f(x))}{2f(x)}$$

Let f(x) = g, as $x \to a$, f(x) = 0

$$\therefore L = \lim_{y\to 0}\frac{\log_e(1+3y)}{2y\times 3}\times 3 = \frac{3}{2}\lim_{y\to 0}\frac{\log_e(1+3y)}{3y}$$

$$L = \frac{3}{2}$$

Sol 29: (B) We have, $X^2 + f(x) - 2x - \sqrt{3}f(x) + 2\sqrt{3} - 3 = 0$

$$f(x) = \frac{(x^2+2x+2\sqrt{3}-3)}{x-\sqrt{3}} = \frac{-(x-\sqrt{3})(x-(2-\sqrt{3}))}{(x-\sqrt{3})}$$

Now, since f(x) is continuous

$$f(\sqrt{3}) = \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}}\frac{(x-\sqrt{3})(x-2+\sqrt{3})}{(x-\sqrt{3})}$$

$$= -(\sqrt{3} - 2 + \sqrt{3}) = 2(1 - \sqrt{3})$$

Sol 30: (C) We have

$$f(x) = \begin{cases} x^2, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$$

for any real number k, in the neighbour of k, for irrational numbers.

LHL = RHL = $\lim_{x\to k} x^2 = k^2$

For rational number,

LHL = RHL = $\lim_{x\to k} 1 = 1$

f(x) = will be continuous if both of these limits are equal,

i. e. $k^2 = 1 \Rightarrow k = \pm 1$

Sol 31: (B) $f(x) = \lim_{n\to\infty} \frac{x^n - \sin x^n}{x^n + \sin x^n} = \lim_{n\to\infty} \frac{1 - \left(\frac{\sin x^n}{x^n}\right)}{1 + \left(\frac{\sin x^n}{x^n}\right)}$

Now, at x = 1,

$$\text{LHL} = \lim_{x\to 1^-} \lim_{n\to\infty} \frac{1 - \left(\frac{\sin x^n}{x^n}\right)}{1 + \left(\frac{\sin x^n}{x^n}\right)}$$

As $x \to 1^-$ and $n \to \infty$, $x^n \to 0$

$$\lim_{x\to 1^-} \lim_{n\to\infty} \left(\frac{\sin x^n}{x^n}\right) = 1$$

$\therefore$ LHL = $\frac{1-1}{1+1}$ = 0

$$\text{RHL} = \lim_{x\to 1^+} \lim_{n\to\infty} \frac{1 - \left(\frac{\sin x^n}{x^n}\right)}{1 + \left(\frac{\sin x^n}{x^n}\right)}$$

Ax $x \to 1^+$ and $n \to \infty$, $x^n \to \infty$ and $\sin x^n \in [-1,1)$

$\therefore \lim_{x\to 1^+} \lim_{n\to\infty} \frac{\sin x^n}{x^n} = 0$

$\therefore$ RHL = $\frac{1-0}{1+0}$ = 1

$\because$ LHL $\neq$ RHL

The function has a finite discontinuity at x = 1

Sol 32: (A) |sin x| is not differentiable at x = 0. Therefore, for f(x) to be differentiable, a = 0

$e^{|x|}$ is not differentiable at x = 0. Therefore for f(x) to be differentiable at x = 0, b = 0.

$|x|^3$ is differentiable at x = 0. Therefore c can be any real number for f(x) to be different.

$\therefore$ a = 0, b = 0, $c \in R$

Sol 33: (D) $f(x) = a^{[x^2]}$, a > 1 will be differentiable at points where $[x^2]$ is not continuous . Now, for $x \in (1, 3)$,

$[x^2]$ will not be continuous at x = $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ (2) , $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, i. e. total 7 points. for which x^2 is an integer.

Sol 34: (D) Noting the definition of [x], f[x] becomes

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ 3x, & x = 3 \end{cases}$$

from def^n of f(x) , we note that f(x) is discontinuous at x = 1, 2, 3 and at these points only, the function will be non-differentiable.

Sol 35: (D) At x = 0,

LHL = $\lim_{x\to 0^-} x + \{x\} + x\sin\{x\}$

at $x \to 0$, {x} = 1

LHL = $\lim_{x\to 0^-} x + 1 + x\sin 1$ = 1

RHL = $\lim_{x\to 0^+} x + \{x\} + x\sin\{x\}$

as $x \to 0^+$, {x} = 0

$\therefore$ RHL = $\lim_{x\to 0^+} x = 0$

$\therefore$ LHL $\neq$ RHL

f(x) is not continuous at x = 0

At, x = 2

LHL = $\lim_{x\to 2^-} x + \{x\} + x\sin\{x\}$

at as $x \to 2^-$, {x} = 1

$\therefore$ LHL = $\lim_{x\to 2^-} x + 1 + x\sin 1$

= 3 + 2 sin 1

RHL = $\lim_{x\to 2^+} x + \{x\} + x\sin\{x\}$

as $x \to 2^+$, {x} = 0

$\therefore$ RHL = $\lim_{x\to 2^+} x + 0 + x\sin 0$ = 2

$\because$ RHL $\neq$ LHL, f(x) is not continuous at x = 0

Therefore, the function is not continuous at x = 0 and x = 2.

Multiple Correct Choice Type

Sol 36: (A, C) f(x) = $\begin{cases} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At x = 0

LHL = $\lim\limits_{x\to 0^-} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}$

x = 0 – h, as x → 0⁻, h → 0⁺

$\therefore$ LHL = $\lim\limits_{h\to 0^+} \dfrac{-h\ln(\cos(-h))}{\ln(1+h^2)}$

$$=-\lim_{h\to 0^+} \frac{\ln\left(1-\left(2\sin^2\frac{h}{2}\right)\right)}{\left(-2\sin^2\frac{h}{2}\right)}.\frac{1}{\frac{\ln(1+h^2)}{h^2}}.\frac{-2\sin^2\frac{h}{2}}{\frac{h^2}{4}}.\frac{h}{4}$$

$= -1 \times 1 \times -2 \times \dfrac{0}{4} = 0 \lim\limits_{h\to 0}\dfrac{\sinh}{h} = 1$

RHL = $\lim\limits_{x\to 0^+} \dfrac{x\ln(\cos x)}{\ln(1+x^2)}$

Proceeding similar to LHL, we get RHL = 0

Since, LHL = Vf(x = 0) = RHL,

f(x) is continuous at x = 0

Now, let L = $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0}\frac{h\ln(\cosh)}{\ln(1+h^2)h} = \lim_{h\to 0}\frac{\ln(\cosh)}{\ln(1+h^2)}$$

$$= \lim_{h\to 0^+}\frac{\ln\left(1-2\sin^2\frac{h}{2}\right)}{-2\sin^2\frac{h}{2}}.\frac{1}{\frac{\ln(1+h^2)}{h^2}}.\frac{-2\left(\sin^2\frac{h}{2}\right)^2}{\left(\frac{h}{2}\right)^2}.\frac{1}{4}$$

