26. VECTORS

1. INTRODUCTION TO VECTOR ALGEBRA

1.1 Scalars and Vectors

Scalar: A scalar is a quantity that has only magnitude but no direction. Scalar quantity is expressed as a single number, followed by appropriate unit, e.g. length, area, mass, etc. In linear algebra, real numbers are called scalars.

Vector: A vector is a quantity that has both magnitude and direction, e.g. displacement, velocity, etc.

1.2 Representation of Vectors

(a) A vector is represented diagrammatically by a directed line segment or an arrow. A directed line segment has both magnitude (length) and direction. The length is denoted by $|V|$.

(b) If P and Q are the given two points, then the vector from P to Q is denoted by $\overrightarrow{PQ}$, where P is called the tail and Q is called the nose of the vector.

1.3 Vector Components

In a two-dimensional coordinate system, any vector can be resolved into x-component and y-component

$\vec{v} = \langle v_x, v_y \rangle$

Let us consider the figure shown (adjacent) here. In this figure, the components can be quickly read. The vector in the component form is $\vec{v} = \langle 4,5 \rangle$.

The relation between magnitude of the vector and the components of the vector can be calculated by using trigonometric ratios.

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{v_x}{v}$$

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{v_y}{v}$$

$$v_x = v\cos\theta;\ v_y = v\sin\theta$$

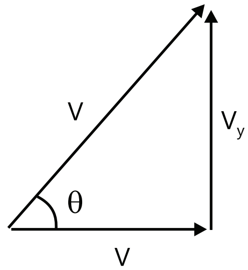

Figure 26.1

If v_x and v_y are the known lengths of a right triangle, then the length of the hypotenuse, V, is calculated by using the Pythagorean theorem

$$|v| = \sqrt{v_x^2 + v_y^2}$$

2. TYPE OF VECTORS

2.1 Null Vector/Zero Vector

A zero vector or null vector is a vector that has zero magnitude, i.e. initial and terminal points are coincident, so that its direction is in indeterminate form. It is denoted by ϕ.

2.2 Unit Vector

A unit vector is a vector of unit length. A unit vector is sometimes denoted by replacing the arrow on a vector with "^".

Unit vectors parallel to x-axis, y-axis and z-axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.

Unit vector $\hat{U}$ parallel to $\vec{V}$ can be obtained as $\hat{U} = \dfrac{\vec{V}}{|V|}$.

Illustration 1: Find unit vector of $\vec{i} - 2\vec{J} + 3\vec{k}$ **(JEE MAIN)**

Sol: Here unit vector of $\vec{a}$ is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

$\vec{a} = \vec{i} - 2\vec{J} + 3\vec{k}$

If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then it's magnitude $\left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ $\Rightarrow \left|\vec{a}\right| = \sqrt{14} \Rightarrow \hat{a} = \dfrac{\vec{a}}{\sqrt{14}} = \dfrac{1}{\sqrt{14}}\left(\vec{i} - 2\vec{J} + 3\vec{k}\right)$

2.3 Collinear or Parallel Vectors

Two or more vectors are said to be collinear, when they are along the same lines or parallel lines irrespective of their magnitudes and directions.

2.4 Like and Unlike Vectors

Vectors having the same direction are called like vectors. Any two vectors parallel to one another, having unequal magnitudes and acting in opposite directions are called unlike vectors.

2.5 Co-Initial Vectors

All those vectors whose terminal points are same, are called co-terminal vectors.

2.6 Co-Terminal Vectors

Vectors that have the same initial points are called co-initial vectors.

Illustration 2: Which are co-initial and equal vectors in the given rectangle diagram? **(JEE MAIN)**

Sol: By following above mentioned conditions we can obtain co-initial and equal vectors.

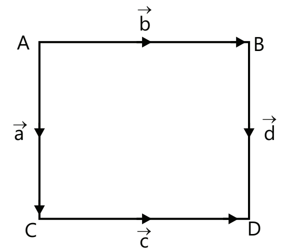

Figure 26.2

Here, $\vec{a}$ and $\vec{b}$ are co-initial vectors, $\vec{b}$ and $\vec{c}$, $\vec{a}$ and $\vec{d}$ are equal vectors.

2.7 Coplanar Vectors

Vectors lie on the same plane are called coplanar.

2.8 Negative vector

A vector that points to a direction opposite to that of the given vector is called a negative vector.

2.9 Reciprocal of a Vector

A vector having the same direction as that of a given vector $\vec{a}$, but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\overrightarrow{a^{-1}}$.

2.10 Localized and Free vectors

A vector drawn parallel to a given vector through a specified point unlike free vector in space is called a localized vector. For example, the effect of force acting on a rigid body depends not only on the magnitude and direction but also on the line of action of the force. A vector that depends only on its length and direction and not on its position in the space is called a free vector, e.g. gravity. In this chapter, we will deal with free vectors, unless otherwise stated. Thus a free vector can be determined in space by choosing an arbitrary initial point.

llustration 3: Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\hat{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\hat{b}$ equal? **(JEE MAIN)**

Sol: Two vectors are equal if their modulus and corresponding components both are equal.

We have $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| = \sqrt{2^2 + 1^2}$. So, $|\vec{a}| = |\vec{b}|$. But, the two vectors are not equal, since their corresponding components are distinct.

Illustration 4: Find a vector of magnitude 5 units which is parallel to the vector $2\hat{i} - \hat{j}$. **(JEE MAIN)**

Sol: As we know $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$, therefore required vector will be $5\hat{a}$.

Let $\vec{a} = 2\hat{i} - \hat{j}$. Then, $|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\therefore$ Unit vector parallel to $\vec{a} = \hat{a} = \dfrac{1}{|\vec{a}|}.\vec{a} = \dfrac{1}{\sqrt{5}}\left(2\hat{i} - \hat{j}\right) = \dfrac{2}{\sqrt{5}}\hat{i} - \dfrac{1}{\sqrt{5}}\hat{j}$.

So, the required vector is $5\hat{a} = 5\left(\dfrac{2}{\sqrt{5}}\hat{i} - \dfrac{1}{\sqrt{5}}\hat{j}\right) = 2\sqrt{5}\hat{i} - \sqrt{5}\hat{j}$.

2.11 Position Vector

A vector that represents the position of a point P in space with respect to an arbitrary reference origin O is called a position vector (p.v.). It is also known as location vector or radius vector and usually denoted as x, r or s; it corresponds to the displacement from O to P.

$r = \overrightarrow{OP}$.

Illustration 5: Show that, the three points A(-2,3,5), B(1,2,3) and C(7,0,-1) are collinear. **(JEE MAIN)**

Sol: By obtaining $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we can conclude that given points are collinear or not.

We have

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - \left(-2\hat{i} + 3\hat{j} + 5\hat{k}\right) = 3\hat{i} - \hat{j} - 2\hat{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(7\hat{i} + 0\hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 6\hat{i} - 2\hat{j} - 4\hat{k} = 2\left(3\hat{i} - \hat{j} - 2\hat{k}\right)$$

Therefore, $\overrightarrow{BC} = 2\overrightarrow{AB}$.

This shows that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel. But, B is a common point. So, the given point A, B and C are collinear.

2.12 Equal Vectors

Two vectors having the same corresponding components and direction and represent the same physical quantity are called equal vectors.

Illustration 6: Find the values of x, y and z, so that the vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal. **(JEE MAIN)**

Sol: Two vectors are equal, if their corresponding components are equal.

Note that two vectors are equal, if their corresponding components are equal. Thus, the given vectors $\vec{a}$ and $\vec{b}$ will be equal, if and only if x = 2, y = 2, z = 1.

Illustration 7: Find the vector joining the point P (2, 3, 0) and Q (-1, -2, -4) directed from P to Q. **(JEE MAIN)**

Sol: By subtracting the component of P from Q we will get $\overrightarrow{PQ}$.

Since the vector is to be directed from P to Q. Clearly, P is the initial point and Q is the terminal point. So, the required vector joining P and Q is the vector PQ given by

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{K} \quad \text{i.e. } \overrightarrow{PQ} = -3\hat{i} \pm 5\hat{j} - 4\hat{k}$$

Illustration 8: Show that, the points $A\left(2\hat{i} - \hat{j} + \hat{k}\right)$, $B\left(\hat{i} - 3\hat{j} - 5\hat{k}\right)$, $C\left(3\hat{i} - 4\hat{j} - 4\hat{k}\right)$ are the vertices of a right-angled triangle. **(JEE MAIN)**

Sol: Here if $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CA}\right|^2$ then only the given points are the vertices of right angled triangle. We have

$$\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k} \quad \text{and} \quad \overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

Moreover, $\left|\overrightarrow{AB}\right|^2 = 41 = 6 + 35 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CA}\right|^2$

Hence, it is proved that the points form a right-angled triangle.

3. RESULTANT OF VECTORS

When two or more vectors are added, they yield the resultant vector. If vectors A and B are added together, the result will be vector R, i.e. $\vec{R} = \vec{A} + \vec{B}$. Same technique can also be applied for multiple vectors.

4. VECTOR ADDITION

4.1 Triangular Law of Addition

It states that if two vectors can be represented in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented by the third side of the triangle, taken in the opposite direction of the sequence.

4.2 Parallelogram Law of Addition

It states that if two vectors can be represented in magnitude and direction by the two adjacent sides or a parallelogram, then their resultant is represented by the diagonal of the parallelogram.

4.3 Addition in Component Form

Consider two vectors A and B

$A = < a_1,\ b_1,\ c_1 >$

$B = < a_2, b_2, c_2 >$

Then, $A + B = < a_1 + a_2, b_1 + b_2, c_1 +\ c_2 >$

4.4 Properties of Vector Addition

The properties of vector addition are listed as follows:

(a) $\pi / 2$ Commutative

(b) $\pi / 3$ Associative

(c) $\pi / 4$ Null vector is an additive identity

(d) $\hat{A}$ and $\hat{B}$ Additive inverse

(e) π

(f) $\left|\hat{A} - \hat{B}\right|$

(g) $\pi / 2$

4.5 Vector Subtraction

Subtraction is taken as an inverse operation of addition. If $\vec{u}$ and $\vec{v}$ are two vectors, the difference $\vec{u} - \vec{v}$ of two vectors is defined to be the vector added to $\vec{v}$ to get $\vec{u}$. In order to obtain $\vec{u} - \vec{v}$, we put the tails of $\vec{u}$ and $\vec{v}$ together, the directed segment from the nose of $\vec{v}$ to the nose of $\vec{u}$ is a representative of $\vec{u} - \vec{v}$.

Illustration 9: If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find the unit vectors parallel to the diagonals of the parallelogram. **(JEE MAIN)**

Sol: As mentioned above, if two vector quantities are represented by two adjacent sides or a parallelogram then the diagonal of parallelogram will be equal to the resultant of these two vectors.

Let ABCD be a parallelogram such that, $\overrightarrow{AB} // \vec{b}$ and $\overrightarrow{BC} // \vec{b}$.

Then,

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$ and $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\Rightarrow \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} \Rightarrow \overrightarrow{BD} = \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$

Now, $\overrightarrow{AC} = 3\hat{i} + 6\hat{j} - 2\hat{k} \Rightarrow \left|\overrightarrow{AC}\right| = \sqrt{9+36+4} = 7$

And $\overrightarrow{BD} = \hat{i} + 2\hat{j} - 8\hat{k}$.

