Hyperbola

Solved Examples

JEE Main/Boards

Example 1: Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol: Calculate the value of 'a', by using the distance between the two foci and eccentricity. Then calculate the value of 'b'. Using these two values find the equation of the hyperbola.

Let S, S′ be the foci and C be the centre of the hyperbola S, S′ and C lie on the line y = 4. The co-ordinates of the centre are (1, 4).

The equation of the hyperbola is

$$\frac{(x-1)^2}{a^2} - \frac{(y-4)^2}{b^2} = 1$$

The distance between the foci is 2ae = 10; $\therefore$ a = $\frac{5}{2}$

$$b^2 = a^2(e^2 - 1) = \frac{25}{4}(4-1) = \frac{75}{4}$$

Hence the equation of the hyperbola is

$$\frac{(x-1)^2}{\frac{25}{4}} - \frac{(y-4)^2}{\frac{75}{4}} = 1$$

Example 2: Obtain the equation of hyperbola whose asymptotes are the straight lines $x + 2y + 3 = 0$ & $3x + 4y + 5 = 0$ and which passes through the point (1, –1)

Sol: Use the following formula:

Equation of hyperbola – Equation of asymptotes = constant.

The equation of the hyperbola, is

$(x + 2y + 3)(3x + 4y + 5) = k$, k being a constant.

This passes through the point (1, –1)

$\therefore (1 + 2(-1) + 3)(3(1) + 4(-1) + 5) = k$

$\Rightarrow k = 2 \times 4 = 8$

$\therefore$ The equation of the hyperbola is

$(x + 2y + 3)(3x + 4y + 5) = 8$

Example 3: If e and e′ are the eccentricities of two hyperbolas conjugate to each other,

show that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol: Start with the standard equation of two hyperbolas and eliminate 'a' and 'b'.

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

be the two hyperbolas with eccentricities e and e′ respectively

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{1}{e^2} = \frac{a^2}{a^2 + b^2}$

$a^2 = b^2(e¢^2 - 1) \Rightarrow \frac{1}{e'^2} = \frac{b^2}{a^2 + b^2}$

$\therefore \frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{(a^2 + b^2)} + \frac{b^2}{(a^2 + b^2)} = 1$

Example 4: If any point P on the rectangular hyperbola $x^2 - y^2 = a^2$ is joined to its foci S, S′ show that SP.S¢P = CP^2, where C is the centre of the hyperbola.

Sol: The eccentricity of a rectangular hyperbola is $\sqrt{2}$. Consider a parametric point on the hyperbola and simplify the LHS.

Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ is P(a secθ, a tanθ) ; eccentricity of a rectangular hyperbola is $\sqrt{2}$.

S is (ae, 0), S′ is (–ae, 0) and C is (0, 0)

$(SP)^2.(S'P)^2 = [(a\sec\theta - ae)^2 + a^2\tan^2 q] \times$

$[(a\sec\theta + ae)^2 + a^2\tan^2 q]$

$= a^4[(\sec^2\theta + \tan^2\theta + e^2)^2 - 4e^2\sec^2 q]$

$= a^4[(2\sec^2\theta - 1 + 2)^2 - 4.2\sec^2 q]$

$= a^4[(2\sec^2\theta + 1)^2 - 8\sec^2 q]$

$= a^4[(2\sec^2\theta - 1)^2]$

$\therefore SP.S'P = a^2(2\sec^2\theta - 1)$

$= a^2(\sec^2\theta + \tan^2\theta)$

$= CP^2$.

Example 5: Find the equation of the hyperbola conjugate to the hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y + 2 = 0$

Sol: Use the formula:

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

Let asymptotes be

$2x^2 + 3xy - 2y^2 - 5x + 5y + \lambda = 0$

The equation above represents a pair of lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore \quad \lambda = -5$

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

$\therefore$ Conjugate Hyperbola

= 2(Asymptotes) – Hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$

Example 6: If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin then the eccentricity of the hyperbola is

Sol: Use the definition of the hyperbola S¢P – SP = 2a.

Let S(5, 12) and S′(24, 7) be the two foci and P(0, 0) be a point on the conic then

$SP = \sqrt{25 + 144} = \sqrt{169} = 13$;

S¢P = $\sqrt{(24)^2 + 7^2} = \sqrt{625} = 25$

and $SS' = \sqrt{(24 - 5)^2 + (7 - 12)^2} = \sqrt{19^2 + 5^2} = \sqrt{386}$

since the conic is a hyperbola, S¢P – SP = 2a, the length of transverse axis and SS′ = 2ae, e being the eccentricity.

$\Rightarrow \quad e = \frac{SS'}{S'P - SP} = \frac{\sqrt{386}}{12}$

Example 7: An equation of a tangent to the hyperbola. $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is

Sol: Write the equation of the hyperbola in the standard form and compare to get the equation of the tangent.

Equation of the hyperbola can be written as

$X^2/5^2 - Y^2/4^2 = 1$ ....(i)

where $X = x - 3$ and $Y = y - 2$.

Equation of a tangent which makes an angle $\pi/4$, with the transverse axis $X = 0$ of (i) is

$$Y = \tan\frac{\pi}{4}X \pm \sqrt{25\tan^2\frac{\pi}{4} - 16}$$

$\Rightarrow \quad y - 2 = x - 3 \pm \sqrt{25-16}$

$\Rightarrow \quad y - 2 = x - 3 \pm 3$

$\Rightarrow \quad y = x + 2$ or $y = x - 4$.

Example 8: If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola, then prove that Gg=2PC.

Sol: In the equation of a normal, find the point of intersection with the axes and find the coordinates of G and g.

Let $P(x_1, y_1)$ be any point on the hyperbola $x^2 - y^2 = 4$ then equation of the normal at P is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$\Rightarrow \quad x_1y + y_1x = 2x_1y_1$.

Then coordinates of G are $(2x_1, 0)$ and of g are $(0, 2y_1)$ so that

$$PG = \sqrt{(2x_1 - x_1)^2 + y_1^2} = \sqrt{x_1^2 + y_1^2} = PC$$

$$Pg = \sqrt{x_1^2 + (2y_1 - y_1)^2} = \sqrt{x_1^2 + y_1^2} = PC$$

and

$$Gg = \sqrt{(2x_1)^2 + (2y_1)^2} = 2\sqrt{x_1^2 + y_1^2} = 2PC$$

Hence proved.

Example 9: The normal to the curve at P(x, y) meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is-

Sol: Similar to the previous question.

Equation of the normal at (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x) \text{ which meets the x-axis at G}$$

$\left(0, x + y\frac{dy}{dx}\right)$, then $x + y\dfrac{dy}{dx} = \pm 2x$

$\Rightarrow \quad x + y\dfrac{dy}{dx} = 2x \Rightarrow \quad y\,dy = x\,dx$

$\Rightarrow \quad x^2 - y^2 = c$

or $\quad y\,dy = -3x\,dx$

$\Rightarrow \quad 3x^2 + y^2 = c$

Thus the curve is either a hyperbola or an ellipse.

Example 10: Find the centre, eccentricity, foci and directrices of the hyperbola

$16x^2 - 9y^2 + 32x + 36y - 164 = 0$

Sol: Represent the equation of the hyperbola in the standard form and compare.

Here,

$16x^2 + 32x + 16 - (9y^2 - 36y + 36) - 144 = 0$

or $\quad 16(x + 1)^2 - 9(y - 2)^2 = 144$

$$\therefore \quad \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

Putting $x + 1 = X$ and $y - 2 = Y$, the equation becomes

$$\frac{X^2}{9} - \frac{Y^2}{16} = 1$$

which is in the standard form.

$\because \quad b^2 = a^2(e^2 - 1)$, here $a^2 = 9$ & $b^2 = 16$

$\therefore \quad e^2 - 1 = \dfrac{16}{9} \Rightarrow \quad e^2 = \dfrac{25}{9}$, i.e., $e = \dfrac{5}{3}$

Now, centre $= (0, 0)_{X,Y} = (-1, 2)$

$$\text{foci} = (\pm ae, 0)_{X,Y} = \left(\pm 3.\frac{5}{3}, 0\right)_{X,Y} = (\pm 5, 0)_{x,Y}$$

$= (-1 \pm 5, 2) = (4, 2), (-6, 2)$

Directrices in X, Y coordinates have the equations

$$X \pm \frac{a}{e} = 0 \quad \text{or} \quad x + 1 \pm \frac{3}{5/3} = 0$$

i.e., $\quad x + 1 \pm \dfrac{9}{5} = 0$

$$\therefore \quad x = -\frac{14}{5} \text{ and } x = \frac{4}{5}$$

JEE Advanced/Boards

Example 1: S is the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. M is the foot of the perpendicular drawn from S on a tangent to the hyperbola. Prove that the locus of M is $x^2 + y^2 = a^2$.

Sol: Use the definition of an auxiliary circle.

Let $M = (x_1, y_1)$ be any point on the locus.

Let the equation of the corresponding tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ be

$y = mx \pm \sqrt{a^2m^2-b^2}$

(the sign is chosen according to the position of M)

But $M(x_1, y_1)$ lies on it

$\therefore \quad y_1 = mx_1 \pm \sqrt{a^2m^2-b^2}$ ... (i)

Segment SM is perpendicular to the given tangent.

$\therefore$ Slope of segment SM is $-\frac{1}{m}$

and $S \equiv (ae, 0)$

$\therefore \quad$ Equation of SM is $(y-0) = -\frac{1}{m}(x-ae)$

But $M(x_1, y_1)$ lies on it

$y_1 = -\frac{1}{m}(x_1-ae)$ ... (ii)

From (i), $(y_1 - mx_1) = \pm\sqrt{a^2m^2-b^2}$

From (ii), $(my_1 + x_1) = ae$

Squaring and adding we get the required locus of M

$y_1^2(1 + m^2) + x_1^2(1 + m^2) = a^2e^2 + a^2m^2 - a^2(e^2 - 1)$

$\therefore \quad x_1^2 + y_1^2 = a^2$

Note: This is the equation of the auxiliary circle

Example 2: PQ is the chord joining the points θ_1 and θ_2 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If $\theta_1 - \theta_2 = 2\alpha$, where α is a constant, prove that PQ touches the hyperbola

$$\frac{x^2\cos^2\alpha}{a^2}-\frac{y^2}{b^2} = 1$$

Sol: Write the equation of the chord passing through the points q_1 and q_2. Represent this equation in the standard form of a tangent to a hyperbola and compare.

Equation of the chord PQ to the hyperbola is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$y = \frac{b}{a}\frac{\cos\alpha}{\sin((\theta_1+\theta_2)/2)}x-\frac{b\cos((\theta_1+\theta_2)/2)}{\sin((\theta_1+\theta_2)/2)} \quad \text{... (i)}$$

For line $y = mx + c$ to be a tangent to

$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ we have

$c^2 = a^2m^2 - b^2$

$$\frac{x^2\cos^2\alpha}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{... (ii)}$$

If (i) is tangent to (ii), then, we must have

$$\left(\frac{b\cos((\theta_1+\theta_2)/2)}{\sin((\theta_1+\theta_2)/2)}\right)^2 = b^2\cot^2\left(\frac{\theta_1+\theta_2}{2}\right)$$

which is true.

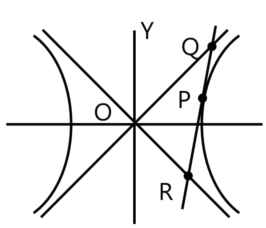
Example 3: Show that the portion of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ intercepted between the asymptotes is bisected at the point of contact. Also show that the area of the triangle formed by this tangent and the asymptotes is constant.

Sol: Calculate the point of intersection of the tangent and the asymptotes and then prove the statement.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{... (i)}$$

equation of the tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 1 \quad \text{... (ii)}$$



Equation of the asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{... (iii)}$$

and $$\frac{x}{a} + \frac{y}{b} = 0 \quad \text{... (iv)}$$

If Q and R are the points of intersection of the tangent at P with the asymptotes, then solving the equation (ii) and (iii), we get

$$Q = \left(\frac{a}{(x_1/a) - (y_1/b)}, \frac{b}{(x_1/a) - (y_1/b)} \right)$$

Solving the equation (ii) and (iv), we get

$$R = \left(\frac{a}{(x_1/a) + (y_1/b)}, \frac{-b}{(x_1/a) + (y_1/b)} \right)$$

The midpoint of QR has coordinate (x_1, y_1) which is also the point of contact of the tangent.

Area of ΔOQR =

$$\frac{1}{2}\left| \left(\frac{a}{(x_1/a) - (y_1/b)} \right)\left(\frac{-b}{(x_1/a) + (y_1/b)} \right) - \left(\frac{b}{(x_1/a) - (y_1/b)} \right)\left(\frac{a}{(x_1/a) + (y_1/b)} \right) \right|$$

=ab sq. units

Example 4: Prove that if a rectangular hyperbola circumscribes a triangle it also passes through the orthocentre of the triangle.

Sol: Take three points on the hyperbola and find the coordinates of the orthocentre. Prove that the orthocentre satisfies the equation of the hyperbola.

Let the equation of the curve referred to its asymptotes be $xy = c^2$ ....(i)

Let the angular points of the triangle be P, Q and R and let their co-ordinates be

$P \equiv \left(ct_1, \frac{c}{t_1} \right)$, $Q \equiv \left(ct_2, \frac{c}{t_2} \right)$ and

$R \equiv \left(ct_3, \frac{c}{t_3} \right)$ respectively.

Equation of QR is $x + t_2t_3\, y = c\,(t_2 + t_3)$

The equation of altitude through P and perpendicular to QR is

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

i.e. $$y + c\,t_1t_2t_3 = t_2t_3\left(x + \frac{c}{t_1t_2t_3} \right) \quad \text{... (ii)}$$

Similarly, the equation of altitude through Q perpendicular to RP is

$$y + ct_1t_2t_3 = t_3t_1\left(x + \frac{c}{t_1t_2t_3} \right) \quad \text{... (iii)}$$

Solving (ii) and (iii), we get

$$\therefore \quad \text{Orthocentre} = \left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3 \right)$$

These co-ordinates satisfy (i)

Hence proved.

Example 5: Find the equation of the hyperbola, whose eccentricity is 5/4, whose focus is (a, 0) and whose directrix is 4x – 3y = a. Find also the coordinates of the centre and the equation to other directrix.

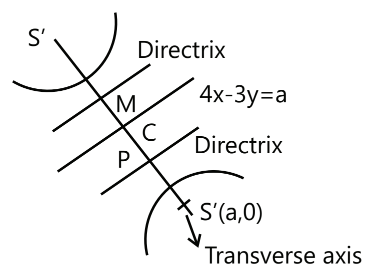
Sol: Use the basic definition of a hyperbola.

$$(x - a)^2 + (y - 0)^2 = e^2\, \frac{(4x - 3y - a)^2}{25}$$

$$x^2 - 2ax + a^2 + y^2 =$$

$$\frac{25}{16}(16x^2 + 9y^2 + a^2 - 24xy - 8ax + 6ay) \times \frac{1}{25}$$

$$7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0 \quad \text{... (i)}$$



Differentiating with respect to 'x'

$$24y - 24a = 0 \quad \text{... (ii)}$$

Differentiating with respect to 'y'

$$14y + 24x - 6a = 0 \quad \text{... (iii)}$$

Solving (ii) and (iii)

$C \equiv (-a/3, a)$

Transverse axis is

$3x + 4y = 3a$

'P' is the point of intersection of the transverse axis and the directrix:

$\therefore \quad P \equiv \left(\frac{13a}{25}, \frac{9a}{25} \right)$ 'C' is mid point of MP

$\therefore \quad M = \left(\frac{-89a}{75}, \frac{41a}{25}\right)$

Equation of the other directrix $4x - 3y = \lambda$, passes through the 'M'

$\therefore \quad 12x - 9y + 29a = 0$

Example 6: Find the centre, eccentricity, foci, directrices and the length of the transverse and conjugate axes of the hyperbola, whose equation is $(x - 1)^2 - 2(y - 2)^2 + 6 = 0$.

Sol: Represent the equation of the hyperbola in the standard form and proceed.

The equation of the hyperbola can be re-written as

$$-\frac{(x-1)^2}{(\sqrt{6})^2} + \frac{(y-2)^2}{(\sqrt{3})^2} = 1$$

$$\Rightarrow \quad \frac{Y^2}{(\sqrt{3})^2} - \frac{X^2}{(\sqrt{6})^2} = 1$$

Where $Y = (y - 2)$ and $X = (x - 1)$ ... (i)

$\therefore$ Centre: $X = 0, Y = 0$ i.e. $(1, 2)$

So $a = \sqrt{3}$ and $b = \sqrt{6}$

so transverse axis $= 2\sqrt{3}$,

and conjugate axis $= 2\sqrt{6}$

Also $b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad 6 = 3(e^2 - 1)$ i.e. $e = \sqrt{3}$

In (X, Y) coordinates, foci are $(0, \pm ae)$

i.e. $(0, \pm 3)$

$\therefore \quad$ foci are $(1 + 0, 2 \pm 3)$

i.e. $\quad (1, 5)$ and $(1, -1)$

Equations of directrices $Y = \pm a/e$

$\therefore \quad$ Directrices are $y - 2 = \pm 1$

$\Rightarrow \quad y = 3, y = 1$.

Example 7: Prove that the locus of a point whose chord of contact touches the circle described on the straight line joining the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as the diameter is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}$.

Sol: Check if the line T = 0 is a tangent to the circle with two foci as the end points of the diameter.

Circle with foci (ae, 0) and (–ae, 0) as diameter is

$(x - ax)(x + ae) + (y - 0)(y - 0) = 0$

i.e. $x^2 + y^2 = a^2e^2 = a^2 + b^2$ ... (i)

$[\because a^2e^2 = a^2 + b^2]$

Let the chord of contact of $P(x_1, y_1)$ touch the circle (i).

Equation of the chord of contact of P is [T = 0]

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

i.e. $b^2x_1x - a^2y_1y - a^2b^2 = 0$ ... (ii)

This equation is tangent to the circle if

$$\frac{a^2b^2}{\sqrt{(b^4x_1^2 + a^4y_1^2)}} = \pm\sqrt{(a^2 + b^2)}$$

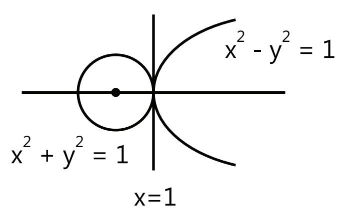
Hence locus of P (x_1, y_1) is $(b^4x^2 + a^4y^2)(a^2 + b^2) = a^4b^4$.

