Solved Examples

JEE Main/Boards

Example 1: Find the differential equation of the family of curves $y = Ae^{x} + Be^{-x}$

Sol: By differentiating the given equation twice, we will get the result.

$$\frac{dy}{dx} = Ae^{x} - Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^{x} + Be^{-x} = y$$

Example 2: Find differential equation of the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.

Sol: By differentiating the given family of curves and then eliminating c we will get the required differential equation.

$y = c(x - c)^2$

$$\Rightarrow \frac{dy}{dx} = 2(x - c)c$$

By division, $\frac{x-c}{2} = \frac{y}{dy/dx}$

or $c = x - \frac{2y}{dy/dx}$

Eliminating c, we get

$$\left(\frac{dy}{dx}\right)^2 = 4c^2(x - c)^2 = 4cy$$

$$= 4y\, 4y\left\{x - \frac{2y}{dy/dx}\right\}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 = 4y\left[x\frac{dy}{dx} - 2y\right]$$

Example 3: Find the differential equation of all parabolas which have their vertex at (a, b) and where the axis is parallel to x-axis.

Sol: Equation of parabola having vertex at (a, b) and axis is parallel to x-axis is $(y - b)^2 = 4L(x - a)$ where L is a parameter. Hence by differentiating and eliminating L we will get required differential equation.

$$\therefore\ 2(y - b)\frac{dy}{dx} = 4L$$

On eliminating L, we get

$$(y - b)^2 = 2(y - b)\ \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)\ (x - a)$$

Differential equation is,

$$2(x - a)\frac{dy}{dx} = y - b.$$

Example 4: Show that the function $y = be^{x} + ce^{2x}$ is a solution of the differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Sol: Differentiating given equation twice we can obtain the required differential equation.

$y = be^{x} + ce^{2x}$

$$\Rightarrow \frac{dy}{dx} = be^{x} + 2ce^{2x} = y + ce^{2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = be^{x} + 4ce^{2x} = y + 3ce^{2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Example 5: Solve: $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$

Sol: By separating x and y term and integrating both sides we can solve it.

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$\Rightarrow \sin^{-1}y = -\sin^{-1}x + c$ or $\sin^{-1}y + \sin^{-1}x = c$

Example 6: Find the equation of the curve that passes through the point P(1, 2) and satisfies the differential equation

$$\int f(x)dx + C = \frac{-2xy}{x^2+1} : y > 0$$

Sol: By integrating both sides we will get general equation of curve and then by substituting point (1, 2) in that we will get value of constant part.

$$\frac{dy}{dx} = -\frac{2xy}{x^2+1} \Rightarrow \int\frac{dy}{y} = -\int\frac{2x}{x^2+1}dx$$

$$\Rightarrow \log|y| = -\log(x^2 + 1) + \log c_0$$

$\Rightarrow$ $\log(|y|(x^2 + 1)) = \log c_0$

$\Rightarrow$ $|y|(x^2 + 1) = c_0$

As point P(1, 2) lies on it,

$2(1 + 1) = c_0$ or $c_0 = 4$

$\therefore$ Curve is $y(x^2 + 1) = 4$

Example 7: Solve: $\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$

Sol: By putting x – y = t and integrating both sides we will obtain result.

Put x – y = t; then, $1 - \frac{dy}{dx} = \frac{dt}{dx}$

Differential equation becomes

$1 - \frac{dy}{dx} = \frac{t+3}{2t+5}$ or $\frac{dt}{dx} = 1 - \frac{t+3}{2t+5} = \frac{t+2}{2t+5}$

$\Rightarrow \int dx = \int \frac{2t \;\; +5}{t+2} dt = 2t + \int \frac{dt}{1+2}$

$\Rightarrow$ $x + c = 2t + \log|t + 2| = 2(x - y) + \log|(x - y + 2)|$

Example 8: $x^2dy + y(x + y)dx = 0: xy > 0$

Sol: We can write the given equation as

$$\left(\frac{dy}{dx}\right)^3 = 4y\left[x\frac{dy}{dx} - 2y\right] = -\frac{dy}{dx}$$

and then by substituting y = vx and integrating we will

get required general equation.

$\frac{dy}{dx} = -\frac{y(x+y)}{x^2}$ (Put y = vx)

$\Rightarrow$ $v + x\frac{dy}{dx} = -v(1 + v)$

$\Rightarrow \frac{dv}{dx} = -2v - v^2$ or $\int \frac{dv}{v(v+2)} = \int \frac{dx}{x}$

$\Rightarrow -\int \frac{dx}{x} = \frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv$

$\Rightarrow -\log|x| = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ $(\log|v| - \log|v+2| + c_0$

or $\left|\frac{V}{V+2}x^2 = c\right|$ or $\frac{\frac{y}{x}x^2}{\frac{y}{x}+2} = c$ or $\frac{yx^2}{y+2x} = c$

Example 9: Solve: $\frac{dy}{dx} + \sec x = \tan x : 0 < x < \frac{dy}{dx}$

Sol: The given equation is in the form of $\frac{dy}{dx} + px = q$ hence by using integration factor method we can solve it.

I.F. = $\frac{-2xy}{x^2 = 1} = \frac{dy}{dx} = -\frac{2xy}{x^2+1}$

= secx + tanx

Solution is y(secx + tanx)

= $\int \tan x(\sec x + \tan)dx$

= $\int \sec x \tan x dx + \int(\sec^2 x - 1)dx$

= secx + tanx – x + c

or (y – 1) (secx + tanx) = c – x

Example 10: Solve:

sinx.cosy.dx + cosx.siny.dy = 0

given, $y = \frac{\pi}{4}$ when x = 0.

Sol: Here by separating variables and taking integration we will get the general equation and then using the given values of x and y we will get value of constant c.