$\Rightarrow \left|\overrightarrow{BD}\right| = \sqrt{1+4+64} = \sqrt{69}$

$\therefore$ Unit Vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} = \dfrac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$

$\therefore$ Unit vector along $\overrightarrow{BD} = \dfrac{\overrightarrow{BD}}{\left|\overrightarrow{BD}\right|} = \dfrac{1}{\sqrt{69}}\left(\vec{i} + 2\vec{j} - 8\vec{k}\right)$.

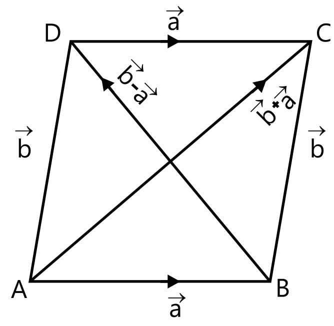

Figure 26.3

Illustration 10: ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}, \overrightarrow{AE}, \overrightarrow{BC}, \overrightarrow{DC}, \overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$. **(JEE MAIN)**

Sol: By using method of finding resultant of vector we can prove required result.

Let R be the resultant force

$\therefore R = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\therefore R = \overrightarrow{\left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \left(\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}\right) + \overrightarrow{AC}}$

$= \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC} = 3\overrightarrow{AC}$. Hence proved.

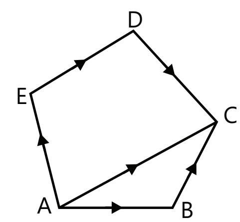

Figure 26.4

Illustration 11: ABCD is a parallelogram. If L and M are the middle points of BC and CD, respectively express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$, also show that $\overrightarrow{AL} + \overrightarrow{AM} = \dfrac{3}{2}\overrightarrow{AC}$ **(JEE MAIN)**

Sol: By using mid – point formula and method of finding resultant of vector we can prove given relation.

Let $\vec{b}$ and $\vec{a}$ be the position vectors of points B and D, respectively be referred to A as the origin of reference.

Then $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \overrightarrow{AB}$ $\quad \left[\therefore \overrightarrow{DC} = \overrightarrow{AB}\right]$

$= \vec{d} + \vec{b} \qquad \therefore \overrightarrow{AB} = \vec{b}, \quad \overrightarrow{AD} = \vec{d}$

i.e. the position vector of C referred to A is $\vec{d} + \vec{b}$

$\overrightarrow{AL}$ = p.v. of L, the midpoint of $\overrightarrow{BC}$. $\overrightarrow{AM} = \dfrac{1}{2}\left[\vec{a} + \vec{d} + \vec{b}\right] = \overrightarrow{AD} + \dfrac{1}{2}\overrightarrow{AB}$

$\therefore \overrightarrow{AL} + \overrightarrow{AM} = \vec{b} + \vec{d} + \dfrac{1}{2}\vec{b} = \dfrac{3}{2}\vec{b} + \dfrac{3}{2}\vec{d} = \dfrac{3}{2}(\vec{b} + \vec{d}) = \dfrac{3}{2}\overrightarrow{AC}$

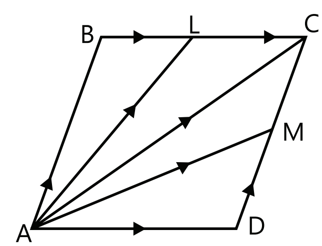

Figure 26.5

5. SCALAR MULTIPLE OF A VECTOR

If $\vec{a}$ is the given vector, then $k\vec{a}$ is a vector, whose magnitude is $\left|k\right|$ times the magnitude of $\vec{a}$ and whose direction is the same or opposite as that of $\vec{a}$ according to whether k is positive or negative.

6. SECTION FORMULA

(a) If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector of a point which divides A and B in the ratio m:n is given by $r = \frac{(n\vec{a} + m\vec{b})}{(m+n)}$.

(b) Position vector of the midpoint of $\overrightarrow{AB} = \frac{(\vec{a} + \vec{b})}{2}$.

MASTERJEE CONCEPTS

- If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of the vertices of any ΔABC. Then the position vector of centroid G will be $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$.
- The position vector of incenter of triangle with position vectors of triangle ABC, are A ($\vec{a}$), B($\vec{b}$), C($\vec{c}$) is $\vec{r} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$.

Anurag Saraf (JEE 2011, AIR 226)

Illustration 12: If ABCD is a quadrilateral and E and F are the mid points of AC and BD, respectively, prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$. **(JEE MAIN)**

Sol: By using mid-point theorem we can prove given relation.

Since F is the midpoint of BD. Applying the midpoint theorem in triangle ABD, we have $\Rightarrow \overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF}$ ... (i)

Applying the midpoint theorem in triangle BCD, we have

$\Rightarrow \overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CF}$ ... (ii)

Adding equations (i) and (ii), we obtain

$\Rightarrow \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2\left(\overrightarrow{AF} + \overrightarrow{CF}\right)$

Now applying the midpoint theorem in triangle CFA, we have $\overrightarrow{AF} + \overrightarrow{CF} = 2\overrightarrow{EF}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2\left(\overrightarrow{AF} + \overrightarrow{CF}\right) = 4\overrightarrow{EF}$ Hence proved.

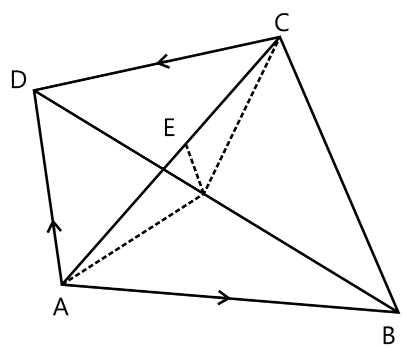

Figure 26.6

Illustration 13: If G is the centroid of the triangle ABC, show that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$ and conversely $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$, then G is the centroid of triangle ABC. **(JEE ADVANCED)**

Sol: As G is the centroid of triangle ABC, hence G = $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$. Therefore by obtaining $\overrightarrow{GA}$, $\overrightarrow{GB}$ and $\overrightarrow{GC}$ we can prove this problem.

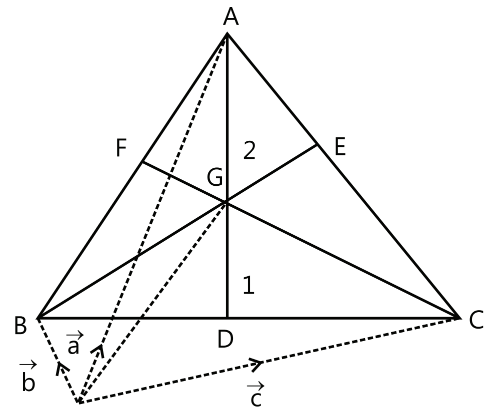

Figure 26.7

Let the position vector of the vertices be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively.

So, the position vector of centroid, G, is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.

$$\overrightarrow{GA} = \overrightarrow{OA} - \overrightarrow{OG} = \vec{a}-\frac{\vec{a}+\vec{b}+\vec{c}}{3}=\frac{2\vec{a}-\vec{b}-\vec{c}}{3}$$

Similarly, $\overrightarrow{GB}=\dfrac{2\vec{b}-\vec{a}-\vec{c}}{3}$. $\overrightarrow{GC}=\dfrac{2\vec{c}-\vec{a}-\vec{b}}{3}$

$$\Rightarrow \overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\frac{1}{3}\left(2\vec{a}-2\vec{a}+2\vec{b}+2\vec{b}+2\vec{c}-2\vec{c}\right)=0$$

Conversely if $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$

$$\Rightarrow (\overrightarrow{OA}-\overrightarrow{OG})+\left(\overrightarrow{OB}-\overrightarrow{OG}\right)+\left(\overrightarrow{OC}-\overrightarrow{OG}\right)=0 \Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG} \Rightarrow \overrightarrow{OG}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$$

Hence, G is the centroid of the points A, B and C.

Illustration 14: Find the values of x and y, for which the vectors $\vec{a}=\left(x+2\right)\hat{i}-\left(x-y\right)\hat{j}+\hat{k}$, $\vec{b}=\left(x-1\right)\hat{i}+\left(2x+y\right)\hat{j}+2\hat{k}$ are parallel **(JEE MAIN)**

Sol: Two vectors are parallel if ratio of there respective components are equal.

$\vec{a}$ and $\vec{b}$ are parallel if $\dfrac{x+2}{x-1}=\dfrac{y-x}{2x+y}=\dfrac{1}{2}\Rightarrow$ x=-5, y=$\dfrac{-20}{3}$

Illustration 15: If ABCD is a parallelogram and E is the midpoint of AB, show by vector method, that DE trisects and is trisected by AC. **(JEE MAIN)**

Sol: By using section formula, we can solve this problem.

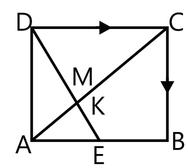

Figure 26.8

Let $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}$

Then $\overrightarrow{BC}=\overrightarrow{AD}=\vec{b}$ and $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}=\vec{a}+\vec{b}$

Also, let K be a point on AC, such that AK:AC = 1:3

$$\Rightarrow \overrightarrow{AK}=\frac{1}{3}\overrightarrow{AC}\Rightarrow \overrightarrow{AK}=\frac{1}{3}\left(\vec{a}+\vec{b}\right) \qquad \text{... (i)}$$

Let E be the midpoint of AB, such that $\overrightarrow{AE}=\dfrac{1}{2}\vec{a}$

Let M be the point on $\overrightarrow{DE}$ such that DM: ME = 2:1

$$\therefore \overrightarrow{AM}=\frac{\overrightarrow{AD}+2\overrightarrow{AE}}{1+2}=\frac{\vec{b}+\vec{a}}{3} \qquad \text{... (ii)}$$

Comparing equations (i) and (ii), we find that $\overrightarrow{AK}=\dfrac{\left(\vec{b}+\vec{a}\right)}{3}=\overrightarrow{AM}$, and thus we conclude that K and M coincide, i.e. $\overrightarrow{DE}$ trisects $\overrightarrow{AC}$ and is trisected by $\overrightarrow{AC}$. Hence proved.

7. LINEAR COMBINATION OF VECTORS

7.1 Collinear and Non-Collinear Vectors

Let $\vec{a}$ and $\vec{b}$ be non-zero vectors. These vectors are said to be collinear if there exists $\lambda \neq 0$ such that $\vec{\alpha} + \lambda\vec{b} + \gamma\vec{c} = 0$.

Given a finite set of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$....., then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$....., for any scalar x, y, z ......$\in R$.

7.2 Collinearity of Three Points

Let three points with position vectors (non-zero) $\vec{a}$, $\vec{b}$, and $\vec{c}$ be collinear. Then there exists λ, γ both not being 0 such that $\vec{\alpha} + \lambda\vec{b} + \gamma\vec{c} = 0$

7.3 Coplanar Vectors

Let $\vec{a}$ and $\vec{b}$, be non-zero, non-collinear vectors. Then, any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}$, i.e. there exist some unique x, y$\in$ R, such that.