$= 1 \times 1 \times (-2) \times \dfrac{1}{4}$

Since L exists, f(x) is differentiable at x = 0 and f ′(x) = $\dfrac{-1}{2}$

Exercise 2

Limits

Single Correct Choice Type

Sol.1: (A)

$$\lim_{n\to\infty}\frac{1^2n+2^2(n-1)+3^2(n-2)+....+n^2(n-(n-1))}{1^3+2^3+....+n^3}$$

$$=\lim_{n\to\infty}\frac{n(1^2+2^2+3^2+....+n^2)-(2^2+2.3^2+3.4^2+...+(n-1)n^2)}{\frac{n^2(n+1)^2}{4}}$$

$$=\lim_{n\to\infty}\frac{n\frac{n(n+1)(2n+1)}{6}-(1\ \ +2^3+3^3+....+n^3)(1+2^2+3^2+...+n^2)}{\frac{n^2(n+1)^2}{4}}$$

$$=\lim_{n\to\infty}\frac{n\sum n^2-\sum n^3+\sum n^2}{\sum n^3}$$

$$=\lim_{n\to\infty}\frac{(n+1)\frac{n(n+1)(2n+1)}{6}-\frac{n^2(n+1)^2}{4}}{\frac{n^2(n+1)^2}{4}}=\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{4}}=\frac{1}{3}.$$

Sol.2: (A) $l=\lim\limits_{n\to\infty}\left[(\sqrt{x^2+2x}-x)\left(\dfrac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)\right]$

$$=\lim_{n\to\infty}\left[\frac{2x}{\sqrt{x^2+2x}+x}\right]=1$$

$$m=\lim_{n\to\infty}\left\{(\sqrt{x^2-2x}+x)\left(\frac{\sqrt{x^2-2x}-x}{\sqrt{x^2-2x}-x}\right)\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{-2x}{\sqrt{x^2-2x}-x}\right\}=0$$

Sol 3: (B) $\lim\limits_{x\to\infty}x^3\left(\sqrt{x^2+\sqrt{x^4+1}}-\sqrt{2}x\right)$

$$=\lim_{x\to\infty}\frac{x^3\left(x^2+\sqrt{x^4+1}-2x^2\right)}{\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x}$$

$$=\lim_{x\to\infty}\frac{x^3\left(\sqrt{x^4+1}-x^2\right)}{\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x}$$

$$= \lim_{x\to\infty} \frac{x^3\left(x^4+1-x^4\right)}{\left(\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x\right)\left(\sqrt{x^4+1}+x^2\right)}$$

$$= \lim_{x\to\infty} \frac{x^3}{\left(\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}x\right)\left(\sqrt{x^4+1}+x^2\right)}$$

$$= \lim_{x\to\infty} \frac{1}{\left(\sqrt{1+\sqrt{\frac{1}{x^4}}}+\sqrt{2}\right)\left(\sqrt{1+\frac{1}{x^4}}+1\right)} = \frac{1}{4\sqrt{2}}$$

Sol.4: (A)

$$\lim_{x\to\pi-0} \tan^{-1}\left(2\tan\frac{x}{2}\right) = \lim_{h\to 0}\tan^{-1}\left(2\tan\left(\frac{\pi-h}{2}\right)\right)$$

$$= \lim_{h\to 0}\tan^{-1}\left(2\cot\frac{h}{2}\right) = \frac{\pi}{2}$$

$$\lim_{x\to\pi+0} \tan^{-1}\left(2\tan\frac{x}{2}\right) = \lim_{h\to 0}\tan^{-1}\left(2\tan\left(\frac{\pi+h}{2}\right)\right)$$

$$= \lim_{h\to 0}\tan^{-1}\left(-2\cot\frac{h}{2}\right) = -\frac{\pi}{2}$$

Sol 5: (D) $x_n = x_{n-1} + x_{n-2}$

$$\Rightarrow \frac{x_n}{x_{n-1}} = 1 + \frac{x_{n-2}}{x_{n-1}} = 1 + \frac{1}{\frac{x_{n-1}}{x_{n-2}}}$$

let $\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = l \Rightarrow l = 1 + \frac{1}{l} \Rightarrow l^2 - l - 1 = 0$

$$\Rightarrow l = \frac{l \pm \sqrt{5}}{2} \Rightarrow l = \frac{\sqrt{5}+1}{2} \quad (\text{as } l > 0)$$

Sol 6: (B) $\lim_{x\to\alpha}(1+ax^2+bx+c)^{\frac{1}{x-\alpha}} = e^k$

$$k = \lim_{x\to\alpha}\frac{1}{x-\alpha}(1+ax^2+bx+c) = \lim_{x\to\alpha}\frac{a(x-\alpha)(x-\beta)}{x-\alpha} = a(\alpha-\beta)$$

Multiple Correct Choice Type

Sol 7: (A,B,C)

(a) $$\lim_{x\to 1}\frac{1-|\cos(x-1)|}{(x-1)^2} = \lim_{x\to 1}\frac{1-\cos^2(x-1)}{(x-1)^2(1+|\cos(x-1)|)}$$

$$= \lim_{x\to 1}\frac{\sin^2(x-1)}{(x-1)^2(1+|\cos(x-1)|)} = \frac{1}{2}$$

(b) $$\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)^{1/x} = e^k; k = \lim_{x\to 0^+}\frac{1}{x}\left(\frac{\tan x - x}{x}\right)$$

$$= \lim_{x\to 0^+}\frac{\sec^2 x - 1}{2x} = \lim_{x\to 0^+}\left(\frac{x}{2}\right)\left(\frac{\tan x}{x}\right)^2 = 0$$

(c) $$\lim_{x\to\infty}\frac{\sin x}{x} = 0\times\sin\infty = 0\times\text{something defined}$$

$$= 0(\text{as } -1 \le \sin x \le 1)$$

(d) $$\lim_{x\to\infty}\frac{\tan x}{x} = 0\times\sin\infty = 0\times\text{something not defined}$$

$$= \text{not defined}$$

Sol 8: (A,B,C) $l = \lim_{x\to 0^+}(\cos x)^{1/x} = e^K$

$$K = \lim_{x\to 0^+}\frac{1}{x}(\cos x - 1) = \lim_{x\to 0^+}\frac{-2\sin^2 x/2}{x^2}.x = 0$$

$$m = \lim_{x\to 0^+}(\cos x)^{1/x^2} = e^a$$

$$a = \lim_{x\to 0^+}\frac{1}{x^2}(\cos x - 1) = \lim_{x\to 0^+}\frac{-2\sin^2 x/2}{x^2}.x = -\frac{1}{2}$$

Sol 9: (A,B,C,D) (a) $\lim_{x\to 0^+}(\cot x)^{\frac{\sin x - x}{x^4}}$

$$\frac{\sin x - x}{x^4} = \frac{1}{x^4}\left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) - x\right]$$

$$= -\frac{1}{x}\left(\frac{1}{3!} - \frac{x^2}{5!} + ...\right)$$

$$\Rightarrow \lim_{x\to 0^+}\frac{\sin x - x}{x^4} = -\infty; \lim_{x\to 0^+}(\cot x)^{\frac{\sin x - x}{x^4}} = \infty^{-\infty} = 0$$

