25. AREA UNDER THE CURVE AND LINEAR PROGRAMMING

AREA UNDER THE CURVE

1. INTRODUCTION

In the previous chapters we have studied the process of integration and its physical interpretation. The most important application of integration is finding the area under a curve. In this topic we will discuss different curves and the area bounded by some simple plane curves taken together. In order to find the area, we need to know the basics of plotting a curve and then use integration with appropriate limits to get the answer. The process of finding area of some plane region is called **Quadrature**.

2. CURVE TRACING

Let us now discuss the basics of curve tracing. Curve tracing is a technique which provides a rough idea about the nature and shape of a plane curve. Different techniques are used in order to understand the nature of the curve, but there is no fixed rule which provides all the information to draw the graph of a given function (say f(x)). Sometimes it is also very difficult to draw the exact curve of the given function. However, the following steps can be helpful in trying to understand the nature and the shape of the curve.

Step 1: Check whether the origin lies on the given curve. Also check for other points lying on the curve by putting some values.

Step 2: Check whether the curve is increasing or decreasing by finding the derivative of the function. Also check for the boundary points of the curve.

Step 3: Check whether the curve $f(x, y) = 0$ is symmetric about

(a) X-axis: If the equation remains same on replacing y by –y i.e. $f(x, y) = f(x, -y)$, or, if all the powers of "y" are even, then the graph is symmetric about the X-axis.

(b) Y-axis: If the equation remains same on replacing x by –x i.e. $f(x, y) = f(-x, y)$, or, if all the powers of "x" are even, then the graph is symmetric about the Y-axis.

(c) Origin: If $f(-x, -y) = -f(x, y)$, then the graph is symmetric about the Origin.

For example, the curve given by $x^2 = y+2$ is symmetrical about y-axis, $y^2 = x+2$ is symmetrical about x-axis and the curve $y = x^3$ is symmetrical about the origin.

Step 4: Find out the points of intersection of the curve with the x-axis and y-axis by substituting $y = 0$ and $x = 0$ respectively.

For example, the curve $\frac{x^2}{9}+\frac{y^2}{4}=1$ intersects the axes at points $(\pm 3, 0)$ and $(0, \pm 2)$.

Step 5: Identify the domain of the given function and the region in which the graph can be drawn. For example, the curve $xy^2 = (8 - 4x)$ or $y = 2\sqrt{\frac{2-x}{x}}$.

Therefore the value of y is defined only when $\frac{2-x}{x} \geq 0$ i.e. $0 < x \leq 2$. Hence, the graph lies between the lines x = 0 and x = 2.

Step 6: Check the behaviour of the graph as $x \to +\infty$ and as $x \to -\infty$. Find all the horizontal, vertical and oblique asymptotes, if any.

Step 7: Determine the critical points, the intervals on which the function (f) is concave up or concave down and the inflection points.

The information obtained from the Steps 1 to 7 are used to trace the curve.

Illustration 1: Trace the curve $y^2(2a - x) = x^3$ **(JEE MAIN)**

Sol: By using curve tracing method as mentioned above.

Given curve: $y^2 = x^3/(2a - x)$ ...(i)

(a) Origin: The point (0 , 0) satisfies the given equation, therefore, it passes through the origin.

(b) Symmetrical about x-axis: On replacing y by –y, the equation remains same, therefore, the given curve is symmetrical about x-axis.

(c) Tangent at the origin: Equation of the tangent is obtained by equating the lowest degree terms to zero.

$\Rightarrow 2ay^2 = 0 \quad \Rightarrow y^2 = 0 \Rightarrow y=0$

(d) Asymptote parallel to y-axis: Equation of asymptote is obtained by equating the coefficient of lowest degree of y to 0. The given equation can be written as $(2a - x)\, y^2 = x^3$

$\therefore$ Equation of asymptote is $2a - x = 0$ or $x = 2a$

(e) Region of absence of curve: The given equation is

$y^2(2a - x) = x^3 \qquad \Rightarrow y^2 = \frac{x^3}{(2a-x)}$.

For x < 0 and x > 2a, RHS becomes negative, therefore the curve exists only for $0 \leq x < 2a$.

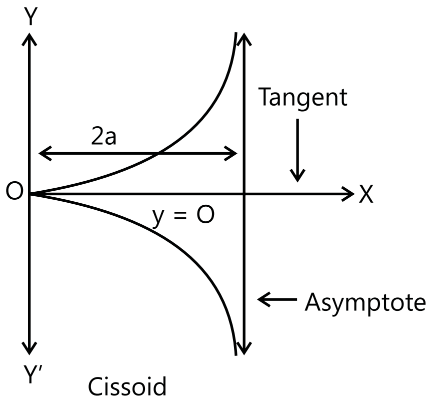

Figure 25.1

Hence the graph of $y^2(2a - x) = x^3$ is as shown in Fig. 25.1. Such a curve is known as a Cissoid.

Illustration 2: Sketch the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **(JEE MAIN)**

Sol: Same as above illustration.

We have, $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...(i)

(a) Origin: The point (0,0) does not satisfy the equation, hence, the curve does not pass through O.

(b) Symmetry: The equation of the curve contains even powers of x and y so it is symmetric about both x and y axes.

(c) Intercepts: Putting y = 0, we get x = ± 2 i.e. the curve passes through the points (2 , 0) and (-2 , 0). Similarly, on substituting x = 0, we get y = ± 3 i.e. the curve passes through the points (0 , 3) and (0 , -3).

(d) Region where the curve does not exist: If $x^2 > 4$, y becomes imaginary. So the curve does not exist for x > 2 and x < –2. Similarly, if $y^2 > 9$, x becomes imaginary. So, the curve does not exist for y > 3 and y < –3.

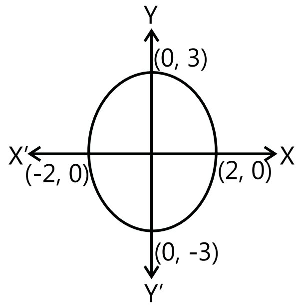

Figure 25.2

(e) Table:

x	– 2	0	1	2
y	0	±3	±2.6	0

Hence the graph of $\frac{x^2}{4}+\frac{y^2}{9}=1$ is as shown in Fig. 25.2.

MASTERJEE CONCEPTS

Using the above rules try to trace the Witch of Agnesi

$xy^2 = a^2 (a - x)$.

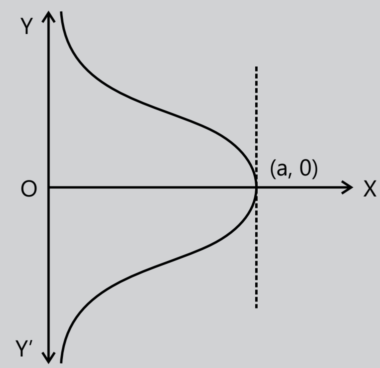

Figure 25.3

Vaibhav Krishnan (JEE 2009 AIR 22)

3. AREA BOUNDED BY A CURVE

3.1 The Area Bounded by a Curve with X-axis

The area bound the curve y=f(x) with the x-axis between the ordinates

x= a and x=b is given by $\text{Area} = \int_a^b y\ dx = \int_a^b f(x)dx$

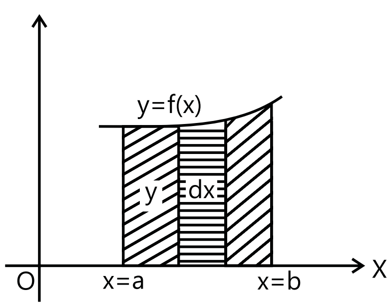

Figure 25.4: Area Bounded By a curve y=f(x) with x-axis

Illustration 3: Find the area bounded by the curve $y = x^3$, x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: By using above formula, we can find out the area under given curve.

Required Area = $\int_1^2 y\,dx = \int_1^2 x^3\,dx = \left(\frac{x^4}{4}\right)_1^2 = \frac{15}{4}$

Illustration 4: Find the area bounded by the curve y = mx x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: Same as above.

Required area $= \int_1^2 y\,dx = \int_1^2 mx\,dx = \left(\frac{mx^2}{2}\right)_1^2 = \frac{m}{2}(4-1) = \frac{3}{2}m$

Illustration 5: Find the area included between the parabola $y^2 = 4ax$ and its latus rectum (x = a).