Example 8: An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. One of its directrices is the common tangent, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$, nearer to P. The equation of the ellipse in the standard form is.

Sol: he circle $x^2 + y^2 = 1$ is the auxiliary circle of the hyperbola $x^2 - y^2 = 1$ and they touch each other at the points $(\pm 1, 0)$. Use the definition of the ellipse to get the final equation.

The common tangent at these points are $x = \pm 1$.

Since $x = 1$ is near to the focus $P\left(\frac{1}{2}, 1\right)$, this is the directrix of the required ellipse.



Therefore, by definition, the equation of the ellipse is

$$\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \left(\frac{1}{2}\right)^2 \left(\frac{x-1}{1}\right)^2$$

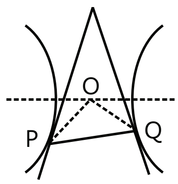
$$\Rightarrow \quad 9\left(x - \frac{1}{3}\right)^2 + 12(y - 1)^2 = 1.$$

Example 9: Prove that the angle subtended by any chord of a rectangular hyperbola at the centre is the supplement of the angle between the tangents at the ends of the chord.

Sol: Using the equation of chord, find the angle subtended at the centre and at the intersection of the tangents.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two ends of a chord of the rectangular hyperbola

$$x^2 - y^2 = 1 \qquad .....(i)$$



Now, 'm' of OP = $\frac{y_1}{x_1}$

'm' of OQ = $\frac{y_2}{x_2}$

$$\therefore \quad \tan\theta = \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1}{x_1}\cdot\frac{y_2}{x_2}} = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2},$$

Where $\angle POQ = \theta$,

The equations of tangents at P and Q are

$xx_1 - yy_1 = 1$ and $xx_2 - yy_2 = 1$.

Their slopes are $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$.

$$\therefore \quad \tan\phi = \frac{\frac{x_1}{y_1} - \frac{x_2}{y_2}}{1 + \frac{x_1}{y_1}\cdot\frac{x_2}{y_2}} = \frac{x_1y_2 - x_2y_1}{y_1y_2 + x_1x_2}$$

$\therefore$ $\tan\theta$ and $\tan\phi$ are equal in magnitude but opposite in sign

$\therefore \tan\theta = -\tan\phi = \tan(\pi - \phi)$

$\therefore \theta + \phi = \pi$. Hence, proved.

Example 10: If a chord of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that the locus of its middle point is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

Sol: Apply the condition of tangency in the equation of the chord.

Let $M(\alpha, \beta)$ be the middle point of the chord PQ of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ....(i)$$

The equation of the chord is

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}$$

$$\Rightarrow \quad y = -\frac{xb^2\alpha}{a^2\beta} + \frac{b^2}{\beta}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)$$

This line is tangent to hyperbola if

$$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad \left(\frac{b^2}{\beta}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)\right)^2 = a^2\left(\frac{b^2\alpha}{a^2\beta}\right)^2 - b^2$$

$$\Rightarrow \quad \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)^2 = \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}$$

$\therefore$ The equation of the required locus of the middle point (α, β) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

JEE Main/Boards

Exercise 1

Q.1 Find the centre, eccentricity and foci of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$

Q.2 Find the equation to the tangent to the hyperbola $4x^2 - 3y^2 = 13$ at the point (2,1).

Q.3 Show that the line $21x + 5y = 116$ touches the hyperbola $7x^2 - 5y^2 = 232$ and find the co-ordinates of the point of contact.

Q.4 Find the locus of the middle points of the portion of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ included between the axes.

Q.5 A point P moves such that the tangents PT_1 and PT_2 from it to the hyperbola $4x^2 - 9y^2 = 36$ are mutually perpendicular. Find the equation of the locus of P.

Q.6 Find the equations of the two tangents to the hyperbola $xy = 27$ which are perpendicular to the straight line $4x - 3y = 7$.

Q.7 Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x+3y+1 = 0$ for its asymptotes and which passes through the origin.

Q.8 Find the equation of chord of contact of tangents drawn from the point (–5, 2) to the hyperbola $xy = 25$.

Q.9 Find the eccentric angle of the point lying in fourth quadrant on the hyperbola $x^2 - y^2 = 4$ whose distance from the centre is 12 units.

Q.10 Find the acute angle between the asymptotes of $4x^2 - y^2 = 16$.

Q.11 If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other axis, shows that $a_1a_2 + b_1b_2 = 0$.

Q.12 Show that the area of the triangle formed by the two asymptotes of the rectangular hyperbola $xy = c^2$ and the normal at (x_1, y_1) on the hyperbola is $\frac{1}{2}\left[\frac{x_1^2 - y_1^2}{c}\right]^2$.

Q.13 PN is the ordinate of any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If Q divides AP in the ratio $a^2: b^2$, show that NQ is perpendicular to A¢P where A'A is the transverse axis of the hyperbola.

Q.14 A normal to the hyperbola $x^2 - 4y^2 = 4$ meets the x and y axes at A and B respectively. Find the locus of the point of intersection of the straight lines drawn through A and B perpendicular to the x and y axes respectively.

Q.15 In any hyperbola, prove that the tangent at any point bisects the angle between the focal distances of the point.

Q.16 If the normals at four points $P_i(x_i, y_i)$ i = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q(h, k), prove that

(i) $x_1 + x_2 + x_3 + x_4 = h$ (ii) $y_1 + y_2 + y_3 + y_4 = k$

(iii) $x_1x_2x_3x_4 = y_1y_2y_3y_4 = -c^4$

Q.17 Find the locus of the points of intersection of two tangents to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if sum of their slopes is a constant λ.

Q.18 A variable tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at Q and to the tangent at the vertex (a, 0) at R. Show that the locus of the mid point of QR is $x(4y^2 + b^2) = ab^2$.

Q.19 A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P and Q. Prove that the middle point of PQ lies on a parabola.

Q.20 Show that the locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Q.21 Given a hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 9$. Find the locus of mid point of chord of contact drawn from a point on the hyperbola to the circle.

Q.22 A rectangular hyperbola whose centre is C, is cut by a circle of radius r in four points P, Q, R, S. Prove that $CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$.

Q.23 The normal at the three points P, Q, R on a rectangular hyperbola, intersect at a point S on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.

Q.24 A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola and one of its diagonals is a chord of the hyperbola, show that the other diagonal passes through the centre.

Q.25 If the straight line $y = mx + 2c\sqrt{-m}$ touches the hyperbola $xy = c^2$ then the co-ordinates of the point contact are (……………..,)

Q.26 If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then t^3t_1 has the value equal to ……………………

Exercise 2

Single Correct Choice Type

Q.1 The line 5x + 12y = 9 touches the hyperbola $x^2 - 9y^2 = 9$ at the point-

(A) (–5, 4/3) (B) (5, –4/3)

(C) (3, –1/2) (D) None of these

Q.2 The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is-

(A) $\frac{2a^2}{b}$ (B) $\frac{2b^2}{a}$ (C) $\frac{b^2}{a}$ (D) $\frac{a^2}{b}$

Q.3 The area of the square whose sides are the directrixes of the hyperbola $x^2 - y^2 = a^2$ and its conjugate hyperbola, is-

(A) a^2 (B) $2a^2$ (C) $4a^2$ (D) $8a^2$

Q.4 The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line 5x + 2y – 10 = 0 is -

(A) Zero (B) 1 (C) 2 (D) 4

Q.5 If m is a variable, the locus of the point of intersection of the lines $\frac{x}{3} - \frac{y}{2} = m$ and $\frac{x}{3} + \frac{y}{2} = \frac{1}{m}$ is a/an -

(A) Parabola (B) Ellipse

(C) Hyperbola (D) None of these

Q.6 The eccentricity of the hyperbola with its principal axes along the co-ordinate axes and which passes through (3, 0) and $(3\sqrt{2}, 2)$ is-

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{\sqrt{5}}{3}$ (D) $\frac{2}{3}$

Q.7 The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is-

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) 1/2

Q.8 An ellipse and a hyperbola have the same centre origin, the same foci and the minor-axis of the one is the same as the conjugate axis of the other. If e_1, e_2 be their eccentricities respectively, then $\frac{1}{e_1^2} + \frac{1}{e_2^2} =$

(A) 1 (B) 2 (C) 4 (D) None of these

Q.9 Which of the following pair may represent the eccentricities of two conjugate hyperbola for all $\alpha \in (0, \pi/2)$?

(A) sin α, cos α (B) tan α, cot α

(C) sec α, cosec α (D) 1 + sinα, 1 + cosα

Q.10 The number of normals to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from an external point is-

(A) 2 (B) 4 (C) 6 (D) 5

Q.11 A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its-

(A) Orthocentre (B) Circum centre

(C) Centroid (D) Incentre

Q.12 If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t′ then-

(A) $t' = \frac{-1}{t^3}$ (B) $t' = \frac{1}{t}$

(C) $t' = \frac{1}{t^2}$ (D) t¢² = $\frac{-1}{t^2}$

Q.13 The centre of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$ is-

(A) (2, 3) (B) (–2, –3) (C) (–2, 3) (D) (2, –3)

Q.14 The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to-

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Q.15 The number of points from where a pair of perpendicular tangents can be drawn to the hyperbola, $x^2\sec^2\alpha - y^2\text{cosec}^2\alpha = 1$, $\alpha \in (0, \pi/4)$ is-

(A) 0 (B) 1 (C) 2 (D) Infinite

Q.16 If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then eccentricity of hyperbola is-

(A) $\sqrt{2}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) None of these

Q.17 If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) and $x^2 - y^2 = c^2$ cut at right angles then-

(A) $a^2 + b^2 = 2c^2$ (B) $b^2 - a^2 = 2c^2$

(C) $a^2 - b^2 = 2c^2$ (D) $a^2b^2 = 2c^2$

Q.18 Two conics $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x^2 = -\frac{1}{b}y$ intersect if -

(A) $0 < b \le \frac{1}{2}$ (B) $0 < a < \frac{1}{2}$

(C) $a^2 < b^2$ (D) $a^2 > b^2$

Q.19 The locus of the mid points of the chords passing through a fixed point (α, β) of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is-

(A) A circle with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(B) An ellipse with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(C) A hyperbola with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(D) Straight line through $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Q.20 If the eccentricity of the hyperbola $x^2 - y^2\sec^2 a = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2 a + y^2 = 25$, then a value of α is-

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.21 For all real values of m, the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to the curve-

(A) $9x^2 + 4y^2 = 36$ (B) $4x^2 + 9y^2 = 36$

(C) $9x^2 - 4y^2 = 36$ (D) $4x^2 - 9y^2 = 36$

Q.22 Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is-

(A) $y + mx = 0$ (B) $y - mx = 0$

(C) $my - mx = 0$ (D) $my + x = 0$

Q.23 The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is-

(A) $3x - 4y = 4$ (B) $3y - 4x + 4 = 0$

(C) $4x - 4y = 3$ (D) $3x - 4y = 2$

Previous Years' Questions

Q.1 The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, |r| < 1$ represents- ***(1981)***

(A) An ellipse (B) A hyperbola

(C) A circle (D) None of these

Q.2 Let P(a sec θ, b tan θ) and Q(a sec ϕ, b tan ϕ), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of the intersection of the normals at P and Q, then k is equal to- ***(1999)***

(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$ (C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

Q.3 If x = 9 is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is- ***(1999)***

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x - 9 = 0$

(D) $9x^2 - 8y^2 + 18x + 9 = 0$

Q.4 For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant with change in 'α' ? ***(2003)***

(A) Abscissa of vertices (B) Abscissa of foci

(C) Eccentricity (D) Directrix

Q.5 If the line $2x + \sqrt{6}\,y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is- ***(2004)***

(A) $(-2, \sqrt{6})$ (B) $(-5, 2\sqrt{6})$

(C) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (D) $(4, -\sqrt{6})$

Q.6 If e_1 is the eccentricity of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and e_2 is the eccentricity of the hyperbola passing through the foci of the ellipse and $e_1e_2 = 1$, then equation of the hyperbola is- ***(2006)***

(A) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (B) $\frac{x^2}{16} - \frac{y^2}{9} = -1$

(C) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (D) None of these

Q.7 A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is- ***(2007)***

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Q.8 Consider a branch of the hyperbola

$x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$

with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is- ***(2008)***

(A) $1-\sqrt{\frac{2}{3}}$ sq. unit (B) $\sqrt{\frac{3}{2}}-1$ sq. unit

(C) $1+\sqrt{\frac{2}{3}}$ sq. unit (D) $\sqrt{\frac{3}{2}}+1$ sq. unit

Q.9 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. If the normal at the point P intersect the x-axis at (9, 0), then the eccentricity of the hyperbola is- ***(2011)***

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Q.10 The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is: ***(2016)***

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) $\frac{4}{3}$

JEE Advanced/Boards

Exercise 1

Q.1 Find the equation to the hyperbola whose directrix is $2x + y = 1$ focus (1, 1) and eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

Q.2 The hyperbola $x^2/a^2 - y^2/b^2 = 1$ passes through the point of inter-section of the lines. $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ and the latus rectum is $32\sqrt{3}/5$. Find 'a' & 'b'.

Q.3 For the hyperbola $x^2/100 - y^2/25 = 1$, prove that the

(i) eccentricity = $\sqrt{5}/2$

(ii) SA.S'A = 25, where S and S' are the foci and A is the vertex.

Q.4 Find the centre, the foci, the directrices, the length of the latus rectum, the length and the equations of the axes and the asymptotes of the hyperbola

$16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

Q.5 If a rectangular hyperbola have the equation, $xy = c^2$, prove that the locus of the middle point of the chords of constant length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$.

Q.6 If m_1 and m_2 are the slopes of the tangents to the hyperbola $x^2/25 - y^2/16 = 1$ through the point (6, 2), find the value of (i) $m_1 + m_2$ and (ii) m_1m_2.

Q.7 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Q.8 If θ_1 and θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, then show that

$$\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}+\frac{e-1}{e+1} = 0.$$

Q.9 If C is the centre of hyperbola $x^2/a^2 - y^2/b^2 = 1$, S, S' its foci and P a point on it. Prove that SP.S'P = $CP^2 - a^2 + b^2$.

Q.10 Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

Q.11 If the tangent at the point (h, k) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ cuts the auxiliary circle in points whose ordinates are y_1 and y_2 then prove that $1/y_1 + 1/y_2 = 2/k$.

Q.12 Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x-axis. If $\tan\theta \,.\, \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

Q.13 Find the number of normal which can be drawn from an external point on the hyperbola $\frac{x^2}{a^2}\frac{-y^2}{b^2}=1$.

Q.14 The perpendicular from the centre upon the normal on any point of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets at R. Find the locus of R.

Q.15 If the normal at a point P to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the x-axis at G, show that SG = e. SP, S being the focus of the hyperbola.

Q.16 Show that the area of the triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$ is a^2.

Q.17 Find the locus of the middle point of the chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

Q.18 The line $y = mx + 6$ is tangent to the hyperbola $\frac{x^2}{10^2}-\frac{y^2}{7^2}=1$ at certain point. Find the value of m.

Q.19 A point P divides the focal length of the hyperbola $9x^2 - 16y^2 = 144$ in the ratio S¢P: SP = 2: 3 where S and S′ are the foci of the hyperbola. Through P a straight line is drawn at an angle of 135° to the axes OX. Find the points of intersection of the line with the asymptotes of the hyperbola.

Q.20 Find the equation of tangent to the hyperbola $x^2 - 2y^2 = 18$ which is perpendicular to the line $y = x$.

Q.21 If a chord joining the points $P(a\sec\theta, a\tan\theta)$ and $Q(a\sec\phi, a\tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$.

Q.22 Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

Q.23 Let 'p' be the perpendicular distance from the centre C of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to the tangent drawn at point R on the hyperbola. If S and S′ are the two foci of the hyperbola, then show that

$(RS + RS')^2 = 4a^2\left(1+\frac{b^2}{p^2}\right)$.