We have,

sinx.cosy.dx + cosx.siny.dy = 0

On separating the variables, we get

$\Rightarrow \frac{dt}{dx}$

Integrating both sides, we get

$\int \frac{\sin x}{\cos x}dx + \int \frac{\sin y}{\cos y}dy = 0$

[Dividing by cosx cosy], we get

$\Rightarrow$ log|secx| + log|secy| = logC

$\Rightarrow$ log|secx| |secy| = logC

$\Rightarrow$ secx.secy = C ... (i)

On putting $y = \frac{\pi}{4}$, x = 0 in (i),

we have C = $\sec 0.\sec\frac{\pi}{4}$

$\Rightarrow$ C (1). $(\sqrt{2}) = \sqrt{2}$

Substituting the value of C in (i) we get

$\sec x. \frac{1}{\cos y} = \sqrt{2} \Rightarrow \cos y = \frac{1}{\sqrt{2}}\sec x$

$\Rightarrow y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sec x\right)$

Example 11: Solve the differential equation

$\frac{dy}{dx} = x^2e^{-3y}$, given that y =0 for x =0.

Sol: Similar to the problem above we can solve it.

Here, $\frac{dy}{dx} = x^2e^{-3y}$ ..... (i)

On separating the variables, we have

$\Rightarrow \quad e^{3y}dy=x^2dx$

Integrating both sides, we get

$\int e^{3y} = \int x^2dx$

$\Rightarrow \quad \frac{e^{3y}}{3} = \frac{x^3}{3} + c$ ..... (ii)

putting: y = 0 for x = 0, in (ii), we obtain

$\frac{e^0}{3} = 0 + C \Rightarrow \frac{1}{3} = C \qquad [e^0 = 1]$

On substituting the value of C in (ii), we get

$\therefore \quad e^{3y} = x^3 + 1$

which is the required particular solution of (i)

Example 12: Solve the following differential equation:

$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

Sol: Here by rearranging the given equation we will get $e^{\log(\sec x + \tan x)} = \int \tan x(\sec x + \tan x)dx$. Now by substituting y = vx and then integrating we can solve the illustration above.

$2x^2\frac{dy}{dx} = 2xy - y^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$ ..... (i)

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2x(vx) - (vx)2}{2x^2}$

$\Rightarrow \quad v + x\frac{dv}{dx} = v - \frac{v^2}{2}$

$\frac{xdv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{dv}{v^2}\frac{dx}{2x}$

Integrating, we have

$\Rightarrow \quad \frac{1}{4} = \frac{1}{4}$ |logx| + c

$\Rightarrow \frac{x}{y} = \frac{1}{2}$ |logx| + c

Example 13: Solve the following differential equation

$\cos^2x\frac{dy}{dx} + y = \tan x$

Sol: Here by reducing the given equation in the form of $\frac{dy}{dx} + py = q$ and then using integration factor we will get the result.

We have, $\cos^2x\frac{dy}{dx} + y = \tan x$

$\Rightarrow \frac{dy}{dx} + y.\sec^2x = \tan x.\sec^2x$

I.F. = $e^{\int \sec^{2x}} = e^{\tan x}$

y.(I.F.) = $\int Q.(\text{I.F.})dx + c$

$y.e^{\tan x} = \int \tan x \sec^2 x e^{\tan x}dx = \int te^t dt$

$te^t - \int e^t dt + c \quad \begin{bmatrix} \tan x = t \\ \sec^2 xdx = dt \end{bmatrix}$

$= te^t - e^t + c$

$= \tan x\, e^{\tan x} - e^{\tan x} + c$

$y = \tan x - 1 + ce^{-\tan x}$

Example 14: Solve $x\frac{dy}{dx} - y = x^2$

Sol: As similar to the problem above, we can reduce the given equation as $\frac{dy}{dx}$ therefore by using integration factor we can solve this.

We have, $x\frac{dy}{dx} - y = x^2$

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = x$ ... (i)

This is a linear differential equation in y

Here, P = $-\frac{2xy - y^2}{2x^2}$ and Q = x

Now, $\qquad$ I.F. = $e^{\int pdx} = e^{\int -\frac{1}{x}dx}$

$= e^{-\log x} = e^{\log^{-1}} = x^{-1} = \frac{1}{x}$

$\therefore$ The solution of (i) is

y(I.F.) = $\int (Q \times \text{IF.})dx + C$ = x + C

$\Rightarrow \quad y = x^2 + Cx$

Example 15: Solve the following differential equation:
$\frac{dv}{dx} + y = \cos x - \sin x$.

Sol: Here given equation is in the form of $\frac{dy}{dx}+Py=Q$, where P = 1 and Q = cosx – sinx hence by using integration factor we will get result.

Given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \qquad \text{... (i)}$$

The given differential equation is a linear differential equation

On comparing with $\frac{dy}{dx} + Py = Q$

$\therefore \quad P = 1, Q = \cos x - \sin x$

I.F. = $e^{\int p dx} = e^x$

$\therefore$ required solution of (i) is

$y\,(\text{I.F.}) = \int Q.(\text{I.F})dx + c$

$\Rightarrow y.e^x = \int (\cos x - \sin x)e^x dx + c$

$\Rightarrow y.e^x = \int \cos x e^x dx - \int \sin x.e^x dx + c$

Integrating by parts, we get

$\Rightarrow y.e^x = \cos x \int e^x dx - \int(-\sin x)e^x dx - \int \sec^2 x\,dx + c$

$\Rightarrow y.e^x = e^x\cos x + \int e^x \sin x dx - \int e^x \sin x dx + c$

$y.e^x = e^x\cos x + c$

$\therefore \quad y = \cos x + ce^{-x}$

JEE Advanced/Boards

Example 1: Solve

$$\left(xe^{y/x} - y\sin\frac{y}{x}\right)dx + x\sin\frac{y}{x}dy = 0;\ x>0$$

Sol: Simply by putting y = vx and integrating we can solve the problem above.

$$\left(e^{\frac{y}{x}} - \frac{y}{x}\sin\frac{y}{x}\right) + \sin\frac{ydy}{xdx} = 0$$

Put y = vx

$\therefore (e^v - v\sin v) + \sin v\left(v + x\frac{dv}{dx}\right) = 0$

$\Rightarrow \int\frac{dx}{x} + \int e^{-v}\sin v dv = 0$

Integrating, we get

$\log x - \frac{1}{2}e^{-v}(\sin v + \cos v) = c$

or $\log x = c + \frac{1}{2}e^{-y/x}.\left(\sin\frac{y}{x} - 4\cos\frac{y}{x}\right)$

Example 2: Solve:

$xdy - ydx = xy^3(1 + \log x)dx$

Sol: We can reduce the given equation in the form of $-\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)dx$. Hence by integrating L.H.S. with respect to $\frac{x}{y}$ and R.H.S. with respect to x we will get the solution.