(JEE ADVANCED)

Sol: Here the curve is $y^2 = 4ax$, latus rectum is x = a, and the curve is symmetrical about the x-axis.

(a) The latus rectum is the line perpendicular to the axis of the parabola and passing through the focus S (a, 0).

(b) The parabola is symmetrical about the x-axis.

∴ The required area AOBSA = 2 × area AOSA

$$= 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{ax}\,dx \quad \left[y^2 = 4ax \Rightarrow y = 2\sqrt{ax}\right]$$

$$= 4\sqrt{a}.\frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{8}{3}\sqrt{a}.a^{3/2} = \frac{8}{3}a^2.$$

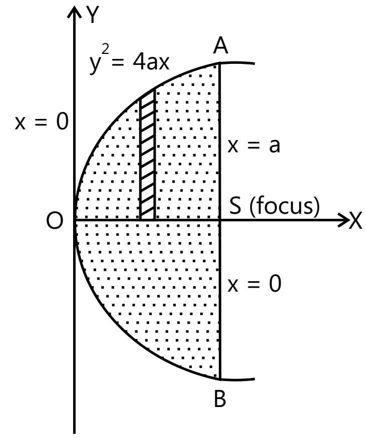

Figure 25.5

Illustration 6: Sketch the region {(x, y): $4x^2 + 9y^2 = 36$} and find its area using integration. **(JEE ADVANCED)**

Sol: The given curve is an ellipse, where a = 3 and b = 2. The X and Y axis divides this ellipse into four equal parts.

Region {(x, y): $4x^2 + 9y^2 = 36$} = Region bounded by $\left(\frac{x^2}{9} + \frac{y^2}{4} = 1\right)$

Limits for the shaded area are x = 0 and x = 3.

∴ The required area of the ellipse

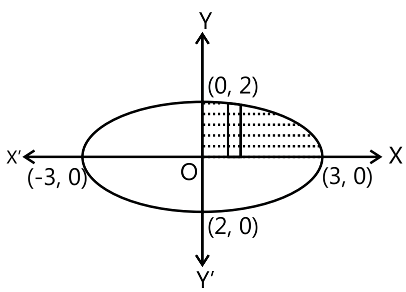

Figure 25.6

$$= 4\int_0^a y\,dx = 4\int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx \quad \left[\because \frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1 - \frac{x^2}{9} \Rightarrow y = 2\sqrt{1-\frac{x^2}{9}}\right]$$

$$= \frac{8}{3}\int_0^3 \sqrt{3^2 - x^2}\,dx = \int_0^3 \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 \quad \left[\text{using} \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{8}{3}\left[0 + \frac{9}{2}\sin^{-1}1 - 0 - 0\right] = \frac{8}{3}\times\frac{9}{2}\times\frac{\pi}{2} = 6\pi \text{ sq. units.}$$

3.2 The Area Bounded by a Curve with y-Axis

The area bound the curve y=f(x) with y-axis between the ordinates

y= a and y=b is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

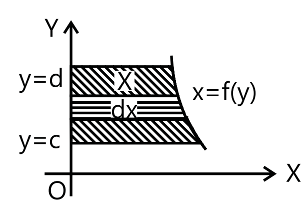

Figure 25.7: Area bounded by a curve with y-axis

Illustration 7: Find the area bounded by the curve $x^2 = \frac{1}{4}y$, y-axis and between the lines y = 1 and y = 4.

(JEE MAIN)

Sol: As we know, area bounded by curve with y – axis is given by $\int_c^d x\,dy = \int_c^d f(y)\,dy$.

Required Area = $\int_1^4 x\,dy = 2\int_1^4 \frac{1}{2}\sqrt{y}\,dy = \frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$ sq. units

Illustration 8: Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line y = 3. **(JEE MAIN)**

Sol: Same as above illustration. $\left(\begin{array}{l}\because y^2 = 4x \\ \frac{y^2}{4} = x\end{array}\right)$

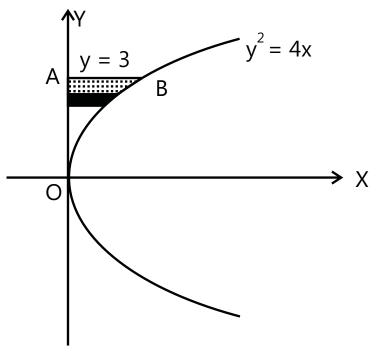

Figure 25.8

Area of region is $A = \int_{y=0}^{y=3} x\,dy = \int_0^3 \frac{y^2}{4}dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{4}\left[\frac{3^3}{3} - \frac{0}{3}\right] = \frac{1}{4}[9] = \frac{9}{4}$ sq. units

Hence, the required area is $\frac{9}{4}$ sq. units.

MASTERJEE CONCEPTS

There is no harm in splitting an integral into multiple components while finding area. If you have any doubt that the integral is changing sign, split the integral at that point.

Vaibhav Gupta (JEE 2009 AIR 54)

3.3 Area of a Curve in Parametric Form

If the given curve is in parametric form say x = f(t), y = g(t), then the area bounded by the curve with x-axis is equal to $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)dt \quad \left[\because dx = d\left(f(t)\right) = f'(t)dt\right]$ Where t_1 and t_2 are the values of t corresponding to the values of a and b of x.

Illustration 9: Find the area bounded by the curve x = a cost, y = b sint in the first quadrant. **(JEE MAIN)**

Sol: Solve it using formula of area of a curve in parametric form.

The given equation is the parametric equation of ellipse, on simplifying we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Required area $= \int_0^a y\,dx = \int_{\pi/2}^0 (b\sin t(-a\sin t)dt) = ab\int_0^{\pi/2} \sin^2 t\,dt = \left(\frac{\pi ab}{4}\right)$.

3.4 Symmetrical Area

If the curve is symmetrical about a line or origin, then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

Illustration 10: Find the area bounded by the parabola $y^2 = 4x$ and its latus rectum. **(JEE MAIN)**

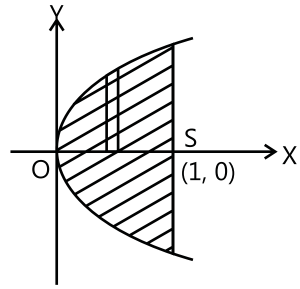

Figure 25.9

Sol: Here the given parabola is symmetrical about x – axis.

Hence required area $= 2\int_0^1 y\,dx$.

Since the curve is symmetrical about x-axis,

$\therefore$ The required Area $= 2\int_0^1 y\,dx = 2\int_0^1 \sqrt{4x}\,dx = 4\cdot\frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{8}{3}$

3.5 Positive and Negative Area

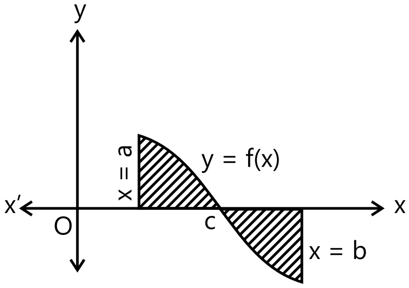

Figure 25.10

The area of a plane figure is always taken to be positive. If some part of the area lies above x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add the numerical values to get the desired area.

If the curve crosses the x-axis at c (see Fig. 25.10), then the area bounded by the curve y = f(x) and the ordinates x = a and x = b, (b > a) is given by

$$A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|; \qquad A = \int_a^c f(x)dx - \int_c^b f(x)\,dx$$

MASTERJEE CONCEPTS

To reduce confusion of using correct sign for the components, take modulus and add all the absolute values of the components.

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Illustration 11: Find the area between the curve y = cos x and x-axis when $\pi/4 < x < \pi$ **(JEE MAIN)**

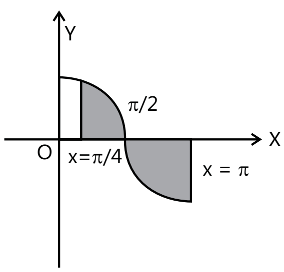


Figure 25.11

Sol: Here some part of the required area lies above x-axis and some part lies below x-axis,. Hence by using above mentioned method we can obtain required area.

