24. DIFFERENTIAL EQUATIONS

1. INTRODUCTION

An equation containing an independent variable, dependent variable and differential coefficients is called a differential equation.

(i) $\frac{dy}{dx} = \sin x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^3 = 0$ (iii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} = 5\cos 3x$

2. ORDER OF DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of the above mentioned differential equations are 1, 2, and 4 respectively.

3. DEGREE OF DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest order derivative when differential coefficients are free from radicals and fractions. For example the degrees of above differential equations are 1, 2, and 3 respectively.

Table 24.1: Degree of differential equation

Differential Equation	Order of D.E.	Degree of D.E
$\frac{dy}{dx} + 4y = \sin x$	1	1
$\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$	2	4
$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$	2	1
$\frac{dy}{dx} = \frac{x^4 - y^4}{xy\left(x^2 + y^2\right)}$	1	1

Differential Equation	Order of D.E.	Degree of D.E
$y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$ $\Rightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$	1	2
$\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$	2	2

4. CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations are first classified according to their order. First-order differential equations are those in which only the first order derivative, and no higher order derivatives appear. Differential equations of order two or more are referred to as higher order differential equations.

A differential equation is said to be linear if the unknown function, together with all of its derivatives, appears in the differential equations with a power not greater than one and not as products either. A nonlinear differential equation is a differential equation which is not linear.

e.g. $y' + y = 0$ is a linear differential equation,

$y'' + yy' + y^2 = 0$ is a non linear differential equation,

Procedure to form a differential equation that represents a given family of curves

Case I:

If the given family F1 of curves depends on only one parameter then it is represented by an equation of the form $F1(x, y, a) = 0$ ... (i)

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form

$f(x, y, a): y^2 = ax$

Differentiating equation (i) with respect to x, we get an equation involving y′, y, x and a.

$g(x, y, y', a) = 0$ ... (ii)

The required differential equation is then obtained by eliminating a from equation (i) and (ii) as

$F(x, y, y') = 0$ ... (iii)

Case II:

If the given family F2 of curves depends on the parameters a, b (say) then it is represented by an equation of the form $F2(x, y, a, b) = 0$ ... (iv)

Differentiating equation (iv) with respect to x, we get an equation involving y′, x, y, a, b.

$g(x, y, y', a, b) = 0$ ... (v)

Now we need another equation to eliminate both a and b. This equation is obtained by differentiating equation (v), wrt x, to obtain a relation of the form $h(x, y, y', y'', a, b) = 0$ ... (vi)

The required differential equation is then obtained by elimination a and b from equations (iv), (v) and (vi) as $F(x, y, y', y'') = 0$ ... (vii)

Note: The order of a differential equation representing a family of curves is the same as the number of arbitrary constants present in the equation corresponding to the family of curves.

5. FORMATION OF DIFFERENTIAL EQUATIONS

If an equation is dependent and dependent variables having some arbitrary constant are given, then the differential equation is obtained as follows:

(a) Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) Hence on eliminating arbitrary constants results a differential equation which involves x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ .... $\frac{d^my}{d_xm}$ (where m=number of arbitrary constants).

Illustration 1: Form the differential equation corresponding to $y^2 = m(a^2 - x^2)$, where m and a are arbitrary constants. **(JEE MAIN)**

Sol: Since the given equation contains two arbitrary constant, we shall differentiate it two times with respect to x and we get a differential equation of second order.

We are given that $y^2 = m(a^2 - x^2)$ ... (i)

Differentiating both sides of (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = m(-2x) \Rightarrow y\frac{dy}{dx} = -mx \quad \text{... (ii)}$$

Differentiating both sides of (ii) w.r.t. to x, we get $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m$ ... (iii)

From (ii) and (iii), we get, $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$

This is the required differential equation.

Illustration 2: Form diff. equation of $ax^2 + by^2 = 1$ **(JEE MAIN)**

Sol: Similar to the above problem the given equation contains two arbitrary constants, so we shall differentiate it two times with respect to x and then by eliminating a and b we get the differential equation of second order.

$$ax^2 + by^2 = 1 \Rightarrow \quad 2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \quad a + b\,(yy'' + (y')^2) = 0$$

Eliminating a and b we get $\frac{y}{x}y' = yy'' + (y')^2 \quad \Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} = 0$

Illustration 3: Form the differential equation corresponding to $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants. **(JEE MAIN)**

Sol: Similar to illustration 1.

We have, $y^2 = a(b^2 - x^2)$ ... (i)

In this equation, there are two arbitrary constants a, b, so we have to differentiate twice, Differentiating the given equation (i) w.r.t. 'x'. We get $2y\dfrac{dy}{dx} = -2x.a \Rightarrow y\dfrac{dy}{dx} = -ax$ ... (ii)

Differentiating (ii) with respect to x, we get $y\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}.\dfrac{dy}{dx} = -a \Rightarrow y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2 = -a$ ... (iii)

Substituting the value of a in (ii), we get

$$y\frac{dy}{dx} = \left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)2\right\}x \Rightarrow y\frac{dy}{dx} = xy\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 \quad \Rightarrow \quad xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Illustration 4: Find the differential equation of the following family of curves: $xy = Ae^x + Be^{-x} + x^2$ **(JEE MAIN)**

Sol: Here in this problem A and B are the two arbitrary constants, hence we shall differentiate it two times with respect to x and then by eliminating constant terms we will get the required differential equation.

Given: $xy = Ae^x + Be^{-x} + x^2$ ... (i)

Differentiating (i) with respect to 'x', we get $x\dfrac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2}+1\frac{dy}{dx}+1.\frac{dy}{dx} = Ae^x + Be^{-x} + 2 \Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx} = xy - x^2 + 2$$

Illustration 5: Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is a general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ **(JEE ADVANCED)**

Sol: Here only one arbitrary constant is present hence we shall differentiate it one time with respect to x and then by substituting the value of c we shall prove the given equation.

Let us find the differential equation for $x^2 - y^2 = c(x^2 + y^2)^2$ ... (i)

Differentiating (i), with respect to 'x', we get $2x - 2y\dfrac{dy}{dx} = c.2\left(x^2+y^2\right)\left(2x+2y\dfrac{dy}{dx}\right)$ ... (ii)

Substituting the value of c from (i) in (ii), we get

$$\Rightarrow x - y\frac{dy}{dx} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) \Rightarrow (x^2+y^2)\left(x-y\frac{dy}{dx}\right) = (x^2-y^2)\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow [2y(x^2-y^2) + y(x^2+y^2)]\frac{dy}{dx} = x(x^2+y^2) - 2x(x^2-y^2) \quad \Rightarrow \quad (3x^2y - y^3)\frac{dy}{dx} = 3xy^2 - x^3$$

$\Rightarrow (x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ As this equation matches the one given in the problem statement. Hence the given equation is the solution for the differential equation.

