13. RELATIONS AND FUNCTIONS

1. INTRODUCTION TO SETS

A collection of well-defined objects that are distinct are known as sets.

Well-defined object clearly defines if the object belongs to a given collection or not.

Well-defined collections

(a) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9.

(b) Rivers of India.

(c) Vowels in the English alphabet a, e, i, o, u.

Not well-defined collections

(a) Collection of bright students in class XI of a Nucleus Academy.

(b) Collection of renowned mathematicians of the world.

(c) Collection of beautiful girls of the world.

(d) Collection of fat people.

MASTERJEE CONCEPTS

- The terms objects, elements and member of a set are synonymous.
- Sets are generally denoted by capital letters A, B, C, ..., X, Y, Z.
- The elements of a set are represented by small letters a, b, c, ..., x, y, z.
- If a is an element of a set A, then we can say that a belongs to A. The Greek symbol $\in$ represents 'belongs to'. Thus we write $a \in A$.

  If b is not an element of a set A, then we write $b \notin A$.

Uday Kiran G (JEE 2012, AIR 102)

2. SOME IMPORTANT SYMBOLS AND THEIR MEANINGS

Symbol	Meaning
$\Rightarrow$	Implies
$\in$	Belongs to
$A \subset B$	A is subset of B
$\Leftrightarrow$	Implies and is implied by
$\notin$	Does not belong to
| or : or S.t.	Such that
$\forall$	For all
$\exists$	There exists
i*ff*	If and only if
&	And

Symbol	Meaning
a/b	a is divisor of b
N	Set of natural numbers
W	Set of whole numbers
I or Z	Set of integers
Q	Set of rational numbers
Q^c	Set of irrational numbers
R	Set of real numbers
C	Set of complex numbers

3. REPRESENTATION OF SETS

The following two methods are used to represent sets:

(a) Roster or Tabular form

(b) Set builder form

3.1 Roster or Tabular Form

In the roster form, all the elements of a set are enclosed within braces { } and each element is separated by a comma.

Few examples are listed as follows:

(a) The set of all even positive number less than 7 is {2, 4, 6}.

(b) The set of vowels in the English alphabets is {a, e, i, o, u}.

(c) The set of odd natural numbers is represented as {1, 3, 5, ...}. The three dots (ellipses) denote that the list is endless.

Note: (i) In the roster form, the elements of a set are not repeated, i.e. all the elements are taken as distinct, e.g. “SCHOOL” $\Rightarrow$ {S, C, H, O, L}

(ii) The order in which the element of a set is written is immaterial.

E.g. The set {1, 2, 3} and {2, 1, 3} are same.

3.2 Set Builder Form

In the set builder form, all the elements of a set possess a common property. This common property does not match with any element outside this set.

E.g. (i) all the elements 'a, e, i, o, u' possess a common property, i.e. each alphabet is a vowel which none other letters possessing this property.

This can be represented in set builder form as follows:

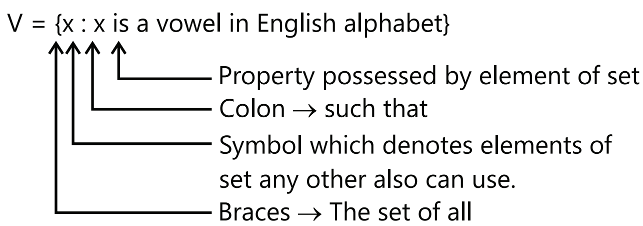

Figure 13.1

i.e. The set of all x such that x is a vowel in English alphabet.

(ii) A = {x: x is a natural number and $3 < x < 10$}

The set of all x such that x is a natural and $3 < x < 10$. Hence the numbers 4, 5, 6, 7, 8 and 9 are the elements of set A.

4. SUBSET

Set A is a subset of B if B has all the elements of A, denoted by $A \subset B$ (read as A is subset of B).

4.1 Number of Subset

If a set X contains n elements $\{x_1, x_2, \ldots, x_n\}$, then total number of subsets of X = 2^n.

Proof: Number of subsets of set X is equal to the number of selections of elements taking any number of them at a time out of the total n elements and it is equal to 2^n.

$\because\ ^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$.

4.2 Types of Subset

Set A is said to be a proper subset of set B if all elements of subset A is present in set B and at least one element in set B is not an element of subset A, i.e. $A \subset B$ and $A \neq B$.

The set A itself is an improper subset of A.

E.g., If X = $\{x_1, x_2, \ldots, x_n\}$, then total number of proper sets = $2^n - 1$ (excluding itself). The statement $A \subset B$ can be written as $B \supset A$, then B is called the superset of A.

5. TYPE OF SETS

5.1 Null or Empty Set

Any set is called empty or null set if no elements are present in that set. It is denoted by { }.

Few examples are as follows:

(a) Set of odd numbers divisible by 2 is null set, as we know that odd numbers are not divisible by 2 and hence the resultant set is a null set.

(b) Set of even prime numbers is not a null set because 2 is a prime number divisible by 2.

(c) {y: y is a point common to any two parallel lines} is null set because two parallel lines do not intersect.

5.2 Finite and Infinite Set

Any set is said to be finite set, if finite number of elements are present in it.

Few examples are listed as follows:

(a) { } [null set is a finite set]

(b) {a, e, i, o, u}

(c) {Jan, Feb,... Dec} etc.

Any set is said to be infinite set if the number of elements are not finite.

Few examples are as follows:

(a) S = {men living presently in different parts of the world} is non-countable. Therefore it is an infinite set.

(b) $S = \{x: x \in R\}$ is infinite set.

(c) $S = \{x: 2 < x > 3, x \in R\}$ is infinite set.

5.3 Equal and Equivalent Sets

Any two sets A and B are said to equal set if all elements of set A are in B and vice versa.

E.g. {a, e, i, o, u} = A and B = {e, i, a, u, o} then A = B.

Any two sets A and B are said to be equivalent sets if their number of elements in both the sets are same.

E.g. A = {1, 2, 3} and B = {a, b, c} are equivalent set, i.e. both the sets have three elements ($A \approx B$).

MASTERJEE CONCEPTS

1. Number of elements in any set is said to be cardinality/cardinal number of that set.

E.g. 1. A = {1, 2, 3, 4}, then cardinality of set A is 4.

2. B = {a, e, i, o, u}, then cardinality of set B is 5.

2. Any set does not change if one or more elements of the set are repeated.

E.g. 1. A = {June, Nov., April, Sept}

2. B = {June, Nov., June, Sept., April, Sept.} are equal.

Shivam Agarwal (JEE 2009, AIR 27)

5.4 Singleton Set

A set with single element is called a singleton set, i.e. n(X) = 1,

E.g. $\{x: x \in N, 1 < x < 3\}$, { { } }, i.e. set of null set, {π} is a set containing alphabet ϕ.

5.5 Universal Set

It is a set which includes all the sets under consideration, i.e. it is a super set of each of the given set. Thus, a set that contains all sets in a given context is called the universal set. It is denoted by U.

E.g. If A = {1, 2, 3}, B = {2, 4, 5, 6} and C = {1, 3, 5, 7}, then U = {1, 2, 3, 4, 5, 6 7} is a universal set which contains all elements of sets A, B and C.

5.6 Disjoint Set

Two sets are said to be disjoint if they have no elements in common, i.e. if A and B are two sets, then $A \cap B = \phi$.

If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

E.g. (i) If A = {1, 2, 3}, B= {4, 5, 6} and C = {4, 7, 9}, then A and B are disjoint sets, and B and C are intersecting sets. (ii) Set of even natural numbers and odd natural numbers are disjoint sets.

5.7 Complementary Set

Complementary set of A is a set containing all those elements of universal except the element in set A. It is denoted by $\overline{A}$, A^C or A′. So A^C = {x: $x \in U$ and $x \notin A$}, e.g. If set A = {1, 2, 3, 4, 5} and universal set U = {1, 2, 3, 4, ..., 50}, then $\overline{A}$ = {6, 7, ..., 50}

Note: All disjoint sets are not complementary sets and vice versa.

5.8 Power Set

The collection of all subsets of set A is called the power set of A and is denoted by P(A) i.e., P(A) = {x: x is a subset of A}. If X = {x_1 x_2 x_3 ..... x_n} then n(P(x)) = 2^n; n(P(P(x))) = 2^{2^n} .

Illustration 1: Which of the two sets are equal?

(i) A = {4, 8, 12, 18}, B = {8, 4, 12, 16}

(ii) A = {2, 4, 6, 8, 10} B = {x : x is positive even integer and $x \leq 10$}

(iii) A = {x: x is a multiple of 10} B = {x: x is a multiple of 5 and $x \geq 10$} **(JEE MAIN)**

Sol: Refer to the definition of different types of sets in the above section.

(i) The elements 4, 8, 12 belong to both sets A and B. But $16 \in B$ and 18 do not belong to both A and B. So $A \neq B$.

(ii) All elements present in set A is present in set B, and vice versa. Therefore A = B.

(iii) Given

A = {10, 20, 30, 40, ...} and B = {10, 15, 20, 25, 30, ...}

Since 15, 25, 35 are not multiples of 10, B does not belong to A.

$\therefore$ $A \neq B$.

Illustration 2: From the following sets, select equal sets:

A = {2, 4, 8, 12}, B = {1, 2, 3, 4}, C = {4, 8, 12, 14}, D = {3, 1, 4, 2}, E = {–1, 1}

F = {0, a}, G = {1, –1}, H = {0, 1} **(JEE MAIN)**

Sol: Similar to the previous question.

Since numbers 2, 4, 8, 12 belongs to set A, but 8, 12 $\notin$ B, 2 $\notin$ C, 8, 12, $\notin$ D, 2, 4, 8, 12, $\notin$ E, F, G and H.

Therefore, $A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$, $A \neq F$, $A \neq G$, $A \neq H$

Elements 1, 2, 3, 4 belong to set B but 1, 2, 3 $\notin$ C; 2, 3, 4 $\notin$ F, G and H. Only D has all elements of B.

Therefore, B $\neq$ C; B $\neq$ E; B $\neq$ F; B $\neq$ G; B $\neq$ H and B = D.

Elements 4, 8, 12, 14 $\notin$ C; elements 8, 12, 14 $\notin$ D, E, F, G and H.

Therefore, C $\neq$ D; C $\neq$ E ; C $\neq$ F ; C $\neq$ G ; C $\neq$ H.

Repeating this procedure completely, we find only one equal set E $\neq$ F, D $\neq$ E, D $\neq$ F, D $\neq$ G, D $\neq$ H, E $\neq$ H G $\neq$ H, F $\neq$ G, F $\neq$ H.

Finally, we have two equal sets from given sets, i.e. B = D and E = G.

6. VENN DIAGRAMS

The diagrams drawn to represent sets and their relationships are called Venn diagrams or Euler–Venn diagrams. Here we represents the universal U as set of all points within rectangle and the subset A of the set U is represented by a circle inside the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then we represent A and B by two intersecting circles.

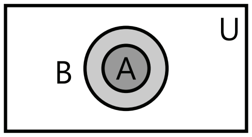

Figure 13.2

E.g., If A is subset of B then it is represent diagrammatically in Fig. 13.2. Let U be the universal set, A is a subset of set B. Then the Venn diagram is represented as follows:

If A is set, then the complement of A is represent as follows (Refer Fig. 13.3):

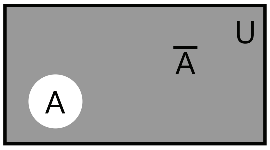

Figure 13.3

7. SET OPERATIONS

7.1 Union of Sets

If A and B are two sets, then union ($\cup$) of A and B is the set of all elements belonging to set A and set B. It is also defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$. It is represented by shaded area in following figures.

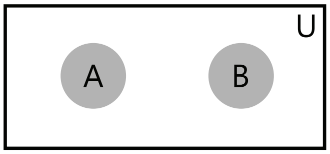

Figure 13.4

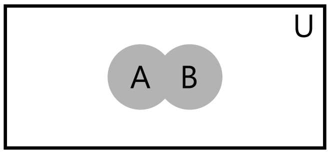

Figure 13.5

7.2 Intersection of Sets

If A and B are two sets, then intersection ($\cap$) of A and B is the set of elements which belongs to both A and B in common, i.e. $A \cap B = \{x: x \in A \text{ and } x \in B\}$ represented with shaded area in Venn diagram (see Fig. 13.6)

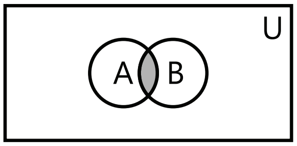

Figure 13.6

7.3 Difference of Two Sets

If A and B are two sets, then the difference of A and B is the set of elements which belongs to A and not B.

Thus, $A - B = \{x: x \in A \text{ and } x \notin B\}$

$A - B \neq B - A$.

It is represented through the Venn diagrams in Fig. 13.7.

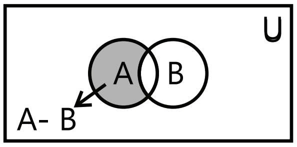

Figure 13.7

7.4 Symmetric Difference of Two Sets

Set of those elements which are obtained by taking the union of the difference of A and B: (A-B) and the difference of B and A: (B-A) is known as the symmetric difference of two sets A and B denoted by ($A \Delta B$).