$\therefore$ Required area $= \int_{\pi/4}^{\pi/2}\cos x\,dx + \left|\int_{\pi/2}^{\pi}\cos x\,dx\right|$

$= [\sin x]_{\pi/4}^{\pi/2} + |[\sin x]_{\pi/2}^{\pi}| = \left(1 - 1/\sqrt{2}\right) + |0 - 1| = \frac{2\sqrt{2}-1}{\sqrt{2}}$

Illustration 12: Using integration, find the area of the triangle ABC, whose vertices are A (4, 1), B (6, 6) and C (8, 4) **(JEE ADVANCED)**

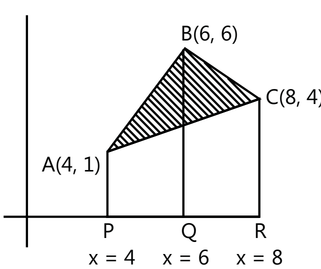


Figure 25.12

Sol: Here by using slope point form we can obtain respective equation of line by which given triangle is made. And after that by using integration method we can obtain required area.

Equation of line AB: $y - 1 = \frac{5}{2}(x-4) \Rightarrow y = \frac{5x}{2} - 9$

Equation of line AC: $y - 1 = \left(\frac{3}{4}\right)(x-4) \Rightarrow y = \frac{3x}{4} - 2$

Equation of line BC: $(y-6)=\left(\frac{-2}{2}\right)(x-6) \Rightarrow y = -x + 12$

$\therefore$ The required area = Area of trapezium ABQP + Area of trapezium BCRQ – Area of trapezium ACRP

$$= \int_4^6\left(\frac{5}{2}x-9\right)dx+\int_6^8(-x+12)dx-\int_4^8\left(\frac{3}{4}x-2\right)dx$$

$$=\left(\frac{5}{4}x^2-9x\right)_4^6+\left(12x-\frac{x^2}{2}\right)_6^8-\left(\frac{3}{8}x^2-2x\right)_4^8 = 7 + 10 - 10 = 7 \text{ sq. units.}$$

3.6 Area between Two Curves

(a) Area enclosed between two curves.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves (where $f_1(x) > f_2(x)$), which intersect at two points, A (x = a) and B(x = b), then the area enclosed by the two curves between A and B is

Common area = $\int_a^b (y_1-y_2)dx = \int_a^b [f_1(x)-f_2(x)]dx$

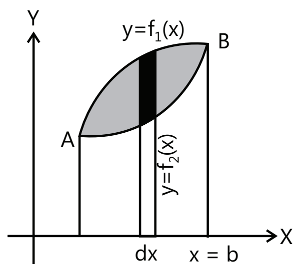


Figure 25.13

Illustration 13: Find the area between two curves $y^2 = 4ax$ and $x^2 = 4ay$. **(JEE MAIN)**

Sol: By using above mentioned formula of finding the area enclosed between two curves, we can obtain required area.

Given, $y^2 = 4ax$ ... (i)

$x^2 = 4ay$ ... (ii)

Solving (i) and (ii), we get x = 4a and y = 4a.

So required area = $\int_0^{4a}\left(\sqrt{4ax}-\frac{x^2}{4a}\right)dx = \left(2\sqrt{a}\,\frac{x^{3/2}}{3/2}-\frac{x^3}{12a}\right)_0^{4a}$

$$= \frac{4\sqrt{a}}{3}\,|4a|^{3/2}-\frac{64a^3}{12a}=\frac{16}{3}a^2$$

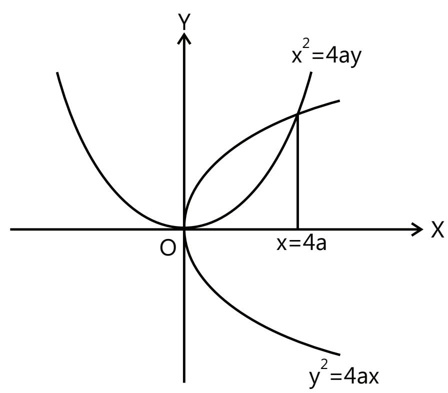


Figure 25.14

(b) Area enclosed by two curves intersecting at one point and the X-axis.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves which intersect at a point P (α, β) and meet x-axis at A (a, 0) and B (b, 0) respectively, then the area enclosed between the curves and x-axis is given by

$$\text{Area} = \int_a^\alpha f_1(x)dx+\int_\alpha^b f_2(x)dx$$

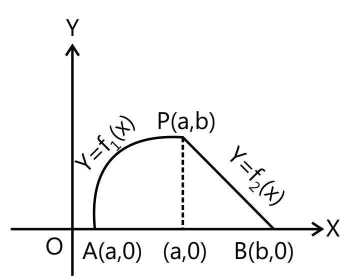


Figure 25.15

(c) Area bounded by two intersecting curves and lines parallel to y-axis.

The area bounded by two curves y = f(x) and y = g(x) (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$, is given by $A = \int_a^b |f(x) - g(x)|\,dx \Rightarrow A = \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

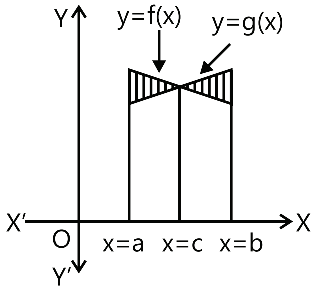


Figure 25.16

Illustration 14: Draw a rough sketch of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$. Using method of integration, find the area of this enclosed region **(JEE ADVANCED)**

Sol: By solving given equations simultaneously, we will be get intersection points of circles and then by using integration method we can obtain required area.

The figure shown alongside is the sketch of the circles

$x^2 + y^2 = 4$ ... (i)

and, $(x-2)^2 + y^2 = 4$ ... (ii)

From (i) and (ii), we have $(x-2)^2 - x^2 = 0$

$\Rightarrow (x - 2 - x)(x - 2 + x) = 0 \quad \Rightarrow x = 1$ ... (iii)

Solving (i) and (iii), we get $y = \pm\sqrt{3}$

Therefore, the circles (i) and (ii) intersect at $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Area of enclosed region = Area OACBO = 2 Area OACO

= 2 [Area OAD + Area ACD]

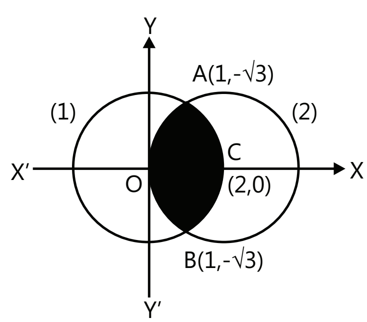


Figure 25.17

$$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= 2\int_1^2 \sqrt{4-x^2}\,dx + 2\int_0^1 \sqrt{4-(x-2)^2}\,dx$$

$$= 2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 + 2\left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 \quad \left[\because \int\sqrt{a^2-x^2}\,dx \Rightarrow \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left(\pi - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right)\right) + 2\left(-\frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \pi\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq. units}$$

Illustration 15: Using integration, find the area of the region given below:

$\{(x, y): 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$ **(JEE ADVANCED)**

Sol: Same as above illustration, by solving given equation $y = x^2 + 1$ and $y = x + 1$ we will be get their points of intersection and after that using integration method and taking these points as limit we can obtain required area.

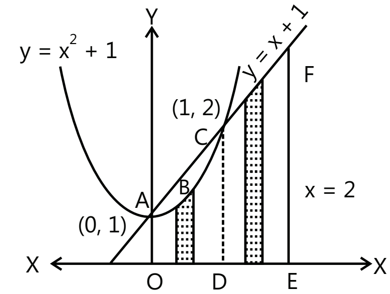


Figure 25.18

The region is shaded as shown in the Fig. 25.18.

Given, $y = x^2 + 1$ ... (i)

$y = x + 1$ ... (ii)

On solving (i) and (ii), we have $x^2 + 1 = x + 1$

$\Rightarrow x = 0,\ 1$ and $y = 1,\ 2$

$\therefore$ The shaded region can be divided into two parts OABCDO and CDEFC.

Limits for the area OABEO are $x = 0$ and $x = 1$.