Exercise 2

Single Correct Choice Type

Q.1 Locus of middle point of all chords of $\frac{x^2}{4}-\frac{y^2}{9}$ = 1. Which are at distance of '2' units from vertex of parabola $y^2 = -8ax$ is-

(A) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)=\frac{xy}{6}$ (B) $\left(\frac{x^2}{4}-\frac{y^2}{9}\right)^2=4\left(\frac{x^2}{16}+\frac{y^2}{81}\right)$

(C) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\left(\frac{x^2}{9}+\frac{y^2}{4}\right)$ (D) None of these

Q.2 Tangents at any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the axes at A and B respectively. If the rectangle OAPB (where O is origin) is completed then locus of point P is given by-

(A) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ (B) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

(C) $\frac{a^2}{y^2}-\frac{b^2}{x^2}=1$ (D) None of these

Q.3 The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is-

(A) $(x^2 - y^2)2 = 4c^2xy$ (B) $(x^2 + y^2)2 = 2c^2xy$

(C) $(x^2 - y^2) = 4x^2xy$ (D) $(x^2 + y^2)^2 = 4c^2xy$

Q.4 The point of intersection of the curves whose parametric equation are $x = t^2 + 1$, $y = 2t$ and $x = 2s$, $y = 2/s$ is given by-

(A) (1, –3) (B) (2, 2) (C) (–2, 4) (D) (1, 2)

Q.5 P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at p meets the transverse axis at T. If O is the centre to the hyperbola, the OT.ON is equal to-

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2

Q.6 The equation to the chord joining two point (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is-

(A) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ (B) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$

(C) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$ (D) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

Q.7 The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci, is-

(A) $\frac{4}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) None of these

Q.8 The equation to the chord of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3) is-

(A) $5x + 3y = 9$ (B) $5x - 3y = 16$

(C) $5x + 3y = 16$ (D) $5x - 3y = 9$

Q.9 The differential equation $\frac{dx}{dy} = \frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity-

(A) $\sqrt{\frac{3}{5}}$ (B) $\sqrt{\frac{5}{3}}$ (C) $\sqrt{\frac{2}{5}}$ (D) $\sqrt{\frac{5}{2}}$

Multiple Correct Choice Type

Q.10 Equation of a tangent passing through (2, 8) to the hyperbola $5x^2 - y^2 = 5$ is-

(A) $3x - y + 2 = 0$ (B) $3x + y - 14 = 0$

(C) $23x - 3y - 22 = 0$ (D) $3x - 23y + 178 = 0$

Q.11 The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represent a hyperbola -

(A) The length of whose transverse axis is $4\sqrt{3}$

(B) The length of whose conjugate axis is 8

(C) Those centre is (1, 2)

(D) Those eccentricity is $\sqrt{\frac{19}{3}}$

Q.12 A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is-

(A) $y = \frac{3}{\sqrt{7}}x + \frac{15}{\sqrt{7}}$ (B) $y = 3\frac{\sqrt{2}}{7}x + \frac{15}{\sqrt{7}}$

(C) $y = 2\frac{\sqrt{3}}{7}x + 15\sqrt{7}$ (D) $y = 3\frac{\sqrt{2}}{7}x - \frac{15}{\sqrt{7}}$

Q.13 Which of the following equation in parametric form can represent a hyperbola, profile, where 't' is a parameter

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$

(D) $x^2 - 6 = 2\cot$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Q.14 Circles are drawn on chords of the rectangular hyperbola $xy = a^2$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose co-ordinates are-

(A) (c, c) (B) (c, –c) (C) (–c, c) (D) (–c, –c)

Q.15 If the normal at (x_i, y_i) i = 1, 2, 3, 4 to its rectangular hyperbola $xy = 2$ meet at the point (3, 4), then-

(A) $x_1 + x_2 + x_3 + x_4 = 3$ (B) $y_1 + y_2 + y_3 + y_4 = 4$

(C) $x_1x_2x_3x_4 = -4$ (D) $y_1y_2y_3y_4 = -4$

Q.16 If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is-

(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

Q.17 The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$ is-

(A) $\sqrt{\left(\frac{17}{20}\right)}$ (B) $-\sqrt{\left(\frac{17}{20}\right)}$ (C) $\sqrt{\left(\frac{20}{17}\right)}$ (D) $-\sqrt{\left(\frac{20}{17}\right)}$

Q.18 The equation $\frac{x^2}{12-k} + \frac{y^2}{k-8} = 1$ represents-

(A) A hyperbola if $k < 8$

(B) An ellipse if $8 < k < 12$, $k \neq 10$

(C) A hyperbola if $8 < k < 12$

(D) Circle if $k = 10$

Q.19 Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is-

(A) $y = x + \sqrt{a^2 - b^2}$ (B) $y = x - \sqrt{a^2 - b^2}$

(C) $y = -x + \sqrt{a^2 - b^2}$ (D) $y = -x - \sqrt{a^2 - b^2}$

Q.20 The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular the line $y = x$ are-

(A) $y = -x + 7$ (B) $y = -x + 3$

(C) $y = -x - 4$ (D) $y = -x - 3$

Q.21 The co-ordinate of a focus of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ are-

(A) (–1, 1) (B) (6, 1) (C) (4, 1) (D) (–6, 1)

Q.22 If $(a \sec\theta, b\tan\theta)$ & $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}$ equal to-

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{e+1}{1-e}$ (D) $\frac{e+1}{e-1}$

Q.23 If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then-

(A) PG = PC (B) Pg = PC (C) PG = Pg (D) Gg = PC

Previous Years' Questions

Q.1 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_1, y_1)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then- ***(1998)***

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1x_2x_3x_4 = c^4$ (D) $y_1y_2y_3y_4 = c^4$

Q.2 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal to that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then ***(2009)***

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are (± 1, 0)

(C) Equation of ellipse is $x^2+2y^2 = 4$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Q.3 Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then ***(2011)***

(A) The equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) A focus of the hyperbola is (2, 0)

(C) The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) The equation of the hyperbola is $x^2 - 3y^2 = 3$

Q.4 For any real t, $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Find the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_2$. ***(1982)***

Q.5 Tangents are drawn from any point o the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid point of the chord of contact. ***(2005)***

Paragraph 6 to 7:

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B. ***(2010)***

Q.6 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is-

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Q.7 Equation of the circle with AB as its diameter is-

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Q.8 The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is …….. ***(2010)***

Q.9 Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is ***(2008)***

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

Q.10 Match the conics in column I with the statements/ expressions in column II. ***(2009)***

Column I	Column II
(A) Circle	(p) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\left\|z+2\right\| - \left\|z-2\right\| = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \left\|z\right\|^2 + 1$

Q.11 The line 2x + y = 1 is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is ***(2010)***

Q.12 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is ***(2011)***

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Q.13 Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then ***(2011)***

(A) The equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) A focus of the hyperbola is (2, 0)

(C) The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) The equation of the hyperbola is $x^2 - 3y^2 = 3$

Q.14 Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line 2x −y = 1. The points of contact of the tangents on the hyperbola are ***(2012)***

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.7 Q.12 Q.15 Q.21
Q.24 Q.25 Q.27

Exercise 2

Q.3 Q.11 Q.18 Q.19

Previous Years' Questions

Q.2 Q.6 Q.8

JEE Advanced/Boards

Exercise 1

Q.5 Q.11 Q.12 Q.15
Q.18 Q.22 Q.25

Exercise 2

Q.3 Q.6 Q.8 Q.11
Q.17 Q.23

Previous Years' Questions

Q.2 Q.3 Q.4 Q.8

Answer Key

JEE Main/Boards

Exercise 1

Q.1 C(1, 2), e = 5/4, (6, 2) and (–4, 2)

Q.2 $8x - 3y - 13 = 0$

Q.3 (6, –2)

Q.4 $a^2y^2 - b^2x^2 = 4x^2y^2$

Q.5 $x^2 + y^2 = 5$

Q.6 $3x + 4y \pm 36 = 0$

Q.7 $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

Q.8 $2x - 5y = 50$

Q.9 $7\frac{\pi}{4}$ rad.

Q.10 $\tan^{-1}\frac{4}{3}$

Q.14 $4x^2 - y^2 = 25$

Q.17 $\lambda(x^2 - a^2) = 2xy$

Q.21 $9x^2 - \frac{81y^2}{4} = (x^2 + y^2)^2$ is the required locus.

Q.25 $\frac{c}{\sqrt{-m}}$, $c\sqrt{-m}$

Q.26 –1

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** A	**Q.3** B	**Q.4** A	**Q.5** C	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** C	**Q.10** B	**Q.11** A	**Q.12** A
Q.13 A	**Q.14** D	**Q.15** D	**Q.16** C	**Q.17** C	**Q.18** B
Q.19 C	**Q.20** B	**Q.21** D	**Q.22** A	**Q.23** A	

Previous Years' Questions

Q.1 B	**Q.2** D	**Q.3** B	**Q.4** B	**Q.5** D	**Q.6** B
Q.7 A	**Q.8** B	**Q.9** B	**Q.10** B		

JEE Advanced/Boards

Exercise 1

Q.1 $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$; $\frac{\sqrt{48}}{5}$

Q.2 $a^2 = 25/2$; $b^2 = 16$

Q.6 (i) 24/11 (ii) 20/11

Q.7 $x + y \pm 3\sqrt{3} = 0$

Q.10 $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$

Q.14 $(x^2+y^2)^2 (a^2y^2-b^2y^2) = x^2y^2(a^2+b^2)^2$

Q.18 $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Q.19 (–4, 3) and $\left(-\frac{4}{7}, -\frac{3}{7}\right)$

Q.22 $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** A	**Q.3** D	**Q.4** B	**Q.5** B	**Q.6** A
Q.7 C	**Q.8** C	**Q.9** B			

Multiple Correct Choice Type

Q.10 A,C	**Q.11** B, C, D	**Q.12** B, D	**Q.13** A, C, D	**Q.14** A, D	**Q.15** A, B, C, D
Q.16 A, D	**Q.17** A, B	**Q.18** A, B, D	**Q.19** A, B, C, D	**Q.20** B, D	**Q.21** C, D
Q.22 B, C	**Q.23** A, B, C				

Previous Years' Questions

Q.1 A, B, C, D	**Q.2** A, B	**Q.3** B, D	**Q.4** t_1	**Q.5** $\frac{x^2}{9}-\frac{y^2}{4} = \frac{(x^2+y^2)^2}{81}$
Q.6 B	**Q.7** A	**Q.8** 2	**Q.9** B	**Q.10** $A \to p; B \to s, t; C \to r; D \to q, s$
Q.11 2	**Q.12** B	**Q.13** B, D	**Q.14** A, B	

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $9(x^2 - 2x + 1) - 16(y^2 - 4y + 4) - 199 - 9 + 64 = 0$

$9(x - 1)^2 - 16(y - 2)^2 = 144$

$\frac{(x-1)^2}{16}-\frac{(y-2)^2}{9} = 1$

so $a = \sqrt{16} = 4$ & $b = \sqrt{9} = 3$

so $e^2 = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{3^2}{4^2}} \Rightarrow e = \frac{5}{4}$

Now centre would be where

$x - 1 = 0$ and $y - 2 = 0$

$\Rightarrow x = 1$ and $y = 2$

and focii distance = $ae = 4 \times \frac{5}{4}$ (in x-direction)

focii = (1 + 5, 2) and (1 – 5, 2)

(6, 2) and (–4, 2)

Sol 2: Tangent $\Rightarrow \frac{x.x_1}{a^2}-\frac{y.y_1}{b^2} = 1$

$\frac{2.x}{13/4}-\frac{1.y}{13/3} = 1 \Rightarrow 8x - 3y = 13$

Sol 3: We have $y = \left(-\frac{21}{5}\right)x+\left(\frac{116}{5}\right)$

[y = mx + c form]

Now, y = mx + c is tangent when

$a^2m^2 - b^2 = c^2$

So $\frac{232}{7}.\left(\frac{21}{5}\right)^2-\left(\frac{232}{5}\right)$

$= \frac{63\times 232}{25}-\frac{232\times 5}{25}\frac{(116)^2}{25} = \frac{(116)^2}{25}$

So LHS = RHS

Hence, the given line is tangent

Now tangent

$\Rightarrow \frac{x.x_1}{282/7}-\frac{y.y_1}{232/5} = 1$

Now, comparing with the given tangent

$\frac{21\times 232}{x_1\times 7} = \frac{5\times 232}{-5y_1} = \frac{116}{1}$

$\Rightarrow x_1 = 6$ and $y_1 = -2$

Sol 4: Tangent = $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Now the tangent cuts the axes at (a cosθ, 0) and (0, b cot θ)

mid points ⇒ $\frac{a\cos\theta}{2} = h$ and $k = \frac{b\cot\theta}{2}$

$\Rightarrow \frac{a}{2h} = \sec\theta$ and $\frac{b}{2k} = \tan\theta$

$\Rightarrow \frac{a^2}{4h^2} - \frac{b^2}{4k^2} = 1 \Rightarrow \frac{a^2}{h^2} - \frac{b^2}{k^2} = 4$

Replacing h and k, we get locus as

$\frac{a^2}{h^2} - \frac{b^2}{k^2} = 4 \Rightarrow a^2y^2 - b^2x^2 = 4x^2y^2$

Sol 5: We have tangents

$\Rightarrow y = mx \pm \sqrt{a^2m^2 - b^2} \Rightarrow y = mx \pm \sqrt{9m^2 - 4}$

$\Rightarrow (y - mx)^2 = (\sqrt{9m^2 - 4})^2$

$y^2 + m^2x^2 - 4mxy = 9m^2 - 4$

$\Rightarrow (9 - x^2)m^2 + (4xy)m\ (4 + y^2) = 0$

Now h, k would satisfy this

$\Rightarrow (9 - h^2)m^2 + (4hk)m\ (4 + k^2) = 0$

So, $m_1m_2 = \frac{-(4+k^2)}{9-h^2} = 1$

$\Rightarrow 4 + k^2 = 9 - h^2 \Rightarrow h^2 + k^2 = 5$

Hence, the locus is $x^2 + y^2 = 5$

Sol 6: We have $m_1 = \frac{4}{3}$ (given line)

Given $m_1.\ m_2 = -1 \Rightarrow m_2 = -\frac{3}{4}$

So $y = -\frac{3}{4}x + c$

⇒ Now putting this in the equation

$x.\left(-\frac{3}{4}x + c\right) = 27 \Rightarrow -\frac{3}{4}x^2 + cx = 27$

$\Rightarrow \frac{3}{4}x^2 - cx + 27 = 0$

has only one solution ⇒ D = 0

$\Rightarrow b^2 - 4ac = 0$

$c^2 - 4 \times \frac{3}{4} \times 27 = 0$

$\Rightarrow c = \pm 3 \times 3 = \pm 9$

$y = -\frac{3}{4}x + 9$ or $y = -\frac{3}{4}x - 9$

equation of asymptotes ⇒ $y = \pm\frac{b}{a}x$

Sol 7: Equation $(3x - 4y + 7)(4x + 3y + 1) + c = 0$

$\Rightarrow 12x^2 - 12y^2 - 7xy + 31x + 17y + (7 + c) = 0$

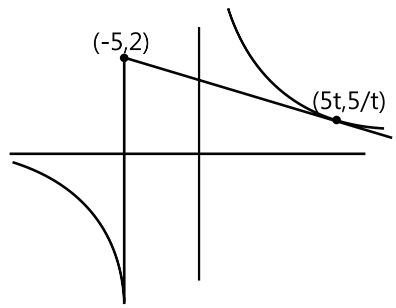
Now, it passes through origin

$\Rightarrow 7 + c = 0 \Rightarrow c = -7$

⇒ equation = $12x^2 - 12y^2 - 7xy + 31x + 17y = 0$

Sol 8: xy = 25 ⇒ parametric ⇒ 5t & $y = \frac{5}{t}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-5)/t^2}{5} = \frac{-1}{t^2}$



Now slope = $\frac{-1}{t^2} = \frac{(5/t - 2)}{(5t+5)}$

$\Rightarrow \frac{-1}{t} = \frac{(5-2t)}{5t(t+1)}$

$\Rightarrow -5(t + 1) = t(5 - 2t)$

$\Rightarrow 2t^2 - 10t - 5 = 0$

Now chord of contact

$\Rightarrow y = \frac{(5/t_1 - 5/t_2)x}{(5t_1 - 5t_2)} + c = \frac{-x}{t_1.t_2} + c$

Now, $\frac{5}{t_1} = \frac{-5t_1}{t_1.t_2} + c$

$\Rightarrow c = 5\left[\frac{1}{t_1} + \frac{1}{t_2}\right] \Rightarrow y = \frac{-x}{t_1 + t_2} + 5\left[\frac{t_1 + t_2}{t_1.t_2}\right]$

$\Rightarrow y = \frac{+x}{(+5/2)} + \frac{5.[5]}{-5/2}$

$\Rightarrow y = \frac{2x}{5} - 10 \Rightarrow 5y = 2x - 50$

Sol 9: 4. $\sec^2\theta + 4\tan^2\theta = 12$

$\Rightarrow \sec^2\theta + \tan^2\theta = 3$

$\Rightarrow 2\tan^2\theta = 2$

$\Rightarrow \tan\theta = \pm 1$

$\Rightarrow \theta = \tan^{-1}(-1)$ [from 4th quadrant]

$\Rightarrow \theta = \frac{7\pi}{4}$

Sol 10: $\frac{x^2}{4} - \frac{y^2}{16} = 1$

asymptotes $\Rightarrow \frac{x}{2} - \frac{y}{4} = 0$ and $\frac{x}{2} + \frac{y}{4} = 0$

$\Rightarrow y = 2x$ and $y = -2x$

Now angle $\Rightarrow \tan^{-1}\frac{(m_1 - m_2)}{1 + m_1.m_2}$

$= \tan^{-1}\frac{[2-(-2)]}{1-4} = \tan^{-1}\frac{4}{3}$

Sol 11: Equation of hyperbola

$\Rightarrow ax\cos q_1 + by\cot q_1 = a^2 + b^2$

[a cos θ, b cot q]

Equation of tangent

$\Rightarrow \frac{x}{a}\sec\theta_2 - \frac{y}{b}\tan\theta_2 = 1$

[a secq_2, b tanq_2]

Intersection of tangents

$\Rightarrow (a\cos q_2, 0)$ and $(0, -b\cot q_2)$

Intersection of normal

$\Rightarrow \left(\frac{\sec\theta_1(a^2+b^2)}{a}, 0\right)$ and $\left(0, \frac{(a^2+b^2)}{a}.\tan\theta_1\right)$

Now, $a_1.a_2 + b_1.b_2 = \frac{a\cos\theta_2.\sec\theta_1.(a^2+b^2)}{a}$

$+ (-b).\cot q_2 \times \frac{(a^2+b^2)}{b}\tan q_1$

$= [\cos q_2.\sec q_1 - \cot q_2.\tan q_1](a^2 + b^2)$

$= \left[\frac{\cos\theta_2}{\cos\theta_1} - \frac{\cos\theta_2.\sin\theta_1}{\cos\theta_1.\sin\theta_2}\right](a^2 + b^2]$

Now if the point is same:

[i. e. , $q_1 = q_2$]

$(1 - 1)(a^2 + b^2) = 0$

Sol 12: The asymptotes are x = 0, y = 0

Let, $x = ct, y = \frac{c}{t}$,

tangent slope $= -\frac{1}{t^2}$

Now normal at x_1, y_1

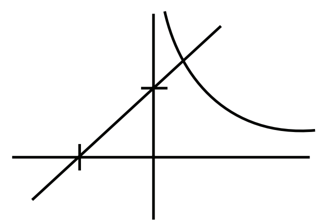
$\Rightarrow$ b has slope $= t^2$

so, $\frac{(y - c/t)}{(x - ct)} = t^2$

$\Rightarrow \left(\frac{y - y_1}{x - x_1}\right) = \left(\frac{x_1}{c}\right)^2$

$\Rightarrow y - y_1 = \left(\frac{x_1^2}{c^2}\right).(x - x_1)$

$y = \frac{x_1^2}{c^2}.x + \left(y_1 - \frac{x_1^3}{c^2}\right)$



Now putting

$x = 0 \Rightarrow \boxed{y = y_1 - \frac{x_1^3}{c^2}}$

and putting y = 0,

$\frac{x_1^2}{c^2}.x = \left(\frac{x_1^3}{c^2} - y_1\right)$

$x = \frac{(x_1^3 - y_1c^2)}{x_1^2}$

$\boxed{x = x_1 - \frac{y_1}{x_1^2}.c^2}$

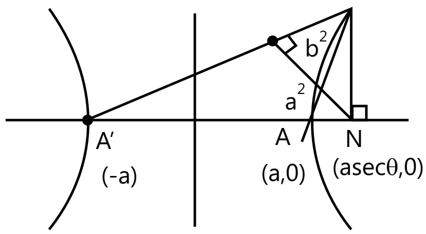
Area $= \frac{1}{2}\left|\left(y_1 - \frac{x_1^3}{c^2}\right)\left(x_1 - \frac{y_1}{x_1^2}c^2\right)\right|$

$= \frac{1}{2}\left|\left[y_1.x_1 + x_1.y_1 - \frac{y_1^2.c^2}{x_1^2} - \frac{x_1^4}{c^2}\right]\right|$

$$= \frac{1}{2}\left|\left[\frac{c^2x^2x_1.y_1}{c^2} - \frac{x_1^4}{c^2} - \frac{y_1^2.c^2}{(c^2/y_1)^2}\right]\right|$$

$$= \frac{1}{2}\left|\frac{x_1^4}{c^2} + \frac{y_1^4}{c^2} - 2(x_1.y_1)^2\right| = \frac{1}{2}\left|\frac{x_1^2 - y_1^2}{c}\right|^2$$

Sol 13:



$$Q = \left(\frac{ab^2 + a^3\sec\theta}{a^2 + b^2}, a^2b\tan\theta\right)$$

slope of NQ

$$= \frac{a^2b\tan\theta}{\frac{ab^2 + a^3\sec\theta}{a^2 + b^2} - a\sec\theta} = \frac{a^2b\tan\theta - 0}{ab^2 - ab^2\sec\theta}$$

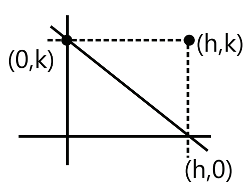
$$= \frac{a\tan\theta}{b - b\sec\theta} = \frac{a\tan\theta}{b(1 - \sec\theta)}$$

slope of A$\Rightarrow$ P = $\frac{(b\tan\theta - 0)}{a(\sec\theta + 1)}$

$$\Rightarrow m_1.m_2 = \frac{b\tan\theta.a\tan\theta}{ab(1 - \sec^2\theta)} = \frac{\tan^2\theta}{-\tan^2\theta} = -1$$

$\Rightarrow$ Hence proved.