$-\frac{ydx - xdy}{y^2} = xy(1 + \log x)dx$

or $-d\left(\frac{x}{y}\right) = xy\,(1 + \log x)dx$

or $-\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)dx$

Integrating, $-\int\frac{x}{y}d\left(\frac{x}{y}\right)$

$= \int x^2(1+\log x)dx$

or $-\frac{1}{2}\left(\frac{x}{y}\right)^2 = (1 + \log x)\frac{x^3}{3} - \int\frac{x^3}{3}.\frac{1}{x}dx$

$\frac{-1}{2}\left(\frac{x}{y}\right)^2 = (1 + \log x)\frac{x^3}{3} - \frac{x^3}{9} + c$

Example 3: Find the equation of the curve passing through (1, 2) whose differential equation is

$y(x + y^3)dx = x(y^3 - x)dy$

Sol: Similar to example 2 we can solve the problem above by reducing the given equation as –

$\frac{y}{x}d\left(\frac{y}{x}\right) + \frac{1}{x^2y^2}d(xy) = 0$.

$(xy + y^4)dx = (xy^3 - x^2)dy$

or $y^3(ydx - xdy) + x(ydx + xdy) = 0$

or $-x^2y^3\frac{xdy - ydx}{x^2} + xd(xy) = 0$

or $-\frac{y}{x}d\left(\frac{y}{x}\right)+\frac{1}{x^2y^2}d\left(xy\right)=0$

Integrating, we get

$$-\frac{1}{2}\left(\frac{y}{x}\right)^2-\frac{1}{xy}=c$$

or $y^3 + 2x - 2cx^2y = 0$

As it passes through (1, 2), condition is

$8 + 2 + 4c = 0 \Rightarrow c = -\frac{5}{2}$

Thus curve is $y^3 + 2x - 5x^2y = 0$

Example 4: Form the differential equation representing the family of curves y = Acos2x + Bsin2x, where A and B are arbitrary constants.

Sol: Here we have two arbitrary constants hence we have to differentiate the given equation twice.

The given equation is:

$y = A\cos 2x + B\sin 2x$ ... (i)

Diff. w.r.t. x,

$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x$

$= -4(A\cos 2x + B\sin 2x) = -4y$ [Using (i)]

Hence $\frac{d^2y}{dx^2} + 4y = 0$, which is the required differential equation.

Example 5: The solution of the differential equation x $\frac{d^2y}{dx^2} = 1$, given that y = 1, $\frac{dy}{dx} = 0$, when x = 1, is

Sol: By integrating $x\frac{d^2y}{dx^2} = 1$ twice we will get its general equation and then by substituting given values of x, y and $\frac{dy}{dx}$ we will get the values of the constants.

$x\frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \log x + C_1$

Again integrating

$y = x\log x - x + C_1x + C_2$

Given y = 1 and $\frac{dy}{dx} = 0$ at x = 1

$\Rightarrow C_1 = 0$ and $C_2 = 2$

Therefore, the required solution is y = x log x – x + 2

Example 6: By the elimination of the constant h and k, find the differential equation of which $(x-h)^2+(y-k)^2=a^2$, is a solution.

Sol: Three relations are necessary to eliminate two constants. Thus, besides the given relation we require two more and they will be obtained by differentiating the given relation twice successively.

Thus we have

$(x - h) + (y - k)\frac{dy}{dx} = 0$ ... (i)

$1 + (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ ... (ii)

From (i) and (ii), we obtained

$$y - k = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$x - h = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

Substitute these values in the given relation, we obtained

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Example 7: Form the differential equations by eliminating the constant(s) in the following problems.

(a) $x^2 - y^2 = c(x^2 + y^2)^2$, (b) $a(y + a)^2 = x^3$

Sol: Given equations have one arbitrary constant, hence by differentiating once and eliminating c and a we will get the required differential equation.

(a) The given equation contains one constant

Differentiating the equation once, we get

$2x - 2yy' = 2c(x^2 + y^2)(2x + 2yy')$

But $c = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$

Substituting for c, we get

$$(x - yy') = \frac{\left(x^2 + y^2\right)\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} .2(x + yy')$$

or $(x^2 + y^2)(x - yy') = 2(x^2 - y^2)(x + yy')$

$\Rightarrow yy'[(x^2 + y^2) + 2(x^2 - y^2)]$

$\Rightarrow x(x^2 + y^2) - 2x(x^2 - y^2)$

$\Rightarrow yy'(3x^2 - y^2) = x(3y^2 - x^2)$

Hence, $y' = \frac{x\left(3y^2 - x^2\right)}{y\left(3x^2 - y^2\right)}$

(b) The given equation contains only one constant. Differentiating once, we get

$2a(y + a)y' = 3x^2$ ... (i)

Multiplying by y + a, we get

$2a(y + a)^2 y' = 3x^2(y + a)$

Using the given equation, we obtain

$2x^3 y' = 3x^2(y + a)$ or $2xy' = 3y + 3a$

or $a = \frac{1}{3}(2xy' - 3y)$

Substituting the value of a in (i) we obtain

$$\frac{2}{3}(2xy' - 3y)\left[y + \frac{1}{3}(2xy - 3y)\right]y' = 3x^2$$

$$\frac{2}{9}(2xy' - 3y)(2xy')y' = 3x^2$$

Cancelling x, we obtain

$8x(y')^3 - 12y(y')^2 - 27x = 0$

Example 8: If $y(x - y)^2 = x$, then show that

$$\int \frac{dx}{(x - 3y)} = \frac{1}{2}\log[x-y)^2 - 1]$$

Sol: As given $y(x - y)^2 = x$, therefore by differentiating it with respect to x we will get the value of $\frac{dy}{dx}$. After that differentiate both sides of equation $\int \frac{dx}{(x-3y)} = \frac{1}{2}$ $\log[x-y)^2 - 1]$ w.r.t. x and then by substituting the value of $\frac{dy}{dx}$ we can prove it.