Hence proved.

Illustration 6: Find the differential equation of the family of curves $y = e^x(a\cos x + b\sin x)$ **(JEE ADVANCED)**

Sol: Since given family of curves have two constants a and b, so we have to differentiate twice with respect to x.

We have, $y = e^x(a\cos x + b\sin x)$ ... (i)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = y + e^x(-a\sin x + b\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} - y = e^x(-a\sin x + b\cos x) \qquad \text{... (ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x) = \frac{dy}{dx} - y - e^x(a\cos x + b\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y \qquad [\because e^x(a\cos x + b\sin x) = y] \Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

This is the required differential equation.

Illustration 7: Find the differential equation of all circles which pass through the origin and whose centers lie on the y axis. **(JEE ADVANCED)**

Sol: As circles passes through the origin and whose centers lie on the y axis hence g = 0 and point (0, 0) will satisfy general equation of given circle.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (i)}$$

Since it passes through origin (0, 0), it will satisfy equation (i)

$$\Rightarrow \quad (0)^2 + (0)^2 + 2g.(0) + 2f.(0) + c = 0 \quad \Rightarrow \quad c = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy = 0$$

This is the equation of a circle with center (–g, –f) and passing through the origin.

If the center lies on the y-axis, we have g = 0,

$$\Rightarrow \quad x^2 + y^2 + 2.(0).x + 2fy = 0 \qquad \Rightarrow \quad x^2 + y^2 + 2fy = 0 \qquad \text{... (ii)}$$

Hence, (ii) represents the required family of circles with center on y axis and passing through origin.

Differentiating (ii) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0 \quad \Rightarrow f = -\left\{\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right\}$$

Substituting this value of f in (2), we get

$$x^2 + y^2 - 2y\left(\frac{x + y.\left(\frac{dy}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right) = 0 \Rightarrow (x^2 + y^2)\frac{dy}{dx} - 2xy - 2y^2\left(\frac{dy}{dx}\right) = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

This is the required differential equation.

MASTERJEE CONCEPTS

Curves representing the solution of a differential equation are called integral curves.

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6. SOLUTIONS OF DIFFERENTIAL EQUATIONS

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between the dependent and independent variables (free from derivatives) such that it satisfies the given differential equation.

Note: The solution of the differential equation is also called its primitive.

There can be two types of solution to a differential equation:

(a) General solution (or complete integral or complete primitive)

A relation in x and y satisfying a given differential equation and involving exactly the same number of arbitrary constants as the order of the differential equation.

(b) Particular solution

A solution obtained by assigning values to one or more than one arbitrary constant of general solution

Illustration 8: The general solution of $x^2\frac{dy}{dx} = 2$ is **(JEE MAIN)**

Sol: First separate out x term and y term and then integrate it, we shall obtain result.

$\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow dy = \frac{2}{x^2}dx$ Now integrate it. We get $y = -\frac{2}{x} + c$

Illustration 9: Verify that the function $x + y = \tan^{-1}y$ is a solution of the differential equation $y^2y' + y^2 + 1 = 0$ **(JEE MAIN)**

Sol: By differentiating the equation $x + y = \tan^{-1}y$ with respect to x we can prove the given equation.

We have, $x + y = \tan^{-1}y$ ... (i)

Differentiating (i), w.r.t. x we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx} \Rightarrow 1 + \frac{dy}{dx}\left(\frac{1+y^2-1}{1+y^2}\right) = 0$$

$$\Rightarrow (1 + y^2) + y^2\frac{dy}{dx} = 0 \Rightarrow y^2y' + y^2 + 1 = 0$$

Illustration 10: Show that the function $y = Ax + \left(2x+2y\frac{dy}{dx}\right)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ **(JEE MAIN)**

Sol: Differentiating $y = Ax + \frac{B}{x}$ twice with respect to x and eliminating the constant term, we can prove the given equation.

We have, $y = Ax + \frac{dy}{dx} \Rightarrow xy = Ax^2 + B$ ... (i)

Differentiation (i) w.r.t. 'x'. we get $\Rightarrow x\frac{dy}{dx} + 1.y = 2Ax$ ... (ii)

Again differentiating (ii) w.r.t., 'x', we get

$$\Rightarrow x.\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 2A \qquad \Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{x\frac{dy}{dx}+y}{x} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Which is same as the given differential equation. Therefore $y = Ax + \frac{dy}{dx}$ is a solution for the given differential equation.

Illustration 11: If $y.\sqrt{x^2+1}=\log\left[\sqrt{x^2+1}\right]$ show that $(x^2 + 1)\dfrac{dy}{dx} + xy + 1 = 0$ **(JEE MAIN)**

Sol: Similar to the problem above, by differentiating $y.\sqrt{x^2+1}=\log\left[\sqrt{x^2+1}-x\right]$ one time with respect to x, we will prove the given equation.

We have, $y.\sqrt{x^2+1}=\log\left[\sqrt{x^2+1}\right]$ ... (i)

Differentiating (i), we get

$$\sqrt{x^2+1}\frac{dy}{dx}+\frac{1}{2}\frac{2x}{\sqrt{x^2+1}}y=\frac{(1/2)\left(2x/\sqrt{x^2+1}\right)-1}{\sqrt{x^2+1}-x}\Rightarrow\sqrt{x^2+1}\frac{dy}{dx}+\frac{x}{\sqrt{x^2+1}}=\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}\left[\sqrt{x^2+1}-x\right]};$$

$$(x^2+1)\frac{dy}{dx}+xy=\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}-x};\qquad(x^2+1)\frac{dy}{dx}+xy=-1;\qquad(x^2+1)\frac{dy}{dx}+xy+1=0$$

Illustration 12: Show that y = acos(logx) + bsin(logx) is a solution of the differential equation:
$x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$ **(JEE ADVANCED)**

Sol: As the given equation has two arbitrary constants, hence differentiating it two times we can prove it.