Sol 14:



$ax\cos\theta + by\cot\theta = a^2 + b^2$

Which should be same as $\frac{x}{h} + \frac{y}{k} = 1$

$$\Rightarrow \frac{a\cos\theta}{1/h} = \frac{b\cot\theta}{1/k} = \frac{a^2 + b^2}{1}$$

$$\Rightarrow h = \frac{a^2 + b^2}{a\cos\theta}, \; k = \frac{a^2 + b^2}{b\cot\theta}$$

$$\Rightarrow \frac{ah}{a^2 + b^2} = \sec\theta, \; \frac{bk}{a^2 + b^2} = \tan\theta$$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \frac{a^2h^2}{(a^2 + b^2)^2} - \frac{b^2k^2}{(a^2 + b^2)^2} = 1$$

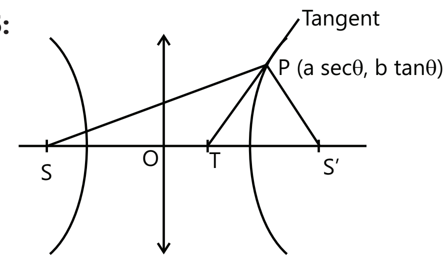
$$\Rightarrow \frac{a^2x^2}{(a^2 + b^2)^2} - \frac{b^2y^2}{(a^2 + b^2)^2} = 1$$

$\Rightarrow a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

here, $a^2 = 4$, $b^2 = 1$

$\Rightarrow 4x^2 - y^2 = 25$

Sol 15:



Coordinates of $S' = (ae, 0)$

$S = (-ae, 0)$

$P = (a\sec\theta, b\tan\theta)$

Tangent at P cut the x axis at point T.

Eq. of target at P $= \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Coordinates of $T = \left(\frac{a}{\sec\theta}, \frac{-b}{\tan\theta}\right)$

$T = (a\text{Cos}\theta, -b\cot\theta)$

$\Rightarrow ST = ae + a\cos\theta$

$\Rightarrow S'T = ae - a\cos\theta$

$$\Rightarrow \frac{ST}{ST} = \frac{ae + a\cos\theta}{ae - a\cos\theta} = \frac{e + \cos\theta}{e - \cos\theta}$$

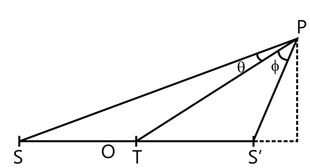
Similarly on evaluating PS & Ps

$$\Rightarrow \frac{PS}{PS'} = \frac{PS}{PS'}$$

$\therefore$ Area of $\Delta PTS' = \frac{S'T \times -h}{2}$

Area of $\Delta PTS = \frac{S'T \times h}{2}$

Using fine rule:



Area of $\Delta PTS' = \frac{PS' \times PT\sin\phi}{2}$

Area of $\Delta PTS' = \frac{PS' \times PT\sin\theta}{2}$

$$\frac{\text{Area of } \Delta PTS'}{\text{Area of } \Delta PTS'} = \frac{PS\sin\theta}{PS'\text{Sin}\phi} = \frac{ST}{S'T}$$

For θ = d the conditions necessary are met & PT bisect the angle sps'

Sol 16: Let x = ct and y = $\frac{c}{t}$

then, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1/t}{t} = \frac{-1}{t^2}$

so normal = t^2

thus, $\frac{(y-c/t)}{(x-ct)} = t^2 \Rightarrow y - \frac{c}{t} = t^2x - ct^3$

$\Rightarrow ty - c = t^3x - ct^4 \Rightarrow ct^4 - t^3x + ty - c = 0$

this satisfies h, k

thus, $ct^4 - ht^3 + kt - c = 0$

thus, $\sum_{i=1}^{h} t_i = \frac{h}{c} \Rightarrow \sum_{i=1}^{h} c.t_i = h$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = h$

similarly we have $t_1.\ t_2.\ t_3.\ t_4 = -1$ and

$(t_1.\ t_2.\ t_3) + t_2.\ t_3.\ t_4 + t_3.\ t_4.\ t_1 + t_4.\ t_1.\ t_2 = \frac{-k}{e}$

dividing by $\prod_{i=1}^{h} t_i$ both sides

$\Rightarrow \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = k \Rightarrow \boxed{\Sigma y_i = k}$

(iii) = –1

$\Rightarrow -c^4 = c^4.$

$-c^4 = x_1.\ x_2.\ x_3.\ x_4$

And = –1

$\Rightarrow -c^4 = \frac{c}{t_1}.\frac{c}{t_2}.\frac{c}{t_3}.\frac{c}{t_4} \qquad \Rightarrow y_1.\ y_2.\ y_3.\ y_4 = -c_1$

Sol 17: tangent $\Rightarrow \frac{a.\sec\theta.x}{a^2} - \frac{b.\tan.y}{b^2} = 1$

$$\boxed{\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1}$$

tangent $\Rightarrow$ y = mx ± $\sqrt{a^2m^2 - b^2}$

tangent passes through h, k

$k = mh + \sqrt{a^2m^2 - b^2}$

$(k - mh)^2 = a^2m^2 - b^2$

$k^2 + m^2h^2 - 2m.\ kh = a^2m^2 - b^2$

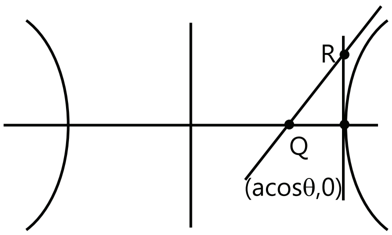
$(a^2 - h^2)m^2 + 2m.\ kh - (b^2 + k^2)$

Now $m_1 + m_2 = \lambda = \frac{-2kh}{a^2 - h^2}$

$\Rightarrow a^2 - h^2 = \frac{-2kh}{\lambda} \Rightarrow \boxed{a^2 - x^2 = \frac{-2xy}{\lambda}}$

Sol 18: x = a secθ, b = tan θ

$\frac{x.\sec\theta}{a} - \frac{y}{b}.\tan\theta = 1$



Now coordinate of Q $\Rightarrow$ x = a cosθ.

Coordinates of R

$\Rightarrow \boxed{\frac{(\sec\theta - 1)b}{\tan\theta} = y}$

Now

2h = a + a cosθ = a(1 + cosθ). . (i)

and $2k = 0 + \frac{(\sec\theta - 1)b}{\tan\theta}$

$2k = \frac{(1-\cos\theta)b}{\sin\theta}$

Now

$4k^2 + b^2 = \frac{(1-\cos\theta)^2b^2 + b^2\sin^2\theta}{\sin^2\theta}$

$(4k^2 + b^2) = \frac{(2 - 2\cos\theta)^2 b^2}{\sin^2\theta}$

$= \frac{2(1-\cos\theta)b^2}{\sin^2\theta} = \frac{2.(b^2)}{(1+\cos\theta)}$ from (i)

$\Rightarrow \quad 4k^2 + b^2 = \frac{2b^2}{(2h/a)}$

$\Rightarrow \quad ab^2 = h(4k^2 + b^2)$

Sol 19: Let x = 2at

y = at^2

then $\frac{dy}{dx} = \frac{2at}{2a} = t$

thus equation of tangent

$y - at^2 = t.(x - 2at)$

$y - at^2 = xt - 2at^2$

$\Rightarrow at^2 - xt + y = 0$

Now $x = \frac{k^2}{y}$ in the above eq^n

$\Rightarrow at^2 - \left(\frac{k^2}{y}\right)t + y = 0$

$\Rightarrow y^2 + yat^2 - k^2t = 0$

Now the $2k = -at^2$

and similarly, $y = \frac{k^2}{x}$ gives

$xat^2 - x^2t + \frac{k^2}{x} = 0$

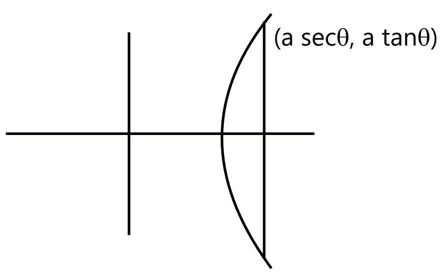
$x^2t - at^2x - k^2 = 0$

so $x_1 + x_2 = \frac{at^2}{t} = at \Rightarrow 2h = at$

so $\frac{4h^2}{a} = at^2 = -2k \Rightarrow \boxed{h^2 = \frac{-ak}{2}}$

Thus, it is a parabola.

Sol 20:



Now $\frac{dy}{dx} = \frac{a.\sec^2\theta}{a\tan\theta.\sec\theta}\ \frac{1}{\sin\theta}$

$\Rightarrow$ slope of normal $= -\sin\theta$

Now $\quad hx - ky = -h^2 - k^2$ (chord of equation)

slope $= \frac{-h}{k} = \sin\theta$

$\cos\theta = \frac{\sqrt{k^2 - h^2}}{k}$ and $\tan\theta = \frac{-h}{\sqrt{k^2 - h^2}}$

so points $A\left(\frac{a.k}{\sqrt{k^2 - h^2}}, \frac{-ah}{\sqrt{k^2 - h^2}}\right)$

this satisfies the line

$\Rightarrow \frac{h.ak}{\sqrt{k^2 - h^2}} + \frac{ahk}{\sqrt{k^2 - h^2}} = (h^2 - k^2)$

$\Rightarrow 2ahk = (h^2 - k^2)(\sqrt{k^2 - h^2})$

$\Rightarrow 4a^2h^2k^2 = (k^2 - h^2)^3$

Sol 21: $(3\sec\theta, 2\tan\theta)$ = point on hyperbola

Now equation of the chord of contact is $hx + ky = h^2 + k^2$

and also $3\sec\theta\ x + 2\tan\theta.\ y = 9$

so $\frac{h}{3\sec\theta} = \frac{h^2 + k^2}{9} = \frac{k}{2\tan\theta}$

$\Rightarrow \sec\theta = \frac{3h}{h^2 + k^2}$ and $\tan\theta = \frac{9k}{2(h^2 + k^2)}$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$

$\boxed{9h^2 - \frac{81k^2}{4} = (h^2 + k^2)^2}$

Sol 22: Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola then its conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

If e_1 and e_2 are their eccentricities, then

$b^2 = a^2\left(e_1^2 - 1\right)$ and $a^2 = b^2\left(e_2^2 - 1\right)$

So $\frac{1}{e_1^2} = \frac{a^2}{\left(a^2 + b^2\right)}$ and $\frac{1}{e_2^2} = \frac{b^2}{\left(a^2 + b^2\right)}$

So $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1, \Rightarrow \frac{1}{e_1^2} = 1 - \frac{1}{e_2^2}$

Sol 23: Let the hyperbola be $xy = c^2$.

Let the x_1, y_1 be point where the other 3 normals intersect.

Now, equation of normal

$\Rightarrow \left(y - \frac{c}{t}\right) = t^2(x - ct)$

$\Rightarrow ty - c = t^3x - ct^4$

$\Rightarrow ct^4 - t^3x + ty - c = 0$

Thus, passes through

(x_1, y_1) or $(cx_1, c/t_1)$

So $ct^4 - t^3. x_1 + ty_1 - c = 0$

Now $\Sigma t_i = \frac{x_1}{c}$ & product of roots $t_i = -1$

$\Rightarrow \Sigma x_i = x_1$ & $\Sigma t_1. t_2. t_3 = \frac{-y_1}{c}$

$\Rightarrow x_2 + x_3 + x_4 = 0 \ \frac{\Sigma 1}{t_i} = \frac{y_1}{c}$

$\Downarrow$

$x_c = 0 \Rightarrow y_1+y_2+y_3+y_4=y_1$

$y_2+y_3+y_4 = 0$

$\Downarrow$

$y_c = 0$

Thus, the centroid of PQR is (0, 0)

Sol 24: Equations of normal at the points

on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ are

$ax\cos\phi + by\cot\phi = a^2 + b^2$ ... (i)

and $ax\cos\phi + by\cot\phi = a^2 + b^2$

i.e. $ax\sin\theta + by\tan\theta = a^2 + b^2$ ... (ii)

$\because \quad \theta + \phi = \frac{\pi}{2}$

Solving (i) and (ii), $y = k = -\frac{(a^2 + b^2)}{b}$.

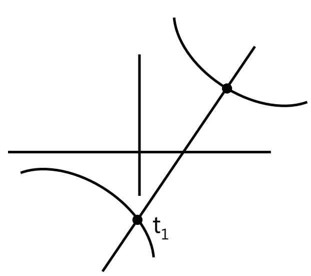
Sol 25: Tangent to the hyperbola $xy = c^2$ at $(ct, c/t)$ will be of the form $y = -\frac{1}{t^2}x + \frac{2c}{t}$

$y = mx + 2c\sqrt{-m} \Rightarrow t = \frac{1}{\sqrt{-m}}$

$\therefore$ Point is $\left(\frac{c}{\sqrt{-m}}, c\sqrt{-m}\right)$

Sol 26: $x = ct$ and $y = \frac{c}{t}$

Now $\frac{dy}{dx} = \frac{-1}{t^2}$



$\Rightarrow$ normal $= t^2$

Now slope $= t^2$

$= \frac{\frac{c}{t} - \frac{c}{t_1}}{c(t - t_1)} = \frac{-1}{t.t_1}$

$\Rightarrow t_1 . t^3 = -1$

Exercise 2

Single Correct Choice Type

Sol 1: (B) We have $x. x_1 - 9y. y_1 = 9$

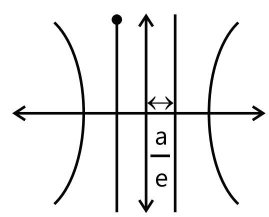
so $\frac{5}{x_1} = \frac{12}{-9y_1} = \frac{9}{9}$

$\Rightarrow y_1 = \frac{-4}{3}$ and $x_1 = 5$

Sol 2: (A) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

latus rectum $= \frac{2a^2}{b}$

Sol 3: (B)



area $= \frac{2a}{e} \times \frac{2a}{e} = \frac{4a^2}{e^2}$

for rectangular hyperbola $e = \sqrt{2}$

area $= 2a^2$

Sol 4: (A) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$y = \frac{2x}{5} + C$

$4\left(\frac{x^2 - 9}{9}\right) = \frac{4x^2}{25} + C^2 + \frac{4Cx}{5}$

$4x^2 \times \frac{16}{225} - \frac{4Cx}{5} - 4 - C^2 = 0$

$64x^2 - 180Cx - 180 - 45C^2 = 0$

$D = 0$

$180C^2 = 4\times64(-180 - 45C^2)$

$\Rightarrow C^2 = 64(-4 - C^2)$

$\Rightarrow C^2 < 0$ no possible tangent

Sol 5: (C) $\frac{x}{3} - \frac{y}{2} = m$

$\frac{x}{3} + \frac{y}{2} = \frac{1}{m}$

$\Rightarrow y^2 = \frac{1}{m^2} + m^2 - 2$

$\Rightarrow x = \frac{3}{2}\left(m + \frac{1}{m}\right)$

$\Rightarrow \frac{4x^2}{9} = m^2 + \frac{1}{m^2} + 2$

$\Rightarrow \frac{4x^2}{9} - y^2 = 4$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$

Sol 6: (B) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{9x^2}{a^2} - \frac{4}{b^2} = 1$

$\Rightarrow a^2 = 3^2 \Rightarrow \frac{4}{b^2} = 1$

$\Rightarrow a = 3 \Rightarrow b^2 = 4$

$\Rightarrow \boxed{\frac{x^2}{9} - \frac{y^2}{4} = 1}$

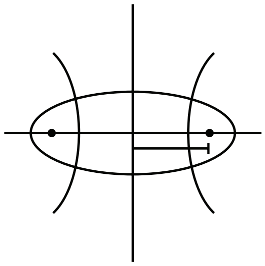
$\Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}} = 1 + \frac{4}{9} = \frac{\sqrt{13}}{3}$

