(c) Transportation problems: In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Example: There are I ports, or production plants, $P_1, ...P_I$, that supply a certain commodity, and there are J markets, $M_1, ...M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity (i=1,2,......I), and market M_j must receive the amount r_j of the commodity (j = 1,........J). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost is

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij} b_{ij} \quad ... (i)$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij} \le y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \text{ for } i = 1,......,I \quad ... (ii)$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \le r_j \text{ for } j = 1,......,I \quad ... (iii)$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \quad \text{for } I = 1,........I \text{ and } j = 1,........J. \quad ... (iv)$$

Our problem is minimize (i) subject to (ii), (iii) and (iv).

FORMULAE SHEET

(a) Area bounded by a curve with x – axis: Area = $\int_a^b y\,dx = \int_a^b f(x)\,dx$

(b) Area bounded by a curve with y – axis: Area = $\int_c^d x\,dy = \int_c^d f(y)\,dy$

(c) Area of a curve in parametric form: Area = $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t) f'(t)\,dt$

(d) Positive and Negative Area: $A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$;

(e) Area between two curves:

(i) Area enclosed between two curves intersecting at two different points.

Area = $\int_a^b (y_1 - y_2)\,dx = \int_a^b [f_1(x) - f_2(x)]\,dx$

(ii) Area enclosed between two curves intersecting at one point and the x – axis.

Area = $\int_a^{\alpha} f_1(x)\,dx + \int_{\alpha}^{b} f_2(x)\,dx$

(iii) Area bounded by two intersecting curves and lines parallel to y – axis.

Area = $\int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

(a) Standard Areas:

(i) Area bounded by two parabolas $y^2 = 4ax$ and $x^2 = 4by$; $a > 0, b > 0$: Area $= \frac{16ab}{3}$

(ii) Area bounded by Parabola $y^2 = 4ax$ and Line $y = mx$: Area $= \frac{8a^2}{3m^3}$

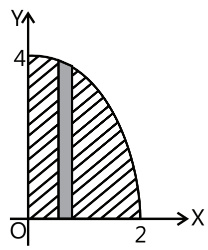
(iii) Area of an Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Area = πab

Solved Examples

JEE Main/Boards

Example 1: Find area bounded by $y = 4 - x^2$, x-axis and the lines $x = 0$ and $x = 2$.

Sol: By using the formula of Area Bounded by the x – axis, we can obtain



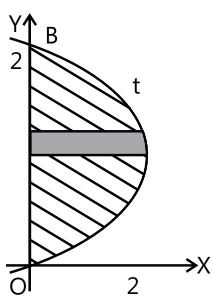
Required Area.

$$= \int_0^2 y\,dx = \int_0^2 (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \text{ sq. units}$$

Example 2: Find the area bounded by the curve $y^2 = 2y - x$ and the y-axis.

Sol: Here given equation is the equation of parabola with vertex (1, 1) and curve passes through the origin.



Curve is $y^2 - 2y = -x$ or $(y - 1)^2 = -(x - 1)$

It is a parabola with

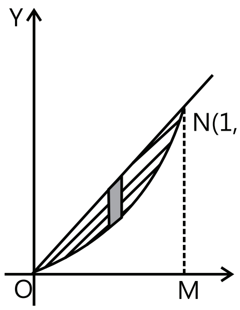
Vertex at (1, 1) and the curve passes through the origin. At B, $x = 0$ and $y = 2$

Area

$$= \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = \left(y^2 - \frac{y^3}{3}\right)_0^2 = \frac{4}{3} \text{ sq. units}$$

Example 3: Find the area of the region $\{(x, y): x^2 \le y \le x\}$

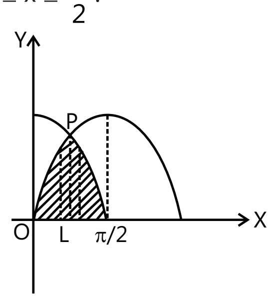
Sol: Consider the function $y = x^2$ and $y = x$ Solving them, we get $x = 0$, $y = 0$ and $x = 1$, $y = 1$;$x^2 \le y \Rightarrow$ area is above the curve $y = x^2$ $y \le x \Rightarrow$ area is below the line $y = x$



$$\text{Area} = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6} \text{ sq. units}$$

Example 4: Find the area of the region enclosed by $y = \sin x$, $y = \cos x$ and x-axis, $0 \le x \le \frac{\pi}{2}$.

Sol: Find point of intersection is P. Therefore after obtaining the co-ordinates of P and then integrating with appropriate limits, we can obtain required Area.



At point of intersection P, $x = \frac{\pi}{4}$ as ordinates of $y = \sin x$ and ; $y = \cos x$ are equal

Hence, P is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ Required area

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = \left(-\cos x\right)_0^{\pi/4} + \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units}$$