We have, $y = a\cos(\log x) + b\sin(\log x)$ ... (i)

Differentiating (i) w.r.t 'x'. we get ; $\dfrac{dy}{dx}=-\dfrac{a\sin(\log x)}{x}+\dfrac{b\cos(\log x)}{x}$

$x\dfrac{dy}{dx} = -a\sin(\log x)+b\cos(\log x)$ ... (ii)

Again differentiating with respect to 'x', we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{a\cos(\log x)}{x}-\frac{b\sin(\log x)}{x}$$

$$\Rightarrow\quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[a\cos(\log x)+b\sin(\log x)]\qquad\Rightarrow\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y\qquad\Rightarrow\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Which is same as the given differential equation

Hence, y = acos(logx) + bsin(logx) is a solution of the given differential equation.

7. METHODS OF SOLVING FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATION

7.1 Equation of the Form dy/dx = f(x)

To solve this type of differential equations, we integrate both sides to obtain the general solution as discussed below

$$\frac{dy}{dx}=f(x)\qquad\Rightarrow\qquad dy=f(x)dx$$

Integrating both sides we obtain $\int dy=\int f(x)dx+c\ \Rightarrow\ y=\int f(x)dx+c$

Illustration 13: The general solution of the differential equation $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$ is **(JEE MAIN)**

Sol: General solution of any differential equation is obtained by integrating it hence for given equation we have to integrate it one time to obtain its general equation.

We have: $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$

Integrating, $y = \int\left(x^5 + x^2 - \frac{2}{x}\right) dx + c = \int x^5 dx + \int x^2 dx - 2\int \frac{1}{x} dx + c \Rightarrow y = \frac{x^6}{6} + \frac{x_3}{3} - 2\log|x| + c$

Which is the required general solution.

Illustration 14: The solution of the differential equation $\cos^2 x \frac{d^2y}{dx^2} = 1$ is **(JEE MAIN)**

Sol: By integrating it two times we will get the result.

$\cos^2 x \frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \sec^2 x$

On integrating, we get $\frac{dy}{dx} = \tan x + c_1$

Integrating again, we get $y = \log(\sec x) + c_1 x + c_2$

7.2 Equation of the form dy/dx = f(x) g(y)

To solve this type of differential equation we integrate both sides to obtain the general solution as discussed below

$\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1} dy = f(x)dx$

Integrating both sides, we get $\int (g(y))^{-1} dy = \int f(x)dx$

Illustration 15: The solution of the differential equation log(dy/dx) = ax + by is **(JEE MAIN)**

Sol: We can also write the given equation as $\frac{dy}{dx} = e^{ax+by}$. After that by separating the x and y terms and integrating both sides we can get the general equation.

$\frac{dy}{dx} = e^{ax+by} \Rightarrow \frac{dy}{dx} = e^{ax+by} \Rightarrow e^{-by} dy = e^{ax} dx \Rightarrow -\frac{1}{b}e^{-by} = \frac{1}{a}e^{ax} + c$

Illustration 16: The solution of the differential equation $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ is **(JEE MAIN)**

Sol: Here first we have to separate the x and y terms and then by integrating them we can solve the problem above.

The given equation is $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$\Rightarrow \frac{dy}{dx} = e^x.e^y + x^2 e^y \Rightarrow e^{-y} dy = (e^x + x^2)dx$, Integrating, $\int e^{-y} dy = \int\left(e^x + x^2\right)dx + c$

$\Rightarrow \frac{e^{-y}}{-1} + e^x + \frac{x^3}{3} + c \Rightarrow -\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c \Rightarrow e^x + \frac{1}{e^y} + \frac{x^3}{3} = C$

7.3 Equation of the Form dy/dx = f (ax+by+c)

To solve this type of differential equation, we put ax + by + c = v and $\frac{dy}{dx}=\frac{1}{b}\left(\frac{dy}{dx}-0\right)$

$\therefore\ \frac{dy}{a+bf(v)}$ = dx

So solution is by integrating $\int\frac{dy}{a+bf(v)}=\int dx$

Illustration 17: $(x + y)^2\frac{dy}{dx}=a^2$ **(JEE MAIN)**

Sol: Here we can't separate the x and y terms, therefore put x + y = t hence $\frac{dy}{dx}=\frac{dt}{dx}-1$. Now we can easily separate the terms and by integrating we will get the required result.

Let x + y = t $\Rightarrow t^2\left(\frac{dt}{dx}-1\right)=a^2$; $\frac{dt}{dx}=\frac{a^2}{t^2}+1=\frac{a^2+t^2}{t^2}\Rightarrow\int\frac{t^2dt}{t^2+a^2}$ = x + c

$\Rightarrow\quad t-a\tan^{-1}\frac{dy}{dx}=x+c\Rightarrow\quad y-a\tan^{-1}\frac{x+y}{a}=c$

Illustration 18: $\frac{dx}{dx}=\frac{x+y-1}{\sqrt{x+y+1}}$ **(JEE MAIN)**

Sol: Put x + y +1 = t^2 and then solve similar to the above illustration.

let x + y +1 = t^2

$\Rightarrow\quad\left(2t\frac{dt}{dx}-1\right)=\frac{t^2-2}{t}\qquad\Rightarrow\quad\frac{2tdt}{dx}=\frac{t^2+t-2}{t}\qquad\Rightarrow\quad\int\frac{2t^2}{(t-1)(t+2)}dt=x+c$

$\Rightarrow\quad 2\int\left(1+\frac{1}{3(t-1)}-\frac{4}{3(t+2)}\right)dt=x+c\qquad\Rightarrow\quad 2t+\frac{2\ln|t-1|}{3}-\frac{8\ln|t+2|}{3}=x+c$

$\Rightarrow\quad 2\sqrt{x+y+1}+\frac{2\ln|\sqrt{x+y+1}-1|}{3}-\frac{8\ln|\sqrt{x+y+1}+2|}{3}=x+c$

Illustration 19: $\frac{dy}{dx}$ = cos(10x + 8y). Find curve passing through origin in the form y = f(x) satisfying differential equations given **(JEE MAIN)**

Sol: Here first put 10x + 8y = t and then taking integration on both sides we will get the required result.
Let 10x + 8y = t

$\Rightarrow\quad 10+8\frac{dy}{dx}=\frac{dt}{dx}\qquad\Rightarrow\quad\frac{dy}{dx}-10=8\cos t\qquad\Rightarrow\quad\int\frac{dt}{8\cos t+10}\int dx=x+c$

$p=\tan t/2\qquad\frac{dp}{dx}=\frac{1+p^2}{2(1)}\frac{dy}{dx}\ \Rightarrow\ \frac{dt}{dx}=\frac{2dp}{1+p^2}$

$$\therefore\int\frac{\frac{2dp}{1+p^2}}{8\left(\frac{1-p^2}{1+p^2}\right)+10}=\int\frac{2dp}{1p^2+18}=\int\frac{dp}{p^2+9}=x+c$$

$\Rightarrow\ \tan^{-1}(P/3)=x+c\ \Rightarrow\ \tan^{-1}\left(\frac{\tan(t/2)}{3}\right)=x+c\Rightarrow\ 3\tan(x+c)=\tan(10x+84)$

7.4 Parametric Form

Some differential equations can be solved using parametric forms.

