12. HYPERBOLA

1. INTRODUCTION

A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.

The fixed point is called the focus, the fixed line is called the directrix. The constant ratio is generally denoted by e and is known as the eccentricity of the hyperbola. A hyperbola can also be defined as the locus of a point such that the absolute value of the difference of the distances from the two fixed points (foci) is constant. If S is the focus, ZZ' is the directrix and P is any point on the hyperbola as show in figure.

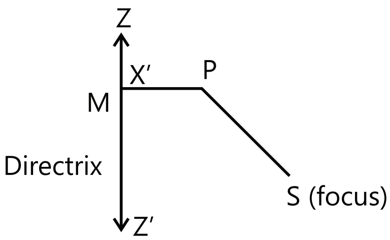


Figure 12.1

Then by definition, we have $\frac{SP}{PM} = e\ (e > 1)$.

Note: The general equation of a conic can be taken as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

This equation represents a hyperbola if it is non-degenerate (i.e. eq. cannot be written into two linear factors)

$\Delta \neq 0$, $h^2 > ab$. Where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

MASTERJEE CONCEPTS

1. The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Degeneracy condition depends on the determinant of the 3x3 matrix and the type of conic depends on the determinant of the 2x2 matrix.

2. Also the equation can be taken as the intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

Vaibhav Gupta (JEE 2009, AIR 54)

2. STANDARD EQUATION OF HYPERBOLA

Let the center O of the hyperbola be at the origin O and the foci F_1 and F_2 be on the x-axis.

The coordinates of foci F_1 and F_2 are (–c, 0) and (c, 0).

By the definition of hyperbola,

Distance between a point P and focus F_1 – Distance between P and focus F_2 = constant (say 2a)

$PF_1 - PF_2 = 2a; \sqrt{(x+c)^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = 2a$

$\Rightarrow \quad \sqrt{(x+c)^2+(y)^2} = 2a + \sqrt{(x-c)^2+(y)^2}$

Squaring both the sides, we get $\quad (x + c)^2 + y^2 = 4a^2 + 2(2a).\sqrt{(x-c)^2+(y)^2} + (x - c)^2 + y^2$

$\Rightarrow \quad x^2 + 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} + x^2 - 2cx + c^2 + y^2$

$\Rightarrow \quad 4cx = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx = a^2 + a.\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx - a^2 = a\sqrt{(x-c)^2+(y)^2}$

Squaring again, we get

$c^2x^2 - 2a^2cx + a^4 = a^2[(x - c)^2 + y^2]$

$c^2x^2 - 2a^2cx + a^4 = a^2[x^2 - 2cx + c^2 + y^2] = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$\Rightarrow \quad (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad$ (taking $b^2 = c^2 - a^2$)

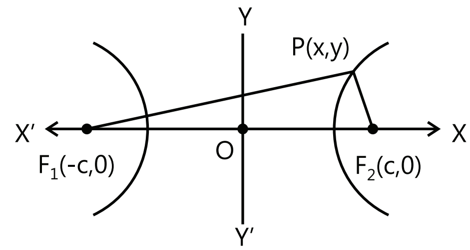


Figure 12.2

Hence, any point P(x, y) on the hyperbola satisfies the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. TERMS ASSOCIATED WITH HYPERBOLA

(a) Focus: The two fixed points are called the foci of the hyperbola and are denoted by F_1 and F_2. The distance between the two foci F_1 and F_2 is denoted by 2c.

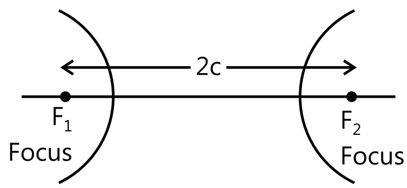


Figure 12.3

(b) Centre: The midpoint of the line joining the foci is called the center of the hyperbola.

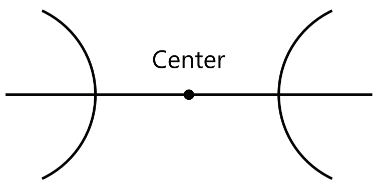


Figure 12.4

(c) Transverse-Axis: The line through the foci is called the transverse axis. Length of the transverse axis is 2a.

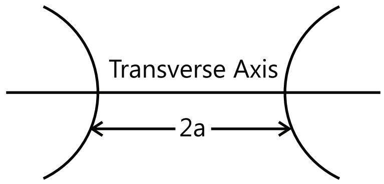


Figure 12.5

(d) Conjugate-Axis: The line segment through the center and perpendicular to the transverse axis is called the conjugate axis. Length of the conjugate axis is 2b.

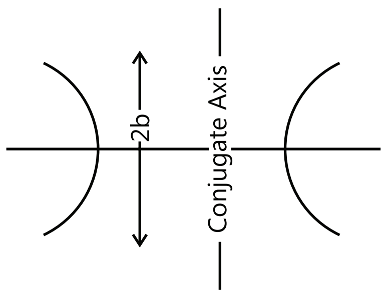


Figure 12.6

(e) Vertices: The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola. The distance between the two vertices is denoted by 2a.

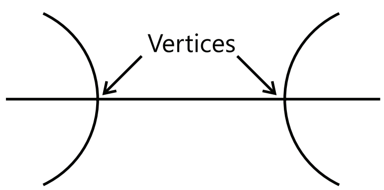


Figure 12.7

(f) Eccentricity: Eccentricity of the hyperbola is defined as $\frac{c}{a}$ and it is denoted by e. And e is always greater than 1 since c is greater than 1.

(g) Directrix: Directrix is a line perpendicular to the transverse axis and cuts it at a distance of $\frac{a^2}{c}$ from the centre.

i.e. $x = \pm\frac{a^2}{c}$ or $y = \pm\frac{a^2}{c}$

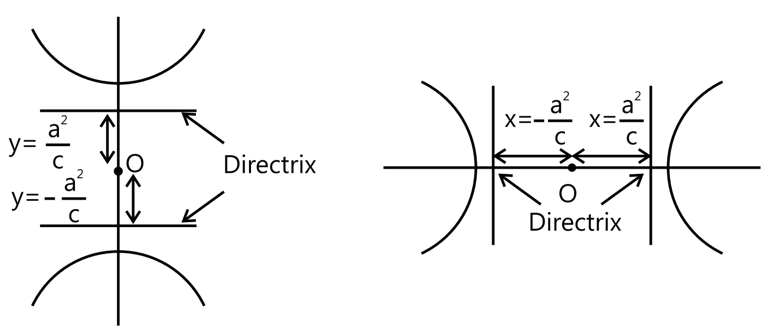


Figure 12.8

(h) Length of The Latus Rectum: The Latus rectum of a hyperbola is a line segment perpendicular to the transverse axis and passing through any of the foci and whose end points lie on the hyperbola. Let the length of LF be ℓ. Then, the coordinates of L are (c, ℓ)

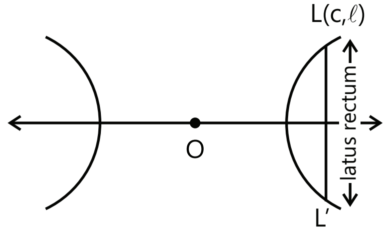


Figure 12.9

Since, L lies on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Therefore, we have $\frac{c^2}{a^2}-\frac{\ell^2}{b^2}=1$

$\Rightarrow \frac{\ell^2}{b^2}-\frac{c^2}{a^2}-1=\frac{c^2-a^2}{a^2} \quad \Rightarrow \quad \ell^2=b^2\left(\frac{b^2}{a^2}\right)=\frac{b^4}{a^2}\Rightarrow \frac{b^2}{a}$

Latus rectum LL′ = LF + L′F = $\frac{b^2}{a}+\frac{b^2}{a}=\frac{2b^2}{a}$

(i) Focal Distance of a Point: Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as shown in figure. Then by definition,

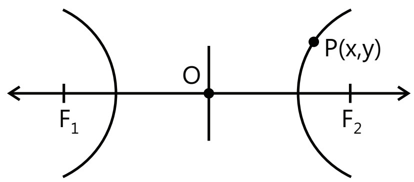


Figure 12.10

We have SP = e.PM and S¢P = e.PM′

$\Rightarrow$ SP = e.NK = e (CN – CK) = $e\left(x-\frac{a}{e}\right)$ = ex – a and S¢P = e(NK′)

= e(CN + CK′) = $e\left(x-\frac{a}{e}\right)$ = ex + a

$\Rightarrow$ S¢P – SP = (ex + a) – (ex – a) = 2a = length of transverse axis

Illustration 1: Find the equation of the hyperbola, where the foci are (±3, 0) and the vertices are (±2, 0). **(JEE MAIN)**

Sol: Use the relation $c^2 = a^2 + b^2$, to find the value of b and hence the equation of the hyperbola.