(b) $\lim_{x\to 0^+}(\cot x)^{\frac{x^3}{\sin x - x}}$

$$\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + ...$$

$$\Rightarrow \lim_{x\to 0^+}\frac{x^3}{\sin x - x} = -6; \lim_{x\to 0^+}(\cot x)^{\frac{x^3}{\sin x - x}} = \infty^{-6} = 0$$

(c) $\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$

$=\lim_{x\to\infty}\frac{2}{\sqrt{1+1/x^2}+1}=0$

(d) $\lim_{x\to\infty}\sqrt{x^2+2x}-x=\lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$

Sol 10: (C, D) $f(x)=\frac{x2^x-x}{1-\cos x}$

$$\lim_{x\to0}\frac{x2^x-x}{1-\cos x}=\lim_{x\to0}\frac{x\left[\left(1+x\ln2+\frac{x^2}{2!}(\ln2)^2+...\right)-1\right]}{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}...\right]}$$

$$=\lim_{x\to0}\frac{\ln2+\frac{x}{2!}(\ln2)^2+...}{\frac{1}{2!}-\frac{x^2}{4!}+...}=2\ln2$$

$g(x)=2x\sin\left(\frac{\ln2}{2^x}\right)$ as $x\to\infty\Rightarrow -x\to-\infty\Rightarrow 2^{-x}\to0$

$\therefore \lim_{x\to\infty}2x\sin\left(\frac{\ln2}{2^x}\right)=\lim_{y\to0}\frac{\sin(y\ln2)}{y}=\ln2$

Sol 11: (A,B,C)

(a) $\lim_{t\to0}\frac{\sin(\tan t)}{\sin t}=\lim_{t\to0}\frac{\sin(\tan t)}{\tan t}\times\frac{t}{\sin t}\times\frac{\tan t}{\sin t}=1$

(b) $\lim_{x\to\pi/2}\frac{\sin(\cos x)}{\cos x}=1\ \left[\because\ \cos x\to0 \text{ as } x\to\pi/2\right]$

(c) $\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to0}\frac{2}{\sqrt{1+x}-\sqrt{1-x}}=\frac{2}{2}=1$

(d) $\lim_{x\to0}\frac{\sqrt{x^2}}{x}=\lim_{x\to0}\frac{|x|}{x}$ (limit doesn't exists)

Sol 12: (A, B, D)

(a) $\lim_{x\to\infty}x^{\frac{1}{4}}\frac{\sin\frac{1}{x}}{\frac{1}{x}}\times\frac{1}{x}=\lim_{x\to\infty}x^{\frac{-3}{4}}=0$

(b) $\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$=\lim_{x\to\frac{\pi}{2}}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2\frac{\sin x}{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right).\sin x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}=0$$

(c) $\lim_{x\to\infty}\frac{2x^2+3}{x^2+x-5}.\text{sgn}(x)$

$$=\lim_{x\to\infty}\frac{2+\frac{3}{x^2}}{1+\frac{1}{x}-\frac{5}{x^2}}.\text{sgn}(x)=2.$$

(d) $\lim_{x\to0}\frac{[3+h]^2-9}{(3+h)^2-9}=0$

Hence A, B and D

Sol 13: (A, B, C, D)

(a) $\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{\tan x}\right)=\lim_{x\to0}\frac{\tan x-x}{x\tan x}$

$$=\lim_{x\to0}\frac{\sec^2x-1}{\tan x+x.\sec^2x}=\lim_{x\to0}\frac{\tan^2x}{\tan x+x\sec^2x}$$

$$=\lim_{x\to0}\frac{\sin^2x}{\sin x.\cos x+x}=\lim_{x\to0}\frac{\frac{\sin x}{x}.\sin x}{\frac{\sin x}{x}.\text{cox}+\frac{x}{x}}=0$$

(b) $\lim_{x\to\infty}\left(\frac{3x^2+1}{2x^2-1}\right)^{x^3/1-x}=\left(\lim_{x\to\infty}\frac{3x^2+1}{2x^2-1}\right)^{\lim_{x\to\infty}\frac{x^3}{1-x}}$

$$=\left(\frac{3}{2}\right)^{\lim_{x\to\infty}\frac{x^3}{1-x}}=0$$

(c) $\lim_{x\to\frac{\pi}{4}^+}\left\{\tan\left(x+\frac{\pi}{8}\right)\right\}^{\tan2x}=\left(\tan\left(\frac{3\pi}{8}\right)\right)^{-\infty}=0$

(d) $\lim_{x\to1}\frac{(x^2-1)^2}{(x-1)(x^2+x+1)}=\lim_{x\to1}\frac{(x-1)^2(x+1)^2}{(x-1)(x^2+x+1)}=0$

Hence A, B, C and D.

Differentiability

Sol 1: (A) $\lim_{x\to0^+}f(x)=\lim_{x\to0^+}f(1-x)=1$

$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}e^x=e^0=1$

$\therefore$ function is continuous at x = 0 for$x < 1$, $f(x) = 1 - x$

$x > 1$, $f(x) = x - 1$

$\therefore$ function is not differentiable at x = 1.

$$f'(0^-) = \lim_{h\to 0} \frac{e^{(-h)} - 1}{-h} = 1$$

$$f'(0^+) = \lim_{h\to 0} \frac{(1-h)-1}{h} = -1$$

$f'(0^-) \neq f'(0^+)$

$\therefore$ Function is not differentiable at x = 0.

Sol 2: (A) $f'(1^-) = \lim_{h\to 0} \dfrac{\sin^{-1}(1-h) - \sin^{-1} 1}{-h}$

$$f'(1^+) = \lim_{h\to 0} \frac{\sin^{-1}(1+h) - \sin^{-1} 1}{h}$$

$\therefore$ Differentiation of $\sin^{-1}x$ is not defined for $x \geq 1$

$\therefore$ $\sin^{-1}x$ is not differentiable at x = 1

All other function are continuous at x = 1 and derivable at x = 1

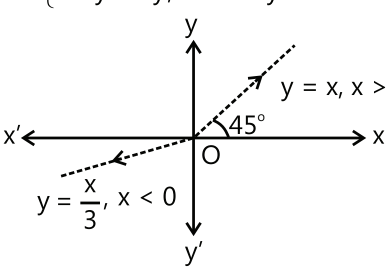
$$\frac{d\tan x}{dx} = \sec^2 x \,;\ \frac{da^x}{dx} = a^x \log a$$

$$\frac{d\cosh x}{dx} = \frac{d\dfrac{e^x + e^{-x}}{2}}{dx} = \frac{e^x}{2} - \frac{e^{-x}}{2} = \frac{e^x - e^{-x}}{2}$$