Case I:

x = rcosθ y = rsinθ

Squaring and adding $x^2 + y^2 = r^2$ ... (i)

$\tan\theta = \int e^{-y}dy = \int (e^x + x^2)dx + c$ ... (ii)

xdx + ydy = rdr ... (iii)

$\sec^2\theta\, d\theta = \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c \quad \Rightarrow xdy - ydx = x^2\sec^2\theta\, d\theta \qquad x = r\cos\theta;\ xdy - ydx = r^2 d\theta$

Case II:

If x = rsecθ, y = rtanθ

$x^2 - y^2 = r^2$ ... (i)

$\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c = \sin\theta$ ... (ii)

$\Rightarrow \quad xdx - ydy = rdr; \qquad xdy - ydx = \cos\theta\, x^2\, d\theta \qquad \Rightarrow \quad xdy - ydx = r^2\sec\theta d\theta$

Illustration 20: Solve xdx + ydy = x(xdy – ydx) **(JEE MAIN)**

Sol: By substituting x = r cosθ and y = r sinθ the given equation reduces to $rdr = r\cos\theta(r^2d\theta)$. Hence by separating and integrating both sides we will get the result.

Let x = rcosθ, y = rsinθ

Hence the given equation becomes $rdr = r\cos\theta(r^2d\theta)$

$$\int \frac{dr}{r^2} = \int \cos\theta d\theta \qquad \Rightarrow \qquad -\frac{1}{r} = \sin\theta + c \qquad \Rightarrow \qquad -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 21: Solve $\frac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$ **(JEE ADVANCED)**

Sol: Similar to the problem above, by substituting x = r cosθ and y = r sinθ the given equation reduces to $\frac{r\,dr}{r^2\,d\theta} = \frac{\sqrt{1-r^2}}{r}$. Hence by integrating both sides we will get the result.

$$\frac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}} \qquad \Rightarrow \qquad \frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$$

Let x = rcosθ, y = rsinθ

$$\frac{rdr}{r^2d\theta} = \frac{\sqrt{1-r^2}}{r} \qquad \Rightarrow \qquad \int \frac{dr}{\sqrt{1-r^2}} = \theta + c \qquad \Rightarrow \qquad \sin^{-1}r = \theta + c$$

$$\Rightarrow \quad \sin^{-1}\sqrt{x^2+y^2} = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} + c$$

Illustration 22: $\frac{xdx+ydy}{\sqrt{x^2+y^2}} = \frac{ydx-xdy}{x}$ **(JEE ADVANCED)**

Sol: Similar to the above illustration.

Let $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow\quad -\frac{rdr}{r^2 d\theta} = \frac{\sqrt{r^2}}{r\cos\theta} \qquad \Rightarrow \quad \int \sec\theta d\theta + \int \frac{dr}{r} = 0$

$\Rightarrow\quad \log(\sec\theta + \tan\theta) + \log r = c \qquad \Rightarrow \quad x^2+y^2+y\left(\sqrt{x^2+y^2}\right)+Cx=0$

7.5 Homogeneous Differential Equations

A differential equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where f(x, y) and g(x, y) are both homogeneous function of the same degree in x and y.

To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,

substitute $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dy}{dx}$

Thus differential reduces to the form $v + x\frac{dy}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v)-v}$

Therefore, solution is $\int \frac{dy}{x} = \int \frac{dv}{f(v)-v} + c$

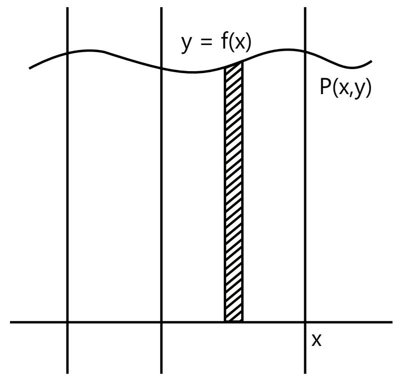

Figure 24.1

Illustration 23: Find the curve passing through (1, 0) such that the area bounded by the curve, x-axis and 2 ordinates, one of which is constant and other is variable, is equal to the ratio of the cube of variable ordinate to variable abscissa. **(JEE MAIN)**

Sol: By differentiating $\int_c^x ydx = \frac{y^3}{x}$, we will get the differential equation.

$A = \int_c^x ydx = \frac{y^3}{x} \qquad \Rightarrow y = \frac{x,3y^2y'-y^3,1}{x^2} \Rightarrow x^2 = 3xyy' - y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2+y^2}{3xy}$

(On differentiating the first integral equation w.r.t x)

Put $y = vx$; $v + x\frac{dt}{dx} = \frac{1+v^2}{3v} \quad \Rightarrow \quad \int \frac{3v}{1-2v^2}dv = \int \frac{1}{x}dx \quad \Rightarrow \quad -\frac{3}{4}\log\left|1-2v^2\right| = \log x + \log c \quad \Rightarrow \quad (x^2-2y^2)^3 = cx^2$

Given this curve passes through (1, 0). So, c=1 Hence the equation of curve is $(x^2-2y^2)^3 = cx^2$

Illustration 24: The solution of differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ is **(JEE MAIN)**

Sol: Here by putting y = xv and then integrating both sides we can solve the problem.

Put $y = xv \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence the given equation becomes $x\frac{dv}{dx} + v = v + \tan v \Rightarrow x\frac{dv}{dx} = \tan v$

$\Rightarrow \quad \frac{dv}{\tan v} = \frac{dx}{x} \quad \Rightarrow \quad \log \sin v = \log x + \log c \Rightarrow \frac{\sin v}{x} = c \Rightarrow \frac{\sin\left(\frac{y}{x}\right)}{x} = c \Rightarrow cx = \sin\left(\frac{y}{x}\right)$

Illustration 25: Solve $\frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$ given y at x = 1 is –1 **(JEE ADVANCED)**

Sol: Similar to the problem above, by putting y = vx, we can solve it and then by applying the given condition we will get the value of c.