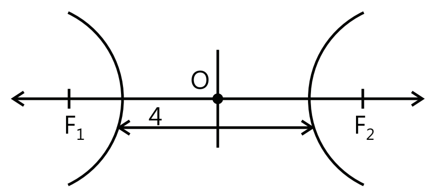


Figure 12.11

We have, foci = (±c, 0) = (±3, 0) $\Rightarrow$ c = 3

and vertices (±a, 0) = (±2, 0)

a = 2

But $c^2 = a^2 + b^2 \Rightarrow 9 = 4 + b^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow b^2 = 5$

Here, the foci and vertices lie on the x-axis, therefore the equation of the hyperbola is of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \Rightarrow \qquad \frac{x^2}{4}-\frac{y^2}{5}=1$$

Illustration 2: Find the equation of the hyperbola, where the vertices are (0, ±5) and the foci are (0, ±8). **(JEE MAIN)**

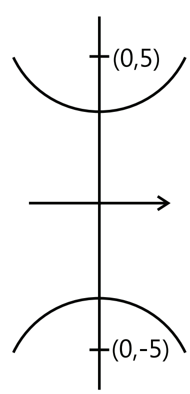


Figure 12.12

Sol: Similar to the previous question.

We have, vertices (0, ±a) = (0, ±5) $\Rightarrow$ a = 5

foci (0, ±c) = (0, ±8) $\Rightarrow$ c = 8

But, we know that $c^2 = a^2 + b^2 \Rightarrow 64 = 25 + b^2$

$\Rightarrow b^2 = 64 - 25 = 39$

Here, the foci and vertices lie on the y-axis, therefore the equation of hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$ i.e., $\frac{y^2}{25}-\frac{x^2}{39}=1$

which is the required equation of the hyperbola.

Illustration 3: If circle c is a tangent circle to two fixed circles c_1 and c_2, then show that the locus of c is a hyperbola with c_1 and c_2 as the foci. **(JEE MAIN)**

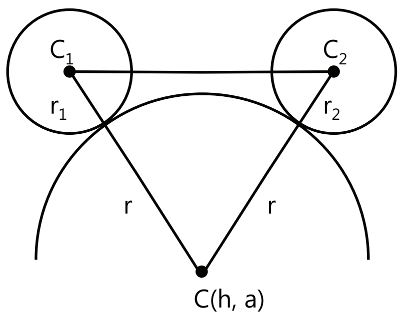


Figure 12.13

Sol: Refer to the definition of a hyperbola.

$cc_1 = r + r_1$; $cc_2 = r + r_1$

$cc_1 - cc_2 = r_1 - r_2$ = constant

Illustration 4: Find the equation of the hyperbola whose directrix is 2x + y = 1 and focus, (1, 2) and eccentricity $\sqrt{3}$. **(JEE MAIN)**

Sol: Use the definition of the hyperbola to derive the equation.

Let P(x, y) be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

They by definition SP = e PM

$\Rightarrow (SP)^2 = e^2(PM)^2 \Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2 \Rightarrow 5(x^2 - y^2 - 2x - 4y + 5) = (4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 5: Find the equation of the hyperbola when the foci are at $(\pm 3\sqrt{5}, 0)$, and the latus rectum is of length 8. **(JEE ADVANCED)**

Sol: Use the formula for the length of the latus rectum to get a relation between a and b. Then use the foci and the relation between a and b to get the equation of the hyperbola.

Here foci are at $(\pm 3\sqrt{5}, 0)$ $\Rightarrow$ $c = 3\sqrt{5}$

Length of the latus rectum $= \frac{2b^2}{a} = 8$

$\Rightarrow$ $b^2 = 4a$ ... (i)

We know that

$c^2 = a^2 + b^2$

$(3\sqrt{5})^2 = a^2 + 4a$

$45 = a^2 + 4a$

$a^2 + 4a - 45 = 0$

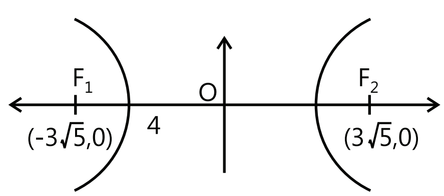


Figure 12.14

$(a + 9)(a - 5) = 0$

$a = -9, a = 5$ (a cannot be –ve)

Putting a = 5 in (i), we get

$b^2 = 5 \times 4 = 20 \Rightarrow b^2 = 20$

Since, foci lie on the x-axis, therefore the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{i.e.,} \quad \frac{x^2}{25} - \frac{y^2}{20} = 1$$

$$\Rightarrow \quad 20x^2 - 25y^2 = 500 \quad \Rightarrow \quad 4x^2 - 5y^2 = 100$$

Which is the required equation of hyperbola.

Illustration 6: Find the equation of the hyperbola when the foci are at $(0, \pm\sqrt{10})$, and passing through (2, 3) **(JEE ADVANCED)**

Sol: Start with the standard equation of a hyperbola and use the foci and the point (2 , 3) to find the equation.

Here, foci are at $(0, \pm\sqrt{10})$

$\Rightarrow \quad c = \sqrt{10}$ Here the foci lie at the y-axis.

So the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{... (i)}$$

Point (ii, iii) lies on (i).

$$\text{So} \quad \frac{9}{a^2} - \frac{4}{b^2} = 1 \quad \Rightarrow \quad \frac{9}{a^2} = 1 + \frac{4}{b^2} \quad \Rightarrow \quad \frac{9}{a^2} = \frac{b^2+4}{b^2} \; a^2 = \frac{9b^2}{b^2+4} \qquad \text{... (ii)}$$

We know that

$$c^2 = a^2 + b^2$$

$$\Rightarrow \quad 10 = \frac{9b^2}{b^2+4} + b^2$$

$$\Rightarrow \quad \frac{9b^2 + b^4 + 4b^2}{b^2+4} = 10$$

$\Rightarrow \quad 10b^2 + 40 = b^4 + 13b^2$

$\Rightarrow \quad b^4 + 3b^2 - 40 = 0$

$\Rightarrow \quad (b^2 + 8)(b^2 - 5) = 0$

$\Rightarrow \quad b^2 + 8 = 0, b^2 - 5 = 0$

$\Rightarrow \quad b^2 = -8$ & $b^2 = 5$ ($b^2 = -8$ not possible)

$\Rightarrow \quad b^2 = 5$ in (ii), we get

$$a^2 = \frac{9 \times 5}{5+4} = \frac{45}{9} = 5$$

Again putting $a^2 = 5$ and $b^2 = 5$ in (i), we get

$$\frac{y^2}{5} - \frac{x^2}{5} = 1 \quad \Rightarrow \quad y^2 - x^2 = 5$$

Which is the required equation of the hyperbola.

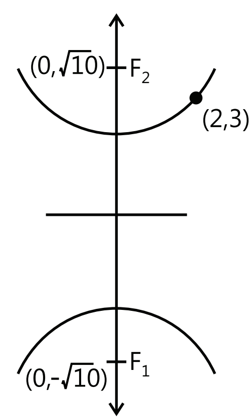


Figure 12.15

Illustration 7: An ellipse and hyperbola are confocal i.e., having same focus and conjugate axis of hyperbola & minor axis of ellipse. If e_1 and e_2 are the eccentricities of the hyperbola and ellipse then find $\frac{1}{e_1^2}+\frac{1}{e_2^2}$.

(JEE ADVANCED)

Sol: Consider the standard equation of an ellipse and hyperbola by taking the eccentricity as e_1 and e_2 respectively. Find the relation between the eccentricities by using the condition that they have the same focus.