Thus $A \Delta B = (A - B) \cup (B - A)$

Venn diagram is represented in Fig. 13.8.

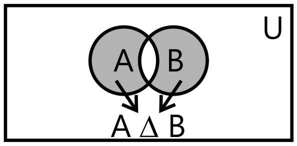

Figure 13.8

8. NUMBER OF ELEMENTS IN DIFFERENT SETS

If A, B and C are finite sets and U be the finite universal set, then

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) $n(A \cup B) = n(A) + n(B)$ (if A and B are disjoint sets)

(c) $n(A - B) = n(A) - n(A \cap B)$

(d) $n(A \Delta B) = n[(A - B) \cup (B - A)] = n(A) + n(B) - 2n(A \cap B)$

(e) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(f) $n(A' \cup B') = n(A \cap B)' = n(\cup) - n(A \cap B)$

(g) $n(A' \cap B') = n(A \cup B)' = n(\cup) - n(A \cup B)$

9. ALGEBRAIC OPERATIONS ON SETS

9.1 Idempotent Operation

For any set A, we have (i) $A \cup A = A$ and (ii) $A \cap A = A$

9.2 Identity Operation

For any set A, we have

(a) $A \cup \phi = A$

(b) $A \cap A = A$, i.e. ϕ and U are identity elements for union and intersection, respectively.

9.3 Commutative Operation

For any set A and B, we have

(a) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

9.4 Associative Operation

If A, B and C are any three sets, then

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $(A \cap B) \cap C = A \cap (B \cap C)$

i.e. union and intersection are associative.

9.5 Distributive Operations

If A, B and C are any three sets, then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union, respectively.

10. DE MORGAN'S PRINCIPLE

If A and B are any two sets, then

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

Proof: (a) Let x be an arbitrary element of $(A \cup B)'$. Then $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow$ $x \notin A$ and $x \notin B$ $\Rightarrow$ $x \in A' \cap B'$

Again let y be an arbitrary element of $A' \cap B'$. Then $y \in A' \cap B'$

$\Rightarrow$ $y \in A'$ and $y \in B'$ $\Rightarrow$ $y \notin A$ and $y \notin B$

$\Rightarrow$ $y \notin (A \cup B)$ $\Rightarrow$ $y \in (A \cup B)'$

$\therefore$ $A' \cap B' \subseteq (A \cup B)'$

Hence $(A \cup B)' = A' \cap B'$

Similarly (b) can be proved.

RELATIONS

1. ORDERED PAIR

A pair of elements written in a particular order is called an ordered pair. Let A and B be two sets. If $a \in A$, $b \in B$, then elements (a, b) denotes an ordered pair, with first component a and second component b. Here, the order in which the elements a and b appear is important. The ordered pair (1, 2) and (2, 1) are different, because they represent different points in the co-ordinate plane.

Equality of ordered pairs: Ordered pair (a_1, b_1) is equal to (a_2, b_2) *iff* $a_1 = a_2$ and $b_1 = b_2$.

2. CARTESIAN PRODUCT OF SETS

2.1 Cartesian Product of Two Sets

The Cartesian product of two sets A and B is the set of all those ordered pair whose first co-ordinate is an element of set A and the second co-ordinate is an element of set B.

It is denoted by A × B and read as 'A cross B' or 'product set of A and B'.

i.e. $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$

E.g. Let $A = \{1, 2, 3\}$

$B = \{3, 5\}$

Then $A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$ and

$B \times A = \{(3, 1), (3, 2), (3, 3), (5, 1), (5, 2), (5, 3)\}$

Hence $A \times B \neq B \times A$

Thus "Cartesian product of sets is not commutative".

E.g. $A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Properties of Cartesian product

(a) $A \times B \neq B \times A$ (non-commutative)

(b) $n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2n(A)n(B)$

(c) $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

(d) If A and B are two non-empty sets with n elements in common, then (A × B) and (B × A) have n^2 element in common.

(e) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(f) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(g) $A \times (B - C) = (A \times B) - (A \times C)$

Illustration 3: If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then find $(A \cap B) \times (A \cup B)$ **(JEE MAIN)**

Sol: Use Cartesian Product of two Sets.

$A \cap B = \{4\}$ and $A \cup B = \{2, 3, 4, 5\}$

$\therefore \quad (A \cap B) \times (B \cup A) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$

2.2 Cartesian Product of More than Two Sets

The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \ldots \times A_n$ and is defined as $A_1 \times A_2 \times \ldots \times A_n = \{ (x_1 x_2, \ldots, x_n): x_{i\in} A$, where $i = 1, 2, \ldots, n\}$

Note:

(i) Elements of A × B are called 2-tuples.

(ii) Elements of A × B × C are called 3-tuples.

(iii) Elements of $A_1 \times A_2 \times \ldots \times A_n$ are also called n-tuples.

E.g. P = {1, 2}, from the set P × P × P.

P × P × P = {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}

2.3 Number of Elements in the Cartesian Product

If A and B are two finite sets then

n(A × B) = n(A) · n(B)

Thus, if A and B have m elements and n elements, respectively, then A × B has mn elements.

Proof: Let A = {x_1, x_2, x_3, ..., x_n}

and B = {y_1, y_2, y_3, ..., y_n}

Then A × B = {(x_1, y_1), (x_2, y_2) ... (x_1, y_n),

(x_2, y_1), (x_2, y_2) ... (x_2, y_n),

⋮

(x_m, y_1), (x_m, y_2) ... (x_m, y_n)}

Clearly each row has n ordered pairs and there are m such rows. Therefore A × B has mn elements.

Similarly, n(A × B × C) = n(A) · n(B) · n(C)

Illustration 4: If n(A) = 7, n(B) = 8 and n(A ∩ B) = 4, then match the following columns: **(JEE MAIN)**

(i) n(A ∪B)	(a) 56
(ii) n(A × B)	(b) 16
(iii) n((A × B) × A)	(c) 392
(iv) n((A × B) ∩ (B × A))	(d) 96
(v) n((A × B) ∪ (B × A))	(e) 11

Sol: Use the formula studied in the above section.

(i) n (A ∪ B) = n (A) + n (B) – n (A ∩ B) = 7 + 8 – 4 = 11

(ii) n(A × B) = n(A) n(B) = 7 * 8 = 56 = n(B × A)

(iii) n((A × B) × A) = n(B × A) · n(A) = 56 × 7 = 392

(iv) n((A × B) ∩ (B × A)) = $(n(A \cap B))^2 = 4^2 = 16$

(v) n((A × B) ∪ (B × A)) = n(A × B) + n(B × A) – n(A × B) ∩ (B × A) = 56 + 56 – 16 = 96

2.4 Representation of Cartesian Product

Let A = {1, 2, 3} and B = {a, b}

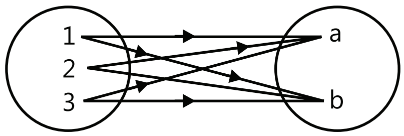

Figure 13.9

Each element of set A to each element of set B is represented by lines from A to B in Fig. 13.9.

3. RELATION

A relation R from one set to another say from set X to set Y (R : X → Y) is a correspondence between set X to set Y through which some or more elements of X are associated with some or more elements of Y. Therefore a relation R, from a non-empty set X to another non-empty set Y, is a subset of X × Y, i.e. R: X → Y is a subset of A × B.

Every non-zero subset of A × B is defined as a relation from set A to set B. Therefore, if R is a relation from A → B then R = {(a, b) | (a, b) ∈ A × B and a R b}.

If A and B are two non-empty sets and R: A → B be a relation such that R: {(a, b) | (a, b) ∈ R, and a ∈ A and b ∈ B}, then

(a) 'b' is an image of 'a' under R.

(b) 'a' is a pre-image of 'b' under R.

For example consider sets X and Y of all male and female members of a royal family of the kingdom Ayodhya. X = {Dashrath, Ram, Bharat, Laxman, Shatrughan} and Y = {Koshaliya, Kaikai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}. A relation R_H is defined as "husband of" from set X to set Y.

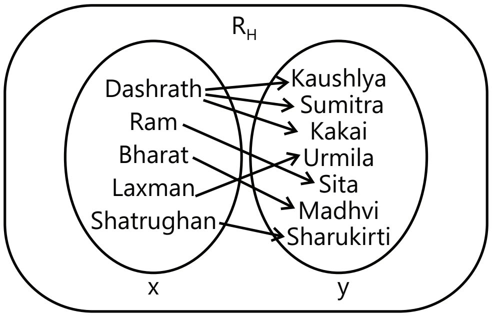

Figure 13.10

R_H = {(Dashrath, Koshaliya), (ram, sita), (Bharat, Mandavi), (Laxman, Urmila), (Shatrughan, Shrutkirti), (Dashrath, Kakai), (Dashrath, Sumitra)}

(a) If a is related to b, it is symbolically written as a R b.

(b) It is not necessary for each and every element of set A to have an image in set B, and set B to have a pre-image in set A.

(c) Elements of set A having image in B are not necessarily unique.

(d) Number of relations between A and B is the number of subsets of A × B.

Number of relations = no. of ways of selecting a non-zero subset of A × B.

$= {}^{mn}C_1 + {}^{mn}C_2 + \ldots + {}^{mn}C_{mn} = 2^{mn} - 1$.

3.1 Domain

Domain of a relation is a collection of elements of the first set participating in the correspondence, i.e. it is set of all pre-images under the relation. For example, from the above example, domain of R_H = {Dashrath, Ram, Bharat, Laxman, Shatrughan}.

3.2 Codomain

All elements of any set constitute co-domain, irrespective of whether they are related with any element of correspondence set or not, e.g. Y = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} is co-domain of R_H.

3.3 Range

Range of relation is a set of those elements of set Y participating in correspondence, i.e. set of all images. Range of R_H = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}.

Illustration 5: A = {1, 2, 3, 4, 5} and B = {2, 4, 5}. Relation between A and B is defined as a R b $\Rightarrow$ a and b are relatively prime or co-prime (i.e., HCF is 1). Find domain and range of R. **(JEE MAIN)**

Sol: Write the elements of the Relation and then write the domain and range.

R = {(1, 2), (1, 4), (1, 5), (2, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 2), (5, 4)}

Domain of R {1, 2, 3, 4, 5}

Range of R {2, 4, 5}

Illustration 6: A = {Jaipur, Patna, Kanpur, Lucknow} and B = {Rajasthan, Uttar Pradesh, Bihar}

a R b $\Rightarrow$ a is capital of b, a $\in$ A and b $\in$ B. Find R. **(JEE MAIN)**

Sol: Use the concept / definition studied above.

R = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh)}

Illustration 7: If A = {1, 3, 5, 7}, B = {2, 4, 6, 8}

Relation is a R b $\Rightarrow$ a > b, a $\in$ A, b $\in$ B. Find domain and range of R. **(JEE MAIN)**

Sol: Similar to the Illustration 3.

R = {(3, 2), (5, 2), (5, 4), (7, 2) (7, 4), (7, 6)}

Domain = {3, 5, 7}

Range = {2, 4, 6}

4. REPRESENTATION OF A RELATION

4.1 Roster Form

In this form we represent set of all ordered pairs (a, b) such that (a, b) $\in$ R where a $\in$ A, b $\in$ B.

4.2 Set Builder Form

Relation is denoted by the rule which co-relates the two set. This is similar to set builder form in sets.

4.3 Arrow-Diagram (Mapping)

It is a pictorial notation of any relation.

E.g. Let A = {–2, –1, 4} and B = {1, 4, 9}

A relation from A to B, i.e. a R b, is defined as a < b.

1. Roster form

R = {(0-2, 1), (–2, 4), (–2, 9), (–1, 1), (–1, 4), (–1, 9), (4, 9)}

2. Set builder notation

R = {(a, b): a $\in$A and b $\in$B, a is less than b}

3. Arrow-diagram (mapping)

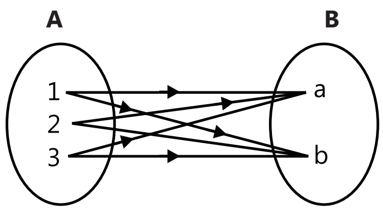

Figure 13.11

5. TYPES OF RELATION

5.1 Reflexive Relation

R: X $\to$ Y is said to be reflexive iff x R x $\forall$ x $\in$ X, i.e. every element in set X must be a related to itself. Therefore $\forall$ x $\in$ X; (x, x) $\in$ R, then relation R is called as reflexive relation.

5.2 Identity Relation

Consider a set X. Then the relation I = {(x, x): x $\in$ X} on X is called the identity relation on X i.e. a relation I on X is identity relation if every element of X related to itself only. For example, y = x.

Note: All identity relations are reflexive but all reflexive relations are not identity.

5.3 Symmetric Relation

R: X $\to$ Y is said to be symmetric iff (x, y) $\in$ R $\Rightarrow$ (y, x) $\in$ R.

For example, perpendicularity of lines in a plane is symmetric relation.

5.4 Transitive Relation

R: X $\to$ Y is transitive iff (x, y) $\in$ R and (y, z) $\in$ R $\Rightarrow$ (x, z) $\in$ R,

i.e. x R y and y R z $\Rightarrow$ x R z.