Let y = vx

$\Rightarrow \quad v + x\frac{dv}{dx} = \left(\frac{v^2 - 2v - 1}{v^2 + 2v - 1}\right) \qquad \Rightarrow x\frac{dv}{dx} = -\frac{(v^3 + v^2 + v + 1)}{v^2 + 2v - 1}$

$\Rightarrow \quad \int \frac{v^2 + 2v - 1}{(v+1)(v^2+1)}dv = c - \log x \Rightarrow \int \frac{2v(v+1) - (v^2+1)}{(v+1)(v^2+1)}dv = c - \log x$

$\Rightarrow \quad \log\left[\frac{(v^2+1)x}{v+1}\right] = \log c \quad \Rightarrow \quad \frac{(v^2-1)x}{(v+1)} = c \Rightarrow \frac{x^2+y^2}{y+x} = c$

$\Rightarrow \quad k(x^2 + y^2) = x + y$

Given at x = 1, y = – 1 $\Rightarrow$ 2k = 0. Hence the required equation is x + y = 0

Illustration 26: Solve $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ given y at x = 1 is $\sqrt{5}$ **(JEE ADVANCED)**

Sol: As we know, when $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Hence from given equation $\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$ so by putting y = vx and integrating both side, we will get the result.

Given $y\left(\frac{dy}{dx}\right) + 2x\frac{dy}{dx} - y = 0$

$\Rightarrow \quad \frac{dY}{dX} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$

Let y = vx

$\Rightarrow \quad x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1} - 1}{v} - v \Rightarrow x\frac{dv}{dx} = \frac{\pm\sqrt{v^2+1} - 1 - v^2}{v}$

$\Rightarrow \quad \int \frac{vdv}{\pm\sqrt{v^2+1} - (1+v^2)} = \log x + C \quad \Rightarrow \quad \int \frac{vdv}{\pm\sqrt{v^2+1}\left(\mp\sqrt{v^2+1} + 1\right)} = \log x + C$

$\Rightarrow \quad -\ln\left(\mp\sqrt{v^2+1} + 1\right) = \log x + C \qquad \Rightarrow \quad x\left(\mp\sqrt{v^2+1} + 1\right) = c$

Given at x = 1, $y = v = \frac{dy}{dx} = \frac{7X - 3Y}{-3X + 7Y} \Rightarrow C = \mp\sqrt{6} + 1$

$\Rightarrow \quad \mp\sqrt{y^2 + x^2} + x = \mp\sqrt{6} + 1$

This is the required equation.

Note: The obtained solution has 4 equations.

7.6 Differential Equations Reducible to Homogenous Form

A differential equation of the form $\frac{dv}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c}$, where $\frac{a_1}{\theta_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by adopting the following procedure

Put x = X + h, y = Y + k, so that $\frac{dy}{dx}=\frac{dy}{dx}$

The equation then transforms to $\frac{dY}{dX}=\frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k the equation becomes

$$\frac{dy}{dx}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

This is a homogeneous equation which can be solved by putting Y = vX and then Y and X should be replaced by y – k and x – h.

Special case: If $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$ and $\frac{a}{a'}=\frac{b}{b'}$ = m say, i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation cannot be solved by the method discussed before because the values of h and k given by the equation will be indeterminate. In order to solve such equations, we proceed as explained in the following example.

Illustration 27: Solve $\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}$ **(JEE MAIN)**

Sol: Here the coefficient of x and y in the numerator and denominator are proportional hence by taking 3 common from 3x – 6y and putting x – 2y = v and after that by integrating we will get the result.

$\frac{dy}{dx}=\frac{3x-6y+7}{x-2y+4}=\frac{3(x-2y)+7}{x-2y+4}$; Put x – 2y = v $\Rightarrow 1-2\frac{dy}{dx}=\frac{dy}{dx}$

Now differential equations reduces to $1-\frac{dv}{dx}=2\left(\frac{3v+7}{v+4}\right)$

$\Rightarrow \frac{dv}{dx}=-5\left(\frac{v+2}{v+4}\right)$ $\Rightarrow \int\left(1+\frac{2}{v+2}\right)dv=-5\int dx$

$\Rightarrow v+2\log|v+2|=-5x+c$ $\Rightarrow 3x-y+\log|x-2y+2|=c$

Illustration 28: Solution of differential equation (3y – 7x + 7)dx + (7y – 3x + 3) dy = 0 is **(JEE MAIN)**

Sol: By substituting x = X + h, y = Y + k where (h, k) will satisfy the equation 3y – 7x + 7 = 0 and 7y – 3x + 3 =0 we can reduce the equation and after that by putting Y = VX and integrating we will get required general equation.

The given differential equation is $\frac{dy}{dx}=\frac{7x-3y-7}{-3x+7y+3}$

Substituting x = X + h, y = Y + k, we obtain

$$\frac{dY}{dX}=\frac{(7X-3Y)+(7h-3k-7)}{(-3X+7Y)+(-3h+7k+3)} \quad \text{... (i)}$$

Choose h and k such that 7h – 3k – 7 = 0 and –3h + 7k + 3 = 0.

This gives h = 1 and k = 0. Under the above transformations, equation (i) can be written as

Let Y = VX so that $\frac{dY}{dX} = V + X\frac{dV}{dX}$, we get $\frac{dY}{dX} = \frac{7X-3Y}{-3X+7Y}$

$V + X\frac{dV}{dX} = \frac{-3V+7}{7V-3} \Rightarrow X\frac{dV}{dX} = \frac{7-7V^2}{7V-3} \qquad \Rightarrow \quad -7\frac{dX}{X} = \frac{7}{2}.\frac{2V}{V^2-1}dV - \frac{3}{V^2-1}dV$

Integrating, we get

$-7\log X = \frac{7}{2}\log(V^2-1) - \frac{3}{2}\log\frac{V-1}{V+1} - \log C \qquad \Rightarrow \quad C = (V+1)^5 (V-1)^2 X^7 \qquad \Rightarrow \quad C = (y+x-1)^5 (y-x+1)^2$

Which is the required solution.

7.7 Linear Differential Equation

A differential equation is linear if the dependent variable (y) and its derivative appear only in the first degree. The general form of a linear differential equation of the first order is

$\frac{dy}{dx} + Py = Q$ ..... (i)

where P and Q are either constants or functions of x.

This type of differential equation can be solved when they are multiplied by a factor, which is called integrating factor.

Multiplying both sides of (i) by $e^{\int pdx}$, we get $e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int pdx}$

On integrating both sides with respect to x, we get

$ye^{\int Pdx} = \int Qe^{\int Pdx} + c$ which is the required solution, where c is the constant and $e^{\int pdx}$ is called the integrating factor.