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1 \Rightarrow$ $ae_1 = Ae_2$ and $B = b$

$\Rightarrow$ $B^2 = b^2$ $\Rightarrow$ $A^2(e_2^{\,2} - 1) = a^2(1 - e_1)$

$\therefore$ $\frac{a^2e_1^2}{e_2^2}(e_2^2-1) = a^2(1 - e_1^{\,2})$ $\therefore$ $\frac{1}{e_1^2}+\frac{1}{e_2^2} = 2$

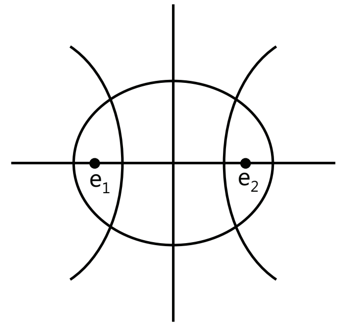


Figure 12.16

Illustration 8: Find the equation of a hyperbola if the distance of one of its vertices from the foci are 3 and 1. Find all the possible equations. **(JEE ADVANCED)**

Sol: Consider two cases when the major axis is parallel to the X – axis and the minor axis is parallel to the Y-axis and vice versa.

Case I:

$ae - a = 1$

$ae + a = 3$

$\Rightarrow$ $e = 2$

$\Rightarrow$ $a = 1$

$\Rightarrow$ $b^2 = 3$

Equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3} = 1$

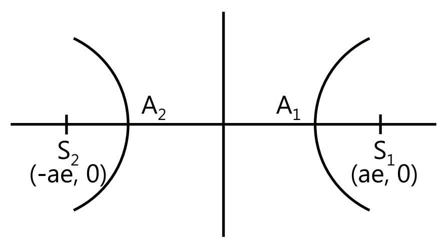


Figure 12.17

Case II

$b(e - 1) = 1$

$b(e + 1) = 3$

$\Rightarrow$ $e = 2,\ b = 1,\ a^2 = 3$

Equation of hyperbola is $\frac{y^2}{3}-\frac{x^2}{1}=1$

4. CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The hyperbola conjugate to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $\frac{y^2}{a^2}-\frac{x^2}{b^2} = 1$

The eccentricity of the conjugate hyperbola is given by a2 = b2(e2 – 1)

and the length of the latus rectum is $\frac{2a^2}{b}$

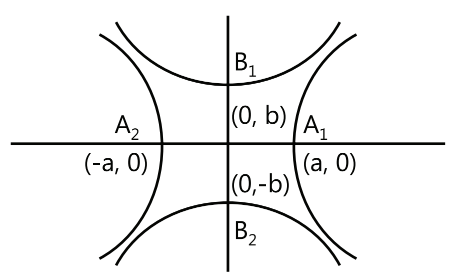
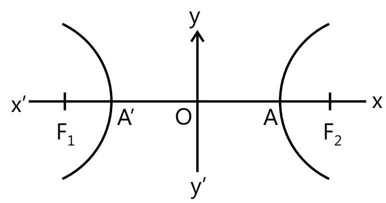
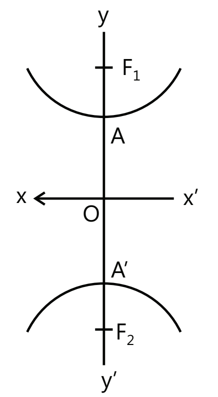


Figure 12.18

Condition of similarity: Two hyperbolas are said to be similar if they have the same value of eccentricity.

Equilateral hyperbola: If a = b or L(T.A.) = L(C.A) then it is an equilateral or rectangular hyperbola.

5. PROPERTIES OF HYPERBOLA/CONJUGATE HYPERBOLA

Equation of the Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Figure	 **Figure 12.19**	 **Figure 12.20**
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Transverse axis	2a	2b
Conjugate axis	2b	2a
Relation between a, b, c	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Foci	(±c, 0)	(0, ±c)
Eccentricity	$e = \frac{c}{a}$	$e' = \frac{c}{b}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

MASTERJEE CONCEPTS

- If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate hyperbola then

$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=1}$$

$$e_1^2 = 1 + \frac{b^2}{a^2} \qquad e_2^2 = 1 + \frac{a^2}{b^2}$$

$$\therefore \quad \frac{1}{e_1^2}+\frac{1}{e_2^2} = 1$$

- The foci of a hyperbola and its conjugate hyperbola are CONCYCLIC and form vertices of square.

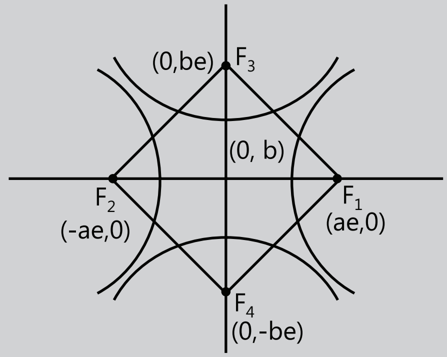


Figure 12.21

Anvit Tawar (JEE 2009, AIR 9)

6. AUXILIARY CIRCLE

A circle described on the transverse axis as diameter is an auxiliary circle and its equation is $x^2 + y^2 = a^2$

Any point of the hyperbola is $P \equiv (a \sec \theta, b \tan \theta)$

P, Q are called corresponding point and θ is eccentric angle of P.

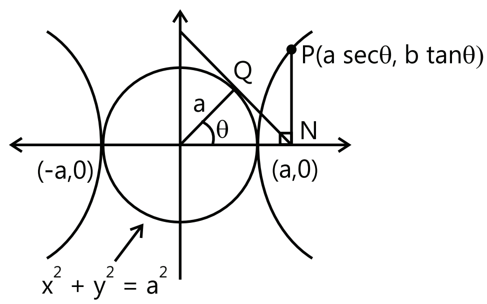


Figure 12.22

MASTERJEE CONCEPTS

1. If $O \in (0, \pi/2)$, P lies on upper right branch.
2. If $O \in (\pi/2, \pi)$, P lies on upper left branch.
3. If $O \in (\pi, 3\pi/2)$, P lies on lower left branch.
4. If $O \in (3\pi/2, 2\pi)$, P lies on lower right branch.

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7. PARAMETRIC COORDINATES

Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PL perpendicular from P on OX and then a tangent LM from L to the circle described on A'A as diameter.

Then, $x = CL = CM \sec\theta = a\sec\theta$

Putting $x = a\sec\theta$ in $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, we obtain $y = b\tan\theta$

Thus, the coordinates of any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ are $(a\sec\theta, b\tan\theta)$, where θ is the parameter such that $0 \le \theta \le 2\pi$. These coordinates are known as the parametric coordinates. The parameter θ is also called the eccentric angle of point P on the hyperbola.

The equation $x = a\sec\theta$ and $y = b\tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$.

Note: (i) The circle $x^2 + y^2 = a^2$ is known as the auxiliary circle of the hyperbola.

Let P $(a\sec\theta_1, b\tan\theta_1)$ and Q $(a\sec\theta_2, b\tan\theta_2)$ be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$.

Then the equation of the chord PQ is

$$y - b\tan q_1 = \frac{b\tan\theta_2 - b\tan\theta_1}{a\sec\theta_2 - a\sec\theta_1}(x - a\sec q_1) \Rightarrow \frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

Illustration 9: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis. **(JEE MAIN)**

Sol: Establish the relation between a and b and then use the eccentricity formula.

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. Then transverse axis = 2a and latus rectum = $\frac{2b^2}{a}$

According to the question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow\ 2b^2 = a^2 \Rightarrow 2a^2(e^2 - 1) = a^2 \Rightarrow 2e^2 - 2 = 1 \Rightarrow e^2 = 3/2 \quad \therefore e = \sqrt{3/2}$

Illustration 10: If the chord joining two points $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ passes through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, then prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$. **(JEE ADVANCED)**

Sol: Obtain a relation between the two given eccentric angles by substituting the point in the equation of chord.

The equation of the chord joining $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If it passes through the focus (ae, 0) then $e\cos\left(\frac{\theta_1-\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$

$$\Rightarrow \frac{\cos((\theta_1-\theta_2)/2)}{\cos((\theta_1+\theta_2)/2)} = 1/e$$

using componendo dividendo rule we get $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$.