Previous Years' Questions

Sol 1: (A, B, D) Since$x + |y| = 2y$

$$\Rightarrow \begin{cases} x + y = 2y, & \text{when } y > 0 \\ x - y = 2y, & \text{when } y < 0 \end{cases}$$



$$\Rightarrow \begin{cases} y = x, & \text{when } y > 0 \Rightarrow x > 0 \\ y = \dfrac{x}{3}, & \text{when } y < 0 \Rightarrow x < 0 \end{cases}$$

which could be plotted as,

Clearly, y is continuous for all x but not differentiable at x = 0,

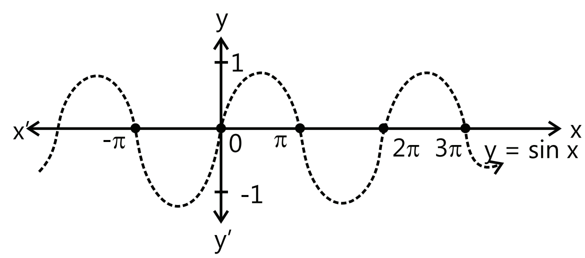
Also, $\dfrac{dy}{dx} = \begin{cases} 1, & x > 0 \\ 1/3, & x < 0 \end{cases}$

Thus, f(x) is defined for all x, continuous at x = 0, differentiable for all

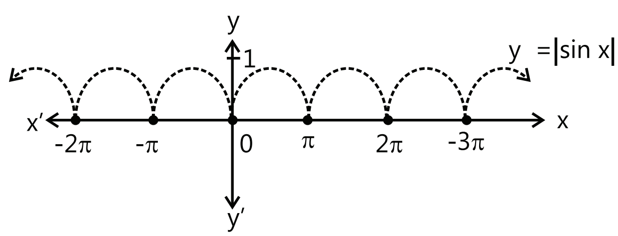
$x \in R - \{0\}$, $\dfrac{dy}{dx} = \dfrac{1}{3}$ for $x < 0$

Sol 2: (B, D) We know, $f(x) = 1 + |\sin x|$ could be plotted as,

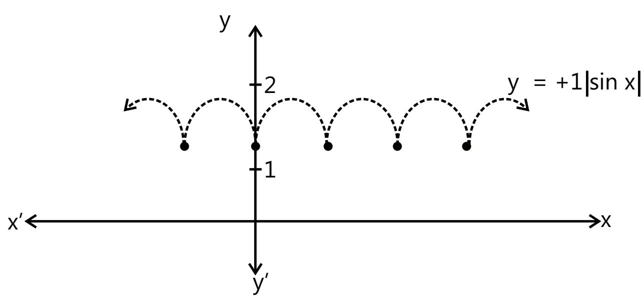
(i) $y = \sin x$ ...(i)



(ii) $y = |\sin x|$..(ii)



(iii) $y = 1 + |\sin x|$...(iii)|



Clearly, $y=1 + |\sin x|$ is continuous for all x, but not differentiable at infinite number of points

Sol 3: (A, B, D) We have, for $-1 < x < 1$

$\Rightarrow 0 \leq x \sin \pi x \leq 1/2 \ \therefore [x \sin \pi x] = 0$

Also, $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by definition of [x] $f(x) = [x \sin \pi x] = -1$, when $1 < x < 1 + h$

Thus, f(x) is constant and equal to 0 in the closed interval (–1, 1) and so f(x) is continuous and differentiable in the open interval (–1, 1)

At x = 1, f(x) is discontinuous, since $\lim_{h\to 0}(1-h)=0$

and $\lim_{h\to 0}(1+h)=1$,

$\therefore$ f(x) is not differentiable at x = 1

Hence, (A) (B) and (D) are correct answers.

Sol 4: (A, B) Here, $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

$\therefore$ RHL at x = 1,

$\Rightarrow \lim_{h\to 0}|1+h-3|=2$

LHL at x = 1

$=\lim_{h\to 0}\frac{(1-h)^2}{4}-\frac{3(1-h)}{2}+\frac{13}{4}=\frac{1}{4}-\frac{3}{2}+\frac{13}{4}=\frac{14}{4}-\frac{3}{2}=2$

$\therefore$f(x) is continuous at x = 1

Again, $f(x)=\begin{cases}-(x-3), & 1\le x<3\\ (x-3), & x\ge 3\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$

$\therefore\ f'(x)=\begin{cases}-1, & 1\le x<3\\ 1, & x\ge 3\\ \frac{x}{2}-\frac{3}{2}, & x<1\end{cases}$

$\left.\begin{array}{l}\text{RHD at } x=1\Rightarrow \quad -1\\ \text{LHD at } x=1\Rightarrow \quad \frac{1}{2}-\frac{3}{2}=-1\end{array}\right]$

differentiable at x = 1

Again $\left.\begin{array}{l}\text{RHD at } x=3 \Rightarrow 1\\ \text{LHD at } x=3 \Rightarrow -1\end{array}\right]$ not

differentiable at x = 3

Sol 5: (B, C) The function f(x) = tan x is not defined at $x=\frac{\pi}{2}$, so f(x) is not continuous on (0, π).

Since, g(x) = x sin $\frac{1}{x}$ is continuous on (0, π) and the integral function of a continuous function is continuous,

$\therefore\ f(x)=\int_0^x t\left(\sin\frac{1}{t}\right)dt$ is continuous on (0, π)

Also, $f(x)=\begin{cases}1, & 0<x\le\frac{3\pi}{4}\\ 2\sin\left(\frac{2x}{9}\right), & \frac{3\pi}{4}<x<\pi\end{cases}$

We have, $\lim_{x\to\frac{3\pi^-}{4}} f(x)=1$

$\lim_{x\to\frac{3\pi^+}{4}} f(x)=\lim_{x\to\frac{3\pi}{4}} 2\sin f\left(\frac{2x}{9}\right)=1$

So, f(x) is continuous at $x=\frac{3\pi}{4}$

$\Rightarrow$f(x) is continuous at all other points

Finally, $f(x)=\frac{\pi}{2}\sin(x+\pi)\Rightarrow f\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}$

$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0}\frac{\pi}{2}\sin\left(\frac{3\pi}{2}-h\right)=\frac{\pi}{2}$

and $\lim_{x\to\left(\frac{x}{2}\right)^+} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0}\frac{\pi}{2}\sin\left(\frac{3\pi}{2}+h\right)=\frac{\pi}{2}$

So, f(x) is not continuous at $x=\frac{\pi}{2}$

Sol 6: A → p; B → r

We know [x] ∈ I, ∀ x ∈ R. Therefore,

sin(π[x]) = 0, ∀ x ⇒R, by theory we know that sin (π[x]) is differentiable everywhere, therefore (A) ↔ (p).