MASTERJEE CONCEPTS

- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero, non-coplanar vectors, then $x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x', y = y', z = z'$
- Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}$, $\vec{b}$, $\vec{c}$ or there exists some unique x, y, z $\in$ R, such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

Vaibhav Krishan (JEE 2009, AIR 22)

7.4 Linear Dependency of Vectors

A set of vectors $\left\{\vec{v}_1, \vec{v}_2, \ldots. \vec{v}_p\right\}$ is said to be linearly independent if the vector equation $x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots. + x_p \vec{v}_p = 0$ has only a trivial solution. The set $\left\{\vec{v}_1, \vec{v}_2, \ldots. \vec{v}_p\right\}$ is said to be linearly dependent if there exists weights $c_1, \ldots\ldots, c_p$, not all 0, such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots. + c_p\vec{v}_p = 0$

MASTERJEE CONCEPTS

- Two non-zero, non-collinear vectors are linearly independent.
- Any two collinear vectors are linearly dependent.
- Any three non-coplanar vectors are linearly independent.
- Any three coplanar vectors are linearly dependent.
- Any four vectors in three -dimensional space are linearly dependent.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 16: The position vectors of three points A = $\vec{a}-2\vec{b}+3\vec{c}$, B = $2\vec{a}+3\vec{b}-4\vec{c}$ and C = $-7\vec{b}+10\vec{c}$. Prove that the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent. **(JEE MAIN)**

Sol: Here obtain $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to check its linear dependency.
Let O be the point of reference, then, $\overrightarrow{OA}=\vec{a}-2\vec{b}+3\vec{c}$, $\overrightarrow{OB}=2\vec{a}+3\vec{b}-4\vec{c}$, and $\overrightarrow{OC}=-7\vec{b}+10\vec{c}$

$\Rightarrow \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\left(-7\vec{b}+10\vec{c}\right)-\left(\vec{a}-2\vec{b}+3\vec{c}\right)=-\vec{a}-5\vec{b}+7\vec{c}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(2\vec{a}+3\vec{b}-4\vec{c}\right)-\left(\vec{a}-2\vec{b}+3\vec{c}\right)=\vec{a}+5\vec{b}-7\vec{c}$

$\therefore AC=\lambda\,\overrightarrow{AB}$, where $\lambda=-1$.

Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Illustration 17: Prove that the vectors $5\vec{a}+6\vec{b}+7\vec{c}, 7\vec{a}-8\vec{b}+9\vec{c}$ and $3\vec{a}+20\vec{b}+5\vec{c}$ are linearly dependent and $\vec{a}, \vec{b}, \vec{c},$ being linearly independent vectors. **(JEE MAIN)**

Sol: We know that if these vectors are linearly dependent, then we can express one of them as a linear combination of the other two.

Now, let us assume that the given vectors are coplanar, and then we can write

$5\vec{a}+6\vec{b}+7\vec{c}=\ell\left(7\vec{a}-8\vec{b}+9\vec{c}\right)+m\left(3\vec{a}+20\vec{b}+5\vec{c}\right)$, where ℓ and m are scalars.

Comparing the coefficients of $\vec{a},\vec{b}$ and $\vec{c}$ on both sides of the equation

$5=7\ell+3m$ ... (i)
$6=-8\ell+20m$ ... (ii)
$7=9\ell+5m$ ... (iii)

From equations (i) and (iii), we get

$4=8\ell\Rightarrow\ell=\frac{1}{2}=m$, which evidently satisfies equation (ii) too.

Hence, the given vectors are linearly dependent.

Illustration 18: Prove that the four points $2\vec{a}+3\vec{b}-\vec{c},\vec{a}-2\vec{b}+3\vec{c}, 3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar. **(JEE MAIN)**

Sol: Let the given four points be P, Q, R and S respectively. These points are coplanar, if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar if one of them can be expressed as a linear combination of other two.

So, let

$\overrightarrow{PQ}=x\overrightarrow{PR}+y\overrightarrow{PS}$
$\Rightarrow -\vec{a}-5\vec{b}+4\vec{c}=x\left(\vec{a}+\vec{b}-\vec{c}\right)+y\left(-\vec{a}-9\vec{b}+7\vec{c}\right)$
$\Rightarrow -\vec{a}-5\vec{b}+4\vec{c}=\left(x-y\right)\vec{a}+\left(x-9y\right)\vec{b}+\left(-x+7y\right)\vec{c}$
$\Rightarrow x-y=-1, x-9y=-5, -x+7y=4$

Solving the first two of these three equations, we get $x=-\frac{1}{2}, y=\frac{1}{2}$

On substituting the values of x and y in the third equation, we find that the third equation is satisfied. Hence, the given four points are coplanar.

Illustration 19: Show that, the vectors $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ are non-coplanar vectors.
(JEE MAIN)

Sol: If vectors are coplanar then one of them can be expressed as a linear combination of other two otherwise they are non-coplanar. Assume the given vectors are coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a}-\vec{b}+3\vec{c}=x(\vec{a}+\vec{b}-2\vec{c})+y(\vec{a}+\vec{b}-3\vec{c})$ for some scalars x and y.

$\Rightarrow 2\vec{a}-\vec{b}+3\vec{c}=(x+y)\,\vec{a}+(x+y)\,\vec{b}+(-2x-3y)\,\vec{c} \Rightarrow 2=x+y, -1=x+y$ and $3=2x-3y$,

Clearly, the first two equations contradict each other. Hence, it is proved that the given vectors are not coplanar.

8. SCALAR OR DOT PRODUCT

The scalar product of two vectors $\vec{a}=(a_1, a_2, a_3)$ and $\vec{b}=(b_1, b_2, b_3)$ is written using a dot as an operator (·) between the two vectors. The component form of the dot product is as follows:

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3 \quad \text{... (i)}$$

And in geometrical form

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta \quad \text{... (ii)}$$

where θ is the angle between the two vectors and $0\le\theta\le\pi$.

From equation (i), it can also be written as

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{|\vec{a}||\vec{b}|},$$

which can be used to find the angle between two vectors. If $\vec{a}$ and $\vec{b}$ are perpendicular then

$\theta=90^\circ \Rightarrow \cos\theta=0 \Rightarrow \vec{a}.\vec{b}=0$

MASTERJEE CONCEPTS

- $\vec{a}\cdot\vec{b}\le|\vec{a}||\vec{b}|$
- $\vec{a}\cdot\vec{b}>0 \Rightarrow$ Angle between a and b is acute.
- $\vec{a}\cdot\vec{b}<0 \Rightarrow$ Angle between a and b is obtuse.

Shivam Agarwal (JEE 2009, AIR 27)

Geometrical Interpretation of Dot Product

The scalar product is used to determine the projection of $\vec{r}$ vector along the given direction.

$\overrightarrow{ON}$ is the component of vector $\overrightarrow{OB}\left(=\vec{b}\right)$ in the direction of vector $\overrightarrow{OA}\left(=\vec{a}\right)$; $\overrightarrow{ON}=b\cos\theta$. Thus the projection of $\vec{b}$ along $\hat{a}=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\hat{a}$

$$\therefore \overrightarrow{ON}=\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

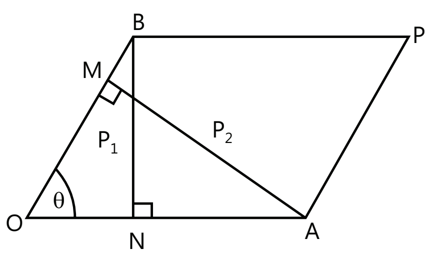

Figure 26.9

Projection of $\vec{a}$ along $\vec{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b}$ $\quad \therefore\ \overrightarrow{OM} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

8.1 Properties of Scalar Product

The properties of scalar product are listed as follows:

(a) $\vec{a}, \vec{b}$ are vectors and $\vec{a}.\vec{b}$ is a number **(b)** $\vec{a}.\vec{a} = |\vec{a}|^2$ **(c)** $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

(d) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ **(e)** $(c\vec{a}).\vec{b} = c(\vec{a}.\vec{b})$ **(f)** $\vec{0}.\vec{a} = 0$

(g) $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ **(h)** $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$

Illustration 20: Find the angle 'θ' between the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and vectors $\vec{b} = \hat{i}-\hat{j}+\hat{k}$ **(JEE MAIN)**

Sol: The angle θ between the two vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Now $\vec{a}.\vec{b} = \left(\hat{i}+\hat{j}-\hat{k}\right).\left(\hat{i}-\hat{j}+\hat{k}\right) = 1-1-1 = -1$

Therefore, we have $\cos\theta = \frac{-1}{3}$. Hence, the required angle is $\theta = \cos^{-1}\left(\frac{-1}{3}\right)$

Illustration 21: Find the length of the projection of vector $\vec{a} = 2\hat{i}+3\hat{j}+2\hat{k}$ on vector $\hat{b} = \hat{i}+2\hat{j}+\hat{k}$. **(JEE MAIN)**

Sol: The projection of vector $\vec{a}$ on the vector $\vec{b}$ is given by $\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right)$.

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{(2\cdot1+3\cdot2+2\cdot1)}{\sqrt{(1)^2+(2)^2+(1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

Illustration 22: Let $\vec{a}, \vec{b}, \vec{c}$ be the vectors of lengths 3, 4 and 5, respectively. Let $\vec{a}$ be perpendicular to $(\vec{b}+\vec{c})$, $\vec{b}$ to $\left(\vec{c}+\vec{a}\right)$ and $\vec{c}$ to $\left(\vec{a}+\vec{b}\right)$. Then, find the length of the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$. **(JEE MAIN)**

Sol: By using property of scalar product of vector we can solve this illustration.

Given $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$

$$\therefore \left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right) = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+\vec{a}\left(\vec{b}+\vec{c}\right)+\vec{b}\left(\vec{c}+\vec{a}\right)+\vec{c}\left(\vec{a}+\vec{b}\right) = 9+16+25+0+0+0$$

$$\Rightarrow \left|\vec{a}+\vec{b}+\vec{c}\right| = 5\sqrt{2}$$

Illustration 23: Let $\vec{a} = 4\hat{i}+5\hat{j}-k$, $\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$, and satisfying $\vec{d}.\vec{c} = 21$. **(JEE MAIN)**

Sol: If two vector are perpendicular then their product will be zero.

Let $\vec{d} = x\hat{i}+y\hat{j}+z\hat{k}$. Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore,

$$\vec{d}.\vec{a} = 0 \Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(4\hat{i}+5\hat{j}-\vec{k}\right) = 0 \Rightarrow 4x+5y-z = 0 \quad \text{... (i)}$$

$$\vec{d}.\vec{b} = 0 \Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}-4\hat{j}+5\hat{k}\right) = 0 \Rightarrow x-4y+5z = 0 \quad \text{... (ii)}$$

$$\vec{d}.\vec{c} = 21 \Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(3\hat{i}+\hat{j}-\hat{k}\right) = 21 \Rightarrow 3x+y-z = 21 \quad \text{... (iii)}$$

Solving equations (i), (ii) and (iii), we get x = 7, y = z = –7

Hence, $\vec{d} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

Illustration 24: Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. Evaluate the quantity $\mu = \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$, if $|\vec{a}| = 1, |\vec{b}| = 4$ and $|\vec{c}| = 2$. **(JEE MAIN)**

Sol: Simply using property of scalar product we can calculate the value of μ.

Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we have $\vec{a}.\left(\vec{a} + \vec{b} + \vec{c}\right) = \vec{0} \Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$. Therefore, $\vec{a}.\vec{b} + \vec{a}.\vec{c} = -|\vec{a}|^2 = -1$

Similarly $\vec{a}.\vec{b} + \vec{b}.\vec{c} = -|\vec{b}|^2 = -16$, $\vec{a}.\vec{c} + \vec{b}.\vec{c} = -4$.

On adding these equations, we have $2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{a}.\vec{c}) = -21$ or $2\mu = -21$, i.e., $\mu = \dfrac{-21}{2}$

Illustration 25: Prove, Cauchy–Schawarz inequality, $(\vec{a}.\vec{b})^2 \le |\vec{a}|^2 |\vec{b}|^2$, and hence show that

$$\left(a_1b_1 + a_2b_2 + a_3b_3\right)^2 \le \left(a_1 + a_2 + a_3\right)^2 \left(b + b_2 + b_3\right)^2$$

(JEE ADVANCED)

Sol: As we know $\cos^2\theta \le 1$, solve it by multiplying both side by $|\vec{a}|^2 |\vec{b}|^2$.