Let $P = \int \frac{dx}{(x-3)} = \frac{1}{2}\log\{(x-y)^2 - 1\}$

$\therefore \quad P = \int \frac{dx}{(x-3y)}$

$\frac{dP}{dx} = \frac{1}{(x-3y)}$ ... (i)

Also $P = \frac{1}{2}\log\{(x-y)^2 - 1\}$

$\therefore \frac{dP}{dx} = \frac{(x-y)\left\{1 - \frac{dy}{dx}\right\}}{\left\{(x-y)^2 - 1\right\}}$ ... (ii)

Given $y(x - y)^2 = x$

Differentiating both sides w.r.t. x

$\therefore \frac{dy}{dx} = \frac{1 - 2y(x-y)}{(x-y)(x-3y)}$ ... (iii)

From (ii) and (iii)

$$\frac{dP}{dx} = \frac{(x-y)\{1 - (1 - 2y(x-y)/(x-y)(x-3y))\}}{\{(x-y)^2 - 1\}}$$

$$= \frac{(x-y)(x-3y) - 1 + 2y(x-y)}{(x-3y)\left\{\left(x-y^2\right) - 1\right\}}$$

$$= \frac{\left\{(x-y)^2 - 1\right\}}{(x-3y)\left\{(x-y)^2 - 1\right\}}$$

$\Rightarrow \frac{dP}{dx} = \frac{1}{(x-3y)}$

It is true from (i)

Hence $\int \frac{dx}{(x-3y)} = \frac{1}{2}\ell n\left\{(x-y)^2 - 1\right\}$

Example 9: Solve: $\cos(x + y)dy = dx$

Sol: Simply by putting x + y = t we can reduce the given equation as $\frac{dt}{dx} = \sec t + 1$ and then by separating the variable and integrating we can solve the problem given above.

We have $\cos(x + y)dy = dx$

$\Rightarrow \frac{dy}{dx} = \sec(x + y)$

On putting x + y = t so that $1 + \frac{dy}{dx} = \frac{dt}{dx}$

or $\frac{dy}{dx} = \frac{dt}{dx} - 1$ we get

$\frac{dt}{dx} - 1 = \sec$

$\Rightarrow \frac{dt}{dx} = 1 + \sec t$

$\frac{dt}{1+\sec t} = dx \Rightarrow \frac{\cos t}{\cos t + 1} dt = dx$

$\int \frac{\cos t}{\cos t + 1} dt = \int dx$

$\Rightarrow \int \left[1 - \frac{1}{\cos t + 1}\right]' dt = x + C$

$\int \left[1 - \frac{1}{2\cos^2(t/2) - 1 + 1}\right] dt = x + C$

$\int \left(1 - \frac{1}{2}\sec^2 \frac{t}{2}\right) dt = x + C$

$\Rightarrow t - \tan\frac{t}{2} = x + C$

$x + y - \tan\frac{x+y}{2} = x + C$

$y - \tan\frac{x+y}{2} = C$

Example 10: Solve: $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol: Similar to example 9.

We have, $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x + y)$

Putting x + y = t, so that

$1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx}$

Now, substituting x + y = t and $\frac{dy}{dx} = \frac{dt}{dx} - 1$ in (i), we get

$\frac{dt}{dx} = \sin t \Rightarrow \frac{dt}{dx} = \sin t + 1 \Rightarrow dx = \frac{dt}{1 + \sin t}$

Integrating both sides, we get

$\int dx = \int \frac{dt}{1 + \sin^2 t} dt + c$

$\Rightarrow \int dx = \int \frac{1 - \sin t}{1 - \sin^2 t} dt + C = \int \frac{1 - \sin t}{\cos^2 t} dt$

$\Rightarrow \int dx = \int \left(\sec^2 t - \tan t \sec t\right) dt$

$\Rightarrow x = \tan t - \sec t$

$\Rightarrow x = \tan(x + y) - \sec(x + y) + C$

Example 11: Solve the equation:

$\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$

Sol: Simply by putting y = vx and integrating we can obtain the general equation of given differential equation.

We have,

$\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$ ... (i)

Put y = vx, so that

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

On putting the value of y and $\frac{dy}{dx}$ in (i), we get

$v + x\frac{dv}{dx} = v + x\sin v$

$\Rightarrow \quad x\frac{dv}{dx} \Rightarrow \frac{dv}{dx} = \sin v$

Separating the variables, we get

$\frac{dv}{\sin v} = dx \Rightarrow \int \operatorname{cosec} v \, dv = \int dx$

$\Rightarrow \log\tan\frac{v}{2} = x + C$ ... (ii)

On putting the value of v in (ii), we have

$\log\tan\frac{y}{2x} = x + C$

This is the required solution

Example 12: Solve:

$2ye^{\frac{x}{y}} dx + \left(y - 2xe^{\frac{x}{y}}\right) dy = 0$

Sol: We can reduce the given equation as $\frac{dy}{dx} = \frac{2xe^{x/y}}{2ye^{x/y}}$ and then by putting x = vy and integrating we can obtain general equation.