Again, f(x) = sin (π(x– [x]))

Now, x – [x] = (x) then π (x – [x]) = π(x)

Which is not differentiable at x ∈ I

Therefore, (B) ↔ (r) is the answer

Sol 7: A → p,q,r; B → p,s; C → r,s; D → p,q

(A)x|x| is continuous, differentiable and strictlyincreasing in (–1, 1)

(B) $\sqrt{|x|}$ is continuous in (–1, 1) and not differentiable at x = 0

(C) x+[x] is strictly increased in (–1, 1) and discontinuous at x = 0

⇒ Not differentiable at x = 0

(D) |x – 1| + |x + 1| = 2 in (–1, 1)

⇒ The function is continuous and differentiable in (–1, 1)

Sol 8: $\lim_{x\to1}\left(\frac{x-1}{(x-1)\,(2x-5)}\right)=\lim_{x\to1}\frac{1}{(2x-5)}=-\frac{1}{3}$

Sol 9: $\lim_{x\to1}(1-x)\tan\left(\frac{\pi x}{2}\right)$

$=\lim_{h\to0}h\tan\frac{\pi}{2}(1-h)\ =\ \lim_{h\to0}h\cot\left(\frac{\pi h}{2}\right)$

$=\lim_{h\to0}\frac{\frac{\pi h}{2}\times\frac{2}{\pi}}{\tan\left(\frac{\pi h}{2}\right)}=\frac{2}{\pi}$

Sol 10: Let y = sin (x^2+ 1)

$\therefore\ \frac{dy}{dx}=\lim_{\delta x\to0}\frac{\sin[(x+\delta x)^2+1]-\sin(x^2+1)}{\delta x}$

$=\lim_{\delta x\to0}\frac{2\cos\left[x^2+x\cdot\delta x+\frac{1}{2}(\delta x)^2+1\right]\sin\left[x\cdot\delta x+\frac{(\delta x)}{2}\right]}{\delta x}$

$=2\cos(x^2+1)\lim_{\delta x\to0}\frac{\sin[x\cdot\delta x+\frac{1}{2}(\delta x)^2]}{x\cdot\delta x+\frac{1}{2}(\delta x)^2}\times\frac{x\cdot\delta x+\frac{1}{2}(\delta x)}{\delta x}$

$=\ 2\cos(x^2+1)\cdot1\lim_{\delta x\to0}\frac{x\cdot\delta x+\frac{1}{2}(\delta x)^2}{\delta x}\ =2x\cos(x^2+1)$

Sol 11: Given f(x) = x $\tan^{-1}x$

using first principle

$f'(1)=\lim_{h\to0}\left[\frac{f(1+h)-f(1)}{h}\right]$

$=\lim_{h\to0}\left[\frac{(1+h)\tan^{-1}(1+h)-\tan^{-1}(1)}{h}\right]$

$=\lim_{h\to0}\left[\frac{\tan^{-1}(1+h)-\tan^{-1}(1)}{h}+\frac{h\tan^{-1}(1+h)}{h}\right]$

$=\lim_{h\to0}\left[\frac{1}{h}\tan^{-1}\left(\frac{h}{2+h}\right)+\tan^{-1}(1+h)\right]$

$=\lim_{h\to0}\left(\frac{\tan^{-1}\left(\frac{h}{2+h}\right)}{(2+h)\cdot\frac{h}{2+h}}\right)+\frac{\pi}{4}$

$=\lim_{h\to0}\frac{1}{2+h}\left(\frac{\tan^{-1}\left(\frac{h}{2+h}\right)}{\frac{h}{(2+h)}}\right)+\frac{\pi}{4}=\frac{1}{2}+\frac{\pi}{4}$

Sol 12: Given that,

$f(x)=\begin{cases}\frac{x-1}{2x^2-7x+5} & ,\text{ when } x\neq1\\ -\frac{1}{3} & ,\text{ when } x=1\end{cases}$

$\text{RHD}=\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$

$=\lim_{h\to0}\frac{\left[\frac{1+h-1}{2(1+h)^2-7(1+h)+5}-\left(-\frac{1}{3}\right)\right]}{h}$

$=\lim_{h\to0}\left[\frac{3h+2(1+h)^2-7(1+h)+5}{3h\{2(1+h)^2-7(1+h)+5\}}\right]$

$=\lim_{h\to0}\left(\frac{2h^2}{3h(-3h+2h^2)}\right)=-\frac{2}{9}$

$\text{LHD}=\lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$

$=\lim_{h\to0}\frac{\left[\frac{1-h-1}{2(1-h)^2-7(1-h)+5}-\left(-\frac{1}{3}\right)\right]}{-h}$

$=\lim_{h\to0}\frac{-3h+2(1+h^2-2h)-7(1-h)+5}{-3h[2(1-h)^2-7(1-h)+5]}$

$=\lim_{h\to0}\frac{2h^2}{-3h(2h^2+3h)}=-\frac{2}{9}\quad\therefore\text{ LHD = RHD}$

Hence, required value of f ′(1) is $\frac{-2}{9}$

Sol 13: $\lim_{x\to0}\sqrt{\frac{x-\sin x}{x+\cos^2x}}=\frac{\lim_{x\to0}(x-\sin x)^{1/2}}{\lim_{x\to0}(x+\cos^2x)^{1/2}}$

$=\frac{\lim_{x\to0}\left[x\left(1-\frac{\sin x}{x}\right)\right]^{1/2}}{\lim_{x\to0}(0+1)^{1/2}}=\frac{0\cdot0}{1}=0$

Sol 14: Here, $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$= \lim_{h\to 0}\frac{a^2[\sin(a+h)-\sin a]}{h}+\frac{h[2a\sin(a+h)+h\sin(a+h)]}{h}$

$= \lim_{h\to 0}\frac{a^2\cdot 2\cos\left(a+\frac{h}{2}\right)\cdot\sin\frac{h}{2}}{2\cdot\frac{h}{2}}+(2a+h)\sin(a+h)$

$= a^2\cos a + 2a\sin a$

Sol 15: Given, $y=\frac{5x}{3|1-x|}+\cos^2(2x+1)$

$$\Rightarrow y=\begin{cases}\frac{5x}{3(1-x)}+\cos^2(2x+1), & x<1\\ \frac{5x}{3(x-1)}+\cos^2(2x+1), & x>1\end{cases}$$

The function is not defined at x = 1

$$\Rightarrow \frac{dy}{dx}=\begin{cases}\frac{5}{3}\left\{\frac{(1-x)-x(-1)}{(1-x)^2}\right\}-2\sin(4x+2), & x\le 1\\ \frac{5}{3}\left\{\frac{(x-1)-x(1)}{(x-1)^2}\right\}-2\sin(4x+2), & x>1\end{cases}$$

$$\Rightarrow \frac{dy}{dx}=\begin{cases}\frac{5}{3(1-x)^2}-2\sin(4x+2), & x<1\\ -\frac{5}{3(x-1)^2}-2\sin(4x+2), & x>1\end{cases}$$

Clearly at x = 1, dy / dx is not defined

Sol 16: (A) $x^{2x}-2x^x\cot y-1=0$ ...(i)

Now $x=1$,

$1-2\cot y-1=0\Rightarrow\cot y=0\Rightarrow y=\frac{\pi}{2}$

Now differentiating eq. (i) w.r.t. 'x'

$2x^{2x}-(1+\log x)-2\left[x^x\left(-\operatorname{coses}^2 y\right)\frac{dy}{dx}+\text{city}\,x^x(1+\log x)\right]=0$

Now at $\left(1\frac{\pi}{2}\right)$

$2(1+\log 1)-2\left(1(-1)\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}+0\right)=0$

$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}=0\Rightarrow\left(\frac{dy}{dx}\right)_{t,\frac{\pi}{2}}=-1$

Sol 17: (A) $g(x+1)=\log(f(x+1))=\log+\log|(f(x))$

$\Rightarrow g''(x+1)-g''(x)=\log x$

$g''\left(1+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4$

$g''\left(2+\frac{1}{2}\right)-g''\left(1+\frac{1}{2}\right)=-\frac{4}{9}$

......