Illustration 29: Solve the following differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x}$ **(JEE MAIN)**

Sol: We can write the given equation as $e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$. By putting e^{-y} = t, we can reduce the equation in the form of $\frac{dt}{dx}$ + Pt = Q hence by using integration factor we can solve the problem above.

We have, $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x} \Rightarrow e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x}$ ... (i)

Put e^{-y} = t. so that $\frac{dy}{dx}$ in equation (i), we get $-\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x} \Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$ ... (ii)

This is a linear differential equation in t.

Here, P = $-\frac{1}{x}$ and Q = $-\frac{1}{x}$ $\therefore$ I.F. = $e^{\int Pdx} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ The solution of (ii) is, t.(I.F.) = $\frac{dy}{dx} = \frac{3x-6y+7}{x-2y+4} = \frac{3(x-2y)+7}{x-2y+4}$

$$t\frac{1}{x} = \int\frac{1}{x}\left(-\frac{1}{x}\right)dx + C \Rightarrow \frac{t}{x} = \frac{1}{x} + c \Rightarrow \frac{e^{-y}}{x} = \frac{1}{x} + C$$

Illustration 30: The function y(x) satisfy the equation $y(x) + 2x\int_0^x \frac{y(x)}{1+x^2}dx = 3x^2 + 2x + 1$. Prove that the substitution $z(x) = \int_0^x \frac{y(x)}{1+x^2}dx$ converts the equation into a first order linear differential equation in z(x) and solve the original equation for y(x) **(JEE MAIN)**

Sol: By putting $z'(x) = \frac{y(x)}{1+x^2}$ we will get the linear differential equation in z form and then by applying integrating factor we get the result.

Let $z'(x) = \frac{d(x)}{1+x^2}$ $\Rightarrow$ $z'(x) \times (1 + x^2) + 2x(z(x)) = 3x^2 + 2x + 1$

$$\Rightarrow \frac{dz}{dx} + \frac{2x}{1+x^2}z = \frac{3x^2+2x+1}{x^2+1} \quad \text{... (i)}$$

This is a first order linear differential equation in z.

$\therefore$ I.F. = $e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx} = 1+x^2$ $\therefore$ Solution of (i) is z(I.F.) = $\int (Q \times \text{I.F}) dx + c$

$\Rightarrow z(1 + x^2) = \int \frac{x^3+x^2+x}{x^2+1}(x^2+1)dx + C$ $\Rightarrow z(1 + x^2) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + C$ and $y = 3x^2 + 2x + 1 - 2xz$

Illustration 31: Solve the differential equation ysin2x.dx – (1 + y^2 + cos2x)dy = 0 **(JEE MAIN)**

Sol: Similar to illustration 28, by putting –cos 2x = t, we can reduce the equation in the form of $\frac{dt}{dx}$ + Pt = Q hence by using integration factor we can solve the problem given above.

We have, ysin2x.dx – (1 + y^2 + cos2x)dy = 0

$$\Rightarrow \sin 2x.\frac{dx}{dy} - \frac{\cos 2x}{y} = \frac{1+y^2}{y} \quad \text{... (i)}$$

Putting –cos 2x = t so that $2\sin 2x \frac{dx}{dy} = \frac{dt}{dy}$ in equation (i), we get $\frac{dt}{dy} + \frac{2}{y}t = 2\left(\frac{1+y^2}{y}\right)$

Here, P = $\frac{2}{y}$ and Q = $2\frac{1+y^2}{y}$

$\therefore$ I.F. = $e^{\int Pdy} = e^{\int \frac{2}{y}dy} = y^2$ $\therefore$ The solution is t.(I.F.) = $\int (Q \times \text{I.F.})dy + C$

$\Rightarrow$ $t.y^2 = 2\int \frac{1+y^2}{y}.y^2dy = 2\int y + y^3\, dy$ $\Rightarrow$ $t.y^2 = y^2 + \frac{y^4}{2} + C$

On putting the value of t, we get –.cos2x = 1 + $\frac{y^2}{2} + Cy^{-2}$

Illustration 32: Solve ylogy $\frac{dx}{dy}$ + x – logy = 0 **(JEE MAIN)**

Sol: By reducing the given equation in the form of $\frac{dx}{dy} + Px = Q$ we can solve this as similar to above illustrations.

We have, ylogy $\frac{dx}{dy}$ +x –logy = 0 $\Rightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$

This is a linear differential equation in x.

Here P = $\frac{1}{y\log y}$, Q = $\frac{1}{y}$; I.F. = $e^{\int \frac{1}{y\log y}dy} = e^{\log(\log y)} = \log y$

The solution is, x(I.F.) = $\int\left(Q \times I.F.\right) + C$; xlogy = $\int \frac{1}{y}\left(\log y\right)dy + c$ = $\frac{1}{2}(\log y)^2 + C$

x = $\frac{1}{2}\log y + C\frac{1}{\log y}$

Illustration 33: Solve $(x + 2y^3)\ \frac{dx}{dy} = y$ **(JEE ADVANCED)**

Sol: By reducing given equation in the form of $\frac{dx}{dy}$ + Px = Q and then using the integration factor we can solve this.

$(x + 2y^3)\ \frac{dx}{dy} = y \Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y} + 2y^2 \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$

I.F = $e^{-\int \frac{1}{y}dy} = \frac{1}{y}$;

Solutions is $x.\frac{1}{y} = y^2 + C$

Alternate method: $xdy + 2y^3dy = ydx$

$\Rightarrow \quad 2ydy = \frac{ydx - xdy}{y^2} \quad \Rightarrow \quad 2ydy = d\left(\frac{x}{y}\right) \Rightarrow y^2 = \frac{x}{y} + C$

Illustration 34: Let g(x) be a differential function for every real x and g′(0) = 2 and satisfying $g(x+y) = e^y g(x) + 2e^x g(y)\ \forall$ x and y. Find g(x) and its range. **(JEE ADVANCED)**

Sol: By using $g'(x) = \lim_{b\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}$ and solving we will get g(x).