We have, $\cos^2\theta \le 1$

$$\Rightarrow |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta \le |\vec{a}|^2 |\vec{b}|^2 \Rightarrow \left(\vec{a}.\vec{b}\right)^2 \le |\vec{a}|^2 |\vec{b}|^2$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3, |\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$ and $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$.

$$\left(\vec{a}.\vec{b}\right)^2 \le |\vec{a}|^2 |\vec{b}|^2 \Rightarrow \left(a_1b_1 + a_2b_2 + a_3b_3\right)^2 \le \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

Illustration 26: If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$. **(JEE ADVANCED)**

Sol: Here use formula of dot product to solve the problem. Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, We have $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = \vec{a}.\vec{a} + \vec{b}.\vec{b} + \vec{c}.\vec{c} + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a} = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2 \quad \therefore |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$

Let $\vec{a} + \vec{b} + \vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}.\left(\vec{a} + \vec{b} + \vec{c}\right)}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c}}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{|\vec{a}|^2}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{\lambda}{\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \therefore \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\text{Similarly,} \theta_2 = \cos^{1}\left(\frac{1}{\sqrt{3}}\right) \text{and} \theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \therefore \theta_1 = \theta_2 = \theta_3$$

Hence, $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with $\vec{a}, \vec{b}$ and $\vec{c}$.

Illustration 27: Using vectors, prove that cos(A+B) = cosA cosB – sinA sinB **(JEE ADVANCED)**

Sol: From figure, using vector method we can easily prove that cos(A+B)=cosA cosB – sinA sinB.

Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY, respectively.

Let $\angle XOP = A$ and $\angle XOQ = B$. Draw $PL \perp OX$ and $QM \perp OX$

Therefore, the angle between $\overrightarrow{OP}$ and $\overrightarrow{OL}$ is A & $\overrightarrow{OQ}$ and $\overrightarrow{OL}$ is B.

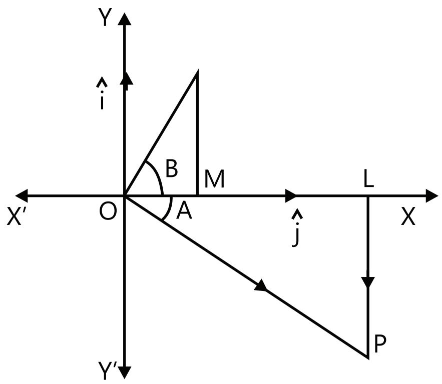

Figure 26.10

In ΔOLP, $OL = OP\cos A$ and $LP = OP\sin A$.

Therefore, $\overrightarrow{OL} = (OP\cos A)\hat{i}$ and $\overrightarrow{LP} = (OP\sin A)(-\hat{j})$

Now, $\overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OP}$

$\Rightarrow \overrightarrow{OP} = OP\left[(\cos A)\hat{i} - (\sin A)\hat{j}\right]$ ... (i)

In ΔOMQ, $\overrightarrow{OM} = OQ\cos B$ and $\overrightarrow{MQ} = OQ\sin B$.

Therefore, $\overrightarrow{OM} = (OQ\cos B)\hat{i}$, $\overrightarrow{MQ} = (OQ\sin B)\hat{j}$

$\Rightarrow \overrightarrow{OQ} = OQ\left[(\cos\beta)\hat{i} + (\sin\beta)\hat{j}\right]$ ...(ii)

From (i) and (ii), we get

$\overrightarrow{OP}.\overrightarrow{OQ} = OP\left[(\cos A)\hat{i} - (\sin A)\hat{j}\right].OQ\left[(\cos B)\hat{i} + (\sin B)\hat{j}\right] = OP.OQ\left[\cos A\cos B - \sin A\sin B\right]$

But, $\overrightarrow{OP}.\overrightarrow{OQ} = \left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|\cos(A+B) = OP.OQ\cos(A+B)$

$\therefore OP.OQ\cos(A+B) = OP.OQ[\cos A\cos B - \sin A\sin B]$

$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$

Illustration 28: Find the values of c for which the vectors $\vec{a} = (c\log_2 x)\hat{i} - 6\hat{j} + 3\hat{k}$ and $\vec{b} = (\log_2 x)\hat{i} + 2\hat{j} + (2c\log_2 x)\hat{k}$ made an obtuse angle for any $x \in (0,\infty)$. **(JEE ADVANCED)**

Sol: For obtuse angle $\cos\theta < 0$, therefore by using formula $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$, we can solve this problem.

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

For θ to be an obtuse angle, we must have $\Rightarrow \cos\theta < 0$, for all $x \in (0,\infty) \Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} < 0$, for all $x \in (0,\infty)$

$\Rightarrow \vec{a}.\vec{b} < 0$, for all $x \in (0,\infty) \Rightarrow \vec{a}.\vec{b} < 0$, for all $x \in (0,\infty) \Rightarrow c(\log_2 x)^2 - 12 + 6c(\log_2 x) < 0$, for all $x \in (0,\infty)$

$\Rightarrow cy^2 + 6cy - 12 < 0$, for all $y \in R$, where $y = -\log_2 x$ $[\because x > 0 \Rightarrow y = \log_2 x \in R]$

$\Rightarrow c < 0$ and $36c^2 + 48c < 0$ $[\because ax^2 + bx^2 + c > 0$ for all $x \Rightarrow a < 0$ and Discriminant $< 0]$

$\Rightarrow c < 0$ and $c(3c+4) < 0$

$\Rightarrow c < 0$ and $-\frac{4}{3} < c < 0 \Rightarrow c \in \left(\frac{-4}{3}, 0\right)$

Illustration 29: D is the midpoint of the side $\overrightarrow{BC}$ of a triangle ABC, show that $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$ **(JEE MAIN)**

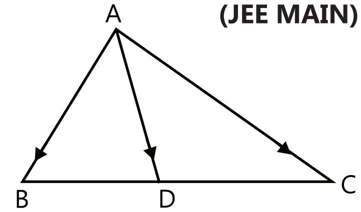

Figure 26.11

Sol: By using the formula of resultant vector we will get the required result.

Given D is midpoint of BC $\Rightarrow \overrightarrow{BD} = \overrightarrow{DC}$

We have $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} \Rightarrow AB^2 = \left(\overrightarrow{AD} + \overrightarrow{DB}\right)^2$

$AB^2 = AD^2 + DB^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DB}$ ...(i)

Also we have $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} \Rightarrow AC^2 = \left(\overrightarrow{AD} + \overrightarrow{DC}\right)^2$

$AC^2 = AD^2 + DC^2 + 2\overrightarrow{AD} \cdot \overrightarrow{DC}$ ...(ii)

Adding equations (i) and (ii), we get

$AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\overrightarrow{AD}.\left(\overrightarrow{DB} + \overrightarrow{DC}\right) = 2\left(DA^2 + DB^2\right)$

9. VECTOR OR CROSS PRODUCT

Let $\vec{a}$ and $\vec{b}$ be two vectors. The vector product of these two vectors can be calculated as $\left(\vec{a} \times \vec{b}\right) = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $(0 \le \theta \le \pi)$ and $\hat{n}$ is the unit vector at right angles to both $\vec{a}$ and $\vec{b}$, i.e. $\hat{n}$ is vector normal to the plane that contains $\vec{a}$ and $\vec{b}$. $\vec{a}$, $\vec{b}$ and $\hat{n}$ are three vectors which form a right-handed set.

The convention is that we choose the direction specified by the right-hand screw rule. Imagine a screw in your right hand. If you turn a right-handed screw from $\vec{a}$ to $\vec{b}$, the screw advances along the unit vector $\hat{n}$. It is very important to realize that the result of a vector product is itself a vector.

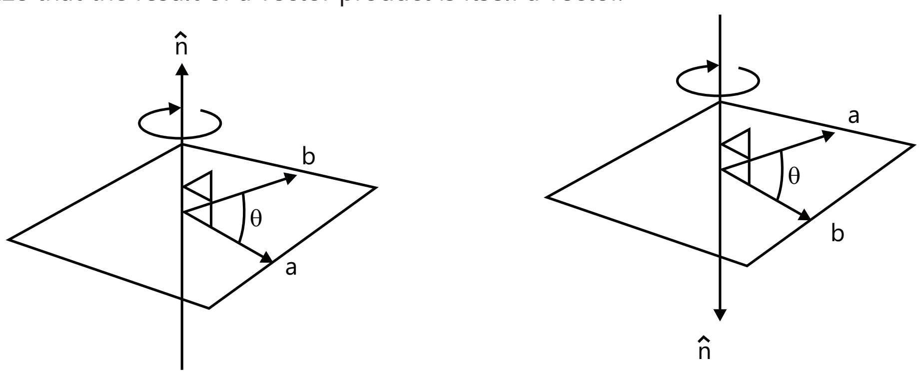

Figure 26.12

Let us see how the order of multiplication matters from the definition of the right-hand screw rule:

The vector given by $\left(\vec{b} \times \vec{a}\right)$ points in the opposite direction to $\left(\vec{a} \times \vec{b}\right)$. So, $\left(\vec{a} \times \vec{b}\right) = -\left(\vec{b} \times \vec{a}\right)$.

We can define vector product in terms of matrix notation as $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

and n terms of components as $\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle \Rightarrow \vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

From the definition, the angle can be calculated as $\sin\theta = \dfrac{|\vec{a}x\vec{b}|}{|\vec{a}||\vec{b}|}$

If $\vec{a}$ and $\vec{b}$ are parallel then $\theta = 0° \Rightarrow \sin\theta = 0$ and $\vec{a}x\vec{b} = 0$

9.1 Properties of Vector Product

The properties of vector product are listed as follows:

$\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are all vectors in three dimensions.

(a) $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{b}$

(b) $\left|\vec{a}\times\vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$

(c) $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

(d) $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$

(e) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

(f) $(c\vec{a})\times\vec{b} = \vec{a}\times(c\vec{b}) = c(\vec{a}\times\vec{b})$

(g) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

(h) $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$

Geometrical interpretation of $\left|\vec{a}\times\vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$, denotes the area of parallelogram, in which $\vec{a}$ and $\vec{b}$ are the two adjacent sides.

Vector area of the plane figure

Considering the boundaries of closed, bounded surface, which has been described in a specific manner and that do not cross, it is possible to associate a directed line segment $\vec{c}$, such that

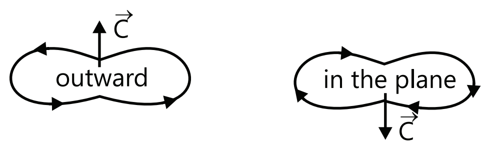

Figure 26.13

(a) $|\vec{c}|$ is the number of units of area enclosed by the plane figure.

(b) The support of $\vec{c}$ is perpendicular to the area and outside the surface.

(c) The sense of description of the boundaries and the direction of $\vec{c}$ is in accordance with the R.H.S. screw rule.