For example, the relation "being sister of" among the numbers of a family is always transitive.

Note:

(i) Every null relation is a symmetric and transitive relation.

(ii) Every singleton relation is a transitive relation.

(iii) Universal and identity relation are reflexive, symmetric as well as transitive.

5.5 Antisymmetric Relation

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff (a, b) $\in$ R and (b, a) $\in$R $\Rightarrow$ a = b for all a, b $\in$A. For example, relations "being subset of"; "is greater than or equal to" and "identity relation on any set A" are anti-symmetric relations.

5.6 Equivalence Relation

A relation R from a set X to set Y ($R \rightarrow X \rightarrow Y$) is said to be an equivalence relation iff it is reflexive, symmetric as well as transitive. The equivalence relation is denoted by, e.g. relation "is equal to" equality similarity and congruence of triangle, parallelism of lines are equivalence relation.

5.7 Inverse Relation

If relation R is defined from A to B then inverse relation would be defined from B to A, i.e.

$R: A \rightarrow B \Rightarrow a\ R\ b$, where $a \in A$, $b \in B$.

$R^{-1}: B \rightarrow A \Rightarrow b\ R\ a$, where $a \in A$, $b \in B$.

Domain of R = Range of R^{-1}

and range of R = Domain of R^{-1}.

$\therefore \quad R^{-1} = \{b, a\} \mid (a, b) \in R\}$

For example, a relation R is defined on the set of 1st ten natural numbers.

$\therefore \quad N = \{1, 2, 3, \ldots, 10\}$ and $a, b \in N$.

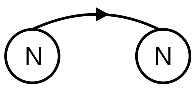

Figure 13.12

$a\ R\ b \Rightarrow a + 2b = 10$

$R = \{(2, 4), (4, 3), (6, 2)\ (8, 1)\}$

$R^{-1} = \{(4, 2), (3, 4), (2, 6)\ (1, 8)\}$

For example, a relation defined on the set of lines.

1. $aR\ b \Rightarrow a \parallel b$

It is a symmetric relation because if line 'a' is || to 'b', the line 'b' is || to 'a'.

where $(a, b) \in L$ {L is a set of || lines}

2. L_1RL_2 $L_{1\perp} L_2$ It is a symmetric relation

$L_{1,} L_{2 \in} L$ {L is a set of lines}

3. a R b $\Rightarrow$ 'a' is brother of 'b' is not a symmetric relation as 'b' may be sister of 'a'.

4. a R b $\Rightarrow$ 'a' is cousin of 'b'. This is a symmetric relation. If R is symmetric.

5. $R = R^{-1}$.

6. Range of R = Domain of R.

FUNCTIONS

1. INTRODUCTION

The concept of function is of fundamental importance in almost all branches of Mathematics. It plays a major role to solve real world problems in mathematics. As a matter of fact, functions are some special type of relations.

General definition:

Definition 1: Consider two sets A and B and let there exist a rule or manner or correspondence 'f' which associates to each element of A to a unique element in B. Then f is called a function or mapping from A to B. It is denoted by symbol f and represented by f: A →B (read as 'f' is a function from A to B or 'f maps A to B').

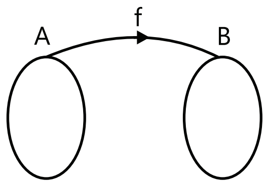

Figure 13. 13

If element a ∈ A is associated with an element b ∈ B, then b is called the 'f image of a' or 'image of a under f' or 'the value of function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

b = f(a) or f: a → b or f: (a, b)

Function as a set of ordered pairs:

A function f: A → B can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function f: A → B can be considered as a set of ordered pairs (a, f(a)), where a ∈ A and f(a) ∈ B, which is the f image of a. Hence, f is a subset of A × B.

Definition 2: A relation R from a set A to a set B is called a function if

(i) each element of A is associated with some element of B.

(ii) each element of A has unique image in B.

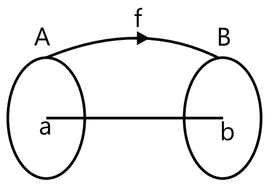

Figure 13. 14

Thus a function 'f' from set A to set B is a subset of A × B in which element a belonging to A appears in only one ordered pair belonging to f. Hence, a function f is a relation from A to B satisfying the following properties:

(i) f ⊂A × B (ii) ∀ a ∈A ⇒(a, (f(a)) ∈ f and (iii) (a, b) ∈f and (a, c) ∈f ⇒b = c.

MASTERJEE CONCEPTS

Every function is a relation, but every relation may not necessarily be a function.

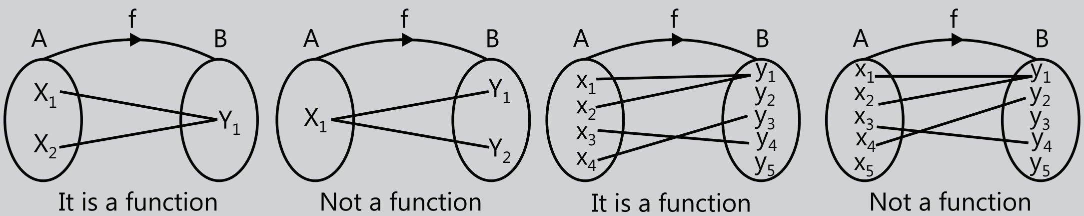

Figure 13.15

Easy way to differentiate function from relation is as follows :

1. Consider every element in set A is a guest and every element in set B is hosting a function at the same time and invited every element from A.

2. None of the elements can be at two functions simultaneously.

3. So if an element is attending two functions at the same time, then it is just a relation and if an element is attending only one function then it is said to be a function.

Chen Reddy Sundeep Reddy (JEE 2012, AIR 63)

2. RELATION VS FUNCTION

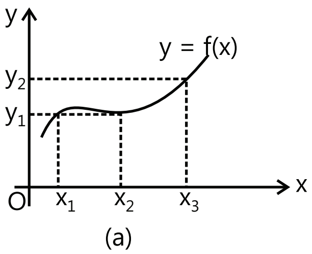

Figure 13.16

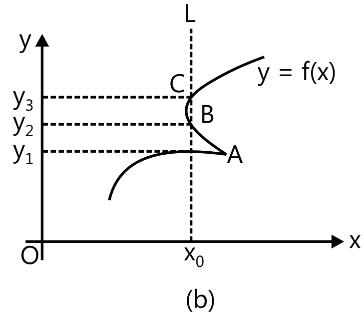

Figure 13.17

These figures show the graph of two arbitrary curves. In figure 16, any line drawn parallel to y-axis would meet the curve at only one point. That means each element of X would have only one image. Thus figure 16 (a) represents the graph of a function.

In figure 17, certain line parallel to y-axis (e.g., line L) would meet the curve in more than one point (A, B and C). Thus element x_0 of X would have three distinct images. Thus, this curve does not represent a function.

Hence, if y = f(x) represents a function, then lines drawn parallel to y-axis through different point corresponding to points of set X should meet the curve at one point.

Equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a relation, which is a combination of two functions

$y=b\sqrt{1-\frac{x^2}{a^2}}$ and $y=-b\sqrt{1-\frac{x^2}{a^2}}$. Similarly, the equation of the parabola $y^2 = x$ is a combination of two functions as shown in Fig. 13.18.

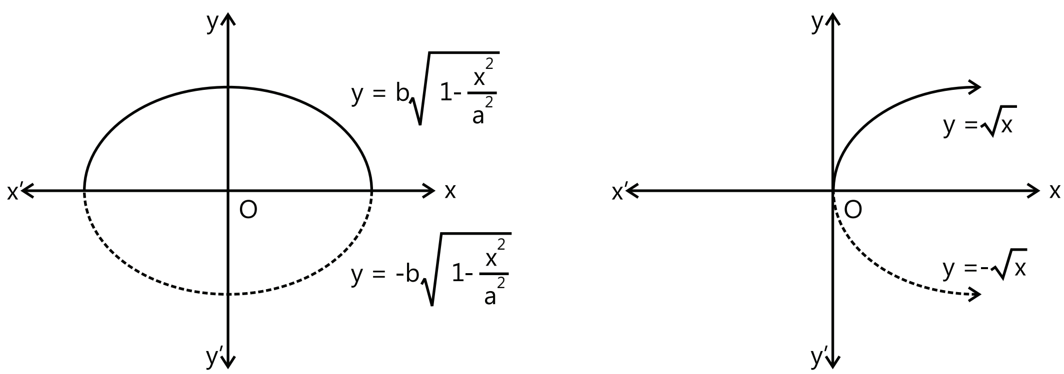

Figure 13.18

3. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let f: A → B, then the set A is known as the domain of f and the set B is known as co-domain of f.

The set of all f images of elements of A is known as the range of f. Thus:

Domain of f = {a |a ∈ A, (a, f(a)) ∈ f} Range of f = {f(a) ∀ a ∈ A and f(a) ∈ B}

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined.

Note: If domain of f(x) is D_1 and domain of g(x) is D_2 then domain of f(x) + g(x) = $D_1 \cap D_2$.

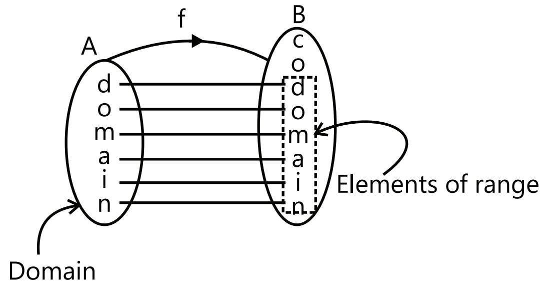

Figure 13.19

4. COMMON FUNCTIONS

4.1 Polynomial Function

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... a_{n-1}x + a_n$ where n is a non-negative integer and a_0, a_1, a_2, ..., a_n are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

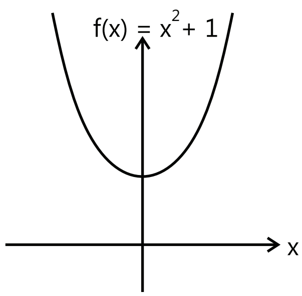

Figure 13.20

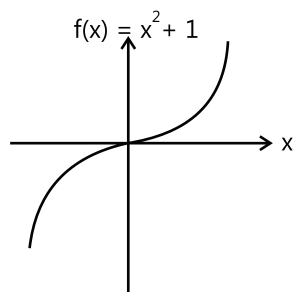

Figure 13.21

MASTERJEE CONCEPTS

(a) A polynomial of degree one with no constant term is called an odd linear function i.e., $f(x) = a_x$, $a \neq 0$

(b) There are two polynomial functions, satisfying the relation ; f(x) f(1/x) = f(x) + f(1/x)

They are :

(i) $f(x) = x^n + 1$ and (ii) $f(x) = 1 - x^n$, where n is positive integer.

(c) Domain of a polynomial function is R

(d) Range for odd degree polynomial is R whereas for even degree polynomial range is a subset of R.

(i) If f(x) + f(y) = f(xy) then f(x) = k log x

(ii) If $f(x) \cdot f(y) = f(x + y)$ then $f(x) = a^{kx}$

(iii) If f(x) + f(y) = f(x+ y) then f(x) = kx

Rohit Kumar (JEE 2012, AIR 79)

4.2 Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x) = y^n + P_1(x)\, y^{n-1} + ... + P_{n-1}(x)\, y + P_n(x) = 0$ where n is a positive integer and $P_0(x)$,

$P_1(x)$ .... are polynomials in x e.g., $x^3 + y^3 - 3xy = 0$ or

$y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

MASTERJEE CONCEPTS

All polynomial functions are Algebraic but not the converse. A function that is not algebraic is called Transcendental function.

Anvit Tawar (JEE 2012, AIR 9)

4.3 Fractional/Rational Function

It is a function of the form $y = f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial and $h(x) \neq 0$.

4.4 Exponential/Logarithmic Function

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. The inverse of exponential function is called the logarithmic function.

i.e. $g(x) = \log_a x$.

MASTERJEE CONCEPTS

(a) $f(x) = e^x$ domain is R and range is R^+.

(b) $f(x) = e^{1/x}$ domain is R – {0} and range is $R^+ - \{1\}$.

(c) f(x) and g(x) are inverse of each other and their graphs are as shown.

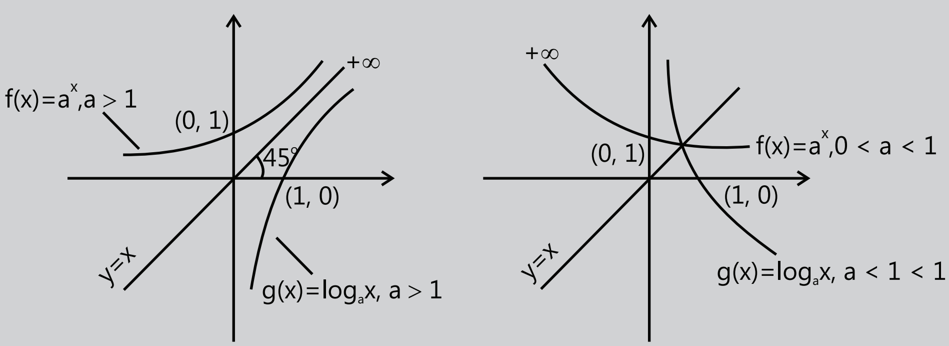

Figure 13.22

Shivam Agarwal (JEE 2009, AIR 27)

4.5 Absolute Value Function

A function $y = f(x) = |x|$ is called the absolute value function or modulus function. It is defined as $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

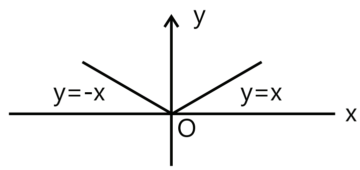

Figure 13.23

Note: (a) f(x) = | x |, domain is R and range is $R^+ \cup \{0\}$.