Sol 7: (B) $(x-2)^2 - (y-2)^2 + 16 = 0$

$\Rightarrow \frac{(y-2)^2}{16} - \frac{(x-2)^2}{16} = 1$

$\Rightarrow e = \sqrt{1 + \frac{16}{16}} = \sqrt{2}$

Sol 8: (B)



Ellipse Hyperbola

$\Rightarrow \frac{x^2}{a_1^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a_2^2} - \frac{y^2}{b^2} = 1$

Now $a_1e_1 = a_2e_2$

also $e_1^2 = 1 - \frac{b^2}{a_1^2}$ $b^2 = (e_2^2 - 1)a_2^2$

$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{1}{1 - \frac{b^2}{a_1^2}} + \frac{1}{1 + \frac{b^2}{a_2^2}}$

$= \frac{a_1^2}{a_1^2 - b^2} + \frac{a_2^2}{a_2^2 + b^2}$ ... (i)

Also we have,

$a_1e_1 = a_2e_2$

$\Rightarrow a_1^2 - b^2 = a_2^2 + b^2$ ... (ii)

$\Rightarrow a_1^2 + a_2^2 = 2(a_2^2 + b^2)$ ... (iii)

Now from (i)

$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a_1^2 + a_2^2}{a_2^2 + b^2} = \frac{2(a_2^2 + b^2)}{a_2^2 + b^2} = 2$

Sol 9: (C)

We have $e_1 = \sqrt{\frac{a^2 + b^2}{a^2}}$ and $e_2 = \sqrt{\frac{b^2 + a^2}{b^2}}$

$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

Sol 10: (B) Equation of normal at any point x′, y′ of the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (i)

$\frac{a^2(x - x')}{x'} + \frac{b^2(y - y')}{y'} = 0$

$\Rightarrow \frac{a^2x}{x'} - a^2 - b^2 + \frac{b^2y}{y'} = 0$

$\frac{a^2x}{x'} + \frac{b^2y}{y'} = a^2 + b^2$ (h, k) satisfy this)

$\Rightarrow \frac{a^2h}{x_1} + \frac{b^2k}{y_1} = a^2 + b^2$

$\Rightarrow a^2h.\, y_1 + b^2k_1x_1 = (a^2 + b^2)(x_1, y_1)$ ... (iii)

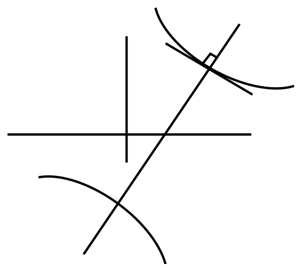
thus, (x_1, y_1) lies on curve (iii) and curve (i) these two points intersect at 4 points.

Sol 11: (A) A rectangular hyperbola circumscribing a triangle ABC always passes through the or the centre.

Sol 12: (A) We have $\frac{dy}{dx} = \frac{(c/t^2)}{c} = \frac{-1}{t^2}$

so normal slope = t^2

Now,



We have $t^2 = \frac{c/t - c/t'}{ct - ct'}$

$\Rightarrow t^2 = \frac{(t'-t)(-1)}{t.t'.(t-t')} \quad \Rightarrow t' = \frac{-1}{t^3}$

Sol 13: (A) $9(x^2 - 4x + 16) - 16(y^2 - 6y + 9)$

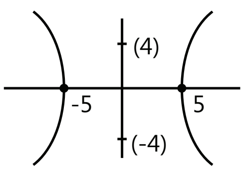
$-252 + 144 - 144 = 0$

$\Rightarrow 9(x-2)^2 - 16(y-3)^2 = 252$

$\Rightarrow$ Centre $\Rightarrow$ (2, 3)

Sol 14: (D) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

tangents $\Rightarrow$



$y = mx + \sqrt{25m^2 - 16}$

$\Rightarrow (y - mx)^2 = 25m^2 - 16$

$\Rightarrow$ the point $(1, 2\sqrt{2})$ satisfy this

$\Rightarrow (1 - 2\sqrt{2}\,m)^2 = 25m^2 - 16$

$\Rightarrow 1 + 8m^2 - 4\sqrt{2}\,m = 25m^2 - 16$

$\Rightarrow 17m^2 + 4\sqrt{2}\,m - 17 = 0$

$\Rightarrow m_1.\, m_2 = -1$

Sol 15: (D) $y = mx + \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = mx + \sqrt{\cos^2\alpha.m^2 - \sin^2\alpha}$

$(k - mh)^2 = \cos^2\alpha.\, m^2 - \sin^2\alpha$

$\Rightarrow k^2 + m^2h^2 - 2mkh$

$= \cos^2\alpha.\, m^2 - \sin^2\alpha$

$\Rightarrow m^2(h^2 - \cos^2\alpha) - 2kh.\, m + (k^2 + \sin^2\alpha)$

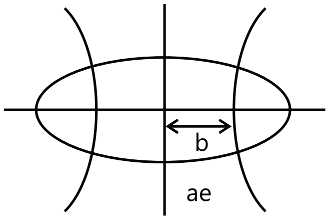
Now we have $m_1.\, m_2 = -1$

$\frac{h^2 - \cos^2\alpha}{k^2 + \sin^2\alpha} = -1$

$\Rightarrow h^2 + k^2 = \cos^2\alpha - \sin^2\alpha$

$h^2 + k^2 = \cos 2\alpha$

Sol 16: (C) We have b = ae



$b = a\sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow \boxed{\frac{b^2}{a^2} = \frac{1}{2}}$

$e_{hyperbola} = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1+2} = \sqrt{3}$

Sol 17: (C) Let any tangent of (x_1, y_1)

then $\frac{x.x_1}{a^2} + \frac{y.y_1}{b^2} = 1$ [1st tangent]

and $x.\, x_1 - y.\, y_1 = c^2$ [2nd tangent]

Now, $m_1.\, m_2 = -1$

$\Rightarrow \left(\frac{-b^2}{a^2}\right)\left(\frac{x_1}{y_1}\right)\left(\frac{x_1}{y_1}\right) = -1$

$\Rightarrow \frac{-b^2}{a^2}.\left(\frac{x_1^2}{y_1^2}\right) = -1 \Rightarrow +b^2\left(\frac{x_1^2}{y_1^2}\right) = +a^2$ ... (i)

Now $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ & $x_1^2 - y_1^2 = c^2$

$\Rightarrow \frac{y_1^2 + c^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow y_1^2\left[\frac{1}{a^2} + \frac{1}{b^2}\right] = \left[1 - \frac{c^2}{a^2}\right]$

$y_1^2 = \frac{[a^2 - c^2]}{a^2\left[\frac{1}{a^2} + \frac{1}{b^2}\right]} = \frac{b^2[a^2 - c^2]}{a^2 + b^2}$

And $x_1^2 = c^2 + \frac{[a^2 - c^2]b^2}{\frac{a^2[b^2 + a^2]}{a^2b^2}}$

$= \frac{b^2c^2 + a^2c^2 + a^2b^2 - c^2b^2}{b^2 + a^2} = \frac{a^2(b^2 + c^2)}{(a^2 + b^2)}$

so $b^2.\ \dfrac{a^2(b^2+c^2)}{(a^2+b^2)} \times \dfrac{1}{\dfrac{b^2(a^2-c^2)}{a^2+b^2}} = a^2$

$\Rightarrow\ a^2 - b^2 = 2c^2$

Sol 18: (B) $\dfrac{y^2}{b^2}+\dfrac{y}{ba^2} + 1 = 0$

$a^2 - 4ac \Rightarrow \left(\dfrac{1}{ba^2}\right)^2 - \dfrac{4\times 1}{b^2} \ge 0$

$\Rightarrow\ \dfrac{1}{b^2a^4}-\dfrac{4}{b^2} \ge 0[b^2 > 0] \Rightarrow\ \dfrac{1}{a^4}-\dfrac{4}{1} \ge 0$

$\Rightarrow\ \dfrac{1}{a^4} \ge 4 \Rightarrow\ \dfrac{1}{a^2} \ge 2 \Rightarrow\ \dfrac{1}{2} \ge a^2$

Sol 19: (C) $\dfrac{hx}{a^2}-\dfrac{ky}{b^2} = \dfrac{h^2}{a^2}-\dfrac{k^2}{b^2}$

$\dfrac{h\alpha}{a^2}-\dfrac{k\beta}{b^2} = \dfrac{h^2}{a^2}-\dfrac{k^2}{b^2}$

$\Rightarrow\ \left(\dfrac{h}{a}-\dfrac{\alpha}{2a}\right)^2 - \left(\dfrac{k}{b}-\dfrac{\beta}{2b}\right)^2 = \dfrac{\alpha^2}{4a^2}-\dfrac{\beta^2}{4b^2}$

$\Rightarrow$ Centre: $\dfrac{h}{a} = \dfrac{\alpha}{2a}$ and $k = \dfrac{\beta}{2}$

Sol 20: (B) $\dfrac{x^2}{5}-\dfrac{y^2}{5\cos^2\alpha} = 1$

so $e_1 = \sqrt{1+\dfrac{b^2}{a^2}} = \sqrt{1+\cos^2\alpha}$

$\dfrac{x^2}{25\cos^2\alpha} + \dfrac{y^2}{25} = 1$

$e_2 = \sqrt{1-\cos^2\alpha}$

$1 + \cos^2\alpha = b.\ (1 - \cos^2\alpha)$

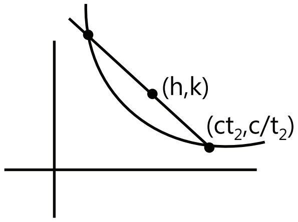
$\Rightarrow\ \cos^2\alpha = \dfrac{1}{2} \Rightarrow\ \cos\alpha = \dfrac{1}{\sqrt{2}}$

Sol 21: (D) $a^2 = 9$ and $b^2 = 4$

$\dfrac{x^2}{9}-\dfrac{y^2}{4} = 1$

$4x^2 - 9y^2 = 36$

Sol 22: (A) We have equation



Now, $2h = c(t_1 + t_2)$

and $2k = \dfrac{c}{t_1}+\dfrac{c}{t_2} = \dfrac{c.(t_1+t_2)}{t_1.t_2}$

$m = \dfrac{c/t_2 - c/t_1}{ct_2 - ct_1} = \dfrac{-1}{t_1t_2}$

$m = \dfrac{-c.(t_1+t_2)}{t_1.t_2c(t_1+t_2)}$

$m = \dfrac{-2k}{2h}$

$\Rightarrow\ k + mh = 0 \Rightarrow\ y + mx = 0$

Sol 23: (A) Let (h, k) be the midpoints of chords having slope 2

$\Rightarrow \tan\theta = 2 \quad \Rightarrow \sin\theta = \dfrac{2}{\sqrt{5}}$ and $\cos\theta = \dfrac{1}{\sqrt{5}}$

Let the two endpoints of the chord be a distance r from (h, k)

$\Rightarrow$ endpoints of the chord are

$(h + r\cos\theta, k + r\sin\theta)$ and $(h - r\cos\theta, k - r\sin\theta)$

$= \left(h+\dfrac{r}{\sqrt{5}}, k+\dfrac{2r}{\sqrt{5}}\right)$ and $= \left(h-\dfrac{r}{\sqrt{5}}, k-\dfrac{2r}{\sqrt{5}}\right)$

Plugging in the equation of the hyperbola

$3\left(h+\dfrac{r}{\sqrt{5}}\right)^2 - 2\left(k+\dfrac{2r}{\sqrt{5}}\right)^2 + 4\left(h+\dfrac{r}{\sqrt{5}}\right) - 6\left(k+\dfrac{2r}{\sqrt{5}}\right) = 0$ ...(i)

and

$3\left(h-\dfrac{r}{\sqrt{5}}\right)^2 - 2\left(k-\dfrac{2r}{\sqrt{5}}\right)^2 + 4\left(h-\dfrac{r}{\sqrt{5}}\right) - 6\left(k-\dfrac{2r}{\sqrt{5}}\right) = 0$ ...(ii)

Subtracting eqn. (ii) from (i),

$\dfrac{12\,hr}{\sqrt{5}}-\dfrac{8kr}{\sqrt{5}}+\dfrac{8r}{\sqrt{5}}-\dfrac{24r}{\sqrt{5}} = 0$

$\Rightarrow 3h - 2k - 4 = 0$

$\Rightarrow$ required locus is

$3x - 4\,y = 4.$

Previous Years' Questions

Sol 1: (B) Given equation is

$\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, where $|r| < 1$

$\Rightarrow$ 1 – r is (+ve) and 1 + r is (+ve)

$\therefore$ Given equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Hence, it represents a hyperbola when $|r| < 1$.

Sol 2: (D) Firstly we obtain the slope of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$

On differentiating w.r.t. x, we get

$\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$

Slope, for normal at the point $(a \sec\theta, b \tan\theta)$ will be

$-\frac{a^2 b \tan\theta}{b^2 a \sec\theta} = -\frac{a}{b}\sin\theta$

$\therefore$ Equation of normal $(a\sec\theta, b\tan\theta)$ is

$y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$\Rightarrow (a\sin\theta)x + by = (a^2 + b^2)\tan\theta$

$\Rightarrow ax + b\,\text{cosec}\theta = (a^2 + b^2)\sec\theta$ ...(i)

Similarly, equation of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b\tan\theta)$ is

$ax + b\,\text{cosec}\phi = (a^2 + b^2)\sec\theta$ ...(ii)

On subtracting eqs.(ii) from (i), we get

$b(\text{cosec}\theta - \text{cosec}\phi)y$

$= (a^2 + b^2)(\sec\theta - \sec\phi)$

$\Rightarrow y = \frac{a^2 + b^2}{b}\frac{\sec\theta - \sec\varphi}{\text{cosec}\theta - \text{cosec}\varphi}$

But $\frac{\sec\theta - \sec\varphi}{\text{cosec}\theta - \text{cosec}\varphi}$

$= \frac{\sec\theta - \sec(\pi/2 - \theta)}{\text{cosec}\theta - \text{cosec}(\pi/2 - \theta)}$

$(\because \phi + \theta = \pi/2)$

$= \frac{\sec\theta - \text{cosec}\theta}{\text{cosec}\theta - \sec\theta} = -1$

Thus, $y = -\frac{a^2 + b^2}{b}$ i.e., $k = -\left(\frac{a^2 + b^2}{b}\right)$

Sol 3: (B) Let (h, k) be point whose chord of contact with respect to hyperbola $x^2 - y^2 = 9$ is $x = 9$.

We know that, chord of contact of (h, k) with respect to hyperbola $x^2 - y^2 = 9$ is $T = 0$

$\Rightarrow h.x + k(-y) - 9 = 0$

$\therefore hx - ky - 9 = 0$

But it is the equation of the line $x = 9$.

This is possible when h = 1, k = 0 (by comparing both equations).

Again equation of pair of tangents is $T^2 = SS_1$.

$\Rightarrow (x - 9)^2 = (x^2 - y^2 - 9)(t^2 - 0^2 - 9)$

$\Rightarrow x^2 - 18x + 81 = (x^2 - y^2 - 9)(-8)$

$\Rightarrow x^2 - 18x + 81 = -8x^2 + 8y^2 + 72$

$\Rightarrow 9x^2 - 8y^2 - 18x + 9 = 0$

Sol 4: (B) Given equation of hyperbola is

$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

Here, $a^2 = \cos^2\alpha$ and $b^2 = \sin^2\alpha$

[We, comparing with standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$]

We know, foci = $(\pm ae, 0)$

where $ae = \sqrt{a^2 + b^2} = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$\Rightarrow$ foci = $(\pm 1, 0)$

whereas vertices are $(\pm \cos\alpha, 0)$

eccentricity, ae = 1 or $e = \frac{1}{\cos\alpha}$

Hence, foci remain constant with change in 'α'.