We have,

$2ye^{\frac{x}{y}} dx + \left(y - 2xe^{\frac{x}{y}}\right) dx = 0$

$\Rightarrow \quad 2ye^{\frac{x}{y}}.\frac{dx}{dy}+\left(y-2xe^{\frac{x}{y}}\right)=0$

$\Rightarrow \quad \frac{dy}{dx}=\frac{2xe^{x/y}}{2ye^{x/y}}$ ... (i)

Clearly, the given differential equation is a homogeneous differential equation. As the right hand side of (i) is expressible as a function of $\left(\frac{x}{y}\right)$. So, we put

$\frac{dt}{dx}=v \Rightarrow x = vy$ and $\frac{dx}{dy}$

$= v + y\frac{dv}{dy}$ in (i), we get

$v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}$

$\Rightarrow \quad y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}-v$

$\Rightarrow y\frac{dv}{dy}=-\frac{1}{2e^{v}}$

$\Rightarrow \quad 2ye^{v}dv = -dy$

$\Rightarrow 2e^{v}dv = -\frac{1}{y}dy$, $y \neq 0$

Integrating both sides, we get

$2\int e^{v}dv=-\int\frac{1}{y}dy+\log c$

$\Rightarrow \quad 2e^{v} = -\log|y| + \log c$

$\Rightarrow \quad 2e^{v} = \log\left|\frac{c}{y}\right|$

$\Rightarrow \quad 2e^{\frac{x}{y}}=\log\left|\frac{c}{y}\right| \quad \left(\because v=\frac{x}{y}\right)$

Example 13: Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$

Sol: Here by reading the above problem, we get that $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$. Hence by putting y = vx and then integrating both sides we can prove the given equation.

We have slope of the tangent

$=\frac{x^2+y^2}{2xy} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

or $\quad \frac{dy}{dx}=\frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}}$ ... (i)

Equation (i) is a homogeneous differential equation.

So we put y = vx and $\frac{dy}{dx}=v+\frac{dv}{dx}$

Substituting the value of $\frac{y}{x}$ and $\frac{dy}{dx}$ in equation (i), we get

$v + x\frac{dv}{dx}=\frac{1+v^2}{2v}$

or $\quad x\frac{dv}{dx}=\frac{1-v^2}{2v}$ ... (ii)

Separating the variables in equation (ii), we get

$\frac{2v}{1-v^2}dv=\frac{dx}{x}$ or $\frac{2v}{v^2-1}dv=-\frac{dx}{x}$ ... (iii)

Integrating both sides of equation (iii), we get $\int\frac{2v}{v^2-1}dv=-\int\frac{1}{x}dx$

or $\quad \log|v^2 - 1| = -\log|x| + \log|C_1|$

or $\quad \log|(v^2 - 1)(x)| = \log|C_1|$ ... (iv)

Replacing v by $\frac{y}{x}$ in equation (iv), we get $\left(\frac{y^2}{x^2}-1\right)$ $x = \pm C_1$

or $(y^2 - x^2) = \pm C_1x$ or $x^2 - y^2 = Cx$

Example 14: Solve: $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Sol: Simply by putting x = X + h; y = Y + k were (h, k) will satisfy the equations x + 2y – 3 =0 and 2x + y – 3 =0 we can solve the problem.

$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Put: x = X + h; y = Y + k

$\Rightarrow \quad dx = dX; dy = dY$

$\Rightarrow \quad \frac{dy}{dx}=\frac{dY}{dX}$

Given equation reduces to

$\frac{dy}{dx}=\frac{(x+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$ ... (i)

Choose h and k such that

h + 2k – 3 = 0; and 2h + k – 3 = 0

$\Rightarrow$ h = 1 ; k = 1

Equation (i) becomes

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \text{... (ii)}$$

Put: Y = VX

$$\Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$$

Now equation (ii) becomes:

$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1-V^2}{2+V}$$

Separating the variables, we have

$$\Rightarrow \frac{2+V}{1-V^2}dV = \frac{dX}{X}$$

Integrating, we get

$$\int\frac{2}{1-V^2}dV + \int\frac{V}{1-V^2}dV = \int\frac{dX}{X}$$

$$\Rightarrow 2.\frac{1}{2}\log\frac{1+V}{1-V} - \frac{1}{2}\log(1-V^2) = \log X + \log c$$

$$\Rightarrow 2\log\left(\frac{1+V}{1-V}\right) - \log(1-V^2) = 2\log cX$$

$$\Rightarrow \log\left[\left(\frac{1+V}{1-V}\right)^2 \times \frac{1}{1-V^2}\right] = \log(cX)^2$$

$$\Rightarrow \left(\frac{1+V}{1-V}\right)^2 \times \frac{1}{\left(1-V^2\right)} = (cX)^2$$

$$\Rightarrow \frac{1+V}{\left(1-V\right)^3} = c^2X^2 \qquad \text{... (iii)}$$

Putting the value of V in (iii), we have

$$\Rightarrow \frac{X+Y}{\left(X-Y\right)^3}X^2 = c^2X^2 \qquad \text{... (iv)}$$

$$\left[\therefore V = \frac{Y}{X}\right]$$

Substituting the value of X and Y in (iv), we get

$$\Rightarrow \frac{x+y-2}{\left(x-y\right)^3} = c^2$$

[$\because$ X = x – h = x – 1; Y = y – 1]

Example 15: Solve the following differential equation: $(x^2-1)\dfrac{dy}{dx} + 2xy = \dfrac{2}{x^2-1}$

Sol: First reduce this into the form of $\dfrac{dy}{dx} + Py = Q$ and then using the integration factor i.e. $e^{\int Pdx}$ we can solve this.

We have, $(x^2-1)\dfrac{dy}{dx} + 2xy = \dfrac{2}{x^2-1}$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2} \qquad \text{.....(i)}$$

This is a linear differential equation in y.