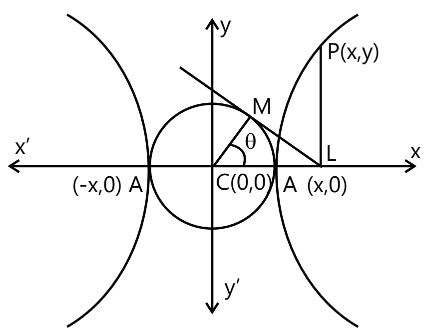

Figure 12.23

8. POINT AND HYPERBOLA

The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ '<' or '=' or '>' 0

Proof: Draw PL perpendicular to x-axis. Suppose it cuts the hyperbola at $Q(x_1, y_2)$.

Clearly, PL > QL

$\Rightarrow \quad y_1 > y_2 \quad \Rightarrow \quad \frac{y_1^2}{b^2} > \frac{y_2^2}{b^2} \quad \Rightarrow \quad -\frac{y_1^2}{b^2} < -\frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$

$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 \qquad \left[\begin{array}{l} \because Q(x_1, y_2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ \qquad\qquad\qquad \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} = 1 \end{array}\right]$

Thus the point (x_1, y_1) lies outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

Similarly, we can prove that the point (x_1, y_1) will lie inside or on the hyperbola according to

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$ or, = 0.

P lies outside/on/inside $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 / = 0 / > 0$

Illustration 11: Find the position of the points (7, –3) and (2, 7) relative to the hyperbola $9x^2 - 4y^2 = 36$.

(JEE MAIN)

Sol: Use the concept of position of a point w.r.t. the hyperbola.

The equation of the given hyperbola is $9x^2 - 4y^2 = 36$ or, $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Now,

$\frac{7^2}{4} - \frac{(-3)^2}{9} - 1 = \frac{41}{4} > 0$ and, $\frac{2^2}{4} - \frac{7^2}{9} \Rightarrow 1 - \frac{49}{9} \Rightarrow 1 = \frac{-49}{9} < 0$.

Hence, the point (7, –3) lies inside the parabola whereas the point (2, 7) lies outside the hyperbola.

Illustration 12: Find the position of the point (5, –4) relative to the hyperbola $9x^2 - y^2 = 1$. **(JEE MAIN)**

Sol: Use the concept of position of a point

Since $9(5)^2 - (4)^2 = 1 = 225 - 16 - 1 = 208 > 0$. So the point (5, –4) inside the hyperbola $9x^2 - y^2 = 1$.

9. LINE AND HYPERBOLA

Consider a line y = mx + c and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving y = mx + c and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \quad b^2x^2 - a^2(mx + c)^2 = a^2b^2 \quad \Rightarrow \quad (b^2 - a^2m^2)\, x^2 - 2a^2cmx - a^2(b^2 + c^2) = 0;$

D > 0	D = 0	D < 0
Secant	Tangent	Neither secant nor tangent
	$\underbrace{c^2 = a^2m^2 - b^2}_{\text{condition of tangency}}$	

$\Rightarrow$ $y = mx + \sqrt{a^2m^2 - b^2}$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

MASTERJEE CONCEPTS

No. of tangents drawn to a hyperbola passing through a given point (h, k)

Let y = mx+c be tangent to the hyperbola

$\Rightarrow \quad c^2 = a^2m^2 - b^2$

Since line passes through (h, k)

$\Rightarrow \quad (k - mh)^2 = a^2m^2 - b^2$

$\Rightarrow \quad (h^2 - a^2)m^2 - 2hkm + k^2 + b^2 = 0$

Hence a maximum of 2 tangents can be drawn to the hyperbola passing through (h, k)

$m_1 + m_2 = \frac{2hk}{h^2 - a^2} \qquad m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2}$

if $m_1m_2 = -1 \qquad \boxed{x^2 + y^2 = a^2 - b^2}$

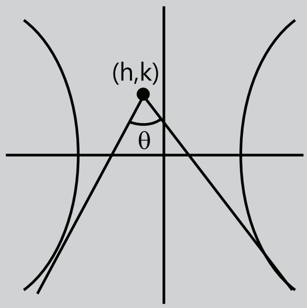

Figure 12.24

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 13: Common tangent to $y^2 = 8x$ and $3x^2 - y^2 = 3$. **(JEE MAIN)**

Sol: Start with the standard equation of a tangent to a parabola and apply the condition for it be a tangent to $3x^2 - y^2 = 3$.

Tangent to the parabola is of the form $y = mx + \frac{2}{m}$. For this line to be tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1 \qquad c^2 = a^2m^2 - b^2$

$\Rightarrow \quad \frac{4}{m^2} = m^2 - 1 \quad \Rightarrow \quad m^2 = 4 \quad \therefore \ \pm y = 2x + 1$ are the common tangents.

10. TANGENT

Point Form: The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Slope Form: The equation of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$y = mx \pm \sqrt{a^2m^2 - b^2}$

The coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Parametric Form: The equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Note:

(i) The tangents at the point P $(a \sec\theta_1, b \tan\theta_1)$ and Q $(a \sec\theta_2, b \tan\theta_2)$ intersect at the point R $\left(\frac{a\cos((\theta_1 - \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}, \frac{b\sin((\theta_1 + \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}\right)$.

(ii) If $|\theta_1 + \theta_2| = \pi$, then the tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touch the hyperbola at the extremities of a diameter.

(iv) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ and the product of these perpendiculars is b^2.

(v) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(vi) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as the diameter of the circle.

Illustration 14: Prove that the straight line lx + my + n = 0 touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2l^2 - b^2m^2 = n^2$. **(JEE MAIN)**

Sol: Apply the condition of tangency and prove the above result.

The given line is lx + my + n = 0 or y = –l/m x – n/m

Comparing this line with y = Mx + c $\quad\therefore$ M = – l/m and c = –n/m ....(i)

This line (i) will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2 = a^2M^2 - b^2$

$\Rightarrow \frac{n^2}{m^2}=\frac{a^2l^2}{m^2}-b^2$ or $a^2l^2 - b^2m^2 = n^2$. **Hence proved.**

Illustration 15: Find the equations of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line x – y + 4 = 0. **(JEE MAIN)**

Sol: Get the slope of the perpendicular line and use it to get the equation of the tangent.

Let m be the slope of the tangent. Since the tangent is perpendicular to the line x – y = 0

$m \times 1 = -1$

$\Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36}-\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \therefore\ a^2 = 36$ and $b^2 = 9$

So the equation of the tangents are $y = (-1)x \pm \sqrt{36\times(-1)^2-9}$

$\Rightarrow y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0$

Illustration 16: If two tangents drawn from any point on hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles θ_1 and θ_2 with the axis then $\tan\theta_1.\tan\theta_2$. **(JEE ADVANCED)**

Sol: Establish a quadratic in m, where m is the slope of the two tangents. Then use the sum and product of the roots to find $\tan\theta_1.\tan\theta_2$.

Let $c^2 = a^2 - b^2$

Any tangent to the ellipse $y = mx \pm \sqrt{a^2m^2+b^2}$

$c\tan\theta = mc\sec\theta \pm \sqrt{a^2m^2+b^2}$)

$c^2(\tan\theta - m\sec\theta)^2 = a^2m^2 + b^2$

$(c^2\sec^2\theta - a^2)m^2 + (......)m + c^2\tan^2\theta - b^2 = 0 \Rightarrow \tan\theta_1.\tan\theta_2$ =product of the roots.

$$= \frac{c^2\tan^2\theta - b^2}{c^2\sec^2\theta - a^2} = 1.$$

11. NORMAL

Point Form: The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

Parametric Form: The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a\,x\cos\theta + b\,y\cot\theta = a^2 + b^2$.

Slope Form: The equation of a normal of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}} \quad \text{at the points} \left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{b^2m}{\sqrt{a^2-b^2m^2}}\right)$$

Note:

(i) At most four normals can be drawn from any point to a hyperbola.

(ii) Points on the hyperbola through which, normal through a given point pass are called co-normal points.

(iii) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This illustrates the reflection property of the hyperbola as "**An incoming light ray**" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common points.

