PROBLEM SOLVING TACTICS

(a) In general convert the given hyperbola equation into the standard form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and compare it with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then solve using the properties of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. So, it is advised to remember the standard results.

(b) Most of the standard results of a hyperbola can be obtained from the results of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ just by changing the sign of b^2.

FORMULAE SHEET

HYPERBOLA

(a) Standard Hyperbola:

Imp. Terms \ **Hyperbola**	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	(±ae, 0)	(0, ±be)
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of L.R.	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a \sec \phi, b \tan \phi)$ $0 \le \phi < 2\pi$	$(a \tan \phi, b \sec \phi)$ $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ S¢P = $ex_1 + a$	$SP = ey_1 - b$ S¢P = $ey_1 + b$
|S¢P – SP|	2a	2b
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

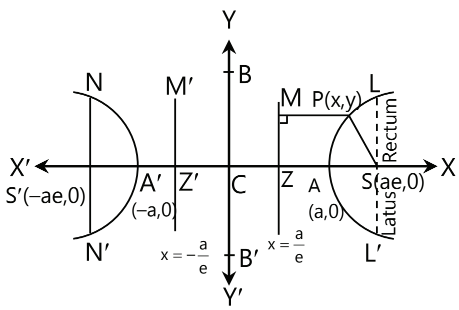

Figure 12.32: Hyperbola

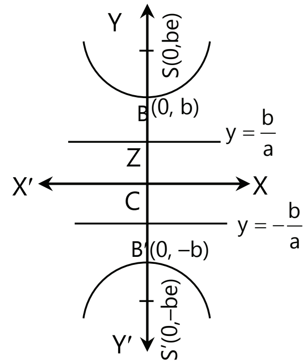

Figure 12.33: Conjugate Hyperbola

(b) Special form of hyperbola: If (h , k) is the centre of a hyperbola and its axes are parallel to the co-ordinate axes, then the equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(c) Parametric equations of a hyperbola: The equation x = a sec ϕ and y = b tan ϕ are known as the parametric equation of the standard hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}$, then $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(d) Position of a point and a line w.r.t. a hyperbola: n The point (x_1, y_1) lies inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ being >, = or < zero.

The line y = mx + c intersects at 2 distinct points, 1 point or does not intersect with the hyperbola according as c^2 >, = or < $a^2m^2 - b^2$.

(e) Tangent:

(i) Point form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(ii) Parametric form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at parametric coordinates (a sec ϕ, b tan ϕ) is $\frac{x}{a}\sec\phi-\frac{y}{b}\phi=1.$

(iii) Slope form: The equation of the tangents having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$y = mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(f) Equation of a pair of tangents from an external point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1 = T^2$.

(g) Normal:

(i) Point form: The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2.$$

(ii) Parametric form: The equation of the normal at parametric coordinates (a sec θ, b tanθ) to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\quad ax\cos\theta + by\cot\theta = a^2 + b^2$.

(iii) Slope form: The equation of the normal having slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$$

(iv) Condition for normality: y = mx + c is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

$$c^2 = \frac{m(a^2 + b^2)^2}{(a^2 - m^2b^2)}$$

(v) Points of contact: Co-ordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2 - b^2m^2}}\right)$.

(h) The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $x^2 + y^2 = a^2 - b^2$.

(i) Equation of the chord of contact of the tangents drawn from the external point (x_1, y_1) to the hyperbola is

given by $\quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$

(j) The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) is $T = S_1$.

(k) Equation of a chord joining points P(a sec f_1, b tan f_1) and Q (a sec f_2, b tan f_2) is

$$\frac{x}{a}\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1 + \phi_2}{2}\right) = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

(l) Equation of the polar of the point (x_1, y_1) w.r.t. the hyperbola is given by T = 0.

The pole of the line lx + my + n = 0 w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\left(-\frac{a^2\ell}{n}, \frac{b^2m}{n}\right)$

(m) The equation of a diameter of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ corresponding to the chords of slope m is $y = \frac{b^2}{a^2m}x$

(n) The diameters $y = m_1x$ and $y = m_2x$ are conjugate if $m_1m_2 = \frac{b^2}{a^2}$

(o) Asymptotes:

- Asymptote to a curve touches the curve at infinity.
- The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$.

- The asymptote of a hyperbola passes through the centre of the hyperbola.
- * The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$
- * The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{a^2}{b^2}$ or $2\sec^{-1} e$.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The bisector of the angles between the asymptotes are the coordinate axes.
- Equation of the hyperbola – Equation of the asymptotes = constant.

(p) Rectangular or Equilateral Hyperbola:

- A hyperbola for which a = b is said to be a rectangular hyperbola, its equation is $x^2 - y^2 = a^2$.
- $xy = c^2$ represents a rectangular hyperbola with asymptotes x = 0, y = 0.
- Eccentricity of a rectangular hyperbola is $\sqrt{2}$ and the angle between the asymptotes of a rectangular hyperbola is 90º.
- Parametric equation of the hyperbola $xy = c^2$ are $x = ct$, $y = \frac{c}{t}$, where t is a parameter.
- Equation of a chord joining t_1, t_2 on $xy = c^2$ is $x + y\,t_1t_2 = c(t_1 + t_2)$
- Equation of a tangent at (x_1, y_1) to $xy = c^2$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$.
- Equation of a tangent at t is $x + yt^2 = 2ct$
- Equation of the normal at (x_1, y_1) to $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.
- Equation of the normal at t on $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

  (i.e. Four normals can be drawn from a point to the hyperbola $xy = c^2$)
- If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.
- Equation of chord of the hyperbola $xy = c^2$ whose middle point is given is $T = S_1$.
- Point of intersection of tangents at t_1 and t_2 to the hyperbola $xy = c^2$ is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$