Limits for the area EBDFE are $x = 1$ and $x = 2$.

Area of the shaded region = Area OABEO + Area EBDFE.

$$= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx = \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2 = \left(\frac{1}{3}+1\right)+\left(\frac{4}{2}+2-\frac{1}{2}-1\right) = \frac{23}{6} \text{ sq. units}$$

Illustration 16: Find the area of the following region: $[(x, y): y^2 \le 4x,\ 4x^2+4y^2 \le 9]$ **(JEE ADVANCED)**

Sol: Similar to above problem, Here the required area is equal to Area AOBA + Area ACBA.

Given $y^2 = 4x$ ... (i)

$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$ ... (ii)

Curves (i) and (ii) intersect at $A\left(\frac{1}{2}, \sqrt{2}\right)$ and $B\left(\frac{1}{2}, -\sqrt{2}\right)$

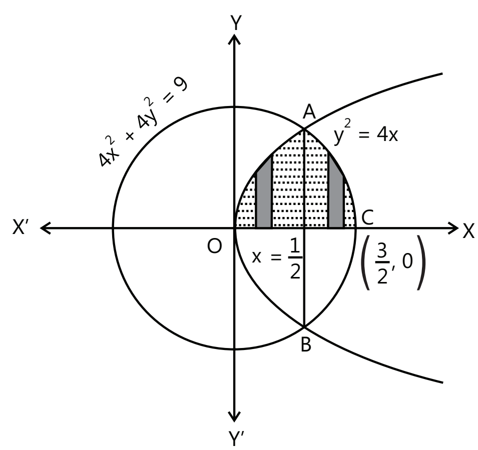


Figure 25.19

Limits for the area OAB are $x = 0,\ x = \frac{1}{2}$

Limits for the area ACB are $x = \frac{1}{2},\ x = \frac{3}{2}$.

The required area = Area AOBA + Area ACBA

$$= 2\left[\int_0^{1/2} y_1\, dx + \int_{1/2}^{3/2} y_2\, dx\right] = 2\left[\int_0^{1/2}\sqrt{4x}\, dx + \int_{1/2}^{3/2}\sqrt{\frac{9}{4}-x^2}\, dx\right]$$

$$= 4\left[\frac{2}{3}x^{3/2}\right]_0^{1/2} + 2\left[\frac{x}{2}\cdot\sqrt{\frac{9}{4}-x^2}+\frac{9}{8}\sin^{-1}\left(\frac{x}{3/2}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} + \left[0-\frac{1}{\sqrt{2}}+\frac{9}{4}\left(\sin^{-1}1-\sin^{-1}\frac{1}{3}\right)\right] = \frac{4}{3\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right) = \frac{1}{3\sqrt{2}}+\frac{9}{4}\cos^{-1}\frac{1}{3}.$$

Illustration 17: Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 8x$ and $x^2 = 8y$, by using method of integration. **(JEE MAIN)**

Sol: As the given two equation is the equation of parabola which intersect at O(0, 0) and A(8, 8), and the required area is equal to Area OBADO – Area OADO.

Given parabolas are $y^2 = 8x$ ... (i)

and, $x^2 = 8y$ ... (ii)

The curves (i) and (ii) intersect at O(0, 0) and A(8, 8).

$\therefore$ Required Area = Area OBADO – Area OADO

$$= \int_0^8 (y_1 - y_2)\,dx$$

$$= \int_0^8 \left(\sqrt{8x} - \frac{x^2}{8}\right)dx = \left[2\sqrt{2}.\frac{x^{3/2}}{3/2} - \frac{1}{8}\frac{x^3}{3}\right]_0^8 = \frac{64}{3} \text{ sq. units.}$$

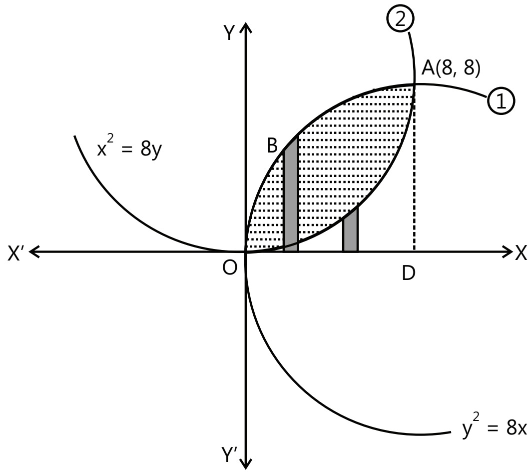


Figure 25.20

Illustration 18: Find the area between the curves y = 2x, x + y = 1 and x-axis. **(JEE MAIN)**

Sol: Here y = 2x and x + y = 1 is a two line intersect at $p\left(\frac{1}{3}, \frac{2}{3}\right)$, therefore using integration method we can obtain required area.

Given $y = 2x$ ... (i)

and, $x + y = 1$ ... (ii)

Solving (i) and (ii), we get $x + 2x = 1 \Rightarrow x = 1/3$.

Line (i) intersects with the x – axis at the origin and the line (ii) intersects with the x – axis at x = 1.

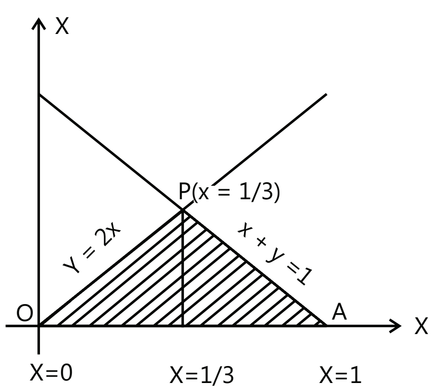


Figure 25.21

So required area = $\int_0^{1/3} 2x\,dx + \int_{1/3}^{1} (1-x)\,dx = \left[x^2\right]_0^{1/3} + \left(x - \frac{x^2}{2}\right)_{1/3}^{1}$

$$= \frac{1}{9} + \left(\frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{18}\right) = \frac{1}{3} \text{ sq. units}$$

Illustration 19: Using the method of integration, find the area of the region bounded by lines: 2x + y = 4, 3x – 2 y = 6 and x – 3y + 5 = 0 **(JEE ADVANCED)**

Sol: Same as above problem.

Given equation of the lines are $2x + y = 4$ ... (i)

$3x - 2y = 6$ ... (ii)

$x - 3y + 5 = 0$ ... (iii)

Solving (i) and (ii), we get (2, 0)

Solving (ii) and (iii), we get (4, 3)

Solving (i) and (iii), we get (1, 2)

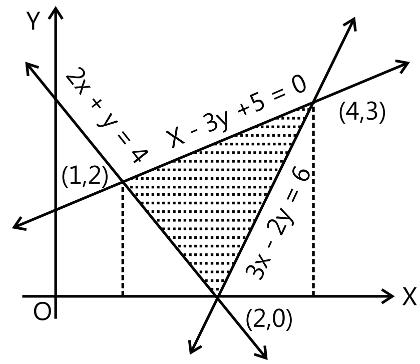


Figure 25.22

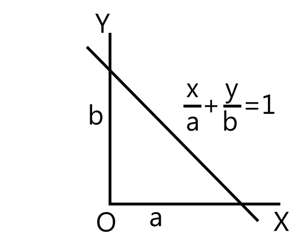
$\therefore$ Required Area $= \int_1^4 \left(\frac{x+5}{3}\right)dx - \int_1^2 (4-2x)\,dx - \int_2^4 \left(\frac{3x-6}{2}\right)dx$

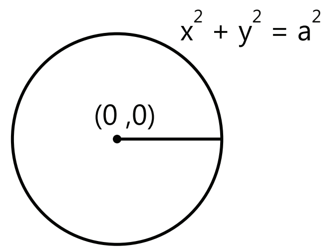
$$= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$$

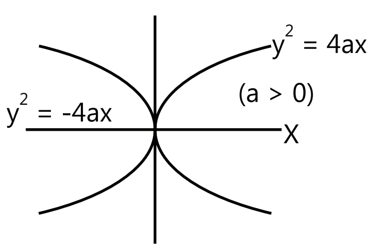
$$= \frac{1}{3}\left[(8+20) - \left(\frac{1}{2}+5\right)\right] - [(8-4)-(4-1)] - \frac{1}{2}[(24-24)-(6-12)]$$