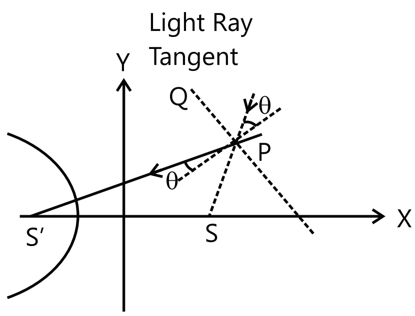

Figure 12.25

(iv) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(v) The sum of the eccentric angles of co-normal points is an odd multiple of π.

(vi) If θ_1, θ_2 and θ_3 are eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The normals at which are concurrent, then $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$

(vii) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then $(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$.

Illustration 17: How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$. Find the equation of these tangents and the angle between them. **(JEE MAIN)**

Sol: Use the concept of Position of a Point w.r.t. the hyperbola to find the number of real tangents.

Given point P = (4, 3)

Hyperbola $S \equiv \frac{x^2}{16}-\frac{y^2}{9}=1=0$

$\because S_1 \equiv \frac{16}{16}-\frac{9}{9}-1=-1<0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the hyperbola

$\therefore$ Two tangents can be drawn from the point P(4, 3). Equation of a pair of tangents is $SS_1 = T^2$.

$$\Rightarrow \left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)(-1) \equiv \left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$$

$$\Rightarrow -\frac{x^2}{16}+\frac{y^2}{9}+1=\frac{x^2}{16}+\frac{y^2}{9}+1-\frac{xy}{6}\ \frac{x}{2}+\frac{2y}{3} \Rightarrow 3x^2-4xy-12x+16y=0 \text{ and } \theta=\tan^{-1}\left(\frac{4}{3}\right)$$

Illustration 18: Find the equation of common tangents to hyperbolas

$$H_1: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ ;\ H_2: \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$ **(JEE MAIN)**

Sol: Compare the equation of the common tangents to H_1 and H_2 and compare the two equations to find the value of m.

Tangent to H_1

$y = mx \pm \sqrt{a^2m^2-b^2}$

$H_2: \frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1$

$a^2m^2-b^2=(-b^2)m^2-(-a^2)$

$\therefore \qquad a^2(m^2-1)=b^2(1-m^2)$

$\qquad m=\pm 1$

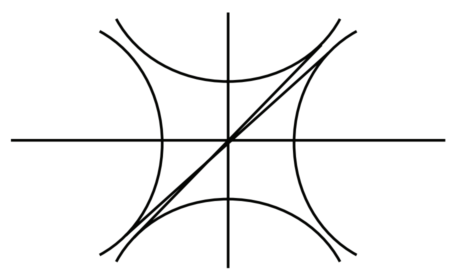

Figure 12.26

Equation of common tangents are $\pm y = x+\sqrt{a^2-b^2}$

Illustration 19: If the normals at (x_r, y_r); r = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q(h, k), prove that the sum of the ordinates of the four points is k. Also prove that the product of the ordinates is $-c^4$. **(JEE ADVANCED)**

Sol: Write the equation of the normal in the parametric form and then use the theory of equations.

Any point on the curve $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$

The equation of the normal to the hyperbola at the point $\left(ct, \frac{c}{t}\right)$ is

$$y-\frac{c}{t}=\frac{-1}{\left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}}}(x-ct).$$

Here, $xy = c^2$; or $y=\frac{c^2}{x'}$ $\quad\therefore\quad \frac{dy}{dx}=\frac{-c^2}{x^2}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}} = \frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

$\therefore$ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$y - \frac{c}{t} = t^2(x - ct)$ or $ty - c = t^3(x - ct)$ or $ct^4 - t^3x + ty - c = 0$

The normal passes through (h, k). So

$ct^4 - t^3h + tk - c = 0$ ... (i)

Let the roots of (i) be t_1, t_2, t_3, t_4. Then $x_r = ct$, $y_r = \frac{c}{t_r}$

$\therefore$ sum of ordinates $= y_1 + y_2 + y_3 + y_4$

$= \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = c\frac{t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3}{t_1t_2t_3t_4}$

$= c \cdot \frac{-k/c}{-c/c} = k$, {from roots of the equation (i)} and, product of the ordinates

$= y_1y_2y_3y_4 = \frac{c}{t_1}.\frac{c}{t_2}.\frac{c}{t_3}.\frac{c}{t_4} = \frac{c^4}{t_1t_2t_3t_4} = \frac{c^4}{-c/c} = -c^4.$

Hence proved.

Illustration 20: The perpendicular from the centre on the normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet at R. Find the locus of R. **(JEE ADVANCED)**

Sol: Solve the equation of the normal and the equation of line perpendicular to it passing through the origin.

Let (x_1, y_1) be any point on the hyperbola.

So, $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ... (i)

The equation of the normal at (x_1, y_1) is $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}$ or $\frac{x_1}{a^2}(y - y_1) + \frac{y_1}{b^2}(x - x_1) = 0$ ... (ii)

'm' of the normal $= -\frac{a^2y_1}{b^2x_1}$

$\therefore$ The equation of the perpendicular from the centre (0, 0) on (ii) is

$y = \frac{b^2x_1}{a^2y_1}.x$ ... (iii)

The intersection of (ii) and (iii) is R and the required locus is obtained by eliminating x_1, y_1 from (i), (ii) and (iii).

From (iii), $\frac{x_1}{a^2y} = \frac{y_1}{b^2x} = t$ (say)

Putting in (ii), $yt(y - b^2xt) + xt(x - a^2yt) = 0$

or $(x^2 + y^2)t - (a^2 + b^2)xyt^2 = 0.$

But $t \neq 0$ for then $(x_1, y_1) = (0, 0)$ which is not true.

$\therefore \quad t = \frac{x^2 + y^2}{xy(a^2 + b^2)}$; $\quad \therefore x_1 = \frac{x^2 + y^2}{xy(a^2 + b^2)}a^2y = \frac{a^2(x^2 + y^2)}{x(a^2 + b^2)}$

and $y_1 = \frac{x^2+y^2}{xy(a^2+b^2)} b^2 x = \frac{b^2(x^2+y^2)}{y(a^2+b^2)}$

$\therefore$ from (i), $\frac{1}{a^2} \cdot \frac{a^4(x^2+y^2)^2}{x^2(a^2+b^2)} - \frac{1}{b^2} \cdot \frac{b^4(x^2+y^2)^2}{y^2(a^2+b^2)^2} = 1$

or $\{x^2 + y^2)^2 . \left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = (a^2 + b^2)^2$.

Illustration 21: A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

(JEE ADVANCED)

Sol: Find the co-ordinates of the point M and N and then eliminate the parameter between the ordinate and abscissae.

The equation of normal at the point Q(a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos\phi + by \cot \phi = a^2 + b^2$ ... (i)

The normal (i) meets the x-axis in M $\left(\frac{a^2+b^2}{a} \sec\varphi, 0\right)$ and y-axis in N $\left(0, \frac{a^2+b^2}{b} \tan\varphi\right)$

$\therefore$ Equation of MP, the line through M and perpendicular to axis, is

$x = \left(\frac{a^2+b^2}{a}\right) \sec\varphi$ or $\sec \phi = \frac{ax}{(a^2+b^2)}$ ... (ii)

and the equation of NP, the line through N and perpendicular to the y-axis is

$y = \left(\frac{a^2+b^2}{b}\right) \tan \phi$ or $\tan \phi = \frac{by}{(a^2+b^2)}$ ... (iii)

The locus of the point is the intersection of MP and NP and will be obtained by eliminating ϕ from (ii) and (iii), so we have $\sec^2\phi - \tan^2\phi = 1$

$\Rightarrow \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$ or $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

Illustration 22: Prove that the length of the tangent at any point of hyperbola intercepted between the point of contact and the transverse axis is the harmonic mean between the lengths of perpendiculars drawn from the foci on the normal at the same point. **(JEE ADVANCED)**

Sol: Proceed according to the question to prove the above statement.