Here, $P = \dfrac{2x}{x^2-1}$, $Q = \dfrac{2}{\left(x^2-1\right)^2}$

$$\text{I.F.} = e^{\int Pdx} = e^{\int\frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = x^2-1$$

$\therefore$ The solution of (i) is

$$y(\text{I.F.}) = \int\left(Q\times \text{I.F}\right)dx + C$$

$$y.(x^2-1) = \int\left(x^2-1\right).\frac{2}{\left(x^2-1\right)^2}dx + C$$

$$= 2\int\frac{1}{x^2-1}dx + C$$

$$= 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow y = \left(\frac{1}{x^2-1}\right)\left[\log\left|\frac{x-1}{x+1}\right| + C\right]$$

JEE Main/Boards

Exercise 1

Q.1 Write the order and degree of the differential equation $x - \cos\left(\frac{dy}{dx}\right) = 0$

Q.2 Solve the differential equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

Q.3 Write the order and degree of the differential equation $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

Q.4 How will you proceed to solve the differential equation $\frac{dy}{dx} = 1 + x + y + xy$?

Q.5 Find the integrating factor for solving the differential equation

$(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$

Q.6 Solve the differential equation $\frac{dr}{d\theta} = \cos\theta$

Q.7 To solve the differential equation $\frac{dy}{dx} + 2y = 6e^x$, how will you proceed?

Q.8 Prove that, the differential equation that represents all parabolas having their axis of symmetry coincident with the axis of x is $yy_2 + y_1^2 = 0$.

Q.9 Form the differential equation representing the family of curves y = Acos2x + Bsin2x, where A and B are constants.

Q.10 Prove that, the function $y = Ax + \frac{B}{x}$ is a solution of the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

Q.11 Prove that, the differential equation of which

$1 + 8y^2\tan x = cy^2$ is a solution is $\cos^2 x\frac{dy}{dx} = 4y^3$

Q.12 Form the differential equation of the family of curves $y = Ae^{BX}$

Q.13 $\frac{dy}{dx} = e^{x-y} + x^3e^{-y}$

Q.14 $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Q.15 $\sqrt{x + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$

Q.16 $\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$

Q.17 $\frac{dy}{dx} = \sin^3x.\cos^3x + xe^x$

Q.18 $(1 - x^2)dy + xydx = xy^2dx$

Q.19 $x\frac{dy}{dx} + y = y^2$

Q.20 $(x - y^3)dy + ydx = 0$

Q.21 $y - x\frac{dy}{dx} = a\left(y^2 + x^2\frac{dy}{dx}\right)$, where x = a, y = a.

Q.22 $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

Q.23 $e^{dy/dx} = x + 1,\ y(0) = 4$

Q.24 $y' + 2y^2 = 0,\ y(0) = \frac{\pi}{2}$

Q.25 $xy' + y = x\cos x + \sin x,\ y\left(\frac{\pi}{2}\right) = 1$

Q.26 $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$, given that when x = 1, y = 1

Q.27 $(1+\sin x^2 x)dy + (1+y^2)\cos x\, dx = 0$, given that when $x = \frac{\pi}{2}$, y = 0

Q.28 xydy = (y + 5)dx, given that y(5) = 0

Q.29 $(x + 2)dx = (x^2 + 4x + 9)dy$, given that y(0) = 0

Exercise 2

Single Correct Choice Type

Q.1 The general solution of the differential equation, y' + yϕ'(x) – ϕ(x).ϕ'(x) = 0 where ϕ(x) is a known function is:

(A) $y = ce^{-\phi(x)} + \phi(x) - 1$

(B) $y = ce^{+\phi(x)} + \phi(x) - 1$

(C) $y = ce^{-\phi(x)} - \phi(x) + 1$

(D) $y = ce^{-\phi(x)} + \phi(x) + 1$

Q.2 The differential equation $y\frac{dy}{dx} + x = C$, where C is any arbitrary constant represents:

(A) A set of circles with centre on x-axis.

(B) A set of circles with centre on y-axis.

(C) A set of concentric circles.

(D) A set of ellipses.

Q.3 The differential equation of all parabolas having their axis of symmetry coinciding with the axis of x is:

(A) $y\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ (B) $x\frac{d^2y}{dy^2} + \left[\frac{dy}{dx}\right]^2 = 0$

(C) $y\frac{d^2y}{dx^2} + \left[\frac{dy}{dx}\right]^2 = 0$ (D) None of these

Q.4 The solution of the differential equation,

$$xy\left[\frac{dy}{dx}\right] = \left[\frac{1+y^2}{1+x^2}\right]\left(1 = x + x^2\right)$$

given that when x = 1, y = 0 is:

(A) $\log\sqrt{1+y^2} = \log x + \tan^{-1}x - \frac{\pi}{2}$

(B) $\log\frac{1+y^2}{x^2} = 2\tan^{-1}x - \frac{\pi}{2}$

(C) $\log\frac{1+y^2}{x^2} = \frac{\pi}{4} - 2\tan^{-1}x$

(D) None of these

Q.5 Given, y = 1 + cosx and y = 1 + sinx are solution of the differential equation $\frac{d^2y}{dx^2} + y = 1$, then its solution will be also:

(A) y = 2(1 + cosx) (B) y = 2 + cosx +sinx

(C) y = cosx – sinx (D) y = 1 + cosx + sinx

Q.6 The solution of the differential equation

$(x + 2y^3)\frac{d^2y}{dx^2} = y$ is:

(A) $\frac{x}{y^2} = y + c$ (B) $\frac{x}{y} = y^2 + c$

(C) $\frac{x^2}{y} = y^2 + C$ (D) $\frac{y}{x} = x^2 + c$

Q.7 A normal is drawn at a point P(x, y) of a curve. It meets the x-axis and y-axis in the points A and B respectively such the $\frac{1}{OA} + \frac{1}{OB} = 1$, where 'O' is the origin. The equation of such a curve passing through (5, 4) denotes:

(A) A line. (B) A circle.

(C) A parabola. (D) Pair of straight line.