......

$g''\left(N+\frac{1}{2}\right)-g''\left(N-\frac{1}{2}\right)=-\frac{4}{(2N-1)^2}$

Summing up all terms

Hence, $g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=4\left(1+\frac{1}{9}+.....+\frac{1}{(2N-1)^2}\right)$

Sol 18: (A)

$f''(X)=\frac{4ax\left(x^2ax+1\right)^2-4ax\left(x^2-1\right)(2x+a)\left(x^2+ax+1\right)}{\left(x^2+ax+1\right)^4}$

$f''(1)=\frac{4a}{(2+a)^2}$ $\quad f'(-1)=\frac{-4a}{(2-a)^2}$

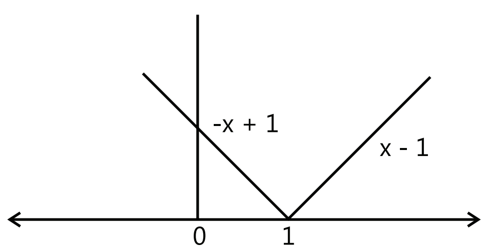
$(2+a)^2 f'(1)+(2-a)^2 f'(-1)=0$

Sol 19: (C) From graph, p = -1

$\Rightarrow \lim_{x\to 1^+} g(x)=-1$

$\Rightarrow \lim_{x\to 0} g(1+h)=-1$

$\Rightarrow \lim_{x\to 0}\left(\frac{h^n}{\log\cos^m h}\right)=-1$



$\Rightarrow \lim_{x\to 0}\frac{n.h^{n-1}}{(-\tan h)}=-\left(\frac{n}{m}\right)\lim_{h\to 0}\left(\frac{h^{n-1}}{\tan h}\right)=-1,$

which holds if $n=m=2$

Sol 20: (B) $f(x) = g(x)\cos x + \sin x . g'(x)$

$\Rightarrow f'(0) = g(0)$

$f'(x) = 2g'(x)\cos x - g(x)\sin x + \sin x g''(x)$

$\Rightarrow f'(0) = 2g'(0) = 0$

But $\lim\limits_{x\to0}\left[g(x)\cot x - g(0)\operatorname{cosec} x\right] = \lim\limits_{x\to0}\dfrac{g(x)\cos x - g(0)}{\sin x}$

$= \lim\limits_{x\to0}\dfrac{g'(x)\cos x - g(x)\sin x}{\cos x} = g'(0) = 0 = f'(0)$

Sol 21: (B) Differentiating the given equation, we get

$3y^2y' - 3y + 1 = 0$

$\Rightarrow y'\left(-10\sqrt{2}\right) = -\dfrac{1}{21}$

Differentiation again we get $6yy^2 + 3y^2y'' - 3y'' = 0$

$\Rightarrow f''\left(-10\sqrt{2}\right) = -\dfrac{6.2\sqrt{2}}{(21)^4} = -\dfrac{4\sqrt{2}}{7^3 3^2}$

Sol 22: (B, C, D) For $f(x) = x\cos\left(\dfrac{1}{x}\right), x \ge 1$

$F' f(x) = x\cos\left(\dfrac{1}{x}\right) + \dfrac{1}{x}\sin\left(\dfrac{1}{x}\right) \to 1$ for $x \to \infty$

Also $f'(x) = \dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right) - \dfrac{1}{x^3} - \cos\left(\dfrac{1}{x}\right)$

$= -\dfrac{1}{x^3}\cos\left(\dfrac{1}{x}\right) < 0 \quad \text{for } x \ge 1$

$\Rightarrow (x)$ is decreasing for $[1, \infty)$

$\Rightarrow f'(x+2) < f'(x)$. Also,

$\lim\limits_{x\to\infty} f(x+2) - f(x) = \lim\limits_{x\to\infty}\left[(x+2)\cos\dfrac{1}{x+2} - x\cos\dfrac{1}{x}\right] = 2$

$\therefore f(x+2) - f(x) > 2 \forall x \ge 1$

Sol 23: Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$p'(1) = p'(2) = 0$

$\lim\limits_{x\to0}\left(\dfrac{x^2 + p'(x)}{x^2}\right) = 2$

$\Rightarrow p(0) = 0 \Rightarrow e = 0$

$\lim\limits_{x\to0}\left(\dfrac{2x + p'(x)}{2x}\right) = 2$

$\Rightarrow p(0) = 0 \Rightarrow d = 0$

$\lim\limits_{x\to0}\left(\dfrac{2 + p''(x)}{2}\right) = 2$

$\Rightarrow c = 1$

On solving, $a = 1/4, b = -1$

So, $p(x) = \dfrac{x^4}{4} - x^3 + x^2$

$\Rightarrow p(2) = 0$

Sol 24: (A, C)

$L = \lim\limits_{x\to0}\dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4} = \lim\limits_{x\to0}\dfrac{1}{x^2\left(a + \sqrt{a^2 - x^2}\right)} - \dfrac{1}{4x^2}$

$= \lim\limits_{x\to0}\dfrac{(4-a) - \sqrt{a^2 - x^2}}{4x^2\left(a + \sqrt{a^{2-x^2}}\right)}$

Numerator $\to 0$ if $a = 2$ and then $L = \dfrac{1}{64}$

Sol 25: (B, C) $f(x) = \ln x + \int_0^2 \sqrt{1 + \sin t}\, dt$

$f'(x) = \dfrac{1}{x} + \sqrt{1 + \sin x}$

$f''(x) = \dfrac{-1}{x^2} + \dfrac{\cos x}{2\sqrt{1 + \sin x}}$

(A) f″ is not defined for $x = \dfrac{-\pi}{2} + n\pi, n \in I$, (A) is wrong

(B) f′(x) always exist for $x > 0$

(C) $|f'| < |f|$

Since f′ > 0 and f > 0 f′ < f

$\dfrac{1}{x} + \sqrt{1 + \sin x} < \ln x + \int_0^x \sqrt{1 + \sin x}\, dx$

LHS is bounded RHS is increasing with range ∞ so there exist some α beyond which RHS is greater than LHS

(d) $|f| + |f'| \le b$ is wrong as f is M1 and its range is not bound while β is finite.

Sol 26: (D) $e^{\ln(1+b^2)} = 2b\sin^2\theta = \dfrac{1+b^2}{2b}$

$\Rightarrow \sin^2\theta = 1 \text{ as} \dfrac{1+b^2}{2b} \geq 1$

$\theta = \pm\pi/2$.

Sol 27: (A, B, C, D) $\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = 0 = f(-\pi/2)$

$\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \cos\left(-\dfrac{\pi}{2}\right) = 0$

$$f'(x) = \begin{cases} -1 & x \leq -\pi/2 \\ \sin x, & -\pi/2 < x \leq 0 \\ 1 & 0 < x \leq 1 \\ ,1/x & x > 1 \end{cases}$$

Clearly, f (x) is not differentiable at x = 0 as f´(0-) = 0 and f´(0+) = 1. f (x) is differentiable at x = 1 as f¢ (1–) = f′ (1+) = 1.