(b) f(x) = $\frac{1}{f|x|}$, domain is R – {0} and range is R^+.

4.6 Signum Function

A function y = f(x) = sgn(x) is defined as follows: y = f(x) = $\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ If is also written as sgn x = | x |/ x or $\frac{x}{|x|}$;x ≠ 0; f(0) = 0

Note: Domain is $x \in R$ and its range = {–1, 0, 1}

sgn x = $\frac{d}{dx}$ | x | when x ≠ 0

= 0 when x = 0

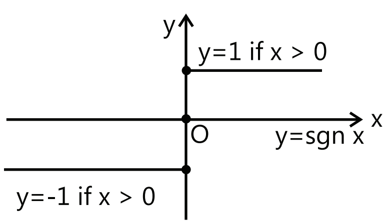

Figure 13.24

4.7 Greatest Integer Function or Step Function

The function y = f(x) = [x] is called the greatest integer function, where [x] denotes the greatest integer less than or equal to x.

Note:

(a) $-1 \le x < 0 \Rightarrow$ [x] = –1

$0 \le x < 1 \Rightarrow$ [x] = 0

$1 \le x < 2 \Rightarrow$ [x] = 1

$2 \le x < 3 \Rightarrow$ [x] = 2 etc.

(b) f(x) = [x], domain is R and range is I.

(c) f(x) = $\frac{1}{[x]}$ domain is R – [0, 1) and range is $\left\{\frac{1}{n} \middle| n \in I - \{0\}\right\}$

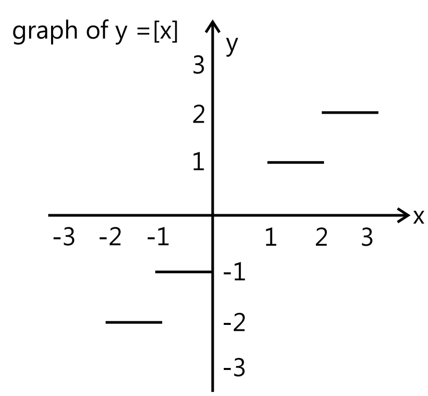

Figure 13.25

Properties of greatest integer function:

(a) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$.

(b) [x + m] = [x] + m if m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$.

(d) [x] + [–x] = 0 if x is an integer= –1 otherwise.

4.8 Fractional Part Function

It is defined as g(x) = {x} = x – [x].

For example, the fractional part of the number 2.1 is 2.1 – 2 = 0.1 and the fractional part of –3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.

Note: (a) f(x) = {x}, domain is R and range is [0, 1)

(b) $f(x) = \frac{1}{\{x\}}$, domain is R – I, range is (1, ∞)

{x + n} = {x}, where n ∈ I

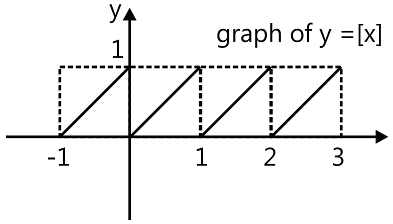

Figure 13.26

5. TRANSFORMATION OF CURVES

(a) Given graph of a function y = f(x) and we have to draw the graph of y = f(x – a) [means replacing x by x–a, a > 0] then shift the entire graph through a distance a units in positive direction of x-axis.

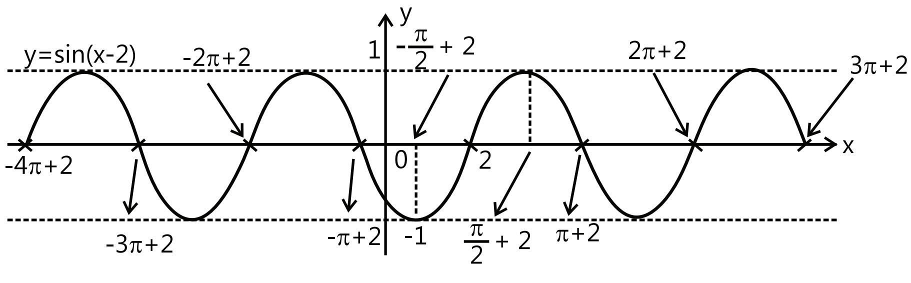

Figure 13.27

(b) Given graph of a function y = f(x), draw a graph of y = f(x + a) [means replacing x by x + a, a > 0]. Shift the entire graph through in negative direction of x-axis.

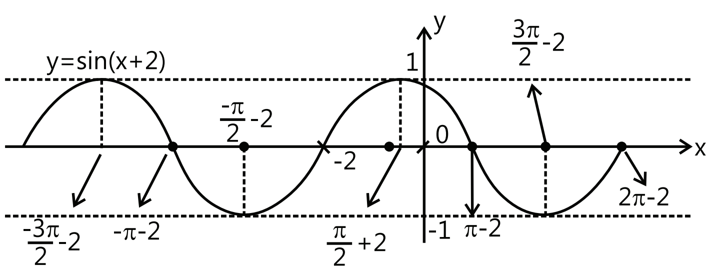

Figure 13.28

(c) Given graph of a function y = f(x), draw a graph of y = af(x) [means replacing y by y/a, a >0]. Then multiply all the values by a on y-axis.

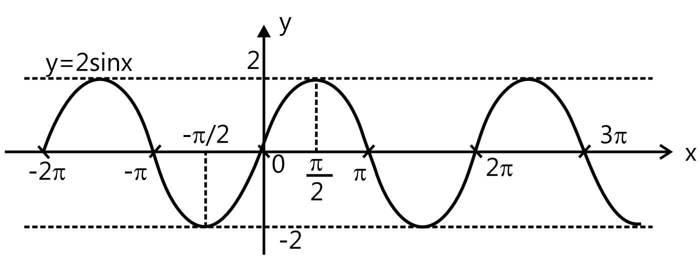

Figure 13.29

(d) Given graph of a function y = f(x), draw a graph of y = f(ax) [means replacing x by ax, a >0]. Then divide all the values by a on x-axis.

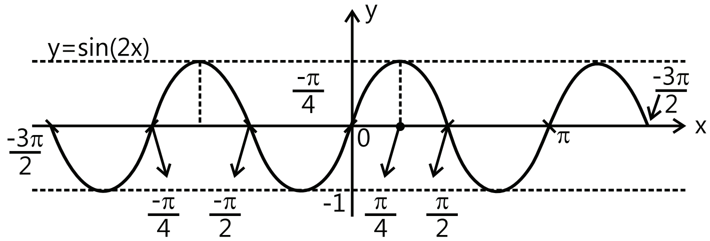

Figure 13.30

(e) Given graph of a function y = f(x), draw the graph of y = f(x) + a [means replacing y by y – a, a >0]. Then shift the entire graph in positive direction of y-axis.

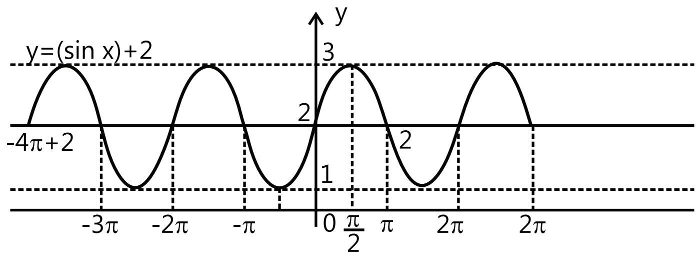

Figure 13.31

(f) Given graph of a function y = f(x), draw the graph of y = f(x) –a [means replacing y by y + a, a > 0]. Then shift the entire graph in negative direction of y-axis.

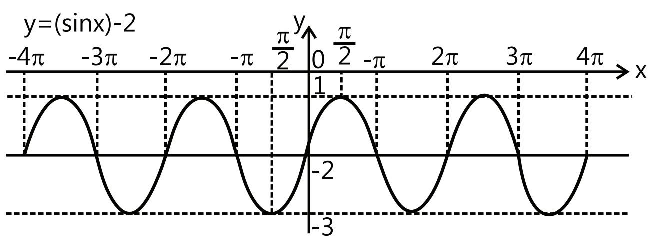

Figure 13.32

(g) Given graph of a function y = f(x), draw the graph of y = f(–x) [means replacing x by –x]. Then take the reflection of the entire curve in y-axis.

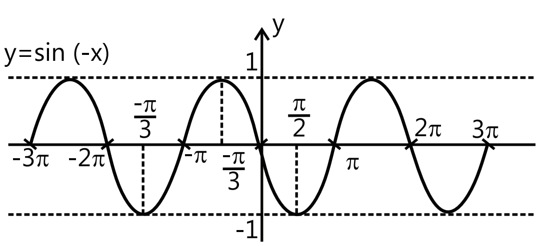

Figure 13.33

(h) Given graph of a function y = f(x), draw the graph of y = –f(x) [means replacing y by –y]. Then take the reflection of the entire curve in x-axis.

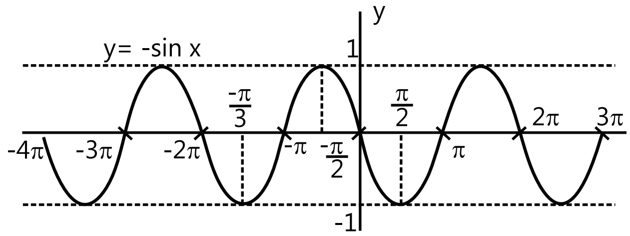

Figure 13.34

(i) Given graph of a function y = f(x), draw the graph of y = f(|x|) [means replacing x by |x|]

(a) Remove the portion of the curve, on left-hand side of y-axis.

(b) Take the reflection of right-hand size on the left-hand side.

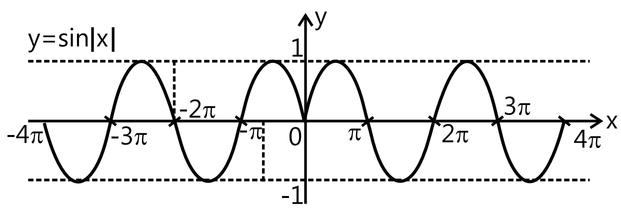

Figure 13.35

(j) Given graph of a function y = f(x), draw the graph of y = | f(x) |.The projection of the curve lying below x-axis will go above the axis.

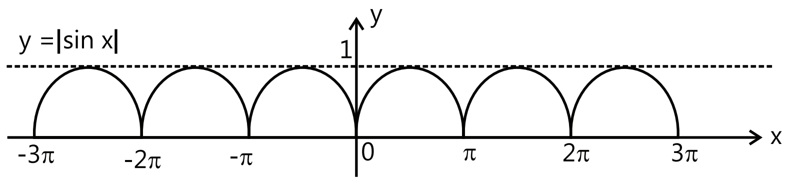

Figure 13.36

(k) Given graph of a function y = f(x), draw the graph of |y| = f(x) [means replacing y by |y|]. Then remove a portion of the curve below x-axis and then take the reflection of the upper part on the lower part.

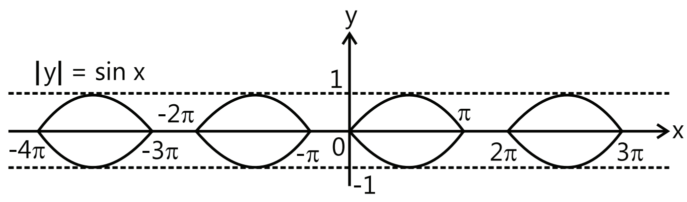

Figure 13.37

6. TIPS FOR PLOTTING THE GRAPH OF A RATIONAL FUNCTION

(a) Examine whether denominator has a root or not. If no root, then graph is continuous and f is non-monotonic.

For example, $f(x) = \dfrac{x}{x^2 - 5x + 9} \Rightarrow$ for $D_rD < 0$, D_r will never be zero.

f(x) is discontinuous, only when dominator has roots and hence non-monotonic.

For example, $f(x) = \dfrac{x^2+2x-3}{x^2+2x-8} = \dfrac{(x+3)(x-1)}{(x+4)(x-2)}$

D = 0 at x = –4, 2.

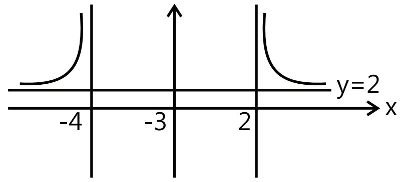

Figure 13.38

(b) If numerator and denominator has a common factor (say x = a) it would mean removable discontinuity at x = a, E.g. $f(x) = \dfrac{(x-2)(x-1)}{(x+3)(x-2)}$. Such a function will always be monotonic, i.e. either increasing or decreasing and removable discontinuity at x = 2.