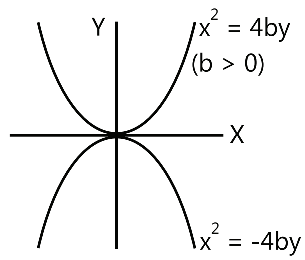
$$= = \frac{7}{2} \text{ sq. units.}$$

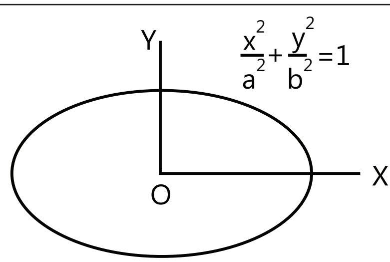
SKETCH OF STANDARD CURVES

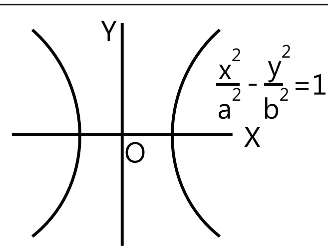


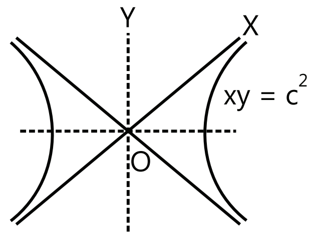
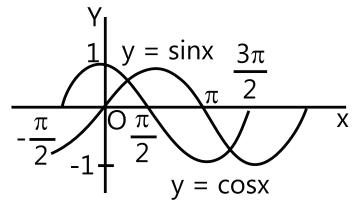
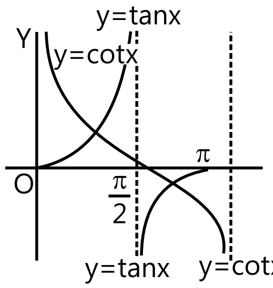
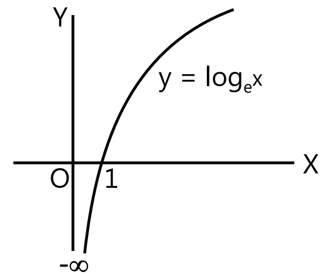
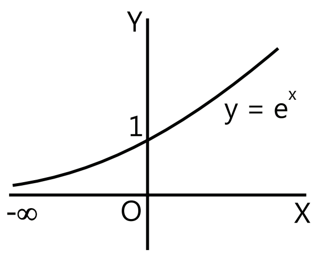










4. STANDARD AREAS

4.1 Area Bounded by Two Parabolas

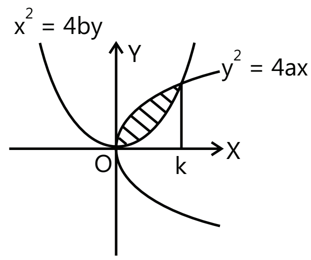


Figure 25.23

Area between the parabolas $y^2 = 4ax$ and $x^2 = 4$; a > 0, b > 0, is

$$|A| = \frac{16ab}{3}$$

Illustration 20: Find the area bounded by $y = \sqrt{x}$ and $x = \sqrt{y}$. **(JEE MAIN)**

Sol: By using above mentioned formula.

Area bounded is shaded in the figure

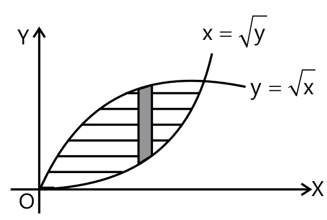


Figure 25.24

Here, $a = \frac{1}{4}$ and $b = \frac{1}{4}$

$\therefore$ Using the above formula, Area = (16 ab)/ 3

$$= \frac{16 \times (1/4) \times (1/4)}{3} = \frac{1}{3}$$

4.2 Area Bounded By Parabola and a Line

Area bounded by $y^2 = 4ax$ and $y = mx$; $a > 0$, $m > 0$ is $A = \frac{8a^2}{3m^3}$

Area bounded by $x^2 = 4ay$ and $y = mx; a > m > 0$

is $y = mx; a > m > 0$ $A = \frac{8a^2}{3m^3}$

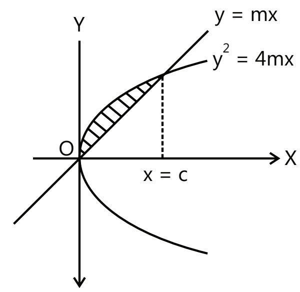


Figure 25.25

Illustration 21: Find the area bounded by, $x^2 = y$ and $y = |x|$. **(JEE MAIN)**

Sol: Using above formula, i.e. $A = \frac{8a^2}{3m^3}$

Area bounded is shaded in the Fig. 25.26.

Here, a = 1/4, m = 1

$\therefore$ Using the above formula, Area = $2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times \left(\frac{1}{4}\right)^2}{3 \times (1)^3} = \frac{1}{3}$

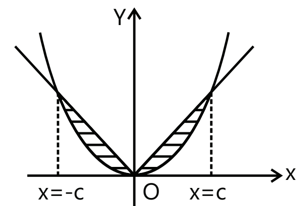


Figure 25.26

Illustration 22: Find the area bounded by $y^2 = x$ and $x = |\, y \,|$. **(JEE MAIN)**

Sol: Here, a = 1/4, m = 1, and required area is divided in to two equal parts at above and below x – axis.

Hence required area will be $2\left(\frac{8a^2}{3m^3}\right)$.

$\therefore$ Using the above formula, Area = $2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times (1/4)^2}{3 \times (1)^3} = \frac{1}{3}$

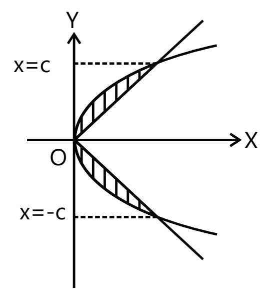


Figure 25.27

4.3 Area Enclosed by Parabola and It's Chord

Area between $y^2 = 4ax$ and its double ordinate at x = a is

Area of AOB = $\frac{2}{3}$ (area ▱ABCD)

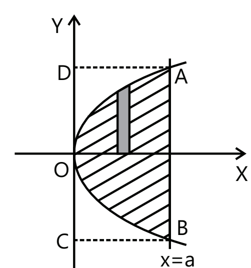


Figure 25.28

Illustration 23: Find the area bounded by $y = 2x - x^2$, $y + 3 = 0$. **(JEE MAIN)**

Sol: Here first obtain area of rectangle ABCD and after that by using above mentioned formula we will be get required area.

Solving $y = 2x - x^2$, $y + 3 = 0$, we get x = –1 or 3

Area (ABCD) = 4 × 4 = 16.

$\therefore$ Required area = $\frac{2}{3} \times 16 = \frac{32}{3}$

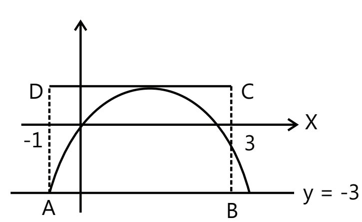


Figure 25.29

4.4 Area of an Ellipse

For an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $A = \pi ab$

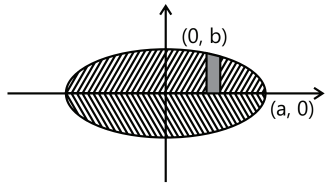


Figure 25.30

MASTERJEE CONCEPTS

Try to remember some standard areas like for ellipse, parabola. These results are sometimes very helpful.

Vaibhav Gupta (JEE 2009 AIR 54)

5. SHIFTING OF ORIGIN

Area remains unchanged even if the coordinate axes are shifted or rotated or both. Hence shifting of origin / rotation of axes in many cases proves to be very convenient in finding the area.
For example: If we have a circle whose centre is not origin, we can find its area easily by shifting circle's centre.