$g'(x) = \lim_{b\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}$

$\Rightarrow \quad g'(x) = \lim_{h\to 0}\frac{e^h g(x) + 2e^x g(h) - g(x)}{h} \quad \Rightarrow \quad g'(x) = g(x)\lim_{h\to 0}\frac{e^h - 1}{h} + 2e^x \lim_{h\to 0}\frac{g(h)}{h} \quad \Rightarrow \quad g'(x) = g(x) + 2e^x$

At x = 0, g(x) = 0 ⇒ g(0) = 0

$\frac{dy}{dx} - y = 2e^x \quad \Rightarrow \quad$ I.F. = e^{-x}

Solution is $y.e^{-x} = 2x + C$

$g(0) = 0 \Rightarrow C = 0 \Rightarrow g(x) = 2xe^x$

$g'(x) = 2e^x + 2xe^x = 2e^x(x + 1)$

g′(x) = 0 at x = -1; g(–1) = –2/e

$\Rightarrow \quad$ Range of g(x) = $\left[-\frac{2}{e}, \infty\right]$

Illustration 35: Find the solution of $(1 - x^2)\ \frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$ **(JEE ADVANCED)**

Sol: By reducing given equation in the form of $\frac{dy}{dx}$ + Py = Q and then by using integration factor i.e. $e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int pdx}$ we can solve the problem.

$$\frac{dy}{dx}+\frac{2x}{(1-x^2)}y=\frac{x\sqrt{1-x^2}}{1-x^2}\ ;\ \text{I.F.} = e^{\int Pdx}=e^{\int \frac{2x}{1-x^2}dx}=\frac{1}{1-x^2}$$

Solution is y. $y\frac{1}{1-x^2}=\int\frac{x}{\sqrt{1-x^2}}\frac{1}{1-x^2}dx+c = \int\frac{x}{(1-x^2)^{3/2}}dx+C=\frac{-1}{2}\int\frac{-2x}{\left(1-x^2\right)^{3/2}}dx+c$

$$y\frac{1}{\left(1-x^2\right)}=\frac{1}{\sqrt{1-x^2}}+c$$

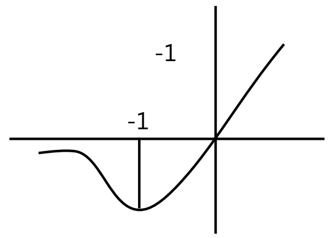

Figure 24.3

MASTERJEE CONCEPTS

Every linear differential equation is of degree 1 but every differential equation of degree 1 is not linear

Shivam Agarwal (JEE 2009, AIR 27)

7.8 Equations Reducible to Linear form

(a) Bernoulli's Equation

A differential equation of the form $\frac{dy}{dx}$ + Py = Qy^n, where P and Q are function of x and y is called Bernoulli's equation. This form can be reduced to linear form by dividing y^n and then substituting y^{1-n} = v

Dividing both sides by y^n, we get, $y^{-n}\frac{dy}{dx}$ + $P.y^{-n+1}$ = Q

Putting y^{-n+1} = v, so that , $(1 - n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, we get $\frac{dv}{dx}$ + (1 – n)P.y = (1 – n)Q

Which is a linear differential equation

(b) If the given equation is of the form $\frac{dy}{dx}$ + P. f(y) = Q.g(y), where P and Q are functions of x alone, we divide the equation by f(y), and then we get $e^{\int Pdx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$

Now substitute $\frac{f(y)}{g(y)}$ = v and solve.

Illustration 36: Solve $\frac{dy}{dx}=xy+x^3y^2$ **(JEE MAIN)**

Sol: By rearranging the given equation we will get $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}x=x^3$ and then by putting $\frac{-1}{y}=t$ and using the integration factor we can solve it.

$$\frac{dy}{dx}=xy+x^3y^2 \Rightarrow \frac{dy}{dx}-xy=x^3y^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}x=x^3$$

put $\frac{-1}{y}=t \Rightarrow \frac{dy}{dx}$ + tx = x^3

This is a linear differential equation with I.F. = $e^{x^2/2} \Rightarrow t\,e^{x^2/2}=\int e^{x^2/2}x^3dx$

Illustration 37: Find the curve such that the y intercept of the tangent is proportional to the square of ordinate of tangent **(JEE MAIN)**

Sol: Here X = 0 and Y= y – mx i.e. $x\frac{dy}{dx}-y=-ky^2$. Hence by putting $\frac{-1}{y}=1$ and applying integration factor we will get the result.

$X = 0 \Rightarrow Y= y - mx \quad \Rightarrow \quad x\frac{dy}{dx}-y=-ky^2$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}.\frac{1}{x}=\frac{-k}{x}$

Put $\frac{-1}{y}=t \qquad \Rightarrow \qquad \frac{dt}{dx}+\frac{t}{x}=\frac{-k}{x}$

$\Rightarrow$ I.f. = x

$\Rightarrow$ Solution is t.x = –kx + C $\Rightarrow \frac{-x}{y}=-kx+C$

7.9 Change of Variable by Suitable Substitution

Following are some examples where we change the variable by substitution.

Illustration 38: Solve $y\sin x\frac{dy}{dx}=\cos x(\sin x-y^2)$ **(JEE MAIN)**

Sol: Here by putting $y^2 = t$, the given equation reduces to $\frac{dt}{dx}+\left(2\cot x\right)t = 2\cos x$ and then using the integration factor method we will get result.

$$y\sin x\frac{dy}{dx}=\cos x(\sin x-y^2)$$

Let $y^2 = t \Rightarrow \frac{1}{2}\sin x\frac{dt}{dx} = \cos x\,(\sin x - t)$

$\Rightarrow \frac{dt}{dx}=2\cos x-(2\cot x)t \qquad \Rightarrow \frac{dt}{dx}+\left(2\cot x\right)t = 2\cos x$

I.F. = $\sin^2 x$

$\Rightarrow$ Solution is $t\sin^2 x = \int 2\cos x.\sin^2 x\,dx$

$y^2\sin^2 x=\frac{2\sin^3 x}{3}+c$

Illustration 39: Solve $\frac{dy}{dx}=e^{x-y}(e^x-e^y)$ **(JEE MAIN)**

Sol: Simply by putting $e^y = t$ and using the integration factor we can solve the above problem.

$$\frac{dy}{dx}=e^{x-y}(e^x-e^y) \qquad \Rightarrow \qquad e^y\frac{dy}{dx}=\left(e^x\right)^2-e^xe^y$$

Put $e^y = t \Rightarrow \frac{dy}{dx} + te^x = (e^x)^2$;

I.F. = $e^{\int e dx} = e^{e^x}$

Solution is $te^{e^x} = \int (e^x)^2.e^{e^x}dx$

MASTERJEE CONCEPTS

If we can write the differential equation in the form

$f(f_1(x, y)\, d(f_1(x, y)) + \phi(f_2(x, y)d(f_2(x,y)) + \ldots\ldots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right) = \frac{xydx - y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x}\right) = \frac{xydx - 2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(xi) $d[\log(xy)] = \frac{xdy + ydx}{xy}$