Q.8 The latus rectum of the conic passing through the origin and having the property that normal at each point (x, y) intersects the x-axis at ((x + 1), 0) is:

(A) 1 (B) 2 (C) 4 (D) None of these

Q.9 The solution of the differential equation

$2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, that given $y(1) = \sqrt{\frac{\pi}{2}}$ is

(A) $\sin x^2y^2 = e^{x-1}$ (B) $\sin(x^2y^2) = x$

(C) $\cos x^2y^2 + x = 0$ (D) $\sin(x^2y^2) = e.e^x$

Q.10 A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, then the time when it would have lost 99.9% of its moisture is: (weather condition remaining same)

(A) More then 100 hours

(B) More than10 hours

(C) Approximately 10 hours

(D) Approximately 9 hours

Q.11 If $y = \frac{c}{\log|x|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ then the function $\phi\left(\frac{x}{y}\right)$ is

(A) $\frac{x^2}{y^2}$ (B) $-\frac{x^2}{y^2}$ (C) $\frac{y^2}{x^2}$ (D) $-\frac{y^2}{x^2}$

Q.12 A tank contains 10000 liters of brine in which 10 kg of salt is dissolved initially at t = 0. Fresh brine containing 20 gms of salt per 100 liters keeps running into the tank at the rate of 50 liters per minute. If the mixture is kept stirring uniformly, then the amount of salt (in kgs) present in the tank at the end of 10 minutes, is (Assume that there is no overflow of brine is the bank)

(A) 11.5 (B) 11.15 (C) 10.1 (D) 10.5

Q.13 Which of the following differential equation is not of degree 1?

(A) $x^3y_2 + (x+x)^2 y_1^2 + e^xy^3 = \sin x$

(B) $y_2^{1/2} + (\sin x)y_1 + xy = x$

(C) $\sqrt{y_1 + y} = x + 1$

(D) None of these

Q.14 If $\frac{dy}{dx} = \frac{xy+y}{xy+y}$, then the solution of the differential equation:

(A) $y = xe^x + c$ (B) $y = e^x + c$

(C) $y = Axe^x$ (D) $y = x + A$

Q.15 The degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^{2-3} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Q.16 The differential equation for all parabolas each of which has a latus rectum '4a' and whose axes are parallel to x-axis is:

(A) of degree 2 and order 1 (B) of order 2 and degree 3

(C) $2a\frac{d^2x}{dy^2} = 1$ (D) $2a\frac{d^2x}{dy^2} + \left(\frac{dy}{dx}\right)^3 = 0$

Previous Years' Questions

Q.1 The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4e^{x+c_5}$. where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is ***(1998)***

(A) 5 (B) 4 (C) 3 (D) 2

Q.2 A solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is: ***(1999)***

(A) y = 2 (B) y = 2x

(C) y = 2x – 4 (D) $y = 2x^2 - 4$

Q.3 If y(t) is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and y(0) = –1, then y(1) is equal to:- ***(2003)***

(A) $-\frac{1}{2}$ (B) $e + \frac{1}{2}$ (C) $e - \frac{1}{2}$ (D) $\frac{1}{2}$

Q.4 If y = y(x) and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ equals ***(2004)***

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) 1

Q.5 If $\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is ***(2005)***

(A) $\log\left(\frac{y}{x}\right) = cx$ (B) $\log\left(\frac{y}{x}\right) = cy$

(C) $y\log\left(\frac{y}{x}\right) = cx$ (D) $x\log\left(\frac{y}{x}\right) = cy$

Q.6 A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0). Find the time after which the cone is empty. ***(2003)***

Q.7 If length of tangent at any point on the curve y = f(x) intercepted between the point and the x-axis is of length 1. Find the equation of the curve. ***(2005)***

Q.8 If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation,

y(1 + xy) dx = xdy, then $f\left(-\frac{1}{2}\right)$ is equal ***(2016)***

(A) $-\frac{4}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $-\frac{2}{5}$

Q.9 Let y(x) be the solution of the differential equal (x log x) $\frac{dy}{dx}+y=2x\log x,\ (x\geq 1)$. Then y(e) is equal to ***(2015)***

(A) e (B) 0 (C) 2 (D) 2e

Q.10 If y = sec($\tan^{-1}$x), then $\frac{dy}{dx}$ at x = 1 is equal to : ***(2013)***

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\sqrt{2}$

Q.11 $\frac{d^2x}{dy^2}$ equals ***(2011)***

(A) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (B) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(C) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (D) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

Q.12 Solution of the differential equation cos x dy = y(sin x – y) dx, $0 < x < \frac{\pi}{2}$ ***(2010)***

(A) y sec x = tan x + c (B) y tan x = sec x + c

(C) tan x = (sec x + c)y (D) sec x = (tan x + c)y

Q.13 The differential equation which represents the family of curves $y=c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants is ***(2009)***

(A) $y' = y^2$ (B) $y'' = y'y$

(C) $yy'' = y'$ (D) $yy'' = (y')^2$

Q.14 The solution of the differential equation $\frac{dy}{dx}=\frac{x+y}{x}$ satisfying the condition y (1) = 1 is ***(2008)***

(A) y = log x + x (B) y = x log x + x^2

(C) $y = xe^{(x-1)}$ (D) y = x log x + x

JEE Advanced/Boards

Exercise 1

Q.1 (i) Solve $\frac{dy}{dx}=\frac{x^2+xy}{x^2+y^2}$

(ii) $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

Q.2 Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

Q.3 The light rays emitting from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solve it.

Q.4 The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1)

Q.5 Use the substitution $y^2 = a - x$ to reduce the equation $y^3.\frac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it.