Sol 5: (D) The equation of tangent at (x_1, y_1) is $xx_1 - 2yy_1 = 4$, which is same as $2x + \sqrt{6}\,y = 2$

$\therefore \frac{x_1}{2} = \frac{-2y_1}{\sqrt{6}} = \frac{4}{2}$

$\Rightarrow x_1 = 4$ and $y_1 = -\sqrt{6}$

Thus, the point of contact is $(4, -\sqrt{6})$

Sol 6: (B) The eccentricity of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is

$e_1 = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

$\therefore e_2 = \frac{5}{3} (\because e_1 e_2 = 1)$

$\Rightarrow$ Foci of ellipse $(0, \pm 3)$

$\Rightarrow$ Equation of hyperbola is

$\frac{x^2}{16} - \frac{y^2}{9} = -1$

Sol 7: (A) The given ellipse is

$\frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow a = 2, b = \sqrt{3}$

$\therefore 3 = 4(1 - e^2) \Rightarrow e = \frac{1}{2}$

$\therefore ae = 2 \times \frac{1}{2} = 1$

Hence, the eccentricity e_1, of the hyperbola is given by

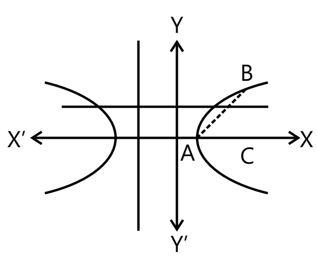
$1 = e_1 \sin\theta \Rightarrow e_1 = \operatorname{cosec}\theta$

$\Rightarrow b^2 = \sin^2\theta(\operatorname{cosec}^2\theta - 1) = \cos^2\theta$

Hence, equation of hyperbola is $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

or $x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$

Sol 8: (B) Given equation can be rewritten as $\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$



For point A(x, y)

$e = \sqrt{1 + \frac{2}{4}} = \sqrt{\frac{3}{2}}$

$x - \sqrt{2} = 2 \Rightarrow x = 2 + \sqrt{2}$

For point C(x, y)

$x - \sqrt{2} = ae = \sqrt{6}$

$x = \sqrt{6} + \sqrt{2}$

Now,

$AC = \sqrt{6} + \sqrt{2} - 2 - \sqrt{2} = \sqrt{6} - 2$

and$BC = \frac{b^2}{a} = \frac{2}{2} = 1$

Area of ΔABC

$= \frac{1}{2} \times (\sqrt{6} - 2) \times 1 = \sqrt{\frac{3}{2}} - 1$ sq. unit

Sol 9: (B) Equation of normal to hyperbola at (x_1, y_1) is

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = (a^2 + b^2)$

$\therefore$ At $(6, 3) \Rightarrow \frac{a^2x}{6} + \frac{b^2y}{3} = (a^2 + b^2)$

It passes through (9, 0)

$\Rightarrow \frac{a^2.9}{6} = a^2 + b^2$

$\Rightarrow \frac{3a^2}{2} - a^2 = b^2 \Rightarrow \frac{a^2}{b^2} = 2$

$\therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2}$

$\Rightarrow e = \sqrt{\frac{3}{2}}$

Sol 10: (B) $2b = \frac{1}{2}.(2ae) \Rightarrow b = \frac{ae}{2}$

$\Rightarrow a^2(e^2 - 1) = \frac{a^2e^2}{4} \Rightarrow 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}}$

JEE Advanced/Boards

Exercise 1

Sol 1: $\frac{SP}{PM} = \sqrt{3}$

$\frac{\sqrt{(x-1)^2 + (y-1)^2}}{\frac{(2x+y-1)}{\sqrt{5}}} = \sqrt{3}$

Squaring

$5[(x - 1)^2 + (y - 1)^2] = 3(2x + y - 1)^2$

$\Rightarrow 7x^2 - 2y^2 + 12xy + 4y - 2x - 7 = 0$

Sol 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$7x + 13y = 87$

$5x - 8y = -7$

$\Rightarrow \frac{87 - 7x}{13} = \frac{5x + 7}{8}$

$\Rightarrow 8.\ 87 - 7.\ 13 = 121\ x$

$\Rightarrow 121x = 605$

x = 5, y = 4

$\frac{2b^2}{a} = \frac{32\sqrt{2}}{5}$

$5b^2 = 16\sqrt{2}a$

$\frac{25}{a^2} - \frac{16}{b^2} = 1$

$\frac{25}{a^2} - \frac{16}{a\sqrt{2}} = 1$

$25\sqrt{2} - 5a = a^2\sqrt{2}$

$a^2\sqrt{2} + 5a - 25\sqrt{2} = 0$

$a = \frac{-5 \pm \sqrt{25+200}}{2\sqrt{5}} = \frac{5}{\sqrt{2}}$ or $\frac{-10}{\sqrt{2}}$

Now, $5b^2 = 16\sqrt{2}a$

$\Rightarrow a > 0 \Rightarrow a = \frac{5}{\sqrt{2}}$

Sol 3: $\frac{x^2}{100} - \frac{y^2}{25} = 1$

$e = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

$S = (ae,0) = \left(\frac{\sqrt{5}}{2}\times 10, 0\right) = (5\sqrt{5}, 0)$

$S' = (-ae, 0) = (-5\sqrt{5}, 0)$

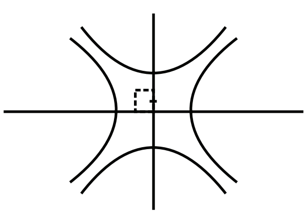
A = (10, 0)

$SA = (10 - 5\sqrt{5})$

$S'A = (10 + 5\sqrt{5})$

SA. S'A = 100 – 75 = 25

Sol 4:



$16x^2 - 9y^2 + 32x + 36y - 164 = 0$

$16x^2 + 32x - 9y^2 + 36y = 164$

$16(x+1)^2 - 9(y-2)^2$

= 164 + 16 – 36 = 144

$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Centre (–1, 2)

$e = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$

foci = (–1 + ae, 2) = (4, 2)

= (–1 – ae, 2) = (–6, 2)

Directrix $x + 1 = \frac{9}{5} \Rightarrow x = \frac{4}{5}$

$x + 1 = \frac{-9}{5} \Rightarrow x = \frac{-14}{5}$

Latus rectum $= \frac{2b^2}{a} = \frac{2.16}{3} = \frac{32}{3}$

Length of major axis = 2×4 = 8

Length of minor axis = 2×3 = 6

Equation of axis is y = 2

Sol 5: $P_1(ct_1, c/t_1)$ $P_2(ct_2, c/t_2)$

$t_1 + t_2 = \frac{2h}{c}$

$\frac{1}{t_1} + \frac{1}{t_2} = \frac{2k}{c} \Rightarrow \frac{2h}{ct_1t_2} = \frac{2k}{c}$ $t_1t_2 = \frac{h}{k}$

$c^2(t_1 - t_2)^2 + c^2\left(\frac{1}{t_1} - \frac{1}{t_2}\right)^2 = 4d^2$

$(t_1 + t_2)^2 - 4t_1t_2 + \left(\frac{1}{t_1} + \frac{1}{t_2}\right)^2$

$-\frac{4}{t_1t_2} = \frac{4d^2}{c^2}$

$\left(\frac{2h}{c}\right)^2 + \left(\frac{2k}{c}\right)^2 - 4t_1t_2 - \frac{4}{t_1t_2} = \frac{4d^2}{c^2}$

$\frac{(2h)^2 + (2k)^2}{c^2} - 4\left(\frac{h}{k} + \frac{k}{h}\right) = \frac{4d^2}{c^2}$

$2\frac{(h^2+k^2)}{c^2} - 2\frac{(h^2+k^2)}{kh} = \frac{2d^2}{c^2}$

$(h^2 + k^2)hk - c^2(h^2 + k^2) = d^2kh$

$(h^2 + k^2)(hk - c^2) = d^2kh$

Hence proved.

Sol 6: y – 2 = m(x – 6)

y = mx + 2 – 6m

$$\frac{x^2}{25}-1 = \frac{(mx+2-6m)^2}{16}$$

$16(x^2 - 25) = 25(m^2x^2 + 4 + 36m^2 + 4mx - 24m - 12m^2x)$

$x^2(16 - 25m^2) + x(-100m + 300m^2) - 400 - 100 - 900 m^2 + 600 m = 0$

$(300m^2 - 100m)^2 = 4(16 - 25m^2)(-900m^2 + 600m - 500)$

$100(3m^2 - m)^2 = 4(16 - 25m^2)(-9m^2 + 6m - 5)$

$25(9m^4 + m^2 - 6m^3) = -144m^2 + 96m - 80 + 225m^4 - 150m^3 + 125m^2$

$25m^2 = -19m^2 + 96m - 80$

$44m^2 - 96m + 80 = 0$

$11m^2 - 24m + 20 = 0$

$m_1 + m_2 = \frac{24}{11}$

$m_1m_2 = \frac{20}{11}$

Sol 7: $y = -x + c$

$x^2 - 4(c - x)^2 = 36$

$x^2 - 4(c^2 + x^2 - 2cx) = 36$

$3x^2 - 8cx + 4c^2 + 36 = 0$

$\Rightarrow x + y = \pm 3\sqrt{3}$

$64c^2 = 12(4c^2 + 36)$

$16c^2 = 12(4c^2 + 36)$

$4c^2 = 3c^2 + 27$

$c^2 = 2 \Rightarrow c = \pm 3\sqrt{3}$

Sol 8: Equation of chord

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\left(\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If it pass through (ae, 0)

$$e = \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}$$

$$\frac{1}{e} = \frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$$

Using componendo rule we get

$$\frac{1-e}{1+e} = \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)$$

Sol 9: $e = (0, 0)$

$S(ae, 0)$

$S'(-ae, 0)$

$P = (a\sec\theta, b\tan\theta)$

SP. S$\Rightarrow$ P =

$$\sqrt{((a\sec\theta-ae)^2+b^2\tan^2\theta)((a\sec\theta+ae)^2+b^2\tan^2\theta)}$$

$$= \sqrt{(a^2\sec^2\theta+a^2e^2+b^2\tan^2\theta)^2-(2a^2e\sec\theta)^2}$$

$= a^2\sec^2\theta + b^2\tan^2\theta - (a^2 + b^2)$

$= CP^2 - (a^2 + b^2)$

Sol 10: $y - \frac{5}{2} = mx$

$y = mx + \frac{5}{2}$

$3x^2 - 25 = 2\left(m^2x^2+\frac{25}{4}+5mx\right)$

$x^2(3 - 2m)^2 - 10mx - \frac{75}{2} = 0$

$100m^2 = 4(3 - 2m^2)\left(-\frac{75}{2}\right)$

$50m^2 = 150m^2 - 225$

$100m^2 = 225$

$m^2 = \frac{9}{4};\quad m = \pm\frac{3}{2}$

$2y = 3x + 5$ or $2y + 3x = 5$

Sol 11: $\frac{y-k}{x-h} = \frac{b^2h}{a^2k}$

$\Rightarrow x^2 + y^2 = a^2$

$$\left[h+\frac{a^2k}{b^2h}(y-k)\right]^2 + y^2 = a^2$$

$$h^2+\frac{a^4k^2}{b^4h^2}(y^2 + k^2 - 2ky)+\frac{2a^2k}{b^2}(y-k) + y^2 = a^2$$

$$y^2\left[\frac{a^4k^2}{b^4h^2}+1\right] + y\left[\frac{-2k^3a^4}{b^4h^2}+\frac{2a^2k}{b^2}\right] + h^2 - a^2 + \frac{a^4k^4}{b^4h^2}-\frac{2a^2k^2}{b^2} = 0$$

$$\frac{y_1+y_2}{y_1y_2} = \frac{\frac{2a^2k}{b^2}-\frac{2k^3a^4}{h^2b^4}}{h^2-a^2+\frac{a^4k^4}{b^4h^2}-\frac{2a^2k^2}{b^2}}$$

$$= \frac{2a^2h^2kb^2-2k^3a^4}{h^4b^4-a^2h^2b^4+a^4k^4-2a^2k^2b^2h^2} = \frac{2a^2ka^2b^2}{k^2a^4b^2} = \frac{2}{k}$$

Sol 12: $\frac{x^2}{2}-\frac{y^2}{3} = 1;\quad y-\beta = m(x-\alpha)$

$$\frac{x^2-2}{\alpha} = \frac{1}{3}(mx - m\alpha + \beta)^2$$

$$3x^2-6 = 2(m^2x^2+m^2\alpha^2+\beta^2-2m^2\alpha x-2m\alpha+2mx\beta)$$

$$x^2(3-2m^2)+2x(2m^2\alpha-2m\beta)-6$$

$$-2m^2\alpha^2-2\beta^2+4m\alpha\beta=0$$

$$(4m^2\alpha-4m\beta)^2 = 4(3-2m)^2(4m\alpha\beta-2m^2\alpha^2-2\beta^2-6)$$

$$2m^2(m\alpha-\beta)^2 = (3-2m^2)(2m\alpha\beta-m^2\alpha^2-\beta^2-3)$$

$$2m^4\alpha^2+2m^2\beta^2-4m^3\alpha\beta = -4m^3\alpha\beta+2m^4\alpha^2+$$

$$6m\alpha\beta-3m^2\alpha^2-3\beta^2+9+2m^2\beta^2+6m^2$$

$$m^2(3\alpha^2-6)-6m\alpha\beta+3\beta^2+9=0$$

$$\frac{3\beta^2+9}{3\alpha^2-6} = 2 \Rightarrow \beta^2+3 = 2\alpha^2-4$$

$$\beta^2 = 2\alpha^2-7$$

Sol 13: Equation of any normal to the hyperbola is

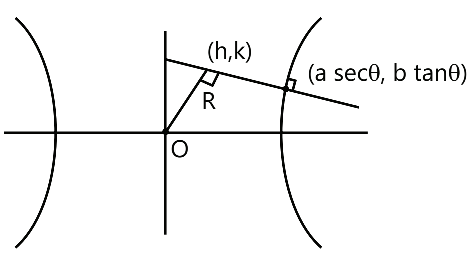
$$y = mx - \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

$$\Rightarrow (a^2-b^2m^2)(y-mx)^2 = m^2(a^2+b^2)^2$$

If it passes through the point (x_1, y_1), then $(a^2-b^2m^2)(y_1-mx_1)^2 = m^2(a^2+b^2)^2$

It is a 4 degree equation in m, so it gives 4 values of m. corresponding to these 4 values, four normal can be drawn from the point (x_1, y_1).

Sol 14:



Slope of normal $= \frac{-h}{k}$

[slope of OR $= \frac{k}{h}$]

that has equation:

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

$$\Rightarrow \text{slope} = -\frac{a^2}{b^2}\times\frac{y_1}{x_1} = \frac{-a^2}{b^2}\times\frac{b-\tan\theta}{a\sec\theta} = \frac{-a}{b}\sin\theta$$

$$\text{so } +\frac{h}{k} = +\frac{a}{b}\sin\theta \Rightarrow \frac{bh}{ak} = \sin\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{a^2k^2-b^2h^2}}{ax}$$

$$\text{and } \tan\theta = \frac{bh}{\sqrt{a^2k^2-b^2h^2}}$$

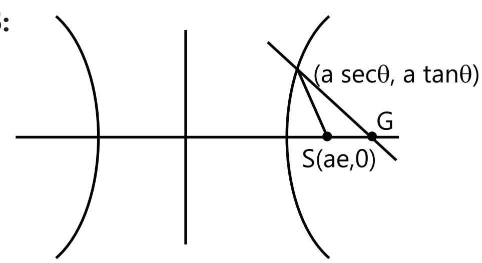
Putting

$$x_1 = b - \frac{bh}{\sqrt{a^2k^2-b^2h^2}},\ y_1 = a\sec\theta = \frac{a^2x}{\sqrt{a^2k^2-b^2h^2}}$$

in equation $\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2+b^2$ and simplifying, we get

locus as $(x^2+y^2)^2(a^2y^2-b^2y^2) = x^2y^2(a^2+b^2)^2$

Sol 15:



Normal: $\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2$

$\Rightarrow$ ax. cosθ + by. cotθ = $a^2 + b^2$

Now for coordinates of G $\Rightarrow$ put y = 0 in above equation

$$\Rightarrow x = \frac{(a^2+b^2)}{a}.\sec\theta$$

$$\text{also } e^2 = 1 + \frac{b^2}{a^2} \Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{a^2+b^2}{a}}$$

Now

$$SG^2 = \left[\frac{(a^2+b^2)\sec\theta}{a} - \sqrt{a^2+b^2}\right]^2$$

and SP² = $(\sqrt{a^2+b^2} - a\sec\theta)^2 + (b\tan\theta)^2$

$SP^2 = a^2 + b^2 + a^2\sec^2\theta - 2a.\ \sqrt{a^2+b^2}\ \sec\theta + b^2\tan^2\theta$

$\Rightarrow\ e^2SP^2 = \dfrac{(a^2+b^2)}{a^2}[(a^2 + b^2) + a^2\sec^2\theta -$

$2a\sqrt{a^2+b^2}\ \sec\theta + b^2\tan^2 q]$

$$= \left[\frac{(a^2+b^2)^2}{a^2} + (a^2+b^2)\sec^2\theta - \frac{2\sqrt{a^2+b^2}\sec\theta}{a} + \frac{b^2(a^2+b^2)}{a^2}\tan 2\theta\right]$$

$$= \left[(a^2+b^2) + (a^2+b^2)\times\frac{b^2}{a^2}\right]\sec^2\theta + \left[\frac{(a^2+b^2)^2}{a^2} - \frac{b^2(a^2+b^2)}{a^2}\ \frac{-2\sqrt{a^2+b^2}\sec\theta}{a}\right]$$

$$= \left[\frac{(a^2+b^2)\sec^2\theta}{a^2} + (a^2+b^2) - \frac{2\sqrt{a^2+b^2}\sec\theta}{a}\right]$$

$$e^2SP^2 = \left[\frac{(a^2+b^2)\sec\theta}{a} - \sqrt{a^2+b^2}\right]^2$$

$e^2SP^2 = SG^2 \Rightarrow\ eSP = SG$

Sol 16: Equation of any tangent to $x^2 - y^2 = a^2$ or $\dfrac{x^2}{a^2}\ \dfrac{-y^2}{a^2} = 1$ is

$\dfrac{x}{a}\tan\theta = \perp$ or $x\sec\theta - y\tan\theta = a$ ... (i)

Equation of other two sides of the triangle are

$x - y = 0$ ...(ii)

$x + y = 0$ ... (iii)

Solving (ii) and (iii), (iii) and (i), (i) and (ii) in pairs, the co-ordinates of the vertices of the triangle are (0, 0); $\left(\dfrac{a}{\sec\theta+\tan\theta}, \dfrac{a}{\sec\theta+\tan\theta}\right)$ and $\left(\dfrac{a}{\sec\theta-\tan\theta}, \dfrac{a}{\sec\theta-\tan\theta}\right)$

∴ Area of triangle =

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & \dfrac{a}{\sec\theta+\tan\theta} & -\dfrac{a}{\sec\theta+\tan\theta} \\ 1 & \dfrac{a}{\sec\theta-\tan\theta} & \dfrac{a}{\sec\theta+\tan\theta} \end{vmatrix}$$

$$= \frac{1}{2}(2a^2) = a^2$$

Sol 17: Let $P(x_1, y_1)$ be the middle point of the chord of the hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$

∴ Equation of the chord is $T = s_1$

$\Rightarrow 3xx_1 - 2yy_1 + 2(x + x_1) - 3(y + y_1)$

$\Rightarrow 3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

$\Rightarrow (3x_1 + 2)x - (2y_1 + 3)y + 2x_1 - 3y_1$

$\Rightarrow 3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

If this chord is parallel to line y = 2x, then

$m_1 = m_2 \Rightarrow\ -\dfrac{3x+2}{-(2y, 3)} = 2$

$\Rightarrow 3x_1 - 4y_1 = 4$

Hence, the locus of the middle point (x_1, y_1) is $3x - 4y = 4$

Sol 18: Eq. of Hyperbola $= \dfrac{x^2}{100} - \dfrac{y^2}{49} = 1$

Eqn. of tangent $= y = mx \pm \sqrt{a^2m^2 - 49}$ ...(i)

$\Rightarrow y = mx I\sqrt{100m^2 - 49}$

Given that $y = mx + 6$ ....(ii)

Equating (i) and (ii)