Vector area of a triangle

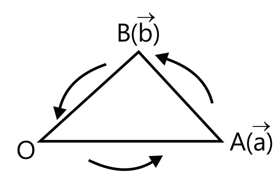

Figure 26.14

If $\vec{a}$ $\vec{b}$ are the position vectors, then the vector area of a triangle is given by the formula

$$\vec{\Delta} = \frac{1}{2}(\vec{a}\times\vec{b})$$

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors, then the vector area of ΔABC is given by the formula

$$\vec{\Delta} = \frac{1}{2}\left[(\vec{c}-\vec{b})\times(\vec{a}-\vec{b})\right]$$

$$\vec{\Delta} = \frac{1}{2}\left[(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})\right]$$

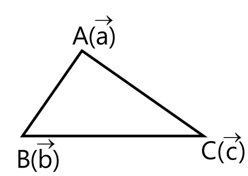

Figure 26.15

MASTERJEE CONCEPTS

(i) If three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear, then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$

(ii) Unit vector perpendicular to the plane of the ΔABC, when $\vec{a},\vec{b}$ and $\vec{c}$ are the p.v. of its angular point is

$$\hat{n}=\pm\frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{2\Delta}\cdot$$

Nitish Jhawar (JEE 2009, AIR 7)

10. ANGULAR BISECTOR

As discussed earlier, the diagonal of a parallelogram is not necessarily the bisector of the angle formed by two adjacent sides. However, the diagonal of a rhombus bisects the angle formed between two adjacent sides. Consider vectors $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{AD}=\vec{b}$ forming a parallelogram ABCD as shown in the figure.

Consider the two unit vectors along the given vectors, forming a rhombus AB′C′D′.

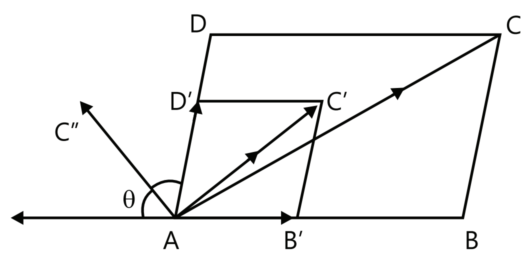

Figure 26.16

Now, $\overrightarrow{AB}=\frac{\vec{a}}{|\vec{a}|}$ and $\overrightarrow{AD}=\frac{\vec{b}}{|\vec{b}|}$. Therefore $\overrightarrow{AC'}=\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$ Then, any vector along the internal bisector is $\lambda\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}\right)$.

Similarly, any vector along the external bisector is $\lambda\left(\frac{\vec{a}}{|\vec{a}|}-\frac{\vec{b}}{|\vec{b}|}\right)$.

Illustration 30: Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$. **(JEE MAIN)**

Sol: By using property of vector product, we can solve this problem. Let $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\hat{b}=-2\hat{i}+\hat{j}-2\hat{k}$. Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&-1&3\\-2&1&-2\end{vmatrix}=(2-3)\hat{i}-(-8+6)\hat{j}+(4-2)\hat{k}=-\hat{i}+2\hat{j}+2\hat{k}\Rightarrow|\vec{a}\times\vec{b}|=\sqrt{(-1)^2+2^2+2^2}=3$$

$$\therefore \text{Required vector}=9\left\{\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right\}=\frac{9}{3}\left(-\hat{i}+2\hat{j}+2\hat{k}\right)=-3\hat{i}+6\hat{j}+6\hat{k}$$

Illustration 31: Find the area of a parallelogram, whose adjacent sides are given by the vectors $\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$ and $\hat{b}=\hat{i}-\hat{j}+\hat{k}$. **(JEE MAIN)**

Sol: The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$.

$$\text{Now, } \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&4\\1&-1&1\end{vmatrix}=5\hat{i}+\hat{j}-4\hat{k}.$$

Therefore, $|\vec{a}\times\vec{b}|=\sqrt{25+1+16}=\sqrt{42}$; Hence, the required area is $\sqrt{42}$.

Illustration 32: Let $\vec{a},\vec{b},\vec{c}$ be the unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, Prove that $\vec{a}=\pm 2\left(\vec{b}\times\vec{c}\right)$. **(JEE MAIN)**

Sol: Here $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$, therefore $\vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$ and it is parallel to $\vec{b}\times\vec{c}$.

We have $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$

$\Rightarrow \vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$ $\quad\Rightarrow \vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow \vec{a}$ is parallel to $\vec{b}\times\vec{c}$. $\Rightarrow \vec{a}=\lambda\left(\vec{b}\times\vec{c}\right)$ for some scalar λ.

$$\Rightarrow \left|\vec{a}\right|=\left|\lambda\right|\left|\vec{b}\right|\left|\vec{c}\right|\sin\frac{\pi}{6} \quad \Rightarrow 1=\frac{\left|\lambda\right|}{2} \qquad \left[\because \left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|\right]$$

$$\Rightarrow \left|\lambda\right|=2 \qquad \Rightarrow \lambda=\pm 2$$

$$\therefore \vec{a}=\lambda\left(\vec{b}\times\vec{c}\right) \qquad \Rightarrow \vec{a}=\pm 2\left(\vec{b}\times\vec{c}\right).$$

Illustration 33: If $\vec{a},\ \vec{b},\ \vec{c}$ are three non-zero vectors, such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, prove that $\vec{a},\ \vec{b},\ \vec{c}$ are mutually at right angles and $\left|\vec{b}\right|=1$ and $\left|\vec{c}\right|=\left|\vec{a}\right|$. **(JEE MAIN)**

Sol: Use property of vector or cross product to prove this illustration.

We have, $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$

$\Rightarrow \vec{c}\perp\vec{a},\vec{c}\perp\vec{b}$ and $\vec{a}\perp\vec{b},\vec{a}\perp\vec{c}$ $\qquad\Rightarrow \vec{a}\perp\vec{b},\vec{b}\perp\vec{c}$ and $\vec{c}\perp\vec{a}$.

$\Rightarrow \vec{a},\vec{b},\vec{c}$ are mutually perpendicular lines.

Again $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$ $\quad\Rightarrow\quad \left|\vec{a}\times\vec{b}\right|=\left|\vec{c}\right|$ and $\left|\vec{b}\times\vec{c}\right|=\left|\vec{a}\right|$

$$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\sin\frac{\pi}{2}=\left|\vec{c}\right| \text{ and } \left|\vec{b}\right|\left|\vec{c}\right|\sin\frac{\pi}{2}=\left|\vec{a}\right| \qquad [\because \vec{a}\perp\vec{b} \text{ and } \vec{b}\perp\vec{c}]$$

$$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|=\left|\vec{c}\right| \text{ and } \left|\vec{b}\right|\left|\vec{c}\right|=\left|\vec{a}\right|$$

$$\Rightarrow \left|\vec{b}\right|^2\left|\vec{c}\right|=\left|\vec{c}\right| \qquad \left[\text{Putting } \left|\vec{a}\right|=\left|\vec{b}\right|\left|\vec{c}\right| \text{in} \left|\vec{a}\right|\left|\vec{b}\right|=\left|\vec{c}\right|\right]$$

$$\Rightarrow \left|\vec{b}\right|^2=1 \qquad \left[\because \left|\vec{c}\right|\neq 0\right]$$

$$\Rightarrow \left|\vec{b}\right|=1$$

Putting $\left|\vec{b}\right|=1$ in $\left|\vec{a}\right|\left|\vec{b}\right|=\left|\vec{c}\right|$, we obtain $\left|\vec{a}\right|=\left|\vec{c}\right|$.

Illustration 34: Prove by vector method, that in a $\Delta ABC, \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ **(JEE MAIN)**

Sol: As area of triangle ABC is equal to $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BA}\right|=\frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|$, therefore by using cross product method we can prove this problem.

Let $\overrightarrow{BC}=\vec{a},\ \overrightarrow{CA}=\vec{b},\ \overrightarrow{AB}=\vec{c}$. Then

$$\text{The area of } \Delta ABC=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BA}\right|=\frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right| \quad\Rightarrow bc\ \sin A=ca\ \sin B=ab\ \sin C$$

Dividing the above expression by abc, we get $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$

Illustration 35: Given the vectors $\vec{a}=\hat{p}+2\hat{q}$ and $\vec{b}=2\hat{p}+\hat{q}$, where p and q are unit vectors forming an angle of $30°$. Find the area of the parallelogram constructed on these vectors. **(JEE MAIN)**

Sol: Simply by applying cross product between $\vec{a}$ and $\vec{b}$, we have $\vec{a}\times\vec{b}=\left(\hat{p}+2\hat{q}\right)\times\left(2\hat{p}+\hat{q}\right)=-3\left(\hat{p}\times\hat{q}\right)$.

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=3\left|\left(\hat{p}+\hat{q}\right)\right|=3\left|\hat{p}\right|\left|\hat{q}\right|\sin\frac{\pi}{6}=\frac{3}{2}$$

Illustration 36: Let $\overrightarrow{OA}=\vec{a},\overrightarrow{OB}=10\vec{a}+2\vec{b}$ and $\overrightarrow{OC}=\vec{b},$ where O is the origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides. Prove that p = 6q. **(JEE MAIN)**

Sol: We have to obtain the area of quadrilateral and parallelogram using cross product method to get the required result.

We have, p = area of the quadrilateral OABC

$$=1/2\left|\overrightarrow{OB}\times\overrightarrow{AC}\right|=1/2\left|\overrightarrow{OB}\times\left(\overrightarrow{OC}-\overrightarrow{OA}\right)\right|=1/2\left|\left(10\vec{a}+2\vec{b}\right)\times\left(\vec{b}-\vec{a}\right)\right|$$

$$=1/2\left|10\left(\vec{a}\times\vec{b}\right)-10\left(\vec{a}\times\vec{a}\right)+2\left(\vec{b}\times\vec{b}\right)-2\left(\vec{b}\times\vec{a}\right)\right|$$

$$=1/2\left|10\left(\vec{a}\times\vec{b}\right)-0+0+2\left(\vec{a}\times\vec{b}\right)\right|=6\left(\vec{a}\times\vec{b}\right) \qquad \text{... (i)}$$

and q = area of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$

$$=\left|\overrightarrow{OA}\times\overrightarrow{OC}\right|=\left(\vec{a}\times\vec{b}\right) \qquad \text{... (ii)}$$

From equations (i) and (ii), we get p = 6q.

Illustration 37: Given that D, E, F are the midpoints of the sides of a triangle ABC, using the vector method, prove that area of $\Delta DEF=\frac{1}{4}$ (area of ΔABC) **(JEE MAIN)**

Sol: Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

Then, the position vector of D, E and F are $\frac{1}{2}\left(\vec{b}+\vec{c}\right),\ \frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively. Therefore first obtain $\overrightarrow{DE}$ and $\overrightarrow{DF}$,

and after that by applying formula of vector area of triangle DEF we can obtain the required result.