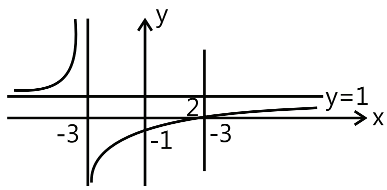

Figure 13.39

(c) Compute point where the curve cuts both x-axis and y–axis by putting y = 0 and x = 0, respectively, and mark points accordingly.

f(x) = x – 1

x = 0, y = –1

y = 0, x = 1

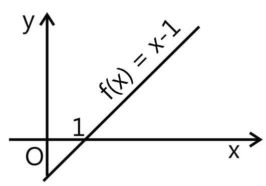

Figure 13.40

(d) Compute $\dfrac{dy}{dx}$ and find the intervals where f(x) increases or decreases and also where it has horizontal tangent.

$y = x^2 - 3x + 2$

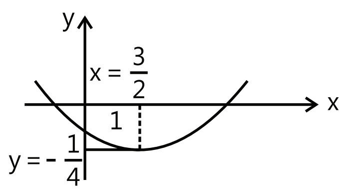

Figure 13.41

(e) In regions where curves is monotonic, compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

$f(x) = \dfrac{x-1}{x-3}$

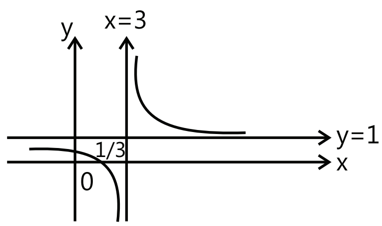

Figure 13.42

(f) If denominator vanishes, say at x = a and (x – a) is not a common factor between numerator and denominator, then examine $\lim_{x\to a^-}$ and $\lim_{x\to a^+}$ to find whether f approaches ∞ or –∞. Plot the graphs of the following function.

Illustration 8: Draw the graph of functions $f(x) = \dfrac{x}{\ell nx}$

Sol: Calculate the domain of the given function. Then use the derivative of the given function to trace the given curve.

Domain of f(x) is $x \in R\ (0, 1) \cup (1, \infty)$, $f'(x) = \dfrac{1.\ell nx - x\cdot(1/x)}{(\ell nx)^2}$

$f'(x) = 0$ at $x = e$

also as x approaches zero f(x) approaches zero from negative side and x approaches ∞ f(x) approaches +∞.

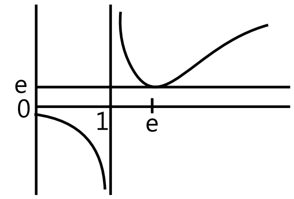

Figure 13.43

From the graph we can observe the range of f(x) is $(-\infty, 0) \cup [e, \infty)$.

7. TO FIND DOMAIN AND RANGE

To calculate domain or range of a function, the following points are considered.

7.1 Domain

(a) Expression under even root (i.e. square root, fourth root, etc.) ≥ 0 and denominator ≠ 0.

(b) If domain of y = f(x) and y = g(x) are D_1 and D_2 respectively, then the domain of f(x) + g(x) or f(x) · g(x) is $D_1 \cap D_2$.

(c) Domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

(d) Domain of $\sqrt{f(x)} = D_1 \cap \{x: f(x) \ge 0\}$

(e) Expression inside logarithm should be positive, i.e. for $\log_a E$ to exist, E should positive.

7.2 Range

(a) For the real valued function, real values of y for x ∈ domain of f are the range of the function.

Therefore, find domain of f and then impose restriction upon y, for the values of x in domain.

(b) If f is a continuous real valued function, then the range of function = [minimum f, maximum f].

Method of finding range:

(a) Range of the function in restricted domain

For the range of y = f(x) in the interval [a, b], retain the portion of the curve y = f(x) below the lines x = a and x = b. Then the required range is the projection of y-axis.

(b) Range of composite function

To find the range of f(g(x)), first find the range of g(x), say A. Then find the range of f(x) in domain A.

(c) If f = (A sin x + B cos x) + C then range of function is $\left[-\sqrt{A^2+B^2}+C,\ \sqrt{A^2+B^2}+C\right]$.

(d) Range of periodic function can be found only for the interval whose length is a period of a function.

(e) Similarly for odd functions, if range on right-hand side on x-axis is (α, β). Then range on left-hand side of x-axis will be (–α, –β), to get the final range, union of both these.

(f) Change of variable

y = f(g(x)); to get the range of f(g(x)) substitute g(x) at t and then find the range of f(t) the domain of f(t).

MASTERJEE CONCEPTS

Whenever we substitute the variable t for g(x), care should be taken that the corresponding condition on t should be written immediately. Further analysis of the function will be according to the condition.

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 9: Find domain and range of f(x) = $\cot^{-1}\log_{\frac{4}{5}}(5x^2-8x+4)$. **(JEE MAIN)**

Sol: Use the definition of Domain and Range.

Consider the quadratic expression P(x) = $5x^2 - 8x + 4$.

For the above quadratic, D < 0 ⇒ the expression always positive.

∴ e expression always ∈ R.

$P_{maximum} = \infty$, $P_{minimum} = 4/5$.

∴ Range of $\log_{\frac{4}{5}}(5x^2-8x+4)$ is (–∞, 1]

Now, draw the graph of $\cot^{-1}(t)$ for t (–∞, 1]

From the graph we can observe that range of f(x) is $\left[\frac{\pi}{4}, \pi\right)$

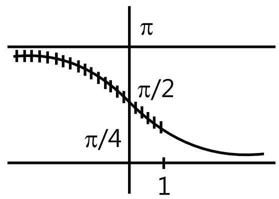

Figure 13.44

Illustration 10: Draw a graph of f(x) = $\frac{x^2-x+1}{x^2+x+1}$ and also evaluate its domain and range. **(JEE MAIN)**

Sol: Find derivative of the given function and use the techniques of curve tracing.

For drawing the graph of f(x) first find out those point where f′(x) = 0

$$f'(x) = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

f′(x) = 0 at x = –1, 1

f(1) = $\frac{1}{3}$; f(–1) = 3

Also when x approach ±∞, f approaches 1.

From the graph, domain is x ∈ R and range $\left[\frac{1}{3}, 3\right]$

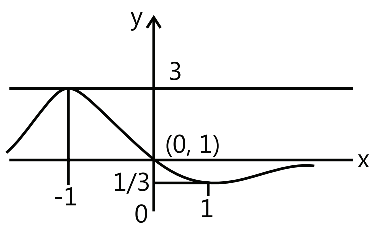

Figure 13.45

Note: Graph of f(x) = $\frac{ax+b}{cx+d}$ is always monotonic.

Illustration 11: If f(x) = $\sin^{-1}x^2 + \left[\ln\sqrt{x-[x]}\right] + \cot\left(\frac{1}{1+\sqrt{2}x^2}\right)$. Then find its domain and range. **(JEE ADVANCED)**

Sol: Follow the steps discussed above.

Domain of function is {–1, 1} – {0}, because x – [x] = 0 for integral value of x; hence, middle term will not be defined.

Also [{f}] = 0, whenever f is meaningful.

Therefore value of f(x) = $\sin^{-1}x^2 + \tan^{-1}(1 + \sqrt{2}x^2)$ $\left(\cot^{-1}x = \tan^{-1}\frac{1}{x} \text{ when } x > 0\right)$

Function is continuous and is even.

Least value of the function will occur when $x \to 0$ and is $\frac{\pi}{4}$.

Maximum value = $\lim_{x\to\pm1} f(x) = \sin^{-1}1 + \tan^{-1}(1+\sqrt{2}) = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$

Therefore, range of f(x) is $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$.

8. DOMAIN AND RANGE OF COMMON FUNCTION

A. ALGEBRAIC FUNCTION

Function	Domain	Range
(i) x^n, $(n \in N)$	R = set of real numbers	R, if n is odd $R^+ \cup \{0\}$, if n is even
(ii) $\frac{1}{x^n}$, $(n \in N)$	R – {0}	R – {0}, if n is odd R^+, if n is even
(iii) $x^{1/n}$ $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
(iv) $\frac{1}{x^{1/n}}$, $(n \in N)$	R – {0}, if n is odd R^+, if n is even	R – {0}, if n is odd R^+, if n is even

B. TRIGONOMETRIC FUNCTION

Function	Domain	Range
(i) sin x	R	[–1, 1]
(ii) cos x	R	[–1, 1]
(iii) tan x	$R - (2k+1)\frac{\pi}{2}$, $k \in I$	R
(iv) sec x	$R - (2k+1)\frac{\pi}{2}$, $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v) cosec x	$R - k\pi$, , $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi) cot x	$R - k\pi$, , $k \in I$	R

C. INVERSE TRIGONOMETRIC FUNCTION

Function	Domain	Range
(i) $\sin^{-1} x$	[–1, 1]	$\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$
(ii) $\cos^{-1} x$	[–1, 1]	$[0, \pi]$
(iii) $\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left[\frac{\pi}{2}\right]$
(v) $\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(vi) $\cot^{-1} x$	R	$(0, \pi)$

D. EXPONENTIAL FUNCTION

Function	Domain	Range
(i) e^x	R	R^+
(ii) $e^{1/x}$	R – {0}	$R^+ - \{1\}$
(iii) a^x, $a > 0$	R	R^+
(iv) $a^{1/x}$, $a > 0$	R – {0}	$R^+ - \{1\}$

E. LOGARITHIMIC FUNCTION

Function	Domain	Range
(i) $\log_a x$, $(a > 0)(a \neq 1)$	R^+	R
(ii) $\log_x a = \frac{1}{\log_a x}$ $(a > 0)\ (a \neq 1)$	$R^+ - \{1\}$	R – {0}

F. INTEGRAL PART FUNCTION

Function	Domain	Range
(i) [x]	R	I
(ii) $\frac{1}{[x]}$	R – [0, 1)	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$

G. FRACTIONAL FUNCTION

Function	Domain	Range
(i) {x}	R	[0, 1)
(ii) $\frac{1}{\{x\}}$	R – I	$(1, \infty)$

H. MODULUS FUNCTION

Function	Domain	Range
(i) $\|x\|$	R	$R^+ \cup \{0\}$
(ii) $\frac{1}{\|x\|}$	R – {0}	R^+

I. SIGNUM FUNCTION

Function	Domain	Range
$\text{sgn}(x) = \frac{\|x\|}{x}$	R	{–1, 0, 1}

J. CONSTANT FUNCTION

Function	Domain	Range
f(x) = c	R	{c}

9. EQUAL OR IDENTICAL FUNCTION

Two functions *f* and g are said to be equal if the following conditions are satisfied:

(i) The domain of *f* is equal to the domain of g

(ii) The range of f is equal to the range of g and

(iii) f(x) = g(x), for every x belonging to their common domain,

E.g. $f(x) = \frac{1}{x}$ and $g(x)\ \frac{x}{x^2}$ are identical functions.

Few examples of equal functions are listed as follows:

(i) $f(x) = \ln x^2$; $g(x) = 2\ln x$ (N.I.)

(ii) $f(x) = \sin^{-1}(3x - 4x^3)$; $g(x) = 3\sin^{-1} x$ (N.I.)

(iii) $f(x) = \sec^{-1}x + \text{cosec}^{-1}x$; $g(x) = \frac{\pi}{2}$ (N.I.)

(iv) $f(x) = \cot^2x.\cos^2x : g(x) = \cot^2x - \cos^2x$ (I)

(v) $f(x) = \text{Sgn}(x^2 + 1)$; $g(x) = \sin^2x + \cos^2x$ (I)

(vi) $f(x) = \tan^2x.\sin^2x$; $g(x) = \tan^2x - \sin^2x$ (I)

(vii) $f(x) = \sec^2x - \tan^2x$; : $g(x) = 1$ (N.I.)

(viii) $f(x) = \tan(\cot^{-1}x)$; $g(x) = \cot(\tan^{-1} x)$ (I)

(ix) $f(x) = \sqrt{x^2 - 1}$; $g(x) = \sqrt{x-1}\sqrt{x+1}$ (N.I.)

(x) f(x) = tan x. cot x ; g(x) = sin x . cosec x (N.I.)

(xi) $f(x) = e^{\ell n e^x}$; $g(x) = e^x$ (I)

(xii) $f(x) = \sqrt{\frac{1-\cos 2x}{2}}$; g(x) = sin x (N.I.)

(xiii) $f(x) = \sqrt{x^2}$; $g(x) = \left(\sqrt{x}\right)^2$ (N.I.)

(xiv) $f(x) = \log(x + 2) + \log(x - 3)$; $g(x) = (x^2 - x - 6)$ (N.I.)

(xv) $f(x) = \frac{1}{|x|}$; $g(x) = \sqrt{x^{-2}}$ (I)

(xvi) $f(x) = x\,|x|$; $g(x) = x^2 \text{ sgn } x$ (I)

(xvii) $f(x) = \underset{n\to\infty}{\text{Limit}} \frac{x^{2n}-1}{x^{2n}+1}$; $g(x) = \text{sgn}(|x|-1)$ (I)

(xviii) $f(x) = \sin(\sin^{-1} x)$; $g(x) = \cos(\cos^{-1} x)$ (I)

(xix) $f(x) = \frac{1}{1+\frac{1}{x}}$; $g(x) = \frac{x}{1+x}$ (N.I.)