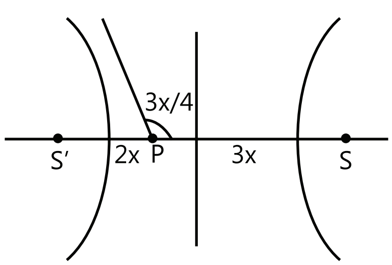
$\Rightarrow \sqrt{100m^2 - 49} = 6$

$\Rightarrow 100m^2 49 = 36$

$\Rightarrow 100m^2 = 85/100 = \dfrac{17}{20}$

$\Rightarrow m = \sqrt{17/20}$

Sol 19:



Now, a = 4, b = 3

$\Rightarrow e^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

so coordinates of S = (5, 0) and

S′ = (–5, 0)

so $P = \left(\frac{3\times(-5)+2\times5}{3+2}, \frac{0\times3+2\times0}{3+2}\right) = (-1, 0)$

Now slope of line through P ⇒ –1

⇒ y = –x + C

⇒ 0 = 1 + c ⇒ c = –1

so line through P = y = – x – 1

Now asymptotes ⇒ $\left(\frac{x}{4}-\frac{y}{3}\right) = 0$

and $\left(\frac{x}{4}+\frac{y}{3}\right) = 0$

Point of intersection ⇒

$\frac{x}{4}+\frac{(x+1)}{3} = 0\ \frac{x}{4}-\frac{(x+1)}{3} = 0$

7x + 4 = 0x = –4

$x = \frac{-4}{7}$ y = 3

and $y = \frac{-3}{7}$

$\left(\frac{-4}{7}, \frac{-3}{7}\right)$ and (–4, 3)

Sol 20: Eq. of Hyperbola; $x^2 - 2y^2 = 18 \Rightarrow \frac{x^2}{18} - \frac{y^2}{9} = 1$

Eq. of tangent $y = mx \pm \sqrt{m^2a^2 - b^2}$

$\Rightarrow y = mx \pm \sqrt{m^2.18 - 9}$

∴ this is perpendicular to y = x

⇒ the value of m =-1

$\Rightarrow y = -1\,x \pm \sqrt{18-9}$

$y = -x \pm \sqrt{9}$

$y = -x \pm 3$

Sol 21: The chord joining the points P(a sec θ, a tan θ) and given by $x\cos\frac{\theta-\varnothing}{2} - y\sin\frac{\theta-\varnothing}{2}$

$= a\cos\frac{\theta-\varnothing}{2}$ ... (i)

And normal to the hyperbola at P(a sec θ, a tan θ) is given by

$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$ ... (ii)

Note that equation (i) and (ii) are the same lines comparing these lines, we get

$$\frac{\cos\frac{\theta-\varnothing}{2}}{\frac{1}{\sec\theta}} = \frac{-\sin\frac{\theta-\varnothing}{2}}{\frac{1}{\tan\theta}} = \frac{a\cos\frac{\theta-\varnothing}{2}}{\frac{1}{2a}}$$

Solving above and simplifying, we get

$\tan\varnothing = \tan\theta\,(4\sec^2\theta - 1)$

Sol 22: $\frac{x^2}{9} - y^2 = 1$

now, line: $y = mx + \sqrt{9m^2-1}$

$\Rightarrow 2 = 3m + \sqrt{9m^2-1}$

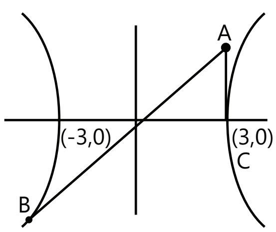
$\Rightarrow (2-3m)^2 = \sqrt{9m^2-1}$

$\Rightarrow 9m^2 - 6m \times 2 + 4 = 9m^2 - 1$

(one m = ∞)

S = 12 m

$\Rightarrow m = \frac{5}{12}$



so one tangent ⇒ x = 3

and one is $y = \frac{5}{12}x + \frac{3}{4}$

12y = 5x + 9

Now tangent at B

$\frac{x.x_1}{9} - y.\,y_1 = 1$

⇒ same –5x + 12y = 9

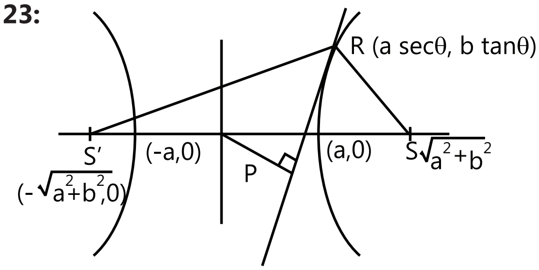
$\Rightarrow \frac{-5}{x_1/9} = \frac{12}{-y_1} = \frac{9}{1} \Rightarrow y_1 = \frac{-4}{3}$

$x_1 = -5$

so $\Delta = \frac{1}{2}$ × AC × height

$= \frac{1}{2} \times 2 \times [(3-(-5))] = 8$ sq. unit

Sol 23:



We have

$(S \Rightarrow R - SR)^2 = S \Rightarrow R^2 + SR^2 - 2S \Rightarrow R.\,RS = (2a)^2$

$\Rightarrow (S\Rightarrow R + SR)^2 = (S\Rightarrow R - SR)^2 + 4S\Rightarrow R.\,SR$

$= 4a^2 + 4.\,S\Rightarrow R \times SR$ ...(i)

Now, tangent

$\Rightarrow \dfrac{x.\sec\theta}{a} - \dfrac{y.\tan\theta}{b} = 1$

$\Rightarrow \dfrac{1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}}$

$\Rightarrow P^2 = \dfrac{a^2b^2}{b^2\sec^2\theta + a^2\tan^2\theta}$ ...(ii)

$SR^2 = \sqrt{(a\sec\theta - \sqrt{a^2+b^2})^2 + b^2\tan^2\theta}$

$SR^2 = a^2\sec^2\theta + a^2 + b^2 + b^2\tan^2\theta - 2a\sec\theta.\ \sqrt{a^2+b^2}$

$= (a^2 + b^2)\sec^2\theta + a^2 - 2a\sec\theta.\ \sqrt{a^2+b^2}$

$SR = (\sqrt{a^2+b^2}\ .\ \sec\theta - a)$

similarly $S\Rightarrow R = (\sqrt{a^2+b^2}\ .\ \sec\theta - a)$

$SR.\ S\Rightarrow R = (a^2 + b^2)\sec^2\theta - a^2$

$= b^2\sec^2\theta + a^2\tan^2\theta$

$SR.\ S\Rightarrow R = \dfrac{a^2b^2}{P^2}$ [(from (ii)]

putting in (i)

$(S\Rightarrow R + SR)^2 = 4a^2 + \dfrac{4a^2b^2}{P^2} = 4a^2\left(1 + \dfrac{b^2}{P^2}\right)$

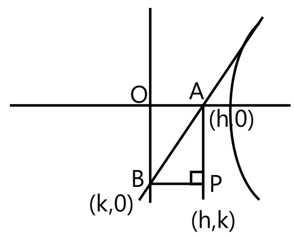
Exercise 2

Single Correct Choice Type

Sol 1: (B) $\dfrac{hx}{4} - \dfrac{ky}{9} = \dfrac{h^2}{4} - \dfrac{k^2}{9}$

Now $d = \dfrac{\dfrac{h^2}{4} - \dfrac{k^2}{9}}{\sqrt{\dfrac{h^2}{16} + \dfrac{k^2}{81}}} = 2$

Sol 2: (A) Equation of tangent,



$\dfrac{x.\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

so for $h \Rightarrow \dfrac{h.\sec\theta}{a} = 1$

$\Rightarrow h = a\cos\theta$

and $h' = -b\cot\theta$

$\Rightarrow \left(\dfrac{a}{h}\right)^2 - \left(\dfrac{-b}{k}\right)^2 = 1$

$$\boxed{\dfrac{a^2}{h^2} - \dfrac{b^2}{k^2} = 1}$$

Sol 3: (D) $\dfrac{h-0}{1} = \dfrac{k-0}{t^2} = \dfrac{-(-2ct)}{1+t^2}$

$h = \dfrac{2ct}{1+t^4}$, $k = \dfrac{2ct^3}{1+t^4}$

$\dfrac{k}{h} = t^2$

$k = \dfrac{2c\left(\dfrac{k}{h}\right)^{3/2}}{1 + \dfrac{k^2}{h^2}}$

$k^2 = \dfrac{4c^2\left(\dfrac{k}{h}\right)^3}{\left(1 + \dfrac{k^2}{h^2}\right)^2}$

$\dfrac{k^2(h^2+k^2)^2}{h^4} = \dfrac{4c^2k^3}{h^3}$

$(x^2 + y^2)^2 = 4c^2\,xy$

Sol 4: (B) We have

$2s = t^2 + 1$and$2t = 2/s$

$\Rightarrow t = 1/s$

$\Rightarrow 2s = \dfrac{1}{s^2} + 1$

$\Rightarrow 2s^3 = 1 + s^3$

$\Rightarrow$ $2s^3 - s - 1 = 0$

$\Rightarrow$ $(s - 1)(2s^2 + s + 1) = 0$

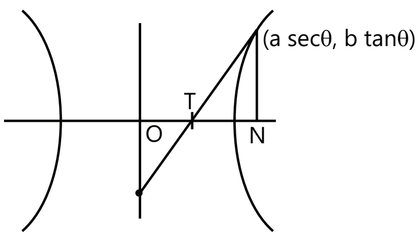
$\Rightarrow$ $s = 1$

$y = \frac{2}{s} = 2$

$x = 2s = 2$

$\Rightarrow$ (2, 2)

Sol 5: (B)



We hae NP = a sec θ and tangent slope:

$$\frac{dy}{dx} = \frac{b.\sec^2\theta}{a.\sec\theta.\tan\theta} = \frac{b}{a\sin\theta}$$

so $\frac{x.\sec\theta}{a} - \frac{y.b\tan\theta}{b} = 1$

so at y = 0

x = a cosθ

so OT = a cosθ

so OT × ON = a cosθ. a secθ = a^2

Sol 6: (A) We have, slope = $\frac{c/t_1 - c/t_2}{ct_1 - ct_2}$

$$= \frac{(t_2 - t_1)}{t_1.t_2(t_1 - t_2)} = \frac{-1}{t_1.t_2}$$

so $y = \frac{-x}{t_1.t_2} + N$

$\Rightarrow$ $y = \frac{-x}{t_1.t_2} + N$

this satisfies,

$$\frac{c}{t_1} = \frac{-c}{t_2} + N$$

$\Rightarrow$ $N = c\left[\frac{1}{t_1} + \frac{1}{t_2}\right]$

Now, $y = \frac{-x}{t_1.t_2} + c.\left[\frac{t_1 + t_2}{t_1.t_2}\right]$

$$\frac{y(t_1t_2)}{c(t_1 + t_2)} - \frac{x}{c(t_1 + t_2)} = 1$$

Now $c(t_1 + t_2) = x_1 + x_2$

and $\frac{c(t_1 + t_2)}{t_1.t_2} = y_1 + y_2$

$\Rightarrow$ $\frac{y}{y_1 + y_2} + \frac{x}{x_2 + x_1} = 1$

Sol 7: (C) We have 2b = ae

$\Rightarrow$ $\frac{b}{a} = \frac{e}{2}$

So $e^2 = 1 + \frac{e^2}{4}$

$\Rightarrow$ $e^2 = \frac{4}{3}$

$\Rightarrow$ $e = \frac{2}{\sqrt{3}}$

Sol 8: (C) $(5)x - (-3)y = (5)^2 - (-3)^2$

$5x + 3y = 16$

Sol 9: (B) We have,

$2\int x.dx = 3\int y.dy$

$\Rightarrow$ $x^2 = \frac{3y^2}{2} + c$

$\Rightarrow$ $x^2 - \frac{3y^2}{2} = c$

$\Rightarrow$ $e^2 = 1 + \frac{2/3}{1} = 1 + \frac{2}{3} = \sqrt{\frac{5}{3}}$

Multiple Correct Choice Type

Sol 10: (A, C) $\frac{x^2}{1} - \frac{y^2}{5} = 1$

tangent $\Rightarrow$ $y = mx \pm \sqrt{1m^2 - 5}$

$\Rightarrow$ $(8 - 2m)^2 = m^2 - 5$

$\Rightarrow$ $4m^2 + 64 - 32m = m^2 - 5$

$\Rightarrow$ $3m^2 - 32m + 69 = 0$

$\Rightarrow$ $3m^2 - 23m - 9m + 69 = 0$

$\Rightarrow$ $m(3m - 23) - 3(3m - 23) = 0$

$\Rightarrow$ m = 3 or m = $\frac{23}{3}$

Now y = 3x + 2(A)

or3y = $\frac{23x}{3} \pm \frac{\sqrt{(23)^2 - 45}}{3}$

$\Rightarrow$ 3y = 23x ± 22

Sol 11: (B, C, D) $16(x^2 - 2x) - 3(y - 4y) = 44$

$16(x - 1)^2 - 3(y - 2)^2 = 44 + 16 - 12$

$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$

Conjugate = 2b = 2 × 4 = 8

Centre = (1, 2)

and $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{16}{9} \Rightarrow e = \sqrt{\frac{12}{3}}$

Sol 12: (B, D) $\frac{x^2}{16} - \frac{y^2}{9} = 0$

Now tangent

$1 \Rightarrow y = mx \pm \sqrt{a^2m^2 - b^2}$

$y = mx \pm \sqrt{16m^2 - 9}$

tangent $2 \Rightarrow y = mx \pm 3\sqrt{m^2 + 1}$

so$16m^2 - 9 = 9(m^2 + 1)$

$\Rightarrow 7m^2 = 18$

$\Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

soy $= 3\sqrt{\frac{2}{7}}x \pm 3\sqrt{\frac{18}{7} + 1}$

$y = 3\sqrt{\frac{2}{7}}x \pm \frac{16}{\sqrt{7}}$

Sol 13: (A, C, D)

(A) $\left(\frac{2x}{a}\right)^2 - \left(\frac{2y}{b}\right)^2 = 4$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(C) $x^2 - y^2 = 4$

(D) $x^2 - 6 = 2\cos t$

and $y^2 + 2 = 2\left(\sin^2\frac{t}{2} - 1\right) + 2$

$y^2 = 2\cos t$

$\Rightarrow x^2 - y^2 = 6$

(B)$t = \frac{b}{y}\left(1 - \frac{x}{a}\right)$

so $\frac{x}{a}.\left(\frac{b}{y}\right).\left(1 - \frac{x}{a}\right) - \frac{b}{y}\left(1 - \frac{x}{a}\right) = 0$

$\frac{bx.(a-x)}{a^2y} - \frac{y}{b} + \frac{b(a-x)}{ay} = 0$

x ba^{2}y

$b^2x.(a - x) - a^2y^2 + ab^2(a - x) = 0$

$ab^2x - b^2x^2 - a^2y^2 + a^2b^2 - ab^2x = 0$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol 14: (A, D) We have equation of circle

$(x - x_1)(x - x_2) + (y_0 - y_1)(y - y_2) = 0$

Now,

$x_1 = ct_1 \& y_1 = c/t_1$

$x_2 = ct_2 \& y_2 = c/t_2$

so slope $\Rightarrow \frac{c/t_2 - c/t_1}{ct_2 - ct_1} = \frac{-1}{t_1.t_2}$

Now, slope = 1

$\Rightarrow -t_1.\frac{1}{t_2}$

$\Rightarrow t_1 = \frac{-1}{t_2}$

putting this above

$(x - ct_1)(x - ct_2) + (y - \frac{c}{t_1})(y - \frac{c}{t_2}) = 0$

$(x - ct_1)(x + \frac{c}{t_1}) + (y - \frac{c}{t_1})(y + ct_1) = 0$

$x^2 - c^2 + 2c\left[\frac{1}{t_1} - t_1\right]x + y^2$

$- c^2 + cy.\left[t_1 - \frac{1}{t_1}\right]$

$(x^2 + y^2 - 2c^2) + c[x - y]\left[\frac{1}{t_1} - t_1\right] = 0$

Now when $x = y$ & $x^2 + y^2 = 2c^2$

this is satisfied for

$x = c$ & $y = c$

$x = -c$ & $y = -c$

Sol 15: (A, B, C, D) $x = \sqrt{2}\,t$ and $y = \sqrt{2}/t$

Now slope of normal = t^2

so $\left(y - \frac{c}{t}\right) = t^2(x - ct)$

$ty - c = t^3x - ct^4$

$\Rightarrow ct^4 - t^3x + ty - c = 0$

passes through (3,4)

Now $ct^4 - 3t^3 + 4t - c = 0$

thus, $St_i = \frac{3}{c}$

$\Rightarrow$ c. Sti = 3

$\Rightarrow$ Sxi = 3(A)

Also $\pi t_i = -1$

$St_1.\,t_2.\,t_3 = \frac{-4}{c}$

$\Rightarrow \frac{\Sigma t_1.t_2.t_3}{\pi t_i} = +\frac{4}{c} \times \frac{1}{(-1)}$

$\Rightarrow c.\,\Sigma\frac{1}{t_i} = 4$

$\Rightarrow Sy_i = 4$(B)

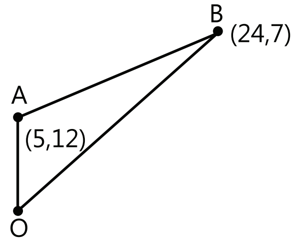
Now $\pi t_i = -1$

$\Rightarrow \pi(ct_i) = -c^4 = -(\sqrt{2})^4 = -4$

and $\frac{1}{\pi t_i} = -1$

$\Rightarrow \pi\left(\frac{c}{t_i}\right) = -c^4 = -4$(C) & (D)

Sol 16: (A, D)



Now AO + BO = 2a

$\sqrt{5^2 + 12^2} + \sqrt{24^2 + 7^2} = 13 + 25 = 38$

So2ae = $\sqrt{19^2 + 5^2}$

38e = $\sqrt{386}$

$\Rightarrow$ e = (0)(if ellipse)

BO – AO = 2a(hyperbola)

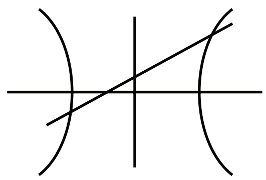
$\Rightarrow \sqrt{24^2 + 72} - \sqrt{52 + 122} = 2a$