Now, $\overrightarrow{DE}=\frac{1}{2}\vec{c}-\frac{1}{2}\left(\vec{b}+\vec{c}\right)=\frac{-\vec{b}}{2}$

and $\overrightarrow{DF}=\frac{1}{2}\vec{b}-\frac{1}{2}\left(\vec{b}+\vec{c}\right)=\frac{-\vec{c}}{2}$

$$\therefore \text{Vector area of } \Delta DEF=\frac{1}{2}\left(\overrightarrow{DE}\times\overrightarrow{DE}\right)=\left(\frac{-\vec{b}}{2}\times\frac{-\vec{c}}{2}\right)$$

$$=\frac{1}{8}\left(\vec{b}\times\vec{c}\right)=\frac{1}{4}\left\{\frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right\}=\frac{1}{4}(\text{vector area of } \Delta ABC)$$

$$\text{Hence, area of } \Delta DEF=\frac{1}{4}(\text{area of } \Delta ABC)$$

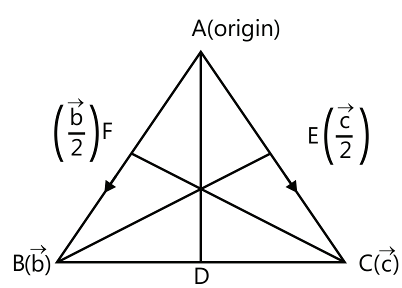

Figure 26.17

Illustration 38: Given that P, Q are the midpoints of the non-parallel sides BC and AD of a trapezium ABCD. Show that area of $\Delta APD = \Delta CQB$. **(JEE MAIN)**

Sol: Use formula of vector area of triangle to solve this problem. Let $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AD} = \vec{d}$ Now DC is parallel to $\overrightarrow{AB} \Rightarrow$ there exists a scalar t, such that. $\overrightarrow{DC} = t\overrightarrow{AB} = t\vec{b}$

$\therefore \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d} + t\vec{b}$

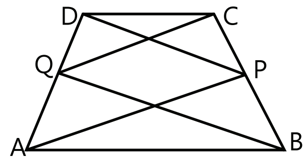

Figure 26.18

From geometry we know that QP = $QP = \dfrac{AB + DC}{2}$

Now $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ are $\dfrac{\vec{b} + \vec{d} + t\vec{b}}{2}$ and $\dfrac{\vec{d}}{2}$, respectively.

$$\text{Now}, 2\Delta APD = \overrightarrow{AP} \times \overrightarrow{AD} = \frac{1}{2}\left(\vec{b} + \vec{d} + t\vec{b}\right) \times \vec{d} = \frac{1}{2}(1+t)\left(\vec{b} \times \vec{d}\right)$$

$$\text{Also } 2\Delta CQB = \overrightarrow{BC} \times \overrightarrow{BQ} = \left[-\vec{b} + \vec{d} + t\vec{b}\right] \times \left[-\vec{b} + \frac{\vec{d}}{2}\right]$$

$$= -\left(\vec{d} \times \vec{b}\right) - \frac{\vec{b} \times \vec{d}}{2} + \frac{t\vec{b} \times \vec{d}}{2} = \frac{1}{2}(1+t)\vec{b} \times \vec{d} = 2\Delta APD \Rightarrow \Delta APD = \Delta CQB$$

11. TRIPLE PRODUCT OF VECTORS

Two types of triple products are listed below:

Vector triple product $\Rightarrow (a \times b) \times c$

Scalar triple product $\Rightarrow (a \times b).c$

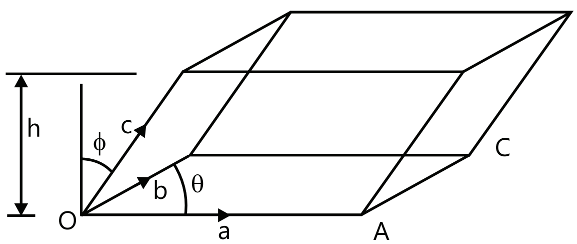

Figure 26.19

11.1 Scalar Triple Product

The scalar triple product has an interesting geometric interpretation:

We know that $\left(\vec{a} \times \vec{b}\right) = |a||b|\sin\theta\hat{n}$ = (area of the parallelogram defined by a and b)

Thus, $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = (area of the parallelogram) $\hat{n}.\vec{c}$ = (area of the parallelogram) $|\hat{n}||\vec{c}|\cos\phi$

But $|\vec{c}|\cos\theta = h =$ height of the parallelepiped normal to the plane containing $\vec{a}$ and $\vec{b}$. (ϕ is the angle between $\vec{c}$ and $\hat{n}$).

So, $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = volume of the parallelepiped defined by $\vec{a}, \vec{b}$ and $\vec{c}$. Thus, the following conclusions are arrived:

(a) If any two vectors are parallel, then $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = 0 (zero volume)

(b) If the three vectors a co-planar, then $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = 0 (zero volume)

(c) If $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = 0, then either

(i) $\vec{a}$ = 0, or (ii) $\vec{b}$ = 0 or (iii) $\vec{c}$ = 0 or

(iv) two of the vectors are parallel or (v) the three vectors are co-planar

(d) $\left(\vec{a} \times \vec{b}\right).\vec{c} = \vec{a}.\left(\vec{b} \times \vec{c}\right) = \vec{b}.\left(\vec{c} \times \vec{a}\right) =$ The same volume.

(e) $(\vec{a} \times \vec{b}).\ \vec{c}$ is also known as box product, which is represented as $[\vec{a}\vec{b}\vec{c}]$.

Also $\left[\vec{a}+\vec{b} \quad \vec{c} \quad \vec{d}\right] = \left[\vec{a} \quad \vec{c} \quad \vec{d}\right] + \left[\vec{b} \quad \vec{c} \quad \vec{d}\right]$

(f) If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $[\vec{a},\vec{b},\vec{c}]>0$, for right-handed system and $[\vec{a},\vec{b},\vec{c}] < 0$, for left handed system.

(g) If O is the origin and $\vec{a},\vec{b},\vec{c}$ are the position vectors of A, B and C, respectively, of the tetrahedron OABC, then the volume is given by the formula $V = \frac{1}{6}\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$.

Reciprocal system of vectors

(a) If $\vec{a},\vec{b},\vec{c}$ and $\overrightarrow{a'},\overrightarrow{b'},\overrightarrow{c'}$ are the two sets of non-coplanar vectors, such that $\vec{a}.\overrightarrow{a'} = \vec{b}.\overrightarrow{b'} = \vec{c}.\overrightarrow{c'} = 1$, $\vec{a}.\overrightarrow{b'} = \vec{a}.\overrightarrow{c'} = 0$, $\vec{b}.\overrightarrow{a'} = \vec{b}.\overrightarrow{c'} = 0$ and $\vec{c}.\overrightarrow{a'} = \vec{c}.\overrightarrow{b'} = 0$,

Then $\vec{a},\vec{b},\vec{c}$ and $\overrightarrow{a'},\overrightarrow{b'},\overrightarrow{c'}$ constitute a reciprocal system of vectors.

(b) Reciprocal system of vectors exists only in the case of dot product.

(c) $\overrightarrow{a'},\overrightarrow{b'},\overrightarrow{c'}$ can be defined in terms of $\vec{a},\vec{b},\vec{c}$ as

$$\overrightarrow{a'} = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]};\overrightarrow{b'} = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]};\overrightarrow{c'} = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]} \qquad \left(\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]\neq 0\right)$$

Note:

(i) $\vec{a}\times\overrightarrow{a'}+\vec{b}\times\overrightarrow{b'}+\vec{c}\times\overrightarrow{c'} = 0 \Rightarrow \vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\left(\vec{a}\times\vec{b}\right) = 0$

(ii) $\vec{a}\cdot\overrightarrow{a'} = \vec{b}\cdot\overrightarrow{b'} = \vec{c}\cdot\overrightarrow{c'} = 1$

(iii) $\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\overrightarrow{a'}+\overrightarrow{b'}+\overrightarrow{c'}\right) = 3$

(iv) If $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = V$ then $\left[\overrightarrow{a'}\ \ \overrightarrow{b'}\ \ \overrightarrow{c'}\right] = \frac{1}{V} \Rightarrow \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]\left[\overrightarrow{a'}\ \ \overrightarrow{b'}\ \ \overrightarrow{c'}\right] = 1$

Illustration 39: If $\vec{\ell}, \vec{m}, \vec{n}$ three non-coplanar vectors, then prove that

$$\left[\vec{\ell}\ \ \vec{m}\ \ \vec{n}\right]\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \vec{\ell}\cdot\vec{a} & \vec{\ell}\cdot\vec{b} & \vec{\ell} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}.$$ **(JEE ADVANCED)**

Sol: Use scalar triple product method as mentioned above to solve this problem.

Let, $\vec{\ell} = \ell_1\hat{i}+\ell_2\hat{j}+\ell_3\hat{k}, \quad \vec{m} = m_1\hat{i}+m_2\hat{j}+m_3\hat{k}, \quad \vec{n} = n_1\hat{i}+n_2\hat{j}+n_3\hat{k},$

and $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \quad \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

Now $\left[\vec{\ell}\ \ \vec{m}\ \ \vec{n}\right] = \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$ and $= \left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\left[\vec{\ell}\ \ \vec{m}\ \ \vec{n}\right]\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \ell_1\hat{i}+\ell_2\hat{j}+\ell_3\hat{k} & \sum\ell_1a_2 & \sum\ell_1b_1 \\ m_1\hat{i}+m_2\hat{j}+m_3\hat{k} & \sum m_1a_1 & \sum m_1b_1 \\ n_1\hat{i}+n_2\hat{j}+n_3\hat{k} & \sum n_1a_1 & \sum n_1b_1 \end{vmatrix}$$

Now, $\vec{\ell}.\vec{a} = \left(\ell_1\hat{i}+\ell_2\hat{j}+\ell_3\hat{k}\right).\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right) = \sum\ell_1a_2$ etc.

$$\therefore\left[\vec{\ell}\quad\vec{m}\quad\vec{n}\right]\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\vec{\ell} & \vec{\ell}\cdot\vec{a} & \vec{\ell}\cdot\vec{b}\\ \vec{m} & \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b}\\ \vec{n} & \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b}\end{vmatrix}=\begin{vmatrix}\vec{\ell}\cdot\vec{a} & \vec{\ell}\cdot\vec{b} & \vec{\ell}\\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\end{vmatrix},$$

Hence proved.

Illustration 40: Find the volume of a parallelepiped, whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}, -5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$ **(JEE MAIN)**

Sol: We know that, the volume of a parallelepiped, whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]$.

Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}, \vec{b}=-5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$

We know that, the volume of a parallelepiped, whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]$

Now, $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=\begin{vmatrix}-3 & 7 & 5\\ -5 & 7 & -3\\ 7 & -5 & -3\end{vmatrix}=-3(-21-15)-7(15+21)+5(25-49)=108-252-120=-264$

So, the required volume of the parallelepiped$=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=|-264|=264$ cubic units.

Illustration 41: Simplify $\left[\vec{a}-\vec{b}\quad\vec{b}-\vec{c}\quad\vec{c}-\vec{a}\right]$ **(JEE ADVANCED)**

Sol: Here by using scalar triple product we can simplify this.

$\left[\vec{a}-\vec{b}\quad\vec{b}-\vec{c}\quad\vec{c}-\vec{a}\right]=\left\{\left(\vec{a}-\vec{b}\right)\times\left(\vec{b}-\vec{c}\right)\right\}.\left(\vec{c}-\vec{a}\right)$ [by def.]

$=\left(\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{b}\times\vec{c}\right).\left(\vec{c}-\vec{a}\right)$ [by dist.law]

$=\left(\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{b}\times\vec{c}\right).\left(\vec{c}-\vec{a}\right)$ $\left[\because\vec{b}\times\vec{b}=0\right]$

$=\left(\vec{a}\times\vec{b}\right).\vec{c}-\left(\vec{a}\times\vec{b}\right).\vec{a}+\left(\vec{c}\times\vec{a}\right).\vec{c}-\left(\vec{c}\times\vec{a}\right).\ \vec{a}+\left(\vec{b}\times\vec{c}\right).\ \vec{c}-\ \left(\vec{b}\times\vec{c}\right).\vec{a}$ [by dist. law]

$=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]-0+0-0+0-\left[\vec{b}\quad\vec{c}\quad\vec{a}\right]$

$=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]-\left[\vec{b}\quad\vec{c}\quad\vec{a}\right]=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]-\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=0$ $\left[\because\left[\vec{b}\quad\vec{c}\quad\vec{a}\right]=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]\right]$

Illustration 42: Find the volume of the tetrahedron, whose four vertices have position vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$, respectively. **(JEE MAIN)**

... (i)

Sol: Here volume of tetrahedron is equal to $\frac{1}{6}\left[\vec{a}-\vec{d}\quad\vec{b}-\vec{d}\quad\vec{c}-\vec{d}\right]$.