(xx) $f(x) = [\{x\}]$; $g(x) = \{[x]\}$ (I) (note that f(x) and g(x) are constant functions)

(xxi) $f(x) = e^{\ell n \cot^{-1} x}$; $g(x) = \cot^{-1} x$ (I)

(xxii) $f(x) = e^{\ell n \sec^{-1} x}$; $g(x) = \sec^{-1} x$ (N.I.) Identical if $x \in (-\infty, -1] \cup (1, \infty)$ (xxiii)

(xxiii) $f(x) = (fog)(x)$; $G(x) = (gof)(x)$ where $f(x) = ex$; $g(x) = \ln x$ (N.I.)

10. CLASSIFICATIONS OF FUNCTIONS

10.1 One–One Function

One–one function (Injective mapping)

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ and $f(x_1) \in A$.

$f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_{1\neq}\, x_2 \Leftrightarrow f(x_1)\ ^1 f(x_2)$

Example: $f_1 \to R \to R$, $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \log x$

Diagrammatically an injective mapping is shown as follows:

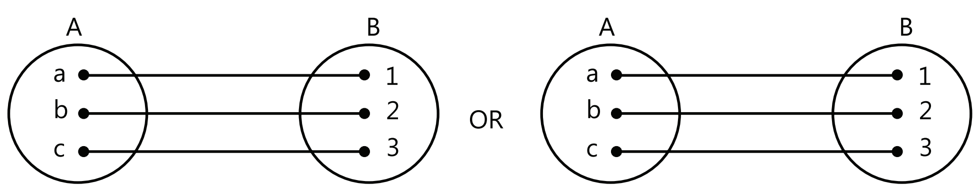

Figure 13.46

Note:

(i) If there is increase or decrease of continuous function in whole domain, then f(x) is one–one.

(ii) If any line parallel to x-axis cuts the graph of the function only at one point, then the function is one–one.

10.2 Many–One Function (Not Injective)

A function f: $A \to B$ is said to be a many–one function if two or more elements of A have the same f image in B. Thus for f: $A \to B$ is many–one if for $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ but $x_{1\neq}\, x_2$

For example, $f_{1\to}$ R → R, f(x) = [x]; f(x) = | x |; f(x) = $ax^2 + bx = c$; f(x) = sin x.

Diagrammatically a many–one mapping is shown as follows:

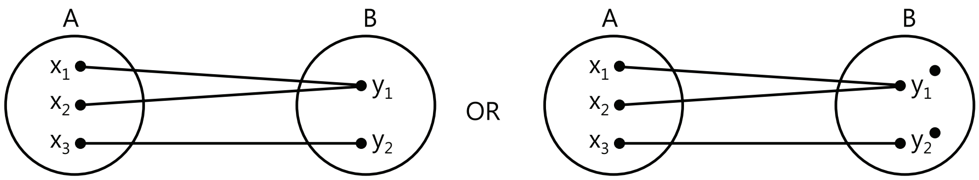

Figure 13.47

Note:

(i) If any continuous function has at least one local maximum or local minimum, then f(x) is many–one. In other words, if a line parallel to x-axis cuts the graph of the function at least at two points, then f is many–one.

(ii) If a function is one–one, it cannot be many–one and vice versa.

One–one + Many–one = Total number of mapping.

10.3 Onto Functions

If the function $f: A \to B$ is such that each element in B (co-domain) is the f image of at least one element in A, then we say that f is a function of A 'onto' B. Thus f: A → B is surjective ∀ b ∈ B, ∃ some a ∈ A such that f(a) = b.

Diagrammatically surjective mapping is shown as follows:

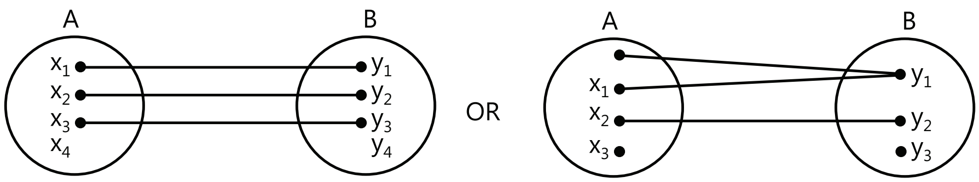

Figure 13.48

Note: if range is equal to co-domain, then f(x) is onto.

10.4 Into Functions

If f: A → B is such that there exists at least one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function is shown as follows:

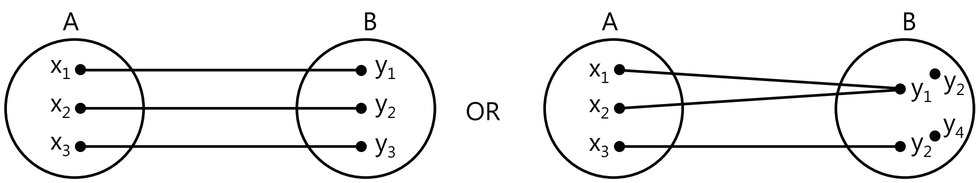

Figure 13.49

Note: If a function is onto, it cannot be into and vice versa. A polynomial of degree even and odd defined from R → R will always be into and onto, respectively.

Thus a function can be one of these four types:

(i) one–one onto (injective & surjective) $(I \cap S)$

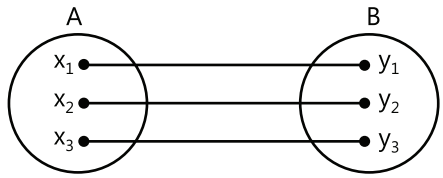

Figure 13.50

(ii) one–one onto (injective but not surjective) $(I \cap \bar{S})$

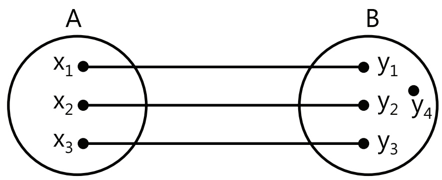

Figure 13.51

(iii) Many–one onto (surjective but not injective) $(S \cap \bar{I})$

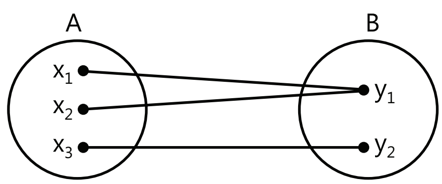

Figure 13.52

(iv) Many–one onto (neither surjective nor injective) $(\bar{I} \cap \bar{S})$

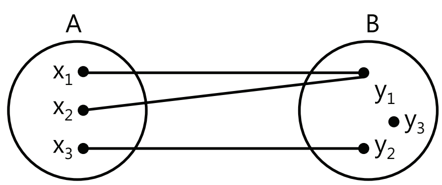

Figure 13.53

Illustration 12: Let A be a finite set. If f: A → A is an onto function, then show that f is one–one. **(JEE MAIN)**

Sol: Use the definition of one-one and onto function.

Let A = $\{a_1, a_2, ..., a_n\}$. To prove that f is a one–one function, we will have to show that $f(a_1), f(a_2), ..., f(a_n)$ are distinct elements of A. We have,

Range of f = $\{f(a_1), \cdot f(a_2), ..., f(a_n)\}$

Since f : A → A is a onto function. Therefore,

Range of f = A.

$\Rightarrow f = \{f(a_1), f(a_2), ..., f(a_n)\} = A$

But, A is a finite set consisting of n elements.

Therefore, $f(a_1), f(a_2), f(a_3), ..., f(a_n)$ are distinct element of A.

Hence, f : A → A is one–one.

Illustration 13: Let C and R denote the set of all complex and all real numbers respectively. Then show that *f*: C → R given by f(z) = | z |, for all z ∈ C is neither one–one nor onto. **(JEE MAIN)**

Sol: Using two complex conjugate numbers, we can prove that the given function is not one-one. For the second part, use the fact that modulus of a number cannot be negative.

Injectivity: We find that z_1 = 1 – i and z_2 = 1 + i are two distinct complex numbers such that | z_1 | = | z_2 |, i.e. z_1,z_2 but f(z_1) = f(z_2).

It is clear that different elements may have the same image. So, f is not an injection.

Surjectivity: f is not a surjection, because negative real numbers in R do not have their pre-image in C. In other words, for every negative real number there is no complex number z ∈ C such that f(z) = | z | = a. So, *f* is not a surjection.

Illustration 14: For function *f*: A →A, fof = f. Prove that f is one–one if and only if f is onto. **(JEE MAIN)**

Sol: Starting with the relation given in the question fof = f and use the definition of one-one and onto function.

Suppose f is one–one.

Then, (fof) (x) = f(x)

f(f(x)) = f(x)

f(x) = x (f is one–one)

Thus, f(x) = x, ∀ x ∈ A.

for each x A, there is an x ∈ A such that f(x) = x

∴ f is onto.

Now suppose f is onto.

Then, for each y ∈ A, there is an x ∈ A such that f(x) = y.

Let x_1, x_2 ∈ A and let f(x_1) = f(x_2).

Then there exist y_1, y_2 ∈ A such that x_1 = f(y_1)·x_2 = f(y_2)

f(f(y_1)) = f(f(y_2))

(fof)(y_1) = (fof)(y_2)

f(y_1) = f(y_2) (fof = f)

x_1 = x_2

∴ f is one–one.

Illustration 15: Show that the function *f*: R → R given by f(x) = x^3 + x is a bijection. **(JEE ADVANCED)**

Sol: Consider two elements x and y in the domain and prove that f(x) = f(y) implies x = y. Use the definition of onto to prove that the function f is a bijection.

Injectivity: Let x, y ∈ R such that

f(x) = f(y)

⇒ $x^3 + x = y^3 + y$

⇒ $x^3 - y^3 + (x - y) = 0$

⇒ $(x - y)(x^2 + xy + y^2 + 1) = 0$

⇒ x – y = 0 [Q$x^2 + xy + y^2 \geq 0$ for all x, y ∈R. $x^2 + xy + y^2 + 1 \geq 1$ for all x,y ∈ R]

⇒ x = y

Thus, f(x) = f(y)

$\Rightarrow$ x = y for all x, y $\in$ R

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of R then

$f(x) = y \Rightarrow x^3 + x = y$

$\Rightarrow x^3 + x - y = 0$

We know that an odd degree equation has at least one real root. Therefore, for every real value of y, the equation $x^3 + x - y = 0$ has a real root α, such that

$a^3 + \alpha - y = 0$

$a^3 + \alpha = y$

$f(\alpha) = y$

Thus, for every y $\in$ R, there exist $\alpha \in$ R such that $f(\alpha) = y$.

So, *f* is a surjective map.

Hence, $f: R \to R$ is a bijection.

Illustration 16: Let f: N $\cup$ {0} $\to$ N $\cup$ {0} be defined by $f(n) = \begin{cases} n+1, & \text{if n is even} \\ n-1, & \text{if n is odd} \end{cases}$. Show that f is a bijective function. **(JEE MAIN)**

Sol: Divide the solution in three cases.

Case I – When both the numbers are even, Case II – When both the numbers are odd & Case III – When one is even and other is odd.

Let f(n) = f(m).

Case I : both n, m are even.

Then, n + 1 = m + 1. So, n = m.

Case II : Both n, m are odd.

Then, n – 1 = m – 1, which implies m = n.

Case III : n is even and m is odd. Then, f(n) is odd and f(m) is even. So, f(n) $\neq$ f(m)

In any case, f(n) = f(m) implies n = m. Thus, f is one–one.

Now f(2n) = 2n + 1, f(2n + 1) = 2n for all n $\in$ N. So, *f* is onto. Hence *f* is a bijective function.

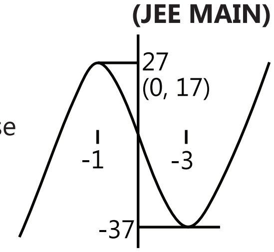

Figure 13.54

Illustration 17: Draw the graph of the function under the following condition and also check whether the function is one–one and onto or not, $f: R \to R$, $f(x) = 2x^3 - 6x^2 - 18x + 17$ **(JEE MAIN)**

Sol: Find the derivative of given function and understand the nature of the curve. Also find the values of f(x) at some particular values of x to trace the given curve.

Domain of the function is x $\in$ R

Here, $f'(x) = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3) = 6(x + 1)(x - 3)$

$f'(x) = 0$ at x = –1, 3

$f(\infty) = \infty$

$f(-\infty) = -\infty$

f(0) = 17, f(–1) = 27, f(3) = –37

From the graph we can say that f is many one onto function.

11. COMPOSITE FUNCTIONS

Let f: A $\to$ B and g: B $\to$ C be two functions. Then the function gof: A $\to$ C defined by (gof) (x) = g (f(x)) $\forall$ x $\in$A is called the composite of the two functions f and g. Diagrammatically

it is shown as follows:

$$\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$$

Thus the image of every x $\in$A under the function gof is g-image of the f-image of x.

Note that gof is defined only if x $\in$ A, f(x) is an element of the domain of g so that we can take its g-image. Hence, for the product gof, the range of f must be a subset of the domain of g. In general, gof not equal to fog.