Q.6 Solve: $\left[x\cos\frac{y}{x}+y\sin\frac{y}{x}\right]y=\left[y\sin\frac{y}{x}-x\cos\frac{y}{x}\right]x\frac{dy}{dx}$

Q.7 Find the curve for which any tangent intersects the y-axis at the point equidistant from the point of tangency and the origin

Q.8 Solve: (x – y)dy = (x + y + 1)dx

Q.9 Solve: $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Q.10 Solve: $\frac{dy}{dx}=\frac{y-x+1}{y+x+5}$

Q.11 Solve: $\frac{dy}{dx}=\frac{x+y+1}{2x+2y+3}$

Q.12 Solve: $\frac{dy}{dx}=\frac{2(y+2)^2}{(x+y-1)^2}$

Q.13 Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2+y^2} = ce^{\pm\tan^{-1}\frac{y}{x}}$

Q.14 If solution of differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value. When $x \to \infty$, then find y_0.

Q.15 Let y = y(t) be a solution to the differential equation $y' + 2t\,y = t^2$, then find $\lim\limits_{t\to\infty} \frac{y}{t}$

Q.16 Solve: $\frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{1}{2x\left(1+x^2\right)}$

Q.17 Solve: $(1 - x^2)\frac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$

Q.18 (i) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point and the tangent at this point equals half the square of its abscissa.

(ii) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2 square units. If the curve passes through (2, 1) find the equation of the curve.

Q.19 Solve: $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x - 1)$

Exercise 2

Single Correct Choice Type

Q.1 A curve passes through the point $\left(1, \frac{\pi}{4}\right)$, and its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$, Then the curve has the equation, y is equal to:

(A) $y = x\tan^{-1}\left(\ell n\frac{e}{x}\right)$

(B) $y = x\tan^{-1}(\log+2)$

(C) $y = \frac{1}{x}\tan^{-1}\left(\ell n\frac{e}{x}\right)$

(D) None of these

Q.2 y = f(x) satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$ with the condition that y is bounded when $x \to +\infty$. The longest interval in which f(x) is increasing in the interval

(A) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{2}\right)$ (C) $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$ (D) $\left(0, \frac{\pi}{6}\right)$

Q.3 The real value of m for which the substitution, $y = u^m$ will transform the differential equation $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is:

(A) m = 0 (B) m = 1

(C) m = 3/2 (D) m = 2/3

Q.4 The solution of the differential equation

$x^2\frac{dy}{dx}.\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$, Where $y \to -1$ as $x \to \infty$ is

(A) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (B) $y = \frac{x+1}{x\sin\frac{1}{x}}$

(C) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (D) $y = \frac{x+1}{x\cos\frac{1}{x}}$

Q.5 The equation of a curve for which the product of the abscissa of point P and the intercept made by a normal at P on the x-axis equals twice the square of the radius vector of the point P, and passes through (1, 0) is:

(A) $x^2 + y^2 = x^4$ (B) $x^2 + y^2 = 2x^4$

(C) $x^2 + y^2 = 4x^4$ (D) None of these

Q.6 The order and the degree of the differential equation whose general solution is, $y = c(x - c)^2$, are respectively:

(A) 1, 1 (B) 1, 2 (C) 1, 3 (D) 2, 1

Q.7 the degree of the differential equation

$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\ell n\left(\frac{d^2y}{dx^2}\right)$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Q.8 Orthogonal trajectories of family of parabolas $y^2 = 4a(x + a)$ where 'a' is an arbitrary constant is

(A) $ax^2 = 3cy$ (B) $x^2 + y^2 = a^2$

(C) $y = ce^{-\frac{x}{2a}}$ (D) $axy = c^2$

Q.9 If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = K$, then the value of K is:-

(A) 4 (B) 6 (C) 9 (D) 12

Q.10 Solution set of the equation $\frac{x\,dy}{dx} - y = x.\frac{f(y/x)}{f'(y/x)}$

(A) $f\left(\frac{x}{y}\right) = cy$ (B) $f\left(\frac{y}{x}\right) = cx$

(C) $f\left(\frac{y}{x}\right) = cxy$ (D) None of these

Q.11 $\frac{dy}{dx} = \frac{x^2 + 2xy + y^2}{x^2 - 2xy + 2y^2}$. Let C_1 and C_2 be two of it's solutions. C_1 passes through, A(1, 2), and line through origin and A meets C_2 at B. Then slope of the tangent to the curve C_2 at B is:

(A) $\frac{5}{9}$ (B) $\frac{9}{5}$ (C) $-\frac{9}{5}$ (D) None of these

Q.12 The solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 4x - 2y - 2$, y = 1 when x = 1 is:-

(A) $2e^{2y+2} = e^{4x} + e^2$

(B) $2e^{2y-2} = e^{4x} + e^4$

(C) $2e^{2y+2} = e^{4x} + e^4$

(D) $3e^{2y+2} = e^{3x} + e^4$

Multiple Correct Choice Type

Q.13 The general solution of the differential equation, $x\left(\frac{dy}{dx}\right) = y\ell n\left(\frac{y}{x}\right)$ is:

(A) $y = xe^{1-cx}$ (B) $y = xe^{1+cx}$

(C) $y = xe.xe^{cx}$ (D) $y = xe^{cx}$

where c is an arbitrary constant.

Previous Years' Questions

Q.1 Le f(x) be differentiable on the interval $(0, \infty)$ such that f(1) = 1, and $\lim_{t\to x}\frac{t^2f(x) - x^2f(t)}{t - x} = 1$ for each x > 0. Then f(x) is: ***(2007)***

(A) $\frac{1}{3x} + \frac{2x^2}{3}$ (B) $-\frac{1}{3x} + \frac{4x^2}{3}$

(C) $-\frac{1}{x} + \frac{2}{x^2}$ (D) $\frac{1}{x}$

Q.2 The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circles with ***(2007)***

(A) Variable radii and a fixed center at (0, 1)

(B) Variable radii and fixed center at (0, –1)

(C) Fixed radius of 1 and variable center along the x-axis

(D) Fixed radius of 1 and variable center a long the y-axis

Q.3 Let y = f(x) be a curve passing through (1, 1) such that the triangle formed by the coordinates axes and the tangent at any point of the curve lies in the first quadrant and has area 2 unit, from the differential equation and determine all such possible