PROBLEM-SOLVING TACTICS

- If the line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then $c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2m^2}$ is the condition of normality of the line to the ellipse.
- The tangent and normal at any point of an ellipse bisect the external and angles between the focal radii to the point. It follows from the above property that if an incident light ray passing through the focus (S) strikes the concave side of the ellipse, then the reflected ray will pass through the other focus (S′).
- If SM and S′M′ are perpendicular from the foci upon the tangent at any point of the ellipse, then SM. S′M′ = b^2 and M, M′ lie on the auxiliary circle.
- If the tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis in T and minor axis in T′, then CN. CT = a^2, CN′. CT′ = b^2

  Where N and N′ are the feet of the perpendicular from P on the respective axis.
- If SM and S′ M′ are perpendicular from the foci S and S′ respectively upon a tangent to the ellipse, then CM and CM′ are parallel to S′P and SP respectively.

FORMULAE SHEET

1. The general equation of second order $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse

if $\Delta \neq 0$, $h^2 < ab$. where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \right)$

2. The sum of the focal distance of any point on an ellipse is a constant and is equal to the length of the major axis of the ellipse i.e. SP + S′P = 2a.

3. Standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Where a = length of semi-major axis,

b = length of semi-minor axis

4.

Imp. Terms \ Ellipse	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	For b > a
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	(±ae , 0)	(0 , ± be)
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2 (1 - e^2)$	$a^2 = b^2 (1 - e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	(a cos φ, b sinφ)	(a cos φ, b sin φ) $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii (SP + S'P =)	2a	2b
Distance between foci	2ae	2be
Distance between directrices	2a/e	2b/e
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

5. The equations $x = a\cos\phi,\ y = b\sin\phi$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.

6. **(i)** If the centre of the ellipse is at (h, k) and the axes are parallel to the coordinate axes, then its equation is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

(ii) If the equation of the ellipse is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $lx+my+n=0$ and $mx-ly+p=0$ are perpendicular lines. Substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$ and $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

7. If $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ are any two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of a chord joining these two points is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

8. The point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0, =0$ or <0 respectively.

9. The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on two distinct points if $a^2m^2+b^2>c^2$, on one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the auxiliary circle is $x^2+y^2=a^2$.

10. The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. The equation of tangent to the ellipse having its slope equal to m is $y = mx \pm \sqrt{a^2m^2+b^2}$ and the point of contact is $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$. The equation of the tangent at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $P(a\cos\theta_1, b\sin\theta_1)$,

and $Q(a\cos\theta_2, b\sin\theta_2)$ is $\left(\frac{a\cos\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}, \frac{b\sin\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}\right)$.

11. Equation of pair of tangents drawn from an outside point $P(x_1, y_1)$ is $SS_1 = T^2$.

12. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of director circle is $x^2+y^2=a^2+b^2$.

13. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point $(a\cos\phi, b\sin\phi)$ is $(ax\sec\phi - by\mathrm{cosec}\phi)=a^2-b^2$.

14. If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the normal is $y = mx \pm \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$. The co-ordinates of the point of contact are $\left(\frac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \frac{\pm mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

15. The properties of conormal points are

(i) Property 1: The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

(ii) Property 2: If θ_1, θ_2 and θ_3 are eccentric angles of three co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$.

(iii) Property 3: Co-normal points lie on a fixed curve called an Apollonian Rectangular Hyperbola $\left(a^2-b^2\right)xy+b^2kx-a^2hy=0$

(iv) Property 4: If the normal at four points $P(x_1y_1)$, $Q(x_2y_2)$, $R(x_3y_3)$ and $S(x_4y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

16. If SM and S′M′ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \times S'M' = b^2$ and M, M′ lie on the auxiliary circle.

17. If the tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major axis at T and minor axis at T′, then CN × CT = a^2, CN′ × CT′ = b^2. Where N and N′ are the feet of the perpendiculars from P on the respectively axis.

18. The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point is (x_1, y_1), is $T = S_1$.

19. The chord of contact from a point $P(x_1, y_1)$ to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0 is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

20. The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$.

21. If m_1 and m_2 are the slopes of two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2=\frac{-b^2}{a^2}$.

22. The eccentric angle of the ends of a pair of conjugate diameters of an ellipse differ by a right angle, i.e., $\phi-\phi'=\frac{\pi}{2}$.

23. The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and is equal to the sum of the squares of the semi axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$.

24. The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point i.e., $SP \times S'P = CD^2$.

25. The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes.

i.e. Area of the parallelogram = (2a)(2b) = Area of the rectangle contained under major and minor axes.

26. Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e., $(CP)^2 = (CD)^2$

$\therefore (CP)=(CD)=\sqrt{\frac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

27. Equation of the polar of the point (x_1, y_1) w.r.t. an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n},\frac{-b^2m}{n}\right)$.

28. The pole of the line $lx + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n},\frac{-b^2m}{n}\right)$.

29. Condition for a conjugate point is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

30. The length of a sub tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2}{x_1}-x_1$.

31. The length of a sub normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{b^2}{a^2}x_1=(1-e^2)x_1$.