Illustration 24: The line $3x + 2y = 13$ divides the area enclosed by the curve $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area. **(JEE ADVANCED)**

Sol: Given $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ ... (i)

and, $3x + 2y = 13$ ... (ii)

$9(x^2 - 2x) + 4(y^2 - 4y) = 11;$

$\Rightarrow 9[(x-1)^2 - 1] + 4[(y-2)^2 - 4] = 11$

$\Rightarrow 9(x-1)^2 + 4(y-2)^2 = 36$

$\Rightarrow \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1 \Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$ (where $X = x - 1$ and $Y = y - 2$)

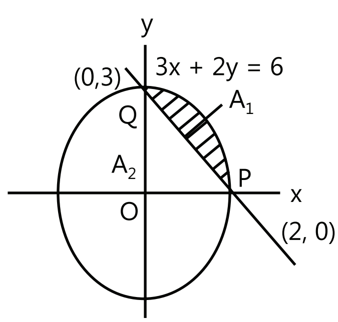


Figure 25.31

Hence $3x + 2y = 13$

$\Rightarrow 3(X + 1) + 2(Y + 2) = 13$

$\Rightarrow 3X + 2Y = 6$

$\Rightarrow \frac{X}{2}+\frac{Y}{3}=1$

$\therefore$ Area of triangle OPQ = $1/2 \times 2 \times 3 = 3$

Also area of ellipse = π (semi major axes) (semi minor axis) = $\pi.3.2 = 6\pi$

$$A_1=\frac{6\pi}{4}-\text{area of }\Delta OPQ=\frac{3\pi}{2}-3$$

$$A_2=3\left(\frac{6\pi}{4}\right)+\text{area of }\Delta OPQ=\frac{9\pi}{2}+3$$

Hence, $$\frac{A_2}{A_1}=\frac{\frac{9\pi}{2}+3}{\frac{3\pi}{2}-3}=\frac{3\pi+2}{\pi-2}$$

6. DETERMINATION OF PARAMETERS

In this type of questions, you will be given area of the curve bounded between some axes or points, and some parameter(s) will be unknown either in equation of curve or a point or an axis. You have to find the value of the parameter by using the methods of evaluating area.

Illustration 25: Find the value of c for which the area of the figure bounded by the curves $y=\frac{4}{x^2}$; x = 1 and y = c is equal to $\frac{9}{4}$. **(JEE MAIN)**

Sol: By using method of evaluating area we can find out the value of c.

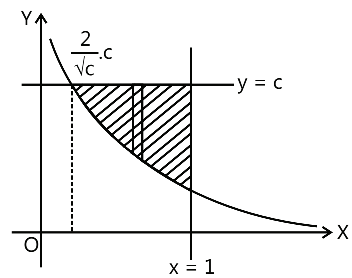


Figure 25.32

$$A=\int_{2/\sqrt{c}}^{1}\left(c-\frac{4}{x^2}\right)dx=\frac{9}{4};\quad \left(cx+\frac{4}{x}\right)_{\frac{2}{\sqrt{c}}}^{1}=\frac{9}{4}$$

$$(c+4)-(2\sqrt{c}+2\sqrt{c})=\frac{9}{4};\ c-4\sqrt{c}+4=\frac{9}{4}$$

$$\Rightarrow\ (\sqrt{c}-2)^2=\frac{9}{4}\Rightarrow(\sqrt{c}-2)=\frac{3}{2}\text{ or }-\frac{3}{2}$$

Hence c = (49/4) or (1/4)

Illustration 26: Consider the two curves:

C_1 : y = 1 + cos x, and C_2 : y = 1 + cos(x – α) for α ∈ (0, π/2) and x ∈ [0, p].

Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and x = 0 is same as that of the area bounded by C_2, y = 1 and x = π. For this value of α, find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 and x = π. **(JEE ADVANCED)**

Sol: Solve C_1 and C_2 to obtain the value of x, after that by following given condition we will be obtain required value of α.

Solving C_1 and C_2 , we get

$$1+\cos x=1+\cos(x-\alpha)\Rightarrow\ x=\alpha-x\quad\Rightarrow x=\frac{\alpha}{2}$$

According to the question,

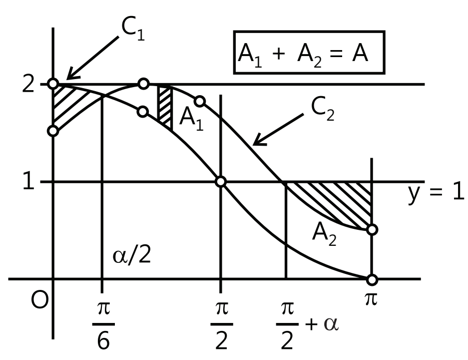


Figure 25.33

$$\int_0^{\alpha/2}(\cos x-\cos(x-\alpha))\,dx=-\int_{\frac{\pi}{2}+\alpha}^{\pi}(\cos(x-\alpha))\,dx$$

$$\Rightarrow\ \left[\sin x-\sin(x-\alpha)\right]_0^{\alpha/2}=\left[\sin(x-\alpha)\right]_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow\ \left[\sin\frac{\alpha}{2}-\sin\left(-\frac{\alpha}{2}\right)\right]-[0-\sin(-\alpha)]\ =\ \sin\left(\frac{\pi}{2}\right)-\sin(\pi-\alpha)$$

$$\Rightarrow\ 2\sin\frac{\alpha}{2}-\sin\alpha=1-\sin\alpha\text{ . Hence, }2\sin\frac{\alpha}{2}=1\qquad\Rightarrow\ \alpha=\frac{\pi}{3}$$

7. AREA BOUNDED BY THE INVERSE FUNCTION

The area of the region bounded by the inverse of a given function can also be calculated using this method. The graph of inverse of a function is symmetric about the line y = x. We use this property to calculate the area. Hence, area of the function between x = a to x = b, is equal to the area of inverse function from f(a) to f(b).

Illustration 27: Find the area bounded by the curve g(x), the x-axis and the lines at y = –1 and y = 4, where g(x) is the inverse of the function $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$. **(JEE MAIN)**

Sol: Here f(x) is a strictly increasing function therefore required area will be

$$A=\int_0^2 (4-f(x))dx+\int_{-2}^{0}(f(x)+1)dx$$

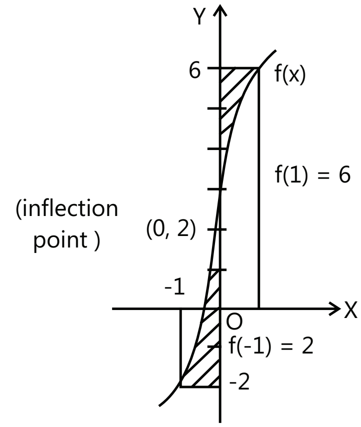


Figure 25.34

Given $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$

$\Rightarrow$ f(0) = 1; f(2) = 4 and f(–2) = –1

Also, $f'(x)=\frac{x^2}{8}+\frac{x}{4}+\frac{13}{12}$,

i.e. f(x) is a strictly increasing function.

$$\therefore A=\int_0^2 (4-f(x))dx+\int_{-2}^{0}(f(x)+1)dx$$

$$A=\int_0^2\left(4-\frac{x^3}{24}-\frac{x^2}{8}-\frac{13x}{12}-1\right)dx+\int_{-2}^{0}\left(\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1+1\right)dx$$

$$\therefore A=\left[\left(3.2-\frac{2^4}{24.4}-\frac{2^3}{8.3}-\frac{13.2^2}{12.2}\right)-(0)\right]+\left[(0)-\left(\frac{2^4}{24.4}-\frac{2^3}{8.3}+\frac{13.2^2}{12.2}-2.2\right)\right]=\frac{16}{3}$$

Illustration 28: Let f(x) = x^3 + 3x + 2 and g(x) is the inverse of it. Find the area bounded by g(x), the x-axis and the ordinate at x = –2 and x = 6. **(JEE ADVANCED)**