12. PROPERTIES OF COMPOSITE FUNCTIONS

(i) The composite of functions is not commutative, i.e. gof $\neq$ fog.

(ii) The composite of functions is associative, i.e. if f, g, h are functions such that fo(goh) and (fog)oh are defined, then fo(goh) = (fog)oh.

Associativity: f: N $\to$ I_0. f(x) = 2x

g: $I_0 \to$ Q, g(x) = $\frac{1}{x}$; h: Q $\to$ R h(x) = $e^{\frac{1}{x}} \Rightarrow$ (hog)of = ho(gof) = e^{2x}

(iii) The composite of two bijections is a bijection, i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

Proof: Let f : A $\to$ B and g: B $\to$ C be two bijections. Then gof exists such that

gof: A $\to$ C

We have to prove that gof is one–one and onto.

One–one: Let a_1, a_2 $\in$ A such that (gof)(a_1) = (gof)(a_2), then

(gof)(a_1) = (gof)(a_2) $\Rightarrow$ g[f(a_1)] = g[f(a_2)]

$\Rightarrow$ f(a_1) = f(a_2) [$\because$ g is one–one]

$\Rightarrow a_1 = a_2$ [$\because$ f is one–one]

$\therefore$ gof is also one–one function.

Onto: Let c $\in$ C, then

c $\in$ C $\Rightarrow \exists$ b $\in$ B s.t. g(b) = c [$\because$ g is onto]

and b $\in$ B $\Rightarrow \exists$ a$\in$A s.t. f(a) = b [$\because$ f is into]

Therefore, we see that

c $\in$ C $\Rightarrow \exists$ a $\in$ A s.t. gof(a) = g[f(a)] = g(b) = c

i.e. every element of C is the gof image of some element of A. As such gof is onto function. Hence gof being one–one and onto is a bijection.

Illustration 18: Let f: R $\to$ R, g: R $\to$ R be defined by f(x) = x^2 + 3x + 1, g(x) = 2x – 3. Find fog and gof.**(JEE MAIN)**

Sol: Use the concept of composite functions.

We have (fog)(x) = f(g(x)) = f(2x – 3)= $(2x – 3)^2$ + 3(2x – 3) + 1= $4x^2$ – 6x+ 1

and (gof)(x) = g(f(x)) = g(x^2 + 3x + 1)= 2(x^2 + 3x + 1) – 3= $2x^2$ + 6x – 1

Hence fog $\neq$ gof.

Illustration 19: If (x) = $\frac{1}{x^2}$ and g(x) = 0 are two real functions, show that fog is not defined. **(JEE MAIN)**

Sol: Find the domain of fog(x).

We have,

Domain (f) = R – {0}, Range (f) = R – {0}

Domain (g) = R and Range (g) = {0}

Clearly, Range (g) $\not\subset$ Domain (f)

$\therefore$ Domain (fog) = {x: x $\in$ Domain (g) and g(x) $\in$ Domain (f)}

$\Rightarrow$ Domain (fog) = {x: x $\in$ R and g(x) $\in$ Domain (f)}

$\Rightarrow$ Doman (fog) = f

[$\because$ g(x) = 0 $\notin$ Domain (f) for any x $\in$ R]

Hence, fog is not defined.

Illustration 20: If f(x) = $\frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that, f(f(x)) = $\frac{2x+1}{2x+3}$, provided that $x \neq -\frac{1}{2}, -\frac{3}{2}$. **(JEE MAIN)**

Sol: Check the domain of f(x) and use the concept of composite functions.

We have, f(x) = $\frac{1}{2x+1}$

Clearly, domain (f) = $R - \left\{-\frac{1}{2}\right\}$

Let, y = $\frac{1}{2x+1} \Rightarrow 2x + 1 = \frac{1}{y} \Rightarrow x = \frac{1-y}{2y}$

Since x is a real number distinct from $-\frac{1}{2}$, y can take any non-zero real value.

So, Range (*f*) = R – {0}

We observe that range

(*f*) = R – {0} $\not\subset$ domain (*f*) = $R - \left\{\frac{1}{2}\right\}$

$\therefore$ Domain (fof) = {x: x $\in$ domain (*f*) and f(x) $\in$ Domain (*f*)}

$\Rightarrow$ Domain (fof) = $\left\{x : x \in R - \left\{-\frac{1}{2}\right\} \text{ and } f(x) \in R - \left\{-\frac{1}{2}\right\}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } f(x) \neq -\frac{1}{2}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } \frac{1}{2x+1} \neq -\frac{1}{2}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } x \neq -\frac{3}{2}\right\}$

$R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\}$

Also, fof(x) = f(f(x)) = $f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(1/(2x+1)\right)+1} = \frac{2x+1}{2x+3}$

Thus, fof : $R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\} \rightarrow R$ is defined by fof(x) = $\frac{2x+1}{2x+3}$

Hence, $f(f(x)) = \frac{2x+1}{2x+3}$ for all $x \in R$. $x \neq -\frac{1}{2}, -\frac{3}{2}$.

Illustration 21: If $f(x) = \log_{100x}\left(\frac{2\log_{10}x+2}{-x}\right)$ and $g(x) = \{x\}$. If the function (fog)(x) exists then find the range of g(x). **(JEE ADVANCED)**

Sol: Find the domain of f(x) and use the given information that fog(x) exists.

To define f(x), the following condition must hold good:

(i) $100x > 0$ and $100x \neq 1 \Rightarrow x \neq \frac{1}{100}$

(ii) $x > 0$ and $\log_{10}x + 1 < 0 \Rightarrow 0 < x < \frac{1}{10}$ and $x \neq \frac{1}{100}$

$\therefore$ Domain of f(x) is $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$.

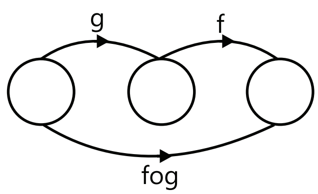

Figure 13.55

Here, $g(x) = \{x\}$, range of g(x) is [0, 1).

But, (fog) (x) exists $\Rightarrow$ range of g(x) $\subset$ domain of f(x).

$\therefore$ Range of g(x) is $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$.

Illustration 22: Consider functions f and g such that composite gof is defined and is one–one. Should f and g necessarily be one–one? **(JEE MAIN)**

Sol: Take an example to prove it.

Consider f: {1, 2, 3, 4} $\to$ {1, 2, 3, 4, 5, 6} defined as $f(x) = x, \ \forall \ x = 1, 2, 3, 4$ and

g: {1, 2, 3, 4, 5, 6} ®{1, 2, 3, 4, 5, 6} as g(x), for x = 1, 2, 3, 4 and g(5) = g(6) = 5.

Then, $gof(x) = x, \ \forall x = 1, 2, 3, 4$, which shows that gof is one–one. But g is clearly not one–one.

14. INVERSE OF A FUNCTION

Let f: A $\to$ B be a one–one and onto function, then there exists a unique function.

g: B $\to$ A such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall x \in A$ and $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

14.1 Properties of Inverse Functions

(i) The inverse of a bijection is unique.

Proof: Let f : A $\to$ B be a bijection, and let g: B $\to$ A and h: B $\to$ A be two inverse functions of f. Also let $a_1, a_2 \in A$ and $b \in B$, such that $g(b) = a_1$ and $h(b) = a_2$. Then

$g(b) = a_1 \Rightarrow f(a_1) = b$

$h(b) = a_2 \Rightarrow f(a_2) = b$

Since f is one–one, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \Rightarrow g(b) = h(b), b \in B$.

(ii) If f: A $\to$ B is a bijection and g: B $\to$ A is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.

Note that the graphs of f and g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x′, y′) corresponding to $y = x^2 (x \geq 0)$ changes to (y′ x′) corresponding to $y = \sqrt{x}$, the changed form of $x = \sqrt{y}$.

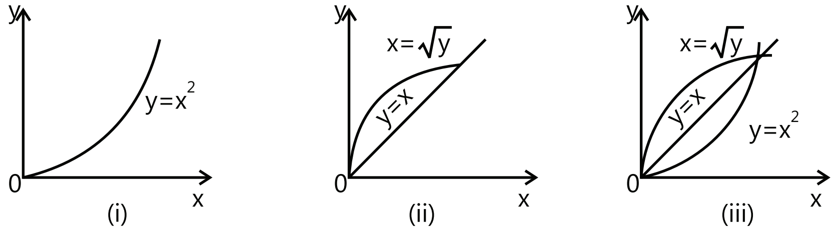

Figure 13.56

(iii) The inverse of a bijection is also a bijection.

Proof: Let f : A $\rightarrow$ B be a bijection and g : B $\rightarrow$ A be its inverse. We have to show that g is one–one and onto.

One-one: Let $g(b_1) = a_2$ and $g(b_2) = a_2$; $a_1, a_2 \in A$ and $b_1, b_2 \in B$

Then $g(b_1) = g(b_2) \Rightarrow a_1 = a_2$

$\Rightarrow$ $f(a_1) = f(a_2)$ [$\because$ f is bijection]

$\Rightarrow$ $b_1 = b_2$ [$\because g(b_1) = a_1 \Rightarrow b_1 = f(a_1); g(b_2) = a_2 \Rightarrow b_2 = f(a_2)$]

Which proves that g is one–one.

Onto: Again, if $a \in A$, then

$a \in A \Rightarrow \exists\ b \in B$ s.t. f(a) = b(by definition of f)

$\Rightarrow \exists\ b \in B$ s.t. a = g(b) [$\because f(a) = b \Rightarrow a = g(b)$]

Which proves that g is onto. Hence g is also a bijection.

(iv) If f and g are two bijections f : A $\rightarrow$ B, g : B $\rightarrow$ C then inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: since f : A $\rightarrow$ B and g : B $\rightarrow$ C are two bijections.

$\therefore$ gof: A $\rightarrow$ C is also a bijection.

[By theorem the composite of two bijection is a bijection.]

As such gof has an inverse function $(gof)^{-1} : C \rightarrow A$. We have to show that

$(gof)^{-1} = f^{-1}og^{-1}$

Now let $a \in A$, $b \in B$, $c \in C$ such that f(a) = b and g(b) = c.

So (gof) (a) = g[f(a)] = g(b) = c

Now f(a) = b $\Rightarrow$ $a = f^{-1}(b)$ ....(i)

g(b) = c $\Rightarrow$ $b = g^{-1}(c)$ .... (ii)

(gof)(a) = c $\Rightarrow$ $a = [gof)^{-1}(c)$ .... (iii)

Also $(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)]$ [by definition]

$= f^{-1}(b)$ [by (ii)]

= a [by (i)]

$= (gof)^{-1}(c)$ [by (iii)]

$\therefore$ $(gof)^{-1} = f^{-1}og^{-1}$, which proves the theorem.

MASTERJEE CONCEPTS

In the line y = x, the graphs of f and g are the mirror images of each other. As shown in the following figure, a point (x′, y′) corresponding to y = ln x(x >0) changes to (y′, x′) corresponding to $y = e^x$, the changed form of $x = e^y$.

The inverse of a bijection is also a bijection.

If f and g are two bijections f: A → B, g: B → C, then inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

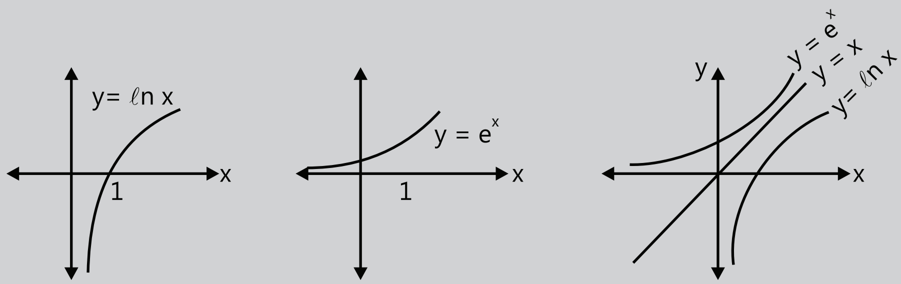

Figure 13.57

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 23: Find inverse of the function $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$ **(JEE MAIN)**

Sol: Put f(x) = y and solve for x.

Graph of f(x)

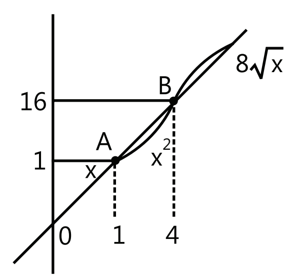

Figure 13.58

Using the above graph $f^{-1}(y) = x = \begin{cases} y & \text{if} \quad y < 1 \\ \sqrt{y} & \text{if} \quad 1 \le y \le 16 \\ \frac{y^2}{164} & \text{if} \quad y > 16 \end{cases}$

or $\quad f^{-1}(x) = \begin{cases} x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{16} & \text{if} \quad x > 16 \end{cases}$.

Illustration 24: Let *f*: N → R be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that f : N → S, where S is the range of f, is invertible. Find the inverse of f. **(JEE MAIN)**

Sol: Put $4x^2 + 12x + 15 = y$ and solve for x. Then put $f^{-1}(y)$ in place of x.

Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in N, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{(\sqrt{y-6}-3)}{2}$, as $y \ge 6$.

Let us define g: S → N by $g(y) = \frac{(\sqrt{y-6}-3)}{2}$.

Now, $\quad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$$= \frac{\sqrt{((2x+3)^2+6-6)}-3}{2} = \frac{(2x+3-3)}{2} = x$$

and $fog(y) = f\left(\frac{\sqrt{y-6}-3}{2}\right) = \left(\frac{2\sqrt{y-6}-3}{2}+3\right) + 6$

$$= (\sqrt{y-6}-3+3)^2 + 6 = (\sqrt{y-6})^2 + 6 = y - 6 + 6 = y$$

Hence, $gof = I_N$ and $fog = I_S$. This implies that f is invertible with $f^{-1} = g$.

Illustration 25: For the function $f: R - \{4\} \rightarrow R - \{-2\} : f(x) = \frac{2x-5}{4-x}$. Find

(a) zero's of f(x)

(b) range of f(x)

(c) intervals of monotonocity

(d) $f^{-1}(x)$

(e) local maxima and minima if any

(f) interval when f(x) is concave upward and concave downward

(g) asymptotes

(h) $\int_1^2 f(x)\,dx$

(i) nature of function whether one–one or onto

(j) graph **(JEE MAIN)**

Sol:

(a) 5/2

(b) $(-\infty, -2) \cup (-2, \infty)$

(c) ↑ in its domain i.e., $(-\infty, 4) \cup (4, \infty)$

(d) $f^{-1}(x) = \frac{4x+5}{x+2}$

(e) no,

(f) $(-\infty, 4)$ upwards and $(4,\infty)$ downwards

(g) (g) $y = -2$

(h) $-\left(2+3\ln\frac{2}{3}\right)$

(i) both one-one and onto

(j)

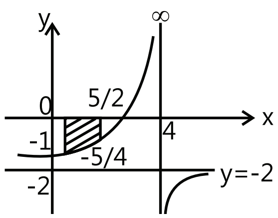

Figure 13.59

Illustration 26: $f: R \rightarrow R$, $f(x) = 3x + 2$. Find the inverse of f, if it exists. **(JEE MAIN)**

Sol: Check whether the given function is one-one onto. Put $3x + 2 = y$ and solve for x. Then put $f^{-1}(y)$ in place of x.

$\forall\, x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow 3x_1 + 2 = 3x_2 + 2 \Rightarrow x_1 = x_2$

$\therefore$ $f: R \rightarrow R$ is one–one.

Now let $y \in R$. Then $y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$

Hence, for every $y \in R$, there is a corresponding $x = \frac{y-2}{3} \in R$ such that $y = f(x)$. Hence, the range of f is R and so f is onto, and the inverse of f exists.

Again, $y = f(x) \Leftrightarrow x = f^{-1}(y)$

$\therefore f^{-1} : R \to R, f^{-1}(y) = \frac{y-2}{3}$ or equivalently, $f^{-1}(x) = \frac{x-2}{3}$.

15. DIFFERENT TYPES OF FUNCTIONS

15.1 Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x and y. Symbolically if,

$f(tx, ty) = t^n. f(x,y)$, then f(x, y) is homogeneous function of degree n.

Examples of Homogeneous function:

$f(x, y) = \frac{x - y\cos x}{y\sin x + x}$ is not a homogeneous function and

$f(x, y) = \frac{x}{y}\ln\frac{y}{x} + \frac{y}{x}\ln\frac{x}{y}$; $\sqrt{x^2 - y^2} + x$; $x + y\cos\frac{y}{x}$ are homogeneous function of degree one.

15.2 Bounded Function

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.

15.3 Implicit and Explicit Function

A function defined by an equation not solved for the dependent variable is called an implicit function. For example, the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of only x, then it is called explicit function.

Examples on implicit and explicit function (x, y) = 0

1. $x\sqrt{1+y} + y\sqrt{1+x} = 0$; explicit

$y = -\frac{x}{1+x}$ or y = x (rejected)

2. $y^2 = x$ represents two separated branches

3. $x^3 + y^3 - 3xy = 0$ folium of desecrates

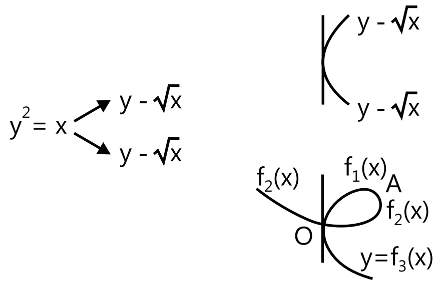

Figure 13.60

15.4 Odd and Even Function

A function f(x) defined on the symmetric interval (–a, a) then,

If $f(-x) = f(x)$ $\forall$ x in the domain of 'f' then f is said to be an even function.

E.g., $f(x) = \cos x$; $g(x) = x^2 + 3$;

If $f(-x) = -f(x)$ $\forall$ x in the domain of 'f' then f is said to be an odd function.

E.g., $f(x) = \sin x$; $g(x) = x^3 + x$.

Examples on odd and even functions:

Odd	**Even**	**Neither odd nor even**
1. $\ell n\left(x + \sqrt{1+x^2}\right)$	1. $x\frac{2^x+1}{2^x-1}$	1. $2x^3 - x + 1k$

2. $\ell n\dfrac{1-x}{1+x}$

3. $x\sin^2 x - x^3$

4. $\sqrt{1+x+x^2}-\sqrt{1+x+x^2}$

5. $\dfrac{1+2^{Kx}}{1-2^{Kx}}$

2. $\sqrt[3]{(1-x)^2}+\sqrt[3]{(1+x)^2}$

3. constant

4. $x^2 - |x|$

5. $\dfrac{(1+2^x)^2}{2^x}$

2. $\sin x + \cos x$

MASTERJEE CONCEPTS

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even and $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined and an even function cannot be strictly monotonic.

(d) Every even function is symmetric about the y-axis and every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even and an odd function.

E.g.,

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}} \qquad 2^x = \underbrace{\frac{2^x+2^{-x}}{2}}_{\text{EVEN}} + \underbrace{\frac{2^x-2^{-x}}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(g) If f and g are either both even or odd, then the function $f \cdot g$ will be even. But if any one of them is odd then $f \cdot g$ will be odd.

f(x)	g(x)	f(x)+g(x)	f(x) – g(x)	f(x) . g(x)	f(x) / g(x)	(gof) / (x)	(fog) (x)
odd	odd	Odd	odd	even	even	odd	odd
even	even	Even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Shrikant Nagori (JEE 2009, AIR 30)

15.5 Periodic Functions

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x.

E.g., Both the functions sin x and cos x are periodic over 2π, and tan x is periodic over π.

Examples on periodic function

(i) $f(x) = \cos\dfrac{2x}{3} - \sin\dfrac{4x}{5}\ (15\pi)$

(ii) $f(x) = \cos(\sin x)$ (π)

(iii) $f(x) = \sin(\cos x)$ (2π)

(iv) $f(x) = \sin^4 x + \cos^4 x$ $\left(\frac{\pi}{2}\right)$

(v) $f(x) = x - [x] = \{x\}$ (one)

MASTERJEE CONCEPTS

- $f(T) = f(0) = f(-T)$, where 'T' is the period.
- Inverse of a periodic function does not exist.
- Every constant function is always periodic, with no fundamental period.
- If f(x) has a period T and g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. For example, $f(x) = |\sin x| + |\cos x|$
- If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
- If f(x) has a period T then f(ax + b) has a period T/a (a >0).

Proof : Let $f(x + T) = f(x)$ and $f[a(x + T') + b] = f(ax + b)$

$f(ax + b + aT') = f(ax + b)$

$f(y + aT') = f(y) = f(y + T) \Rightarrow T = aT' \Rightarrow T' = \frac{T}{a}$

Vaihbav Gupta (JEE 2009, AIR 54)

15.6 Special Functions

If x, y are independent variables, then

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$

(b) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n$, $n \in R$

(c) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant

Illustration 27: Which of the following function(s) is(are) bounded on the intervals as indicated

(A) $f(x) = 2^{\frac{1}{x-1}}$ on (0, 1) (B) $g(x) = x \cos \frac{1}{x}$ on $(-\infty, \infty)$

(C) $h(x) = xe^{-x}$ on $(0, \infty)$ (D) $l(x) = \arctan 2^x$ on $(-\infty, \infty)$

Sol: Check for the continuity of the given functions. If the function is continuous then to find the value of f(x) at the boundary points.

(A) $\underset{x \to 0^+}{\text{Limit}} f(x) = \underset{h \to 0}{\text{Limit}}\, 2^{\frac{1}{h-1}} = \frac{1}{2}$; $\underset{x \to 1^-}{\text{Limit}} f(x) = \underset{h \to 0}{\text{Limit}}\, 2^{\frac{1}{-h}} = 0$

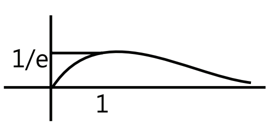

Figure 13.61

$\Rightarrow$ $f(x) \in \left(0, \frac{1}{2}\right) \Rightarrow$ bounded

(C) $\underset{h\to 0}{\text{Limit}}\ xe^{-x} = \underset{h\to 0}{\text{Limit}}\ he^{-h} = 0; \underset{x\to\infty}{\text{Limit}}\ x\,e^{-x} = \underset{x\to\infty}{\text{Limit}}\ \frac{x}{e^x} = 0$

$\Rightarrow$ also $y = \frac{x}{e^x} \Rightarrow y' = \frac{e^x - xe^x}{e^{2x}}\ e^{x(1-x)} \Rightarrow h(x) = \left(0, \frac{1}{e}\right]$

FORMULAE SHEET

Table: Domain and range of some standard functions-

Functions	**Domain**	**Range**
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	(K)
Reciprocal function $\frac{1}{x}$	R_0	R_0
$X^2, \lvert x \rvert$ (modulus function)	R	$R^+ \cup \{x\}$
$x^3, x\lvert x \rvert$	R	R
Signum function $\frac{\lvert x \rvert}{x}$	R	{-1,0,1}
$X + \lvert x \rvert$	R	$R^+ \cup \{x\}$
$x - \lvert x \rvert$	R	$R^- \cup \{x\}$
[x] (greatest integer function)	R	1
x-{x}	R	[0,1]
$\sqrt{x}$	(0, ∞)	[0,∞]
a^x(exponential function)	R	R^+
Log x(logarithmic function)	R^+	R

Inverse Trigo Functions	Domain	Range
$\sin^{-1}x$	(-1,1]	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	[-1,1]	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	R-(-1,1)	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\mathrm{cosec}^{-1}x$	R-(-1,1)	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Inverse function: f^{-1} exists iff f is both one–one and onto.

$f^{-1}:B\to A$, $f^{-1}(b)=a \Rightarrow f(a)=b$

Even and odd function: A function is said to be

(a) Even function if f(x)=f(x) and

(b) Odd function if f(–x)= –f(x)

Properties of even & odd function:

(a) The graph of an even function is always symmetric about y-axis.

(b) The graph of an odd function is always symmetric about origin.

(c) Product of two even or odd function is an even function.

(d) Sum & difference of two even (odd) function is an even (odd) function.

(e) Product of an even or odd function is an odd function.

(f) Sum of even and odd function is neither even nor odd function.

(g) Zero function, i.e. f(x) = 0, is the only function which is both even and odd.

(h) If f(x) is an odd (even) function, then $f'(x)$ is even (odd) function provided f(x) is differentiable on R.

(i)A given function can be expressed as sum of even and odd function.

i.e. $f(x)=\frac{1}{2}\left[f(x)+f(-x)\right]+\frac{1}{2}\left[f(x)-f(-x)\right]$ =even function + odd function.

Increasing function: A function f(x) is an increasing function in the domain, D if the value of the function does not decrease by increasing the value of x.

Decreasing function: A function f(x) is a decreasing function in the domain, D if the value of function does not increase by increasing the value of x.

Periodic function: Function f(x) will be periodic if a +ve real number T exists such that

$f(x+T)=f(x), \forall x \in \text{Domain}.$

There may be infinitely many such real number T which satisfies the above equality. Such a least +ve number T is called period of f(x).

(i) If a function f(x) has period T, then period of f(xn+a)=T/n and period of (x/n+a)=nT.

(ii) If the period of f(x) is T_1 & g(x) has T_2 then the period of f(x) ± g(x) will be L.C.M. of T_1& T_2 provided it satisfies definition of periodic function.

(iii) If period of f(x) & f(x) are same T, then the period of af(x)+bg(x) will also be T.

Function	Period
sin x, cos x	2π
sec x, cosec x	
tan x, cot x	π
sin (x/3)	6π
tan 4x	$\pi/4$
cos $2\pi x$	1
$\lvert \cos x \rvert$	π
$\sin^4 x+\cos^4 x$	$\pi/2$
$2\cos\left(\frac{x-\pi}{3}\right)$	6π
sin3 x + $\cos^3 x$	$2\pi/3$
$\text{Sin}^3 x + \cos^4 x$	2π
$\frac{\sin x}{\sin 5x}$	2π
$\tan^2 x - \cot^2 x$	π
x-[x]	1
[x]	1