(xii) $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$,

(xiii) $d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$

(xvii) $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$

(xviii) $d(x^my^n) = x^{m-1}y^{n-1}(mydx + nxdy)$

(xix) $d\frac{dt}{dx} + \frac{t}{x}$

(xx) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{xdy - ydx}{x^2 - y^2}$

(xxi) $\frac{d\left[f(x,y)\right]^{1-n}}{1-n} = \frac{f'(x,y)}{\left(f(x,y)\right)^n}$

(xxii) $d\left(\frac{1}{y} - \frac{1}{x}\right) = d\left(\frac{1}{y}\right) - d\left(\frac{1}{x}\right) = \frac{dx}{x^2} - \frac{dy}{y^2}$

Shrikant Nagori (JEE 2009, AIR 30)

8. EXACT DIFFERENTIAL EQUATION

The differential equation Mdx + Ndy = 0, where M and N are functions of x and y, is said to be exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Rule for solving Mdx + Ndy = 0 when it is exact

(a) First integrate the terms in M w.r.t. x treating y as a constant.

(b) Then integrate w.r.t. y only those terms of N which do not contain x.

(c) Now, sum both the above integrals obtained and quote it to a constant i.e. $\int M dx + \int N dy = k$, where k is a constant.

(d) If N has no term which is free from x, the $\int M dx = c$ (y constant)

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d(\log xy)$

(v) $\dfrac{dx + dy}{x + y} = d\log(x + y)$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-2}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydy - xdx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(x) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dy - e^x dy}{y^2}$

9. ORTHOGONAL TRAJECTORY

Definition 1: Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical then we say that the family is self-orthogonal

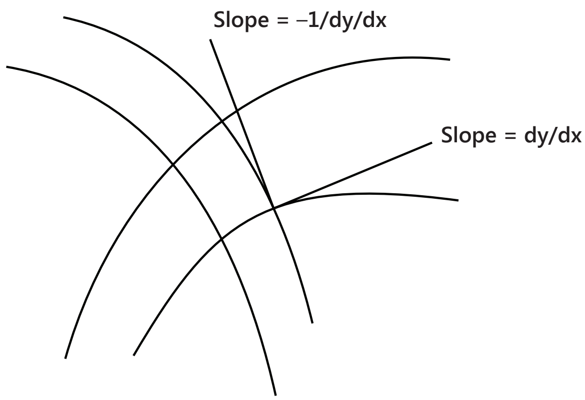

Figure 24.4: Orthogonal trajectories

MASTERJEE CONCEPTS

Orthogonal trajectories have important application in the field of physics. For example, the equipotential lines and the streamlines in an irrotational 2D flow are orthogonal.

Ravi Vooda (JEE 2009, AIR 71)

9.1 How to Find Orthogonal Trajectories

Suppose the first family of curves $F(x, y, c) = 0$ ... (i)

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (i) w.r.t. x to find $G(x, y, y', c) = 0$ ... (ii)

Now eliminate c between (i) and (ii) to find the differential equation $H(x, y, y') = 0$ ... (iii)

The differential equation for the other family is obtained by replacing y′ with –1/y′. Hence, the differential equation the orthogonal trajectories is H(x, y, –1/y′) = 0 ... (iv)

General solution of (iv) gives the required orthogonal trajectories.

Illustration 40: Find the orthogonal trajectories of a family of straight lines through the origin. **(JEE MAIN)**

Sol: Here as we know, a family of straight lines through the origin is given by y = mx.

Hence by differentiating it with respect to x and eliminating m we will get an ODE of this family and by putting –1/y′ in place of y′ we will get an ODE for the orthogonal family.

The ODE for this family is xy′ – y = 0

The ODE for the orthogonal family is x + yy′ = 0

Integrating we find $x^2 + y^2 = c$, which are family of circles with center at the origin.

10. CLAIRAUT'S EQUATION

The differential equation

y = mx + f(m), ... (i)

where $m = \frac{dy}{dx}$ is known as Clairaut's equation.

To solve (i), differentiate it w.r.t. x, which gives

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{df(m)}{dx}$$

$$\Rightarrow \quad x\frac{dm}{dx} + f'(x)\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ... (ii)

or $x + f'(x) = 0$ ... (iii)

MASTERJEE CONCEPTS

- If m is eliminated between (i) and (ii),the solution obtained is a general solution of (i)
- If m is eliminated between (i) and (iii), then the solution obtained does not contain any arbitrary constants and is not the particular solution of (i). This solution is called singular solution of (i)

Chinmay S Purandare (JEE 2012, AIR 698)

PROBLEM SOLVING TACTICS

Think briefly about whether you could easily separate the variables or not. Remember that means getting all the x terms (including dx) on one side and all the y terms (including dy) on the other. Don't forget to convert y′ to dy/dx or you might make a mistake.

If it's not easy to separate the variables (usually it isn't) then we can try putting our equation in the form y′ + P(x)y = Q(x).

In other words, put the y′ term and the y term on the left and then you may divide so that the coefficient of y′ is 1.

Then we can use the trick of the integrating factor in which we multiply both sides by . $d\left(\frac{e^x}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$. This makes things much simpler, but it's best to see why from doing problems, not from memorizing formulas.

FORMULAE SHEET

(a) Order of differential equation: Order of the highest derivative occurring in the differential equation

(b) Degree of differential equation: Degree of the highest order derivative when differential coefficients are free from radicals and fractions.

(c) General equation : $\frac{dy}{dx} = f(x) \Rightarrow \quad y = \int f(x)dx + c$

(d) $\frac{dy}{dx} = f(ax + by + c)$, then put ax + by + c = v

(e) If $\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1}dy = f(x)dx$ then $\int (g(y))^{-1}dy = \int f(x)dx$

(f) Parametric forms

Case I: x = rcosθ, y = rsinθ ⇒ $x^2 + y^2 = r^2$; tanθ = $\frac{y}{x}$; xdx + ydy = rdr; xdy – ydx = $r^2d\theta$

Case II: x = rsecθ, y = rtanθ ⇒ $x^2 - y^2 = r^2$; $\frac{y}{x}$ = sinθ; xdx – ydy = rdr; xdy – ydx = $r^2\sec\theta d\theta$

(g) If $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, then substitute y = vx ⇒ $\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

(h) If $\frac{dv}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, then substitute x = X + h, y = Y + k

$$\Rightarrow \frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

(i) If the equation is in the form of $\frac{dy}{dx}$ + Py = Q then $ye^{\int Pdx} = \int Qe^{\int Pdx} + c$