Sol: Let $A=\int_{-2}^{6}\left|f^{-1}(x)\right|dx$

Substitute $x=f(u)$ or $u=f^{-1}(x)$

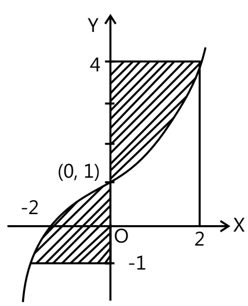


Figure 25.35

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|u|\,f^{-1}(u)\,du$$

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|4|\left(3u^2+3\right)du$$

We have, $f(-1)=2$ and $f(1)=6$

$$=\int_{-1}^{1}|u|\left(3u^2+3\right)du=2\int_0^1\left(3u^3+3u\right)du$$

$$=\left[\frac{3}{2}u^4+3u^2\right]_0^1=\frac{9}{2}\text{ Sq. units.}$$

8. VARIABLE AREA

If y = f(x) is a monotonic function in (a, b), then the area of the function y = f(x) bounded by the lines at x = a, x = b, and the line y = f(c), [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

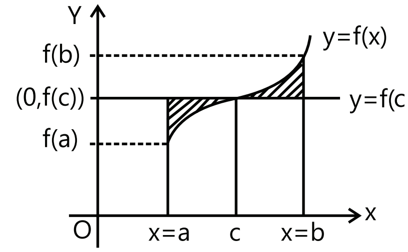


Figure 25.36

Proof: A = $\int_a^c f(c) - f(x)dx + \int_c^b (f(x) - f(c))dx$

$= f(c)\,(c-a) - \int_a^c (f(x))dx + \int_c^b (f(x))dx - f(c)(b-c)$

$= \{(c-a) - (b-c)\}\, f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

$A = [2c - (a+b)]\ f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

For maxima and minima $\frac{dA}{dc} = 0 \Rightarrow$ f ′ (c) = [2c – (a + b)] = 0 (as f ′ (c) = 0) hence $c = \frac{a+b}{2}$ also $c < \frac{a+b}{2}$, $\frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}$, $\frac{dA}{dc} > 0$ Hence A is minimum when c = $\frac{a+b}{2}$

9. AVERAGE VALUE OF A FUNCTION

In this section, we would study the average of a continuous function. This concept of average is frequently applied in physics and chemistry.

Average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

MASTERJEE CONCEPTS

(a) Average value can be positive, negative or zero .

(b) If the function is defined in (0, ∞), then $y_{av} = \lim_{b \to \infty} \frac{1}{b}\int_0^b f(x)dx$ provided the limit exists

(c) Root mean square value (RMS) is defined as $\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)\,dx\right]^{\frac{1}{2}}$

(d) If a function is periodic then we need to calculate average of function in particular time period that is its overall mean.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 29: Find the average value of y^2 w.r.t. x for the curve ay = b$\sqrt{a^2 - x^2}$ between x = 0 & x = a. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$. **(JEE MAIN)**

Sol: As average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Let $f(x) = y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ Now $f(x)|_{av} = \frac{b^2}{a^2(a-0)}\int_0^a (a^2 - x^2)dx = \frac{2b^2}{3}$

Again y_{av} w.r.t. x^2 as $f(x)|_{av} = \frac{1}{(a^2-0)}\int_0^{a^2} y\, d(x^2) = \frac{b}{a^2 a}\int_0^{a^2}\sqrt{a^2 - x^2}\, dx^2 = \frac{b}{a^3}\int_0^{a^2} 2t^2\, dt = \frac{2ba^3}{3}$

10. DETERMINATION OF FUNCTION

Sometimes the area enclosed by a curve is given as a variable function and we have to find the function. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$ i.e. derivative of the area function is the function itself. Thus we can easily find f(x) by differentiating area function.

MASTERJEE CONCEPTS

If F(x) is integral of f(x) then, $A_a^x = \int f(x)dx = [F(x) + c]$

And since, $A_a^a = 0 = F(a) + c$ $\Rightarrow c = -F(a)$.

$\therefore$ $A_a^x = F(x) - F(a)$. Finally by taking x = b we get, $A_a^b = F(b) - F(a)$

Note that this is true only if the function doesn't have any zeroes between a and b.

If the function has zero at c then area = |F(b) – F(c)| + |F(c) – F(a)|

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 30: The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1,\ x \ge 0$;

(a) Find the average rate of change of A w.r.t. x and x increases from 1 to 8.

(b) Find the instantaneous rate of change of A w.r.t. x at x = 5.

(c) Find the ordinate (height) y of the graph as a function of x.

(d) Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8. **(JEE ADVANCED)**

Sol: Here by differentiating given area function we can obtain the main function.

(a) A(1) = 1, A(8) = 4 ; $\frac{A(8) - A(1)}{8-1} = \frac{3}{7}$

(b) $\left.\frac{dA}{dx}\right|_{x=5} = \left.\frac{1.\,3}{2\sqrt{1+3x}}\right|_{x=5} = \frac{3}{8}$

(c) $y = \frac{3}{2\sqrt{1+3x}}$

(d) $\frac{1}{(8-1)}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{1}{7}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{3}{7}$

Illustration 31: Let C_1 & C_2 be the graphs of the function $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graphs of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P(on C_1), the area of the shaded regions OPQ & ORP are equal, determine the function f(x). **(JEE ADVANCED)**

Sol: Similar to the above mentioned method.

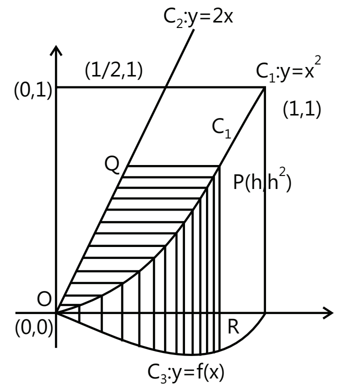


Figure 25.37

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy=\int_0^{h}(x^2-f(x))\,dx \text{ differentiate both sides w.r.t. h}$$

$$\left(h-\frac{h^2}{2}\right)2h = h^2 - f(h)$$

$$f(h) = h^2 - \left(h-\frac{h^2}{2}\right)2h$$

$$= h^2 - h(2h - h^2) = h^2 - 2h^2 + h^3$$

$$f(h) = h^3 - h^2$$

$$f(x) = x^3 - x^2 = x^2(x - 1)$$

11. AREA ENCLOSED BY A CURVE EXPRESSED IN POLAR FORM

$r = a(1 + \cos\theta)$ (Cardioid)

$$A=\frac{1}{2}\int_0^{2\pi} r^2 d\theta=\frac{a^2}{2}\int_0^{2\pi} 4\cos^4\frac{\theta}{2}d\theta$$

Substitute $\frac{\theta}{2}=t$, $d\theta = 2dt$

$$A=a^2\int_0^{\pi}4\cos^4 t\,dt=8\times\frac{3\pi a^2}{16}$$

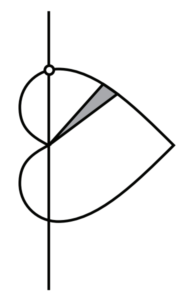


Figure 25.38

Illustration 32: Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$. **(JEE MAIN)**

Sol:

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \quad \text{and } dx=3a\sin^2 t.\cos t.dt$$

$$A=4\int_0^a y\,dx \;;\; A=4a^2\int_0^{\pi/2}3\cos^3 t\sin^2 t\cos t\,dt$$

$$A=12a^2\int_0^{\pi/2}\sin^2 t\cos^4 t\,dt=(12a^2)\,.\,\frac{1.3.1}{6.4.2}\,.\,\frac{\pi}{2}=\frac{12a^2\pi}{32}=\frac{3\pi a^2}{8}$$

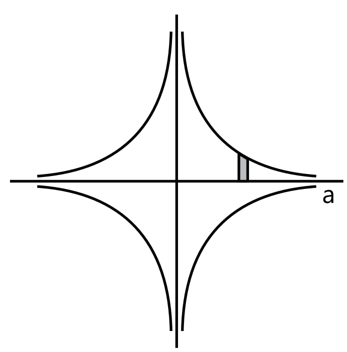


Figure 25.39

Linear Programming

1. INTRODUCTION

Linear Programming was developed during World War II, when a system with which to maximize the efficiency of resources was of utmost importance.

2. LINEAR PROGRAMMING

Linear programming may be defined as the problem of maximising or minimising a linear function subject to linear constraints. The constraints may be equalities or inequalities. Here is an example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$x_1 + 2x_2 \leq 4$$

$$4x_1 + 2x_2 \leq 12$$

$$-x_1 + x_2 \leq 1$$

Here we have two unknowns and five inequalities (constraints). Notice that these constraints are all linear functions of the variables. The first two constraints, $x_1 \geq 0$ and $x_2 \geq 0$, are special. These are called no negativity constraints and are often found in linear programming problems. The other constraints are called the main constraints. The function to be maximised (or minimized) is called the objective function. In the above example the objective function is $x_1 + x_2$.