Sol 28: $y'(x) + y(x)g'(x) = g(x)g'(x)$

$\Rightarrow e^{g(x)}y'(x) + e^{g(x)}g'(x) = e^{g(x)}g(x)g'(x)$

$\Rightarrow \dfrac{d}{dx}\left(y(x)e^{e(x)}\right) = e^{g(x)}g(x)g'(x)$

$\therefore y(x) = e^{g(x)} = \int e^{g(x)}g(x)g'(x)dx$

= $\int e^t t dt$, where g(x) = t

= (t – 1) et + c

$\therefore y(x)e^{g(x)} = (g(x)-1)e^{g(x)} + c$

Put x = 0 _ 0 = (0 – 1) . 1 + c ⇒ c = 1

Put x = 2 ⇒ y (2) . 1 = (0 – 1) . (1) + 1

Y(2)=0

Sol 29: (B, C) $\because f(0) = 0$

And $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x\_+h)-f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{f(h)}{h} = f'(0) = k$ (say)

$\Rightarrow f(x) = lx + x \Rightarrow f(x) = kx\,(\because f(0) = 0)$

Sol 30: (B) Given $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1} - ax - b\right) = 4$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{x^2+x+1-ax^2-ax-bx-b}{(x+1)} = 4$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)} = 4$

⇒ 1 – a = 0 and 1 – α – b = 4 b = –4, a = 1.

Sol 31: (B) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$= \lim\limits_{h\to 0}\dfrac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h} \quad = \lim\limits_{h\to 0} h\cos\left(\dfrac{\pi}{h}\right) = 0$

so, f(x) is differentiable at x = 0

$f'(2^+) = \lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h^2)\left|\cos\frac{\pi}{2+h}\right| - 0}{h} \quad = \lim\limits_{h\to 0}\dfrac{(2+h^2)\cos\left(\frac{\pi}{2+h}\right)}{h}$

$f'(2^+) = \lim\limits_{h\to 0}\dfrac{(2+h)^2}{h}\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{2+h}\right)$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{\pi h}\sin\left[\dfrac{\pi.h}{2(2+h)}\right]$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{\pi h}\sin\dfrac{\pi.h}{2(2+h)} \times \dfrac{\pi}{2(2+h)} = \pi$

Again $f'(2^-) = \lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{-h} \quad \lim\limits_{h\to 0}\dfrac{(2-h)^2\left|\cos\frac{\pi}{2-h}\right|}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h} \quad = \lim\limits_{h\to 0}\dfrac{(2-h)^2\sin\left[\frac{\pi}{2} - \frac{\pi}{2-h}\right]}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2}{\frac{\pi h}{2(2-h)}}.\sin\dfrac{\pi h}{2(2-h)} \times \dfrac{\pi}{2(2-h)} = -\pi$

Sol 32:(B, D) Required limit is

$$= \frac{\int_0^1 x^a dx}{\int_0^1 (a+x)dx} = \frac{2}{(2a+1)(a+1)} = \frac{2}{120}$$

$\Rightarrow a = 7 \text{ or} -\frac{17}{2}$.

Sol 33: (D) Let g (x) = e–x f (x)

and g″ (x) > 1 > 0

So, g (x) is concave upward and g (0) = g (1) = 0

Hence, g (x) < 0 ∀ x ∈ (0, 1)

$\Rightarrow e^{-x} f(x) < 0$

$f(x) < 0 \forall x \in (0,1)$

Alternate solution

$f(x) - 2f'(x) + f(x) \ge e^x$

$$\Rightarrow \left(f(x)e^{-x} - \frac{x^2}{2} \right)'' \ge 0$$

Let $g(x) = f(x)e^{-x} - \frac{x^2}{2}$

$g(0) = 0, g(1) = -\frac{x^2}{2}$

Since g is concave up so it will always lie below the chord joining the extremities which is y = $y = -\frac{x}{2}$

$$\Rightarrow f(x)e^{-x} - \frac{x^2}{2} < -\frac{x}{2}$$

$$\Rightarrow f(x) < \frac{(x^2 - x)e^x}{2} < 0 \forall x \in (0,1)$$

Sol 34: (D) Given f′ (x) – 2f (x) < 0

$\Rightarrow f(x) < ce^{2x}$

Put $x = \frac{1}{2} \Rightarrow c > \frac{1}{e}$

Hence f (x) $< e^{2x-1}$

$$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1}dx$$

$$0 < \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$$

Sol 35: (B) $F'(x) = 2xf(x) = f(x)$

$f(x) = e^{x^2 + c}$

$f(x) = e^{x^2} \; (\because f(0) = 1)$

$$F(x) = \int_0^{x^2} e^x dx$$

$F(x) = e^{x^2} - 1 (\because F(0) = 0)$

$F(2) = e^4 - 1$

Sol 36: (A) $g\left(\frac{1}{2}\right) = \lim_{h \to 0^+} \int_h^{1-h} t^{-1/2} dt$

$$= \int_0^1 \frac{dt}{\sqrt{t - t^2}} = \int_0^1 \frac{dt}{\sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}} \right)_0^1$$

$= \sin^{-1} 1 - \sin^{-1}(-1) = \pi$.

Sol 37: (D) We have $g(a) = g(1-a)$ and g is differentiable

Hence $g'\left(\frac{1}{2}\right) = 0$

Sol 38: (B, C) Let H (x) = f (x) – 3g (x)

H (-1) = H (0) = H (2) = 3.

Applying Rolle's Theorem in the interval $[-1,0]$

$H'(x) = f'(x) - 3g'(x) = 0$ for atleast one $c \in (-1,0)$

As H″(x) never vanishes in the interval

⇒ Exactly one $c \in (-1,0)$ for which H′(x) = 0

Similarly, apply Rolle's Theorem in the interval [0,2].

⇒ Exactly one $c \in (-1,0)$ for which H′(x) = 0

Similarly, apply Rolle's Theorem in the interval [0, 2].