$\frac{P_1}{P} = \frac{S_1G}{TG} = \frac{e^2x_1 - ae}{e^2x_1 - a\cos\theta} = \frac{ae^2 - ae\cos\theta}{ae^2 - a\cos^2\theta}$

$\therefore \frac{P_1}{P} = \frac{e(e-\cos\theta)}{(e-\cos\theta)(e+\cos\theta)} \Rightarrow \frac{P}{P_1} = \frac{e+\cos\theta}{e} = 1 + \frac{\cos\theta}{e}$

Similarly we get $\frac{P}{P_1} = 1 - \frac{\cos\theta}{e}$ $\therefore \frac{P}{P_1} + \frac{P}{P_2} = 2 \Rightarrow \frac{1}{P_1} + \frac{P}{P_2} = \frac{2}{P}$

Hence Proved.

12. DIRECTOR CIRCLE

The locus of the intersection point of tangents which are at right angles is known as the Director Circle of the hyperbola. The equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real.

If $b^2 = a^2$ (rectangular hyperbola) the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no pair of tangents at right angles can be drawn to the curve.

Or we can say that

If L(T.A) > L(C.A) $\Rightarrow$ circle is real.

If L(T.A) < L(C.A) $\Rightarrow$ No real locus, Imaginary circle.

If L(T.A) = L(C.A) $\Rightarrow$ point circle

13. CHORD

13.1 Chord of Contact

It is defined as the line joining the point of intersection of tangents drawn from any point. The equation to the chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

13.2 Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ ($T = S_1'$), where T and S_1' have their usual meanings.

13.3 Chord of Hyperbola (Parametric Form)

Note: Chord of ellipse $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

For a hyperbola it is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

Passing through (d, 0) $\frac{d}{a}\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

$$\frac{d}{a} = \frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)}$$

$$\frac{d+a}{d-a} = \frac{-2\cos\alpha/2\ \cos\beta/2}{2\cos\alpha/2\ \ \sin\beta/2}$$

$$\frac{a-d}{a+d} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

if $\quad d = ae \Rightarrow \frac{1-e}{1+e} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

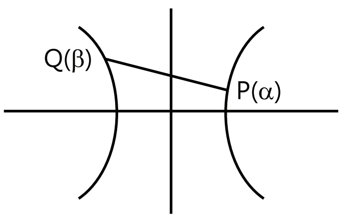

Figure 12.27

MASTERJEE CONCEPTS

Point of intersection of tangents at P(α) and Q(β) can be obtained by comparing COC with the chord at P(α) & Q(β)

Equation of PQ

COC ⇒ $\frac{xh}{a^2}-\frac{yk}{b^2} = 1$

PQ ⇒ $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

$\therefore\ h = a\frac{\cos((\alpha-\beta)/2)}{\cos((\alpha+\beta)/2)},\ k = b\frac{\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)}$

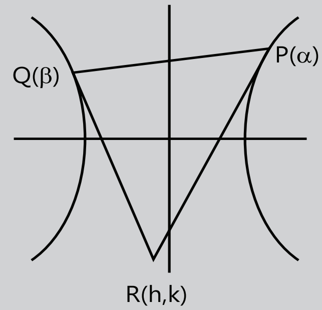

Figure 12.28

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 23: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B. **(JEE MAIN)**

Sol: The point of intersection of the tangents at A and B is nothing but the point for which AB is the chord of contact. Use this information to find the locus.

Let P ≡ (h, k) be the point of intersection of tangent at A and B

∴ Equation of the chord of contact AB is $\frac{xh}{a^2}-\frac{yk}{b^2} = 1$ ....(i)

Which touches the parabola. Equation of the tangent to the parabola $y^2 = 4ax$

$y = mx - a/m \Rightarrow mx - y = -a/m$ ....(ii)

equation (i) and (ii) must be same

$\therefore\quad \frac{m}{\left((h/a^2)\right)} = \frac{-1}{\left(-(k/b^2)\right)} = \frac{-a/m}{1} \quad\Rightarrow\quad m = \frac{h}{k}\frac{b^2}{a^2} \text{ and } m = -\frac{ak}{b^2}$

$\therefore\quad \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \Rightarrow$ locus of P is $y^2 = -\frac{b^4}{a^3}x$.

Illustration 24: A point P moves such that the chord of contact of a pair of tangents from P to $y^2 = 4x$ touches the rectangular hyperbola $x^2 - y^2 = 9$. If locus of 'P' is an ellipse, find e. **(JEE MAIN)**

Sol: Write the equation of the chord of contact to the parabola w.r.t. a point (h , k). Then solve this equation with the equation of the hyperbola.

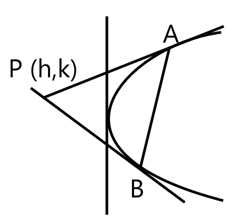

Figure 12.29

$yy_1 = 2a(x + x_1)$; $yk = 2(x + h) \quad\Rightarrow\quad y = \frac{2x}{k}+\frac{2h}{k}$; $\frac{4h^2}{k^2} = 9.\frac{4}{k^2} - 9$

$4h^2 = 36 - 9k^2$ $\quad \frac{x^2}{9} + \frac{y^2}{4} = 1$ $\quad e^2 = 1 - \frac{4}{9}$ $\quad e = \frac{\sqrt{5}}{3}$

Illustration 25: Find the locus of the mid-point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the formula $T = S_1$ to get the equation of the chord and substitute the co-ordinates of the focus.

Let $P \equiv (h, k)$ be the mid-point

$\therefore$ Equation of the chord whose mid-point (h, k) is given $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$ since it is a focal chord.

$\therefore$ It passes through the focus, either (ae, 0) or (–ae, 0)

$\therefore$ Locus is $\pm \frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Illustration 26: Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line x + y = b. **(JEE ADVANCED)**

Sol: Consider a point on the line x + y = b and then find a chord with this point as the mid-point. Then substitute the point in the equation of the chord to get the condition between 'a' and 'b'.

Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ Equation of the chord whose mid-point is $P(\alpha, b - \alpha)$ is:

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b) $\therefore \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0 \quad \Rightarrow \quad a = b$$

Illustration 27: Locus of the mid points of the focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is another hyperbola whose eccentricity is e. **(JEE ADVANCED)**

Sol: Use the formula $T = S_1$ and proceed further.

$T = S_1;\ \frac{xh}{a^2} - \frac{yb}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

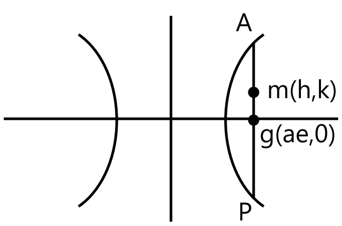

Figure 12.30

It passes through focus $\Rightarrow \quad \frac{eh}{a} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{ex}{a} = \frac{y^2}{b^2} \Rightarrow \quad \frac{1}{a^2}[x^2 - eax] = \frac{y^2}{b^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left[\left(x - \frac{ea}{2}\right)^2 - \frac{e^2a^2}{4}\right] = \frac{y^2}{b^2} \Rightarrow \quad \frac{\left(x - (ea/2)\right)^2}{a^2} - \frac{y^2}{b^2} = \frac{e^2}{4}$$

Hence the locus is a hyperbola of eccentricity e.

Illustration 28: Find the locus of the midpoint of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtends a right angle at the origin. **(JEE ADVANCED)**

Sol: Use the formula T = S_1 and then homogenise the equation of the hyperbola using the equation of the chord to find the locus.

Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2} \qquad \text{... (i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (i) is obtained by making a homogeneous hyperbola with the help of (i)

$$\therefore \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{\left((hx/a^2)-(ky/b^2)\right)^2}{\left((h^2/a^2)-(k^2/b^2)\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 x^2-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 y^2=\frac{h^2}{a^4}x^2+\frac{k^2}{b^4}y^2-\frac{2hk}{a^2b^2}xy \qquad \text{... (ii)}$$

The lines represented by (ii) will be at right angles if the coefficient of x^2 + the coefficient of y^2 = 0

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{h^2}{a^4}-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{k^2}{b^4}=0 \quad \Rightarrow \quad \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{h^2}{a^4}+\frac{k^2}{b^4}$$

hence, the locus of (h, k) is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^4}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}$

14. DIAMETER

The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter. The point where a diameter intersects the hyperbola is known as the vertex of the diameter.

14.1 Equation of Diameter

The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}x$.

14.2 Conjugate Diameters

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chords parallel to the other.