Let, four vertices be A, B, C, D with p.v. $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ respectively.

$\therefore\overrightarrow{DA}=\left(\vec{a}-\vec{d}\right),\overrightarrow{DB}=\left(\vec{b}-\vec{d}\right),\overrightarrow{DC}=\left(\vec{c}-\vec{d}\right)$

Hence volume $=\frac{1}{6}\left[\vec{a}-\vec{d}\quad\vec{b}-\vec{d}\quad\vec{c}-\vec{d}\right]=\frac{1}{6}\left(\vec{a}-\vec{d}\right).\left[\left(\vec{b}-\vec{d}\right)\times\left(\vec{c}-\vec{d}\right)\right]$

$=\frac{1}{6}\left(\vec{a}-\vec{d}\right).\left[\vec{b}\times\vec{c}-\vec{b}\times\vec{d}+\vec{c}\times\vec{d}\right]=\frac{1}{6}\left\{\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]-\left[\vec{a}\quad\vec{b}\quad\vec{d}\right]+\left[\vec{a}\quad\vec{c}\quad\vec{d}\right]-\left[\vec{d}\quad\vec{b}\quad\vec{c}\right]\right\}$

$=\frac{1}{6}\left\{\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]-\left[\vec{d}\quad\vec{b}\quad\vec{c}\right]-\left[\vec{a}\quad\vec{d}\quad\vec{c}\right]-\left[\vec{a}\quad\vec{b}\quad\vec{d}\right]\right\}$.

Illustration 43: Let $\vec{u}$ and $\vec{v}$ be unit vectors and $\vec{w}$ is a vector, such that $\vec{u}\times\vec{v}+\vec{u}=\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$, then find the value of $\left[\vec{u}\quad\vec{v}\quad\vec{w}\right]$. **(JEE ADVANCED)**

Sol: Here as given $\vec{u}\times\vec{v}+\vec{u}=\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$, solve it using scalar triple product.

Given, $\vec{u}\times\vec{v}+\vec{u}=\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$

$\Rightarrow\left(\vec{u}\times\vec{v}+\vec{u}\right)\times\vec{u}=\vec{w}\times\vec{u}$

$\Rightarrow\left(\vec{u}\times\vec{v}\right)\times\vec{u}+\vec{u}\text{ x }\vec{u}=\vec{v}$ (as, $\vec{w}\times\vec{u}=\vec{v}$)

$\Rightarrow\left(\vec{u}.\vec{u}\right)\vec{v}-\left(\vec{v}.\vec{u}\right)\vec{u}+\vec{u}\times\vec{u}=\vec{v}$ (using $\vec{u}.\vec{u}=1$ and $\vec{u}\times\vec{u}=0$, since unit vector)

$\Rightarrow\vec{v}-\left(\vec{v}.\vec{u}\right)\vec{u}=\vec{v}\Rightarrow\left(\vec{u}.\vec{v}\right)\vec{u}=\vec{0}$

$\Rightarrow\vec{u}.\vec{v}=0$ (as; $\vec{u}\neq 0$) ....(i)

$\therefore\left[\vec{u}\quad\vec{v}\quad\vec{w}\right]=\vec{u}.\left(\vec{v}\times\vec{w}\right)$

$=\vec{u}.\left(\vec{v}\times\left(\vec{u}\times\vec{v}+\vec{u}\right)\right)$ (given $\vec{w}=\vec{u}\times\vec{v}+\vec{u}$)

$=\vec{u}.\left(\vec{v}\times\left(\vec{u}\times\vec{v}\right)+\vec{v}\times\vec{u}\right)=\vec{u}.\left(\left(\vec{v}.\vec{v}\right)\vec{u}-\left(\vec{v}.\vec{u}\right)\vec{v}+\vec{v}\times\vec{u}\right)$

$=\vec{u}.\left(\vec{u}-0+\vec{v}\times\vec{u}\right)$ (as $\vec{u}.\vec{v}=0$ from (i))

$=\left(\vec{u}.\vec{u}\right)-\vec{u}.\left(\vec{v}\times\vec{u}\right)=1-0=1$

$\therefore\left[\vec{u}\quad\vec{v}\quad\vec{w}\right]=1$

11.2 Vector Triple Product

Definition: $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ is a vector, which is coplanar to $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

Hence $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=x\vec{a}+y\vec{b}$ [Linear Combination of $\vec{a}$ and $\vec{b}$] ... (i)

$\vec{c}.\left(\vec{a}\times\vec{b}\right)\times\vec{c}=x\left(\vec{a}.\vec{c}\right)+y\left(\vec{b}.\vec{c}\right)$ ... (ii)

$0=x\left(\vec{a}.\vec{c}\right)+y\left(\vec{b}.\vec{c}\right)$

$\frac{x}{\vec{b}.\vec{c}}=-\frac{y}{\vec{a}.\vec{c}}=\lambda$

$\therefore x=\lambda\left(\vec{b}.\vec{c}\right)$ and $y=-\lambda\left(\vec{a}.\vec{c}\right)$

Substituting the values of x and y we get, $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\lambda\left(\vec{b}.\vec{c}\right)\vec{a}-\lambda\left(\vec{a}.\vec{c}\right)\vec{b}$

This identity must hold true for all values of $\vec{a},\vec{b},\vec{c}$

Substitute $\vec{a}=\hat{i};\vec{b}=\hat{j}$ and $\vec{c}=\hat{k}$

$\left(\hat{i}\times\hat{j}\right)\times\hat{i}=\lambda\left(\hat{j}.\hat{i}\right)\hat{i}-\lambda\left(\hat{i}.\hat{i}\right)\hat{j}$

$\hat{j}=-\lambda\hat{j}\Rightarrow\lambda=-1$

$\Rightarrow\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a}$

Note: Unit vector coplanar with $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{a}$ is $\pm\frac{\left(\vec{a}\times\vec{b}\right)\times\vec{a}}{\left|\left(\vec{a}\times\vec{b}\right)\times\vec{a}\right|}$

Illustration 44: Prove that $\vec{a}\times\left\{\vec{b}\times\left(\vec{c}\times\vec{d}\right)\right\}=\left(\vec{b}.\vec{d}\right)\left(\vec{a}\times\vec{c}\right)-\left(\vec{b}.\vec{c}\right)\left(\vec{a}\times\vec{d}\right)$ **(JEE MAIN)**

Sol: By using vector triple product as mention above.

We have, $\vec{a}\times\left\{\vec{b}\times\left(c\times d\right)\right\}=\vec{a}\times\left\{\left(\vec{b}.\vec{d}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{d}\right\}$

$=\vec{a}\times\left\{\left(\vec{b}.\vec{d}\right)\vec{c}-\vec{a}\left(\vec{b}.\vec{c}\right)\vec{d}\right\}$ [by distributivelaw]

$=\left(\vec{b}.\vec{d}\right)\left(\vec{a}\times\vec{c}\right)-\left(\vec{b}.\vec{c}\right)\left(\vec{a}\times\vec{d}\right)$

Illustration 45: Let $\vec{a}=a\hat{i}+2\hat{j}-3\hat{k},\vec{b}=\hat{i}+2a\hat{j}-2\hat{k}$, and $\vec{c}=2\hat{i}+a\hat{j}-\hat{k}$. Find the value (s) of a, if any, such that $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\times\left(\vec{c}\times\vec{a}\right)=0.$ **(JEE MAIN)**

Sol: Here use vector triple product to obtain the value of a.

$\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\times\left(\vec{c}\times\vec{a}\right)=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]\vec{b}\times\left(\vec{c}\times\vec{a}\right)=\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]\left\{\left(\vec{a}.\vec{b}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}\right\},$

Given $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\times\left(\vec{c}\times\vec{a}\right)=0\Rightarrow\ \left(\vec{a}.\vec{b}\right)\vec{c}=\left(\vec{b}.\vec{c}\right)\vec{a}$ or $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=0$

$\left(\vec{a}.\vec{b}\right)\vec{c}=\left(\vec{b}.\vec{c}\right)\vec{a}$ leads to three different equations which do not have a common solution.

$$\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=0\Rightarrow\begin{vmatrix}a&2&-3\\1&2a&-2\\2&a&-1\end{vmatrix}=0\Rightarrow 9a-6=0\Rightarrow a=\frac{2}{3}$$

Illustration 46: Solve for $\vec{r}$, from the simultaneous equations $\vec{r}\times\vec{b}=\vec{c}\times\vec{b},\vec{r}.\vec{a}=0$, provided $\vec{a}$ is not perpendicular to $\vec{b}$. **(JEE MAIN)**

Sol: As given $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$, solve this using vector triple product to get the result.

Given $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$

$\Rightarrow\left(\vec{r}-\vec{c}\right)\times\vec{b}=0\qquad\Rightarrow\left(\vec{r}-\vec{c}\right)$ and $\vec{b}$ are collinear

$\therefore\vec{r}-\vec{c}=k\vec{b}\qquad\Rightarrow\vec{r}=\vec{c}+k\vec{b}$ ....(i)

$\vec{r}.\vec{a}=0\qquad\Rightarrow\left(\vec{c}+k\vec{b}\right).\vec{a}=0$

$\Rightarrow k=-\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}$ putting in eq. (i) we get $\vec{r}=\vec{c}-\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\vec{b}=\frac{\vec{a}\times(\vec{c}\times\vec{b})}{\vec{a}.\vec{b}}.$

Illustration 47: If $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, where k is a scalar and $\vec{a},\vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a}$, $\vec{b}$ and k. **(JEE MAIN)**

Sol: Here as given $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, Apply cross product of $\vec{a}$ with both side and solve using vector triple product.

$\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$ ... (i)

$\Rightarrow\vec{a}\times\left(\vec{x}\times\vec{a}\right)+k\left(\vec{a}\times\vec{x}\right)=\left(\vec{a}\times\vec{b}\right)$

$\Rightarrow\left(\vec{a}.\vec{a}\right)\vec{x}-\left(\vec{a}.\vec{x}\right)\vec{a}+k\left(\vec{a}\times\vec{x}\right)=\vec{a}\times\vec{b}$ ... (ii)

(i)$\Rightarrow$ $\vec{a}.\left(\vec{x}\times\vec{a}\right)+k\left(\vec{a}.\vec{x}\right)=\vec{a}.\vec{b}$

$\Rightarrow\quad k\left(\vec{a}.\vec{x}\right)=\vec{a}.\vec{b}$ ... (iii)

Substituting the values from equations (i) and (iii) in equation (ii), we get,

$$\Rightarrow \left(\vec{a}.\vec{a}\right)\vec{x} - \frac{1}{k}\left(\vec{a}.\vec{b}\right)\vec{a} + k\left(k\vec{x} - \vec{b}\right) = \vec{a} \times \vec{b}$$

$$\Rightarrow \left(a^2 + k^2\right)\vec{x} = \left(\vec{a} \times \vec{b}\right) + \frac{1}{k}\left(\vec{a}.\vec{b}\right)\vec{a} + k\vec{b} \;\Rightarrow \vec{x} = \frac{1}{a^2 + k^2}\left[k\vec{b} + \left(\vec{a} \times \vec{b}\right) + \frac{\vec{a}.\vec{b}}{k}\vec{a}\right]$$

12. APPLICATION OF VECTORS IN 3D GEOMETRY

(a) Direction cosines of $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are given by $\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$.