$\Rightarrow H'(x) = 0$ has exactly one solution in (0, 2)

Sol 39: (A, B, C) (A) $f(x) = F(x) + xF'(x)$

$F'(1) = F(1) + F'(1)$

$f'(1) = F'(1) < 0$

$f'(1) < 0$

(B) $f'(2)=2F(2)$

$f(x)$ is decreasing and $F(1)=0$

Hence $F(2)<0$

$\Rightarrow f(2)<0$

(C) $f(x)=F(x)+xF'(x)$

$F(x)<0\forall x\in(1,3)$

$F'(x)<0\forall x\in(1,3)$

Hence, $F(x)<0\forall x\in(1,3)$

Sol 40: (A, B) $p(\text{RedBall})=p(1).p(R\,|\,I)+p(II).p(R\,|\,II)$

$$p(II\,|\,R)=\frac{1}{3}=\frac{p(II).p(R\,|\,II)}{p(I).p(R\,|\,(R\,|\,I)+p(II).p(R\,|\,II))}$$

$$\frac{1}{3}=\frac{\frac{n_3}{n_3+n_3}}{\frac{n_1}{n_1+n_2}+\frac{n_3}{n_3+n_4}}$$

Of the given options, A and B satisfy above condition

Sol 41: (A, D) Differentiability of $f(x)$ at $x=0$

$$\text{LHD } f(0^-)=\lim_{\sigma\to0}\left(\frac{f(0)-f(0-\sigma)}{\sigma}\right)=\lim_{\sigma\to0}\frac{0+g(-\sigma)}{\sigma}=0$$

$$\text{RHD } f'(0^+)=\lim_{\sigma\to0}\frac{f(0+\sigma)-f(0)}{\sigma}(\quad)=\lim_{\sigma\to0}\frac{g(\sigma)}{\sigma}=0$$

$\Rightarrow f(x)$ differentiable at $x=0$

Differentiability of $h(x)x=0$

$h'(0)=1g\left(e^{|x|}\right)\forall\in R$

LHD

$$f(h(0^-))=\lim_{\sigma\to0}\frac{f(h(0))-f(h(0-\sigma))}{\sigma}=\lim_{\sigma\to0}\frac{g(1)-g(e^{\sigma})}{\sigma}=-g'(1)$$

RHD

$$f(h(0^+))=\lim_{\sigma\to0}\frac{f(h(0+\sigma))-f(h(0))}{\sigma}=\lim_{\sigma\to0}\frac{g(e^{\sigma})-g(1)}{\sigma}=-g'(1)$$

Since $g(1)\neq0\Rightarrow f(h(x))$ is non diff. at x = 0

Differentiability of $h(f(x))$ at $x=0$

$$h(f(x))=\begin{cases}e^{|f(x)|} & x\neq0\\ 1 & x=0\end{cases}$$

$$\text{LHD } h'(f(0-\sigma))=\lim_{\sigma\to0}\frac{h(f(0))-h(f(0-\sigma))}{\sigma}$$

$$=\lim_{\sigma\to0}\frac{1-e^{|e-\sigma|}}{\left|g^{(\sigma)}\right|}.\frac{|g(-\sigma)|}{\sigma}=0$$

$$\text{RHD } h'(f(0+\sigma))=\lim_{\sigma\to0}\frac{h(f(0+\sigma))-h(f(0))}{\sigma}$$

$$=\lim_{\sigma\to0}\frac{e^{|-\sigma|}-1}{\left|g^{(\sigma)}\right|}.\frac{|g(-\sigma)|}{\sigma}=0$$

Sol 42: (A, B) $f(x)=a\cos\left(\left|x^3-x\right|\right)+b|x|\sin\left(\left|x^3\right|+x\right)$

[A] If $a=0,b=1,\ f(x)=|x|\sin\left(\left|x^3=x\right|\right)$

$\Rightarrow f(x)=|x|\sin\left(\left|x^3+x\right|\right)$

Hence f(x) is differentiable.

[B] If $a=1,b=0\ f(x)=\cos\left(\left|x^3\right|-x\right)$

$\Rightarrow f(x)=\cos\left(x^3-x\right)$ Which is differentiable at x = 1 and x = 0.

Sol 43: (B, C)

$$\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{n}{r}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{n^2}{r^2}\right)}.\frac{1}{\prod_{r=1}^{n}\left(\frac{n}{r}\right)}\right]$$

$$\ln f(x)=\lim_{n\to\infty}\frac{x}{n}\ln\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{\frac{r}{n}}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{\left(\frac{r}{n}\right)^2}\right)}.\frac{1}{\prod_{r=1}^{n}\left(\frac{n}{r}\right)}\right]$$

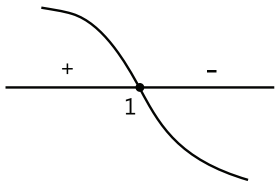
$$=x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\ln\left(\frac{x\left(\frac{r}{n}\right)+1}{\left(x+\frac{r}{n}\right)^2+1}\right)$$

$$= x\int_0^1 \ln\left(\frac{1+tx}{1+t^2x^2}\right)dt$$

Put, tx = p, we get

$$\ln f(x) = \int_0^x \ln\left(\frac{1+p}{1+p^2}\right)dp$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$$



sign scheme of f′ (x)

Also, f′ (1) = 0

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1), f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right), f'(2) < 0$$

$$\text{Also, } \frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ln\left(\frac{4}{10}\right) - \ell n\left(\frac{3}{5}\right)$$

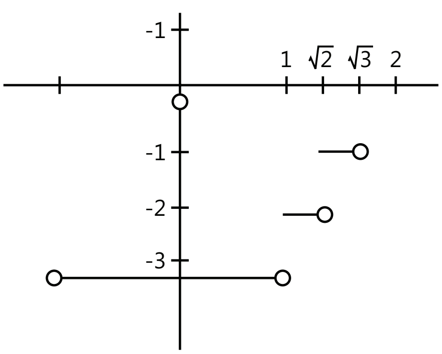
$$= \ln\left(\frac{4}{6}\right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

Sol 44: (B, C)

$$f(x) = \left[x^2 - 3\right] = \left[x^2\right] - 3$$

f(x) is discontinuous at x = 1, $\sqrt{2}$, $\sqrt{3}$, 2

$$g(x) = \left(|x| + |4-7|\right)\left(\left[x^2\right] - 3\right)$$



g(x)		
	15x - 21	$x < 0$
	9x - 21	$0 \le x \le 1$
	6x - 14	$1 \le x < \sqrt{2}$
	3x - 7	$\sqrt{2} \le x < \sqrt{3}$
	0	$\sqrt{3} \le x < 2$
	3	x = 2

$\therefore g(x)$ is not differentiable, at x = 0, 1, $\sqrt{2}$, $\sqrt{3}$

Sol 45: (A) $f'(x) + \frac{f(x)}{x} = 2$

$$\Rightarrow xf'(x) + f(x) = 2x \quad \Rightarrow \int d(x.f(x)) = \int 2x dx$$

$$\Rightarrow xf(x) = x^2 + c; \qquad f(x) = x + \frac{x}{c}\left(c \ne 0 \text{ as } f(1 \ne 1)\right)$$

For this function, only (A) is correct.

Sol 46: (B, C)

$$f(x) = x^3 + 3x + 2, \ \ f(1) = 6 \ \ g(6) = 1$$

$$g(f(x)) = x \Rightarrow g'f(x) \times f'(x) = 1$$

$$x = 0, \quad g'(f(0))f(0) = 1$$

$$g'(2) = \frac{1}{f(0)} = \frac{1}{3}$$

$$f(3) = 38$$

$$\therefore g(38) = 3$$

$$f(2) = 16 \Rightarrow g(16) = 2$$

$$\therefore h(g(g16)) = h(0)$$

$$\therefore 16 = h(g(g16)) = h(0)`$$

$$\therefore (c) \text{iscorrect}$$

$$f'(6) = 111, f(1) = 6 \Rightarrow g'(6)\frac{1}{2}$$

$$h(g(g(x))) = x$$

$$\Rightarrow h'(g(g(x))) \times g'(x) = 1$$