Let $y = m_1x$ and $y = m_2x$ be conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Then, $y = m_2x$ bisects the system of chords parallel to $y = m_1x$. So, its equation is

$$y=\frac{b^2}{a^2m}x \qquad \text{... (i)}$$

Clearly, (i) and $y = m_2x$ represent the same line. Therefore, $m_2=\frac{b^2}{a^2m_1} \Rightarrow m_1m_2=\frac{b^2}{a^2}$

Thus, $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $m_1m_2=\frac{b^2}{a^2}$

MASTERJEE CONCEPTS

- In a pair of conjugate diameters of a hyperbola, only one meets the hyperbola on a real point.
- Let P(a sec θ, b tan θ) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 such that CP and CD are conjugate diameters of the hyperbola. Then, the coordinates of D are (a tan θ, b sec θ)
- If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D respectively then $CP^2 - CD^2 = a^2 - b^2$.

Shivam Agarwal (JEE 2009, AIR 27)

15. POLE AND POLAR

Let $P(x_1, y_1)$ be any point inside the hyperbola. A chord through P intersects the hyperbola at A and B respectively. If tangents to the hyperbola at A and B meet at Q(h, k) then the locus of Q is called the polar of P with respect to the hyperbola and the point P is called the pole.

If $P(x_1, y_1)$ is any point outside the hyperbola and tangents are drawn, then the line passing through the contact points is polar of P and P is called the pole of the polar.

Note: If the pole lies outside the hyperbola then the polar passes through the hyperbola. If the pole lies inside the hyperbola then the polar lies completely outside the hyperbola. If pole the lies on the hyperbola then the polar becomes the same as the tangent.

Equation of polar: Equation of the polar of the point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$, i.e., T = 0

Coordinates of Pole: The pole of the line $lx+my+n=0$ with respect to hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$.

Properties of pole and polar:

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points. Condition for conjugate points is $\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}=1$.
2. If the pole of line $l_1x+m_1y+n_1=0$ lies on another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.
3. Pole of a given line is the same as the point of intersection of the tangents at its extremities.
4. Polar of focus is its directrix.

16. ASYMPTOTES

An asymptote to a curve is a straight line, such that distance between the line and curve approaches zero as they tend to infinity.

In other words, the asymptote to a curve touches the curves at infinity i.e. asymptote to a curve is its tangent at infinity.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 are y = $\pm \frac{b}{a}$x or $\frac{x}{a}\pm\frac{y}{b}$ = 0

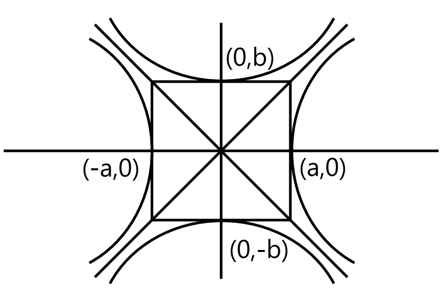

Figure 12.31

Combined equation of asymptote $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

Note: If the angle between the asymptotes of the hyperbola is θ, then its eccentricity is sec θ.

MASTERJEE CONCEPTS

- The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.
- When b = a, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The equation of the pair of asymptotes differ from the hyperbola and the conjugate hyperbola by the same constant, i.e. Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola
- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angles between the asymptotes are the coordinates axes.
- The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.
- Asymptotes are the tangents to the hyperbola from the centre.
- The tangent at any point P on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with the centre C meets asymptotes at Q, R and cut off ΔCQR of constant area = ab.
- The parts of the tangent intercepted between the asymptote is bisected at the point of contact.
- If f(x, y) = 0 is an equation of the hyperbola then the centre of the hyperbola is the point of intersection of $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

Ravi Vooda (JEE 2009, AIR 71)

Illustration 29: Find the asymptotes of $xy - 3y - 2x = 0$. **(JEE MAIN)**

Sol: Proceed according to the definition of asymptotes.

Since the equation of a hyperbola and its asymptotes differ in constant terms only

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$ ...(i)

where λ is any constant such that represents two straight lines

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ $0 + 2x - 3/2x - 1 + 1/2 - 0 - 0 - \lambda\,(1/2)^2 = 0$

$\therefore$ $\lambda = 6$

From (i) the asymptotes of given hyperbola are given by $xy - 3y - 2x + 6 = 0$ or $(y - 2)(x - 3) = 0$

$\therefore$ Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

Illustration 30: Find the equation of that diameter which bisects the chord $7x + y - 20 = 0$ of the hyperbola $\frac{x^2}{3}-\frac{y^2}{7}=1$. **(JEE ADVANCED)**

Sol: Consider a diameter y = mx and solve it with the equation of the hyperbola to form a quadratic in x. Find the midpoint of the intersection of the chord and hyperbola. Use this point to find the slope of the diameter.

The centre of the hyperbola is (0, 0). Let the diameter be $y = mx$ ... (i)

The ends of the chord are found by solving

$$7x + y - 20 = 0 \quad \text{... (ii)}$$

and $$\frac{x^2}{3} - \frac{y^2}{7} = 1 \quad \text{... (iii)}$$

Solving (ii), (iii) we get $\frac{x^2}{3} - \frac{1}{7}(20 - 7x)^2 = 1$

or $7x^2 - 3(400 - 280x + 49x^2) = 21$ or $140x^2 - 840x + 1221 = 0$

Let the roots be x_1, x_2

Then $x_1 + x_2 = \frac{840}{140} = 6$ ... (iv)

If $(x_1, y_1), (x_2, y_2)$ be ends then $7x_1 + y_1 - 20 = 0$, $7x_2 + y_2 - 20 = 0$

Adding, $7(x_1 + x_2) + (y_1 + y_2) - 40 = 0$

or $42 + y_1 + y_2 - 40 = 0$, using (iv) ; $\therefore$ $y_1 + y_2 = -2$

$\therefore$ The middle point of the chord $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6}{2}, \frac{-2}{2}\right) = (3, -1)$

This lies on (i). So $-1 = 3m$; $\therefore$ $m = -\frac{1}{3}$ $\therefore$ the equation of the diameter is $y = -\frac{1}{3}x$.

Illustration 31: The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3) show that its equation is $(2x + 3y - 8)(3x + 2y + 7) = 154$. **(JEE ADVANCED)**

Sol: With the information given, find out the equation of the asymptotes and then use the fact that the point (5, 3) lies on the hyperbola to find the equation of the hyperbola.

Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since the asymptote passes through (1, 2) then $\lambda = -8$ and $\mu = -7$

Thus the equation of the asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Let the equation of the hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + v = 0$ ... (i)

It passes through (5, 3), then $(10 + 9 - 8)(15 + 6 - 7) + v = 0$

$\Rightarrow$ $11 \times 14 + v = 0$

$\therefore$ $v = -154$

putting the value of v in (i) we obtain $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$

which is the equation of the required hyperbola.

17. RECTANGULAR HYPERBOLA

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = \pm \frac{b}{a}x$.

The θ angle between these two asymptotes is given by

$$\tan\theta\left(\frac{(b/a)-(-(b/a))}{1+(b/a)(-(b/a))}\right) = \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}$$

If the asymptotes are at right angles, then $\theta = \pi/2 \Rightarrow \tan\theta = \tan\pi/2 \Rightarrow \frac{2ab}{a^2-b^2} = \tan\frac{\pi}{2} \Rightarrow a = b$.

Thus, the transverse and conjugate axes of a rectangular hyperbola are equal and the equation of the hyperbola is $x^2 - y^2 = a^2$.

Remarks: Since the transverse and conjugate axis of a rectangular hyperbola are equal. So, its eccentricity e is given by

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{2}$$

17.1 With Asymptotes as Coordinate Axes

Equation of the hyperbola referred to the transverse and conjugate axes along the axes of co-ordinates, the equation of the rectangular hyperbola is $x^2 - y^2 = a^2$ ....(i)

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of $-\pi/4$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and, we have

$x = X\cos(-\pi/4) - Y\sin(-\pi/4) = \frac{X+Y}{\sqrt{2}}$ and $y = X\sin(-\pi/4) + Y\cos(-\pi/4) = \frac{Y-X}{\sqrt{2}}$

Substituting the values of x and y in (i), we obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2$

$\Rightarrow XY = \frac{a^2}{2} \Rightarrow XY = c^2$, where $c^2 = \frac{a^2}{2}$

Thus, the equation of the hyperbola referred to its asymptotes as the coordinates axes is

$xy = c^2$, where $c^2 = \frac{a^2}{2}$

Remark: The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy = c^2$.