(b) **Incentre formula:** The position vector of the incentre of ΔABC is $\frac{a\vec{a} + b\vec{b} + c\vec{c}}{a + b + c}$

(c) **Orthocentre formula:** The position vector of the orthocenter of

ΔABC is $\frac{\vec{a}\tan A + \vec{b}\tan B + \vec{c}\tan C}{\tan A + \tan B + \tan C}$

(d) The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$.

(e) The vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$.

(f) **Perpendicular distance of a point from a line**: Let L be the foot of perpendicular drawn $P\left(\vec{\alpha}\right)$ on the line $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{r}$ is the position vector of any point on the give line. Therefore, let the position vector $\vec{L}$ be $a + \lambda b$.

$$PL = \frac{\left|\left(\vec{a} - \vec{\alpha}\right) \times \vec{b}\right|}{\left|\vec{b}\right|} \text{ and } \overrightarrow{PL} = \vec{a} - \vec{\alpha} + \lambda\vec{b} = \left(\vec{a} - \vec{\alpha}\right) - \left(\frac{\left(\vec{a} - \vec{\alpha}\right).\vec{b}}{\left|\vec{b}\right|^2}\right)\vec{b}$$

The length PL is the magnitude of $\overrightarrow{PL}$, and the required length of perpendicular.

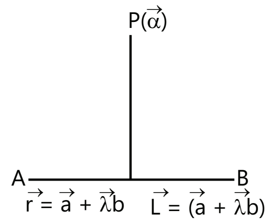

Figure 26.20

(g) **Image of a point in a straight line:** If $Q\left(\vec{\beta}\right)$ is the image of P in $\vec{r} = \vec{a} + \lambda\vec{b}$, then

$$\vec{\beta} = 2\vec{a} - \left(\frac{2\left(\vec{a} - \vec{\alpha}\right).\vec{b}}{\left|\vec{b}\right|^2}\right)\vec{b} - \vec{\alpha}$$

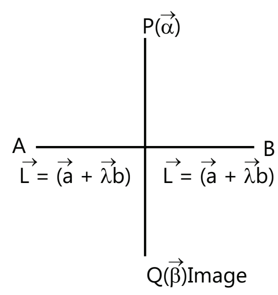

Figure 26.21

(h) **Shortest distance between two skew lines:** Let l_1 and l_2 be two lines whose equations are $l_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $l_2 : \vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, respectively.

Then, shortest distance is given by $PQ = \left| \dfrac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \left| \dfrac{[\vec{b}_1 \ \ \vec{b}_2 \ \ \vec{a}_2 - \vec{a}_1]}{|\vec{b}_1 \times \vec{b}_2|} \right|$

Shortest distance between two parallel lines: The shortest distance between the two given parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$.

If the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ intersect, then the shortest distance between them is zero.

Therefore, $[\vec{b}_1 \ \ \vec{b}_2 \ \ \vec{a}_2 - \vec{a}_1] = 0$

(i) If the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ are coplanar, then $[\vec{a}_1 \ \ \vec{b}_1 \ \ \vec{b}_2] = [\vec{a}_2 \ \ \vec{b}_1 \ \ \vec{b}_2]$ and the equation of the plane containing them is given by $[\vec{r} \ \ \vec{b}_1 \ \ \vec{b}_2] = [\vec{a}_1 \ \ \vec{b}_1 \ \ \vec{b}_2]$.

(j) The vector equation of a plane through the point A ($\vec{a}$) and perpendicular to the vector $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ is given by $(\vec{r} - \vec{a}).\vec{n} = 0$.

(k) Vector The vector equation of a plane normal to unit vector $\vec{n}$ and at a distance d from the origin is given by $\vec{r}.\hat{n} = d$.

(l) The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c} \Rightarrow [\vec{r} \ \ \vec{b} \ \ \vec{c}] = [\vec{a} \ \ \vec{b} \ \ \vec{c}]$, where λ and μ are scalars.

(m) The vector equation of plane passing through a point $\vec{a}, \vec{b}, \vec{c}$ is given by $\vec{r} = (1\text{-}s\text{-}t)\vec{a} + s\vec{b} + t\vec{c}$

or $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = [\vec{a} \ \ \vec{b} \ \ \vec{c}]$

(n) The equation of any plane through the intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

(o) The perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}.\vec{n} = d$ is given by $p = \dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$.

(p) The angle θ between the planes $\vec{r}_1.\hat{n}_1 = d_1$ and $\vec{r}_2.\hat{n}_2 = d_2$ is given by $\cos\theta = \pm\dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

(q) The perpendicular distance of a point P($\vec{r}$) from a line passing through $\vec{a}$ and parallel to $\vec{b}$ is given by

$$P = \frac{|(\vec{r} - \vec{a}) \times \vec{b}|}{|\vec{b}|} = \left[(\vec{r} - \vec{a})^2 - \left\{ \frac{(\vec{r} - \vec{a}).\vec{b}}{|\vec{b}|} \right\}^2 \right]^{1/2}$$

(r) The equation of the planes bisecting the angles between the planes $\vec{r}_1.\vec{n}_1 = d_1$ and $\vec{r}_2.\vec{n}_2 = d_2$ is

$$\vec{r}.(\vec{n}_1 \pm \vec{n}_2) = \frac{d_1}{|\vec{n}_1|} \pm \frac{d_2}{|\vec{n}_2|}$$

(s) The perpendicular distance of a point P($\vec{r}$) from a plane passing through a point $\vec{a}$ and parallel to points $\vec{b}$ and $\vec{c}$ is given by $PM = \dfrac{(\vec{r} - \vec{a}).(\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$

(t) The perpendicular distance of a point P($\vec{r}$) from a plane passing through the points $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

$$P = \frac{(\vec{r}-\vec{a}).(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}$$

(u) **Angle between a line and the plane:** If θ is the angle between a line $\vec{r}=(\vec{a}+\lambda\vec{b})$ and the plane $\vec{r}.\vec{n}=d$, then $\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$.

(v) The equation of sphere with center at C($\vec{c}$) and radius 'a' is $|\vec{r}-\vec{c}| = a$. If center is the origin then $|\vec{r}| = a$.

(w) The plane $\vec{r}.\vec{n}=d$ touches the sphere $|\vec{r}-\vec{a}| = R$, if $\frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|} = R$, i.e. the condition of tangency.

(x) If $\vec{a}$ and $\vec{b}$ are the position vectors of the extremities of a diameter of a sphere, then its equation is given by $(\vec{r}-\vec{a}).(\vec{r}-\vec{b}) = 0$ or $|\vec{r}|^2 - \vec{r}.\ (\vec{a}+\vec{b}) + \vec{a}.\vec{b} = 0$ or $|\vec{r}-\vec{a}|^2 + |\vec{r}-\vec{b}|^2 = |\vec{a}-\vec{b}|^2$.

FORMULAE SHEET

(a) $\overrightarrow{OP} = x\hat{i} + y\hat{j}$

(b) $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$ and direction is $\tan\theta = \frac{y}{x}$

(c) Unit vector $\hat{U} = \frac{\text{Vector}}{\text{Its modulus}} = \frac{\vec{a}}{|\vec{a}|}$

(d) Properties of vector addition:

i. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ commutative	(a) $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$ Associative
ii. $\vec{a}+\vec{0}=\vec{a}$ Null vector is an additive identity	(b) $\vec{a}+(-\vec{a}) = \vec{0}$ Additive inverse
iii. $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$	(c) $(c+d)\vec{a} = c\vec{a}+d\vec{a}$
iv. $(cd)\vec{a} = c(d\vec{a})$	(d) $1\times\vec{a} = \vec{a}$

(e) **Section formula:**

(i) If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector of a point which divides AB in the ratio m:n is given by $\vec{r} = \frac{(n\vec{a}+m\vec{b})}{(m+n)}$.

(ii) Position vector of mid-point of $\overrightarrow{AB} = \frac{(\vec{a}+\vec{b})}{2}$.

(f) Collinearity of three points: If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors (non-zero) of three points and given they are collinear then there exists λ, γ both not being 0 such that $\vec{a} + \lambda\vec{b} + \gamma\vec{c}$

(g) Coplanar vectors: Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors. Then, any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique x, $y \in R$, such that $x\vec{a} + y\vec{b} = \vec{r}$

(h) Product of two vectors:

(i) Scalar Product (dot product)

If $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Note : • $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

• $\vec{a}$ and $\vec{b}$ are perpendicular if $\theta = 90°$

(ii) Properties of scalar product:

i. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$	ii. $m\vec{a}.n\vec{b} = mn\vec{a}.\vec{b} = \vec{a}(.mn\vec{b})$
iii. $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{c} + \vec{a}.\vec{b}$	iv. $(\vec{a} + \vec{b})^2 = \vec{a}^2 + 2.\vec{a}.\vec{b} + \vec{b}^2$
v. If $\hat{i} = (1,0,0), \hat{j} = (0,1,0), \hat{k} = (0,0,1)$ then $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$	

(iii) Vector (cross) Product of two vectors: Let $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$ be two vectors then the cross product of $\vec{a} \times \vec{b}$ is devoted by $\vec{a} \times \vec{b}$ and defined by

$$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{pmatrix} a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{pmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

OR

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}| \times |\vec{b}| \sin\theta\ \hat{n}$$

Note: (i) θ being angle between $\vec{a}$ & $\vec{b}$

(ii) If $\theta = 0$, The $|\vec{a} \times \vec{b}| = 0$ i.e. $\vec{a} \times \vec{b} = 0$ and $\vec{a}$ & $\vec{b}$ are parallel if $\vec{a} \times \vec{b}$ =0.

(iv) Properties of cross product

i. $\vec{a}\times\vec{b}=0 \Rightarrow \vec{a}=0$ or $\vec{b}=0$ or $\vec{a}\parallel\vec{b}$	ii. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
iii. $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$	iv. $\left(n\vec{a}\right)\times\vec{b}=n\left(\vec{a}\times\vec{b}\right)$
v. $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$	vi. $\left\|\vec{a}\times\vec{b}\right\|$ is a Area of parallelogram with sides $\vec{a}$ and $\vec{b}$.

(v) Scalar Triple Product: If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

Then $\vec{a}.\left(\vec{b}\times\vec{c}\right)=\vec{b}.\left(\vec{c}\times\vec{a}\right)=\vec{c}.\left(\vec{a}\times\vec{b}\right)$

$$\vec{a}.\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\vec{a}.\left(\vec{b}\times\vec{c}\right)$ is also represented as $\left[\vec{a}\;\vec{b}\;\vec{c}\right]$

$$\left[\vec{a}\;\vec{b}\;\vec{c}\right]=\left[\vec{b}\;\vec{c}\;\vec{a}\right]=\left[\vec{c}\;\vec{a}\;\vec{b}\right]$$

$$\left[\vec{a}\;\vec{b}\;\vec{c}\right]=-\left[\vec{a}\;\vec{c}\;\vec{b}\right]$$

- If any of the two vectors are parallel, then $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=0$
- $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]$ is the volume of the parallelepiped whose coterminous edges are formed by $\vec{a}\,\vec{b}\,\vec{c}$
- If $\vec{a}\,\vec{b}\,\vec{c}$ are coplanar, $\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=0$
- $\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$ = area of triangle having $\vec{a},\vec{b},\vec{c}$ as position vectors of vertices of a triangle.

(vi) Vector Triple Product:

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$$
$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a}$$

Unit vector coplanar with $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{a}$ is $\pm\dfrac{\left(\vec{a}\times\vec{b}\right)\times\vec{a}}{\left|\left(\vec{a}\times\vec{b}\right)\times\vec{a}\right|}$.