$\Rightarrow 25 - 13 = 2a$

$\Rightarrow 12 = 2a$

So2ae = $\sqrt{386}$ $\Rightarrow$ e = $\boxed{e = \sqrt{386}/12}$

Sol 17: (A, B)



Now,

$6 = \sqrt{100m^2 - 49}$

$\Rightarrow 36 + 49 = 100\,m^2$

$\Rightarrow \pm\sqrt{\frac{85}{100}} = m \Rightarrow m = \pm\sqrt{\frac{17}{20}}$

Sol 18: (A, B, D)

k < 8 and k > 12 hyperbola (A)

8 < k < 12 ellipse and

if k = 10 circle

Sol 19: (A, B, C, D) $y = mx + \sqrt{a^2bm^2 - b^2}$

and $y = mx + \sqrt{a^2 - b^2m^2}$

so $\sqrt{a^2m^2 - b^2} = \sqrt{a^2 - b^2m^2}$

$\Rightarrow a^2m^2 - b^2 = a^2 - b^2m^2$

$\Rightarrow a^2(m^2 - 1) = (m^2 - 1)(-b^2)$

$\Rightarrow m = \pm 1$

So, $y = x \pm \sqrt{a^2 - b^2}$ or $y = -x \pm \sqrt{a^2 - b^2}$

Sol 20: (B, D) $\frac{x^2}{18}-\frac{y^2}{9}=1$

Now m = –1

So $y = mx \pm \sqrt{a^2m^2-b^2}$

$\Rightarrow y = -x \pm \sqrt{18(+1)-9}$

$\Rightarrow y = -x \pm 3$

$\Rightarrow y = -x \pm 3$

$\Rightarrow x + y = 3$ and $x + y = -3$

Sol 21: (C, D) $9(x^2 + 2y) - 16(y^2 - 2y) = 151$

$9(x + 1)^2 - 16(y - 1)^2 = 151 + 9 - 16$

$\frac{(x+1)^2}{16}-\frac{(y-1)^2}{9}=1$

Now $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

So distance from centre

$ae = 4 \times \frac{5}{4} = 5$

$\Rightarrow$ (–1 + 5, 1) and (–1–5, 1)

(4, 1) and (–6, 1)

Sol 22: (B, C) Equation of chord connecting the points (a secθ, b secθ) and (a tanϕ, b tanϕ) is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right) \quad \text{... (i)}$$

If it passes through (ae, 0); we, have

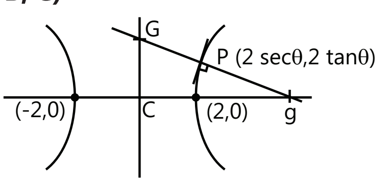
$$e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad e=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1-e}{1+e}$$

Similarly if (i) passes through (-ae, 0), tan.tan = $\frac{1+e}{1-e}$

Sol 23: (A, B, C)



Normal: $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

$\Rightarrow g \Rightarrow x_1 = x = \frac{x_1.(a^2+b^2)}{a^2}$

$\boxed{x=\frac{\sec\theta.(a^2+b^2)}{a}} = 4\sec\theta$

and $G \Rightarrow y = \frac{\tan\theta(a^2-b^2)}{b} \Rightarrow 4\tan\theta$

$PC = 2\sqrt{\sec^2\theta+\tan^2\theta}$

$Og = \sqrt{\frac{\sec^2\theta.(a^2+b^2)^2}{a^2}+\frac{\tan^2\theta(a^2+b^2)}{b^2}}$

$= (a^2+b^2)\sqrt{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}} = \frac{8}{2}\sqrt{\sec^2\theta+\tan^2\theta}$

$PG = \sqrt{(2\sec\theta)^2+(2\tan\theta)^2} = 2.\sqrt{\sec^2\theta+\tan^2\theta}$

$Pg = \sqrt{(4\sec\theta-2\sec\theta)^2+(2\tan\theta)^2}$

$= 2\sqrt{\sec^2\theta+\tan^2\theta}$

Previous Years' Questions

Sol 1: (A, B, C, D) It is given that

$x^2 + y^2 = +a^2$ ... (i)

and $xy = c^2$ ... (ii)

We obtain $x^2 = c^4/x^2 = a^2$

$\Rightarrow x^4 - a^2x^2 + c^4 = 0$ ... (iii)

Now, x_1, x_2, x_3, x_4 will be roots of Eq. (iii)

Therefore, $Sx_2 = x_1+x_2+2x_3+x_4 = 0$

and product of the roots $x_1x_2x_3x_4=c^4$

Similarly, $y_1 + y_2 + y_3 + y_4 = 0$ and $y_3y_2y_1y_4 = c^4$

Hence, all options are correct.

Sol 2: (A, B) Given, $2x^2 - 2y^2 = 1$

$\Rightarrow \frac{x^2}{\left(\frac{1}{2}\right)}-\frac{y^2}{\left(\frac{1}{2}\right)}=1$ ... (i)

Eccentricity of hyperbola = $\sqrt{2}$ So eccentricity of ellipse

$= 1/\sqrt{2}$

Let equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ (a > b)

$\therefore \frac{1}{\sqrt{2}} = \sqrt{1-\frac{b^2}{a^2}}$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow a^2 = 2b^2$

$\therefore x^2 + 2y^2 = 2b^2$ ... (ii)

Let ellipse and hyperbola intersect as

$A\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)$

On differentiating Eq. (i),

$4x - 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\therefore$ At (6, 3) $\Rightarrow \frac{a^2x}{6}+\frac{b^2y}{3} = (a^2+b^2)$

It passes through (9, 0)

$\Rightarrow \frac{a^2.9}{6} = a^2 + b^2$

$\Rightarrow \frac{3a^2}{2}-a^2 = b^2 \Rightarrow \frac{a^2}{b^2} = 2$

$\therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2} \Rightarrow \sqrt{\frac{3}{2}}$

$\left.\frac{dy}{dx}\right|_{at\ A} = \frac{\sec\theta}{\tan\theta} = \text{cosec}\theta$

and differentiating Eq. (ii)

$2x + 4y\frac{dy}{dx} = 0$

$\left.\frac{dy}{dx}\right|_{at\ A} = -\frac{x}{2y} = -\frac{1}{2}\text{cosec}\theta$

Since, ellipse and hyperbola are orthogonal

$\therefore -\frac{1}{2}\text{cosec}^2\theta = -1$

$\Rightarrow \text{cosec}^2\theta = 2 \Rightarrow \theta = \pm\frac{\pi}{4}$

$\therefore A\left(1, \frac{1}{\sqrt{2}}\right)$ or $\left(1, -\frac{1}{\sqrt{2}}\right)$

$\therefore$ From Eq. (i),

$1 + 2\left(\frac{1}{\sqrt{2}}\right)^2 = 2b^2$

$\Rightarrow b^2 = 1$

Equation of ellipse is $x^2 + 2y^2 = 2$

Coordinate of foci

$(\pm ae, 0) = \left(\pm\sqrt{2}.\frac{1}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

Hence, option (A) and (B) are correct.

If major axis is along y-axis, then

$\frac{1}{\sqrt{2}} = \sqrt{1-\frac{a^2}{b^2}}$

$\Rightarrow b^2 = 2a^2$

$\therefore 2x^2 + y^2 = 2a^2$

$\Rightarrow y' = -\frac{2x}{y}$

$\Rightarrow y'_{\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)} = \frac{-2}{\sin\theta}$

As ellipse and hyperbola are orthogonal

$\therefore \frac{-2}{\sin\theta}.\text{cosec}\theta = -1$

$\Rightarrow \text{cosec}^2\theta = 1 \Rightarrow \theta = \pm\frac{\pi}{4}$

$\therefore 2x^2 + y^2 = 2a^2$

$\Rightarrow 2 + \frac{1}{2} = 2a^2$

$\Rightarrow a^2 = \frac{5}{4}$

$\Rightarrow 2x^2 + y^2 = \frac{5}{2}$, corresponding foci are (0, ± 1)

Sol 3: (B, D) Here, equation of ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$

$\Rightarrow e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore e = \frac{\sqrt{3}}{2}$ and focus (± ae, 0)

$\Rightarrow (\pm\sqrt{3}, 0)$

For hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

$e_1^2 = 1 + \frac{b^2}{a^2}$

where, $e_1^2 = \frac{1}{e^2} = \frac{4}{3}$

$\Rightarrow 1 + \frac{b^2}{a^2} = \frac{4}{3}$

$\therefore \frac{b^2}{a^2} = \frac{1}{3}$ ... (i)

and hyperbola passes through $(\pm\sqrt{3}, 0)$

$\Rightarrow \frac{3}{a^2} = 1$

$\Rightarrow a^2 = 3$ ... (ii)

From Eqs.(i) and (ii), we get

$b^2 = 1$ ... (iii)

$\therefore$ Equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

Focus is $(\pm ae_1, 0)$

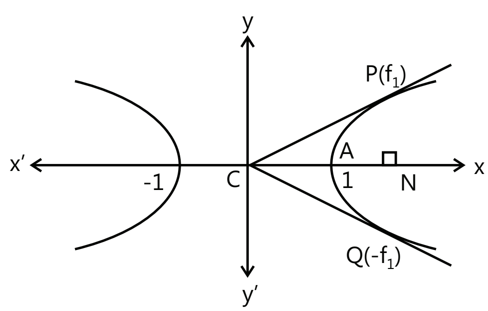
$\Rightarrow \left(\pm\sqrt{3}.\frac{2}{\sqrt{3}}, 0\right) \Rightarrow (\pm 2, 0)$

$\therefore$ (B) and (D) are correct answers.

Sol 4: Let P = $\left(\frac{e^{t_1}+e^{-t_1}}{2}, \frac{e^{t_1}-e^{-t_1}}{2}\right)$

and Q = $\left(\frac{e^{-t_1}+e^{t_1}}{2}, \frac{e^{-t_1}-e^{t}}{2}\right)$

We have to find the area of the region bounded by the curve $x^2 - y^2 = 1$ & the lines joining the centre x = 0, y = 0 to the points (t_1) and $(-t_1)$



Required area

$= 2\left[\text{area of } \Delta PCN = \int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dy\right]$

$= 2\left[\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) - \int_1^{t_1} y\frac{dx}{dy}.dt\right]$

$= 2\left[\frac{e^{2t_1}-e^{-2t_1}}{8} - \int_0^{t_1}\left(\frac{e^t-e^{-t}}{2}\right)^2.dt\right]$

$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\int_0^{t_1}(e^{2t}+e^{-2t}-2)dt$

$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} - 2t\right]_0^{t_1}$

$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{4}(e^{2t_1} - e^{-2t_1} - 4t_1) = t_1$

Sol 5: Let any point on the hyperbola is $(3\sec\theta, 2\tan\theta)$

$\therefore$ Chord of contact of the circle $x^2+y^2 = 9$ with respect to the point $(3\sec^t\theta, 2\tan\theta)$ is,

$(3\sec\theta)x + (2\tan\theta)y = 9$ ... (i)

Let (x_1, y_1) be the mid point of the chord of contact

$\Rightarrow$ Equation of chord in mid point form is

$xx_1 + yy_1 = x_1^2 + y_1^2$ ... (ii)

Since, Eqs. (i) and (ii) are identically equal

$\therefore \frac{3\sec\theta}{x_1} = \frac{2\tan\theta}{y_1} = \frac{9}{x_1^2+y_1^2}$

$\Rightarrow \sec\theta = \frac{9x_1}{3(x_1^2+y_1^2)}$

and $\tan\theta = \frac{9y_1}{2(x_1^2+y_1^2)}$

Thus, eliminating 'θ' from above equation, we get

$\frac{81x_1^2}{9(x_1^2+y_1^2)^2} - \frac{81y_1^2}{4(x_1^2+y_1^2)^2} = 1$

$(\because \sec^2\theta - \tan^2\theta = 1)$

$\therefore$ Required locus is $\frac{x^2}{9} - \frac{y^2}{4} = \frac{(x^2+y^2)^2}{81}$

Sol 6: (B) Equation of tangents to hyperbola having slope m is

$y = mx + \sqrt{9m^2-4}$ ... (i)

Equation of tangent to circle is

$y = m(x-4) + \sqrt{16m^2+16}$ ... (ii)

Eqs. (i) and (ii) will be identical for $m = \frac{2}{\sqrt{5}}$ satisfy.

$\therefore$ Equation of common tangent is $2x - \sqrt{5}y + 4 = 0$.

Sol 7: (A) The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$

and that of circle is $x^2 + y^2 - 8x = 0$

For their points of intersection $\frac{x^2}{9}+\frac{x^2-8x}{4}=1$

$\Rightarrow 4x^2 + 9x^2 - 72x = 36$

$\Rightarrow 13x^2 - 72x - 36 = 0$

$\Rightarrow 13x^2 - 78x + 6x - 36 = 0$

$\Rightarrow 13x(x - 6) = 6(x - 6) = 0$

$\Rightarrow x = 6, x = -\frac{13}{6}$

$x = -\frac{13}{6}$ not acceptable

Now, for $x = 6$, $y = \pm 2\sqrt{3}$

Required equation is $(x - 6)^2 + (y + 2\sqrt{3})(y - 2\sqrt{3})=0$

$\Rightarrow x^2 - 12x + y^2 + 24 = 0$

$\Rightarrow x^2 + y^2 - 12x + 24 = 0$

Sol 8: On substituting $\left(\frac{a}{e},0\right)$ in $y = -2x + 1$,

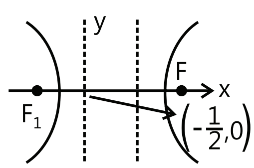
we get $0 = -\frac{2a}{e} + 1$

$\Rightarrow \frac{a}{e} = \frac{1}{2}$

Also, $y = -2x + 1$ is tangent to hyperbola

$\therefore 1 = 4a^2 - b^2$

$\Rightarrow \frac{1}{5} = 4 - (e^2 - 1)$



$\Rightarrow \frac{4}{e^2} = 5 - e^2$

$\Rightarrow e^4 - 5e^2 + 4 = 0$

$\Rightarrow (e^2 - 4)(e^2 - 1) = 0$

$\Rightarrow e = 2, e = 1$

e = 1 gives the conic as parabola. But conic is given as hyperbola, hence e = 2.

Sol 9: (B) Hyperbola is $\frac{\left(x-\sqrt{2}\right)^2}{4}-\frac{\left(y+\sqrt{2}\right)^2}{2}=1$

$a = 2, b = \sqrt{2}$

$e = \sqrt{\frac{3}{2}}$

Area $= \frac{1}{2}a(e-1)\times\frac{b^2}{a} = \frac{1}{2}\frac{\left(\sqrt{3}-\sqrt{2}\right)\times 2}{\sqrt{2}} = \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{2}}$

$\Rightarrow$ Area $= \left(\sqrt{\frac{3}{2}}-1\right)$.

Sol 10: A→p; B→s, t; C→ r; D→ q, s

(p) $\frac{1}{k^2} = 4\left(1+\frac{h^2}{k^2}\right)$

$\Rightarrow 1 = 4\left(k^2+h^2\right)$

$\therefore h^2+k^2 = \left(\frac{1}{2}\right)^2$ which is a circle.

(q) If $\left|z-z_1\right|-\left|z-z_2\right| = k$ where $k < \left|z_1-z_2\right|$ the locus is a hyperbola.

(r) Let $t = \tan\alpha$

$\Rightarrow x = \sqrt{3}\cos 2\alpha$ and $\sin 2\alpha = y$

or $\cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \frac{x^2}{3}+y^2 = \sin^2 2\alpha+\cos^2 2\alpha = 1$ which is an ellipse.

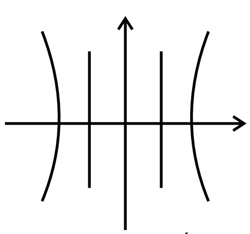
(s) If eccentricity is $\left[1,\infty\right)$, then the conic can be a parabola (if e = 1) and a hyperbola if $e\in\left(1,\infty\right)$.

(t) Let $z = x+iy;\ x,y\in R$

$\Rightarrow \left(x+1\right)^2-y^2 = x^2+y^2+1$

$\Rightarrow y^2 = x$; which is a parabola.

Sol 11: $y = -2x+1$



$0 = -\frac{2a}{e}+1$

$\Rightarrow \frac{a}{e} = \frac{1}{2}$

$e = 2a$

$c^2 = a^2m^2-b^2$

$\Rightarrow 1 = 4a^2-b^2$

$\Rightarrow 1+b^2-4a^2 = 0$

$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{\left(4a^2-1\right)}{a^2}$

$e^2 = 1+4-\frac{1}{a^2}$

$e^2 = 5-\frac{1}{e^2}$

$\Rightarrow e^4-5e^2+4 = 0$

$\Rightarrow \left(e^2-1\right)\left(e^2-4\right) = 0$

$e^2 - 1 \neq 0 \;\; e = 2$

Sol 12: (B) Equation of normal is

$$\left(y-3\right) = \frac{-a^2}{2b^2}\left(x-6\right) \Rightarrow \frac{a^2}{2b^2} = 1 \Rightarrow e = \sqrt{\frac{3}{2}}.$$

Sol 13: (B, D) Ellipse is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

$$1^2 = 2^2\left(1-e^2\right) \Rightarrow e = \frac{\sqrt{3}}{2}$$

$\therefore$ Eccentricity of the hyperbola is

$$\frac{2}{\sqrt{3}} \Rightarrow b^2 = a^2\left(\frac{4}{3}-1\right) \Rightarrow 3b^2 = a^2$$

Foci of the ellipse are $\left(\sqrt{3},0\right)$ and $\left(-\sqrt{3},0\right)$.

Hyperbola passes through $\left(\sqrt{3},0\right)$

$$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3 \text{ and } b^2 = 1$$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$

Focus of hyperbola is (ae, 0)

$$\left(ae,0\right) \equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) \equiv \left(2, 0\right)$$

Sol 14: (A, B) Slope of tangent = 2

The tangents are $y = 2x \pm \sqrt{9\times 4 - 4}$

i.e., $2x - y = \pm 4\sqrt{2}$

$$\Rightarrow \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1 \text{ and } -\frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$

We get point of contact as $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.