If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then its equation is

$(x-\alpha)(y-\beta) = c^2$ or $xy - ay - bx + \lambda = 0$; $(\lambda \le \alpha\beta)$

17.2 Tangent

Point Form

The equation of the tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or, $\frac{x}{x_1}+\frac{y}{y_1} = 2$.

Parametric Form

The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$.

Note: Tangent at P $\left(ct_1, \frac{c}{t_1}\right)$ and Q $\left(ct_2, \frac{c}{t_2}\right)$ to the rectangular hyperbola $xy = c^2$ intersect at $\left(\frac{2ct_1t_2}{t_1+t_2}. \frac{2c}{t_1+t_2}\right)$

17.3 Normal

Point Form

The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$

Parametric Form

The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt - \frac{y}{t} = ct^2 - \frac{c}{t^2}$

Note:

(i) The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree equation in t. So, in general, at most four normals can be drawn from a point to the hyperbola $xy = c^2$.

(ii) The equation of the polar of any point $P(x_1, y_1)$ with respect to $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

(iii) The equation of the chord of the hyperbola $xy = c^2$ whose midpoint (x,y) is $xy_1 + yx_1 = 2x_1y_1$ or, T = S′. where T and S′ have their usual meanings.

(iv) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the rectangular $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

Illustration 32: A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC

(ii) The area of the triangle formed by the tangents A, B and C **(JEE ADVANCED)**

Sol: Use parametric co-ordinates and the formula for the area to get the desired result.

Let co-ordinates of A, B and C on the hyperbola $xy = c^2$ be $\left(ct_1, \frac{c}{t_1}\right).\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

(i) Area of triangle ABC $= \frac{1}{2}\left(\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right) = \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$

$$= \frac{c^2}{2t_1t_2t_3}\,|\, t_3^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2 \,| = \frac{c^2}{2t_1t_2t_3}\,|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

(ii) Equation of tangents at A, B, C are $x + yt_1^2 - 2ct_1 = 0$, $x + yt_2^2 - 2ct_2 = 0$ and $x + yt_3^2 - 2ct_3 = 0$

$$\therefore \quad \text{Required Area} = \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2 \qquad \text{... (i)}$$

where $C_1 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$

$\therefore \quad C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (i) $$= \frac{1}{2|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|}4c^2(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2 = 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$$

$\therefore \quad$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

PROBLEM SOLVING TACTICS

(a) In general convert the given hyperbola equation into the standard form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and compare it with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then solve using the properties of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. So, it is advised to remember the standard results.

(b) Most of the standard results of a hyperbola can be obtained from the results of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ just by changing the sign of b^2.

FORMULAE SHEET

HYPERBOLA

(a) Standard Hyperbola:

Imp. Terms \ **Hyperbola**	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of L.R.	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a \sec \phi, b \tan \phi)$ $0 \le \phi < 2\pi$	$(a \tan \phi, b \sec \phi)$ $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ $S¢P = ex_1 + a$	$SP = ey_1 - b$ $S¢P = ey_1 + b$
$\lvert S¢P - SP \rvert$	2a	2b
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

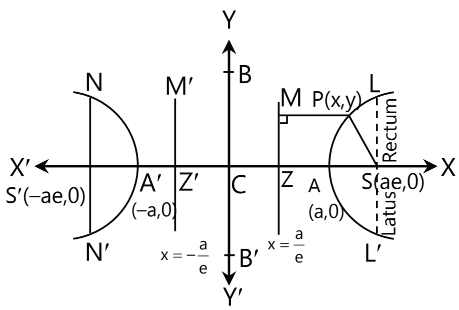

Figure 12.32: Hyperbola

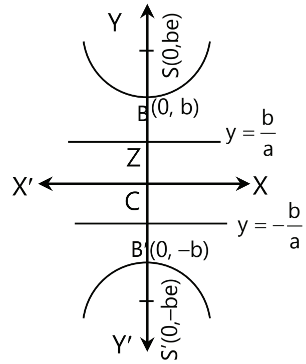

Figure 12.33: Conjugate Hyperbola

(b) Special form of hyperbola: If (h , k) is the centre of a hyperbola and its axes are parallel to the co-ordinate axes, then the equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(c) Parametric equations of a hyperbola: The equation x = a sec ϕ and y = b tan ϕ are known as the parametric equation of the standard hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}$, then $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(d) Position of a point and a line w.r.t. a hyperbola: n The point (x_1, y_1) lies inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ being >, = or < zero.

The line y = mx + c intersects at 2 distinct points, 1 point or does not intersect with the hyperbola according as c^2 >, = or < $a^2m^2 - b^2$.

(e) Tangent:

(i) Point form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(ii) Parametric form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at parametric coordinates (a sec ϕ, b tan ϕ) is $\frac{x}{a}\sec\phi-\frac{y}{b}\phi=1.$

(iii) Slope form: The equation of the tangents having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$y = mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(f) Equation of a pair of tangents from an external point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1 = T^2$.

(g) Normal:

(i) Point form: The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2.$$

(ii) Parametric form: The equation of the normal at parametric coordinates (a sec θ, b tanθ) to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 is $\quad$ ax cos θ + by cot θ = $a^2 + b^2$.

(iii) Slope form: The equation of the normal having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 is

$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

(iv) Condition for normality: y = mx + c is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 if

$$c^2 = \frac{m(a^2+b^2)^2}{(a^2-m^2b^2)}$$

(v) Points of contact: Co-ordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$.

(h) The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 is given by $x^2 + y^2 = a^2 - b^2$.

(i) Equation of the chord of contact of the tangents drawn from the external point (x_1, y_1) to the hyperbola is

given by $\quad \frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 1.$

(j) The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 whose mid point is (x_1, y_1) is T = S_1.

(k) Equation of a chord joining points P(a sec f_1, b tan f_1) and Q (a sec f_2, b tan f_2) is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

(l) Equation of the polar of the point (x_1, y_1) w.r.t. the hyperbola is given by T = 0.

The pole of the line lx + my + n = 0 w.r.t. $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 is $\left(-\frac{a^2\ell}{n}, \frac{b^2m}{n}\right)$

(m) The equation of a diameter of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 corresponding to the chords of slope m is $y = \frac{b^2}{a^2m}x$

(n) The diameters y = m_1x and y = m_2x are conjugate if $m_1m_2 = \frac{b^2}{a^2}$

(o) Asymptotes:

- Asymptote to a curve touches the curve at infinity.
- The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 are $y = \pm\frac{b}{a}x$.

- The asymptote of a hyperbola passes through the centre of the hyperbola.
- * The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$
- * The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{a^2}{b^2}$ or $2\sec^{-1}e$.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The bisector of the angles between the asymptotes are the coordinate axes.
- Equation of the hyperbola – Equation of the asymptotes = constant.

(p) Rectangular or Equilateral Hyperbola:

- A hyperbola for which a = b is said to be a rectangular hyperbola, its equation is $x^2 - y^2 = a^2$.
- $xy = c^2$ represents a rectangular hyperbola with asymptotes $x = 0$, $y = 0$.
- Eccentricity of a rectangular hyperbola is $\sqrt{2}$ and the angle between the asymptotes of a rectangular hyperbola is 90º.
- Parametric equation of the hyperbola $xy = c^2$ are $x = ct$, $y = \frac{c}{t}$, where t is a parameter.
- Equation of a chord joining t_1, t_2 on $xy = c^2$ is $x + y\,t_1t_2 = c(t_1 + t_2)$
- Equation of a tangent at (x_1, y_1) to $xy = c^2$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$.
- Equation of a tangent at t is $x + yt^2 = 2ct$
- Equation of the normal at (x_1, y_1) to $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.
- Equation of the normal at t on $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

  (i.e. Four normals can be drawn from a point to the hyperbola $xy = c^2$)
- If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.
- Equation of chord of the hyperbola $xy = c^2$ whose middle point is given is $T = S_1$.
- Point of intersection of tangents at t_1 and t_2 to the hyperbola $xy = c^2$ is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$