3. GRAPHICAL METHOD

As we have only two variables, we can solve this problem by plotting the constraints with x_1 and x_2 as axes. The intersection region of these inequalities is called feasible region for the objective function. This is the region which satisfies all the constraints. Now from this feasible region we have to select point(s) such that objective function is maximized or minimized.

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let Z=ax + by be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem and let Z=ax + by be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at corner point (vertex) of R.

Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists it must occur at a corner point of R. (By Theorem 1).

So for the above example

Corner point (x_1, x_2)	Z (= x_1 + x_2) value
0,1	1
3,0	3
8/3,2/3	10/3
2/3,5/3	7/3

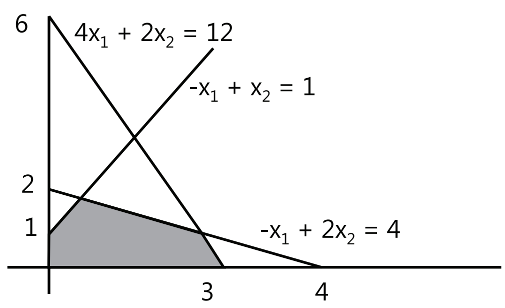


Figure 25.40

Hence (8/3, 2/3) is the optimal solution.

Note that z has also minimum value in the feasible region at (0, 1).

This method of solving is generally called as corner point method. Note that a function can have more than one optimal points.

4. MODELS

There are few important linear programming models which are more frequently used and some of them we encounter in our daily lives.

(a) Manufacturing/Assignment problems: In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of product, warehouse space per unit of the output. In order to make maximum profit.

Example: There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij}, for i = 1,......,I, and j = 1,.....,J. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers, x_{ij}, for i = 1,......,I, and j=1,......,J, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \le 1 \quad \text{For } i = 1,......, I \qquad ...\text{(i)}$$

$$\sum_{i=1}^{I} x_{ij} \le 1 \quad \text{For } j=1,......, J \qquad ...\text{(ii)}$$

And $\qquad x_{ij} \ge 0 \qquad$ for i = 1,......,I, and j=1,......,J $\qquad$ ... (iii)

Equation (i) reflects the fact that a person cannot spend more than 100% of his time working, (ii) means that only one person is allowed on a job at a time, and (iii) says that no one can work a negative amount of time on any job, Subject to (i), (ii) and (iii) , we wish to maximize the total value of $\sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}x_{ij}$

(b) Diet problems: In these problems, we determine the amount of different kinds of nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each nutrients.

Example: There are m different types of food, $F_1,........,F_m$, that supply varying quantities of the n nutrients , $N_1,........,N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1y_1 + b_2y_2 + ... + b_my_m \qquad ...\text{(i)}$$

The amount of nutrient N_j contained in this diet is

$$a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m$$

For j = 1,........, n. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m \ge c_j \qquad \text{For } j = 1,........, n \qquad ...\text{(ii)}$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \ge 0,\ y_2 \ge 0, ..., y_m \ge 0 \qquad ...\text{(iii)}$$

Our problem is: minimize (i) subject to (ii) and (iii). This is exactly the standard minimum problem.

(c) Transportation problems: In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Example: There are I ports, or production plants, $P_1,.........P_I$, that supply a certain commodity, and there are J markets, $M_1,.........M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity (i=1,2,......I), and market M_j must receive the amount r_j of the commodity (j = 1,........J). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost is

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}b_{ij} \qquad \text{... (i)}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij} \le y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \text{ for } i = 1,......,I \qquad \text{... (ii)}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \le r_j \text{ for } j = 1,......,I \qquad \text{... (iii)}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \quad \text{for } I = 1,........I \text{ and } j = 1,........J. \qquad \text{... (iv)}$$

Our problem is minimize (i) subject to (ii), (iii) and (iv).

FORMULAE SHEET

(a) Area bounded by a curve with x – axis: Area = $\int_a^b y\,dx = \int_a^b f(x)\,dx$

(b) Area bounded by a curve with y – axis: Area = $\int_c^d x\,dy = \int_c^d f(y)\,dy$

(c) Area of a curve in parametric form: Area = $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)\,dt$

(d) Positive and Negative Area: $A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$;

(e) Area between two curves:

(i) Area enclosed between two curves intersecting at two different points.

Area = $\int_a^b (y_1 - y_2)dx = \int_a^b [f_1(x) - f_2(x)]dx$

(ii) Area enclosed between two curves intersecting at one point and the x – axis.

Area = $\int_a^{\alpha} f_1(x)dx + \int_{\alpha}^{b} f_2(x)dx$

(iii) Area bounded by two intersecting curves and lines parallel to y – axis.

Area = $\int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$

(a) Standard Areas:

(i) Area bounded by two parabolas $y^2 = 4ax$ and $x^2 = 4by$; $a > 0$, $b > 0$: Area $= \dfrac{16ab}{3}$

(ii) Area bounded by Parabola $y^2 = 4ax$ and Line $y = mx$: Area $= \dfrac{8a^2}{3m^3}$

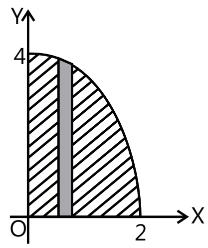
(iii) Area of an Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: Area $= \pi ab$

Solved Examples

JEE Main/Boards

Example 1: Find area bounded by $y = 4 - x^2$, x-axis and the lines $x = 0$ and $x = 2$.

Sol: By using the formula of Area Bounded by the x – axis, we can obtain



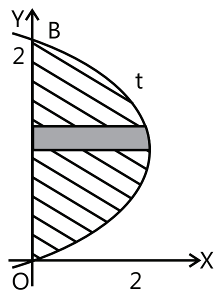
Required Area.

$$= \int_0^2 y\,dx = \int_0^2 (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \text{ sq. units}$$

Example 2: Find the area bounded by the curve $y^2 = 2y - x$ and the y-axis.

Sol: Here given equation is the equation of parabola with vertex (1, 1) and curve passes through the origin.



Curve is $y^2 - 2y = -x$ or $(y - 1)^2 = -(x - 1)$

It is a parabola with

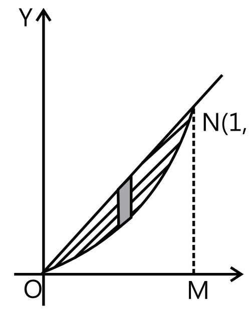
Vertex at (1, 1) and the curve passes through the origin. At B, $x = 0$ and $y = 2$

Area

$$= \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = \left(y^2 - \frac{y^3}{3}\right)_0^2 = \frac{4}{3} \text{ sq. units}$$

Example 3: Find the area of the region $\{(x, y): x^2 \le y \le x\}$

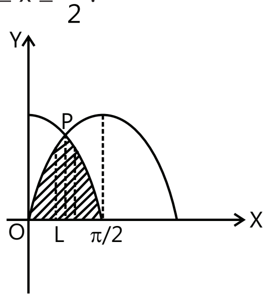
Sol: Consider the function $y = x^2$ and $y = x$ Solving them, we get $x = 0$, $y = 0$ and $x = 1$, $y = 1$;$x^2 \le y \Rightarrow$ area is above the curve $y = x^2$ $y \le x \Rightarrow$ area is below the line $y = x$



$$\text{Area} = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6} \text{ sq. units}$$

Example 4: Find the area of the region enclosed by $y = \sin x$, $y = \cos x$ and x-axis, $0 \le x \le \dfrac{\pi}{2}$.

Sol: Find point of intersection is P. Therefore after obtaining the co-ordinates of P and then integrating with appropriate limits, we can obtain required Area.



At point of intersection P, $x = \dfrac{\pi}{4}$ as ordinates of $y = \sin x$ and ; $y = \cos x$ are equal

Hence, P is $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ Required area

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = \left(-\cos x\right)_0^{\pi/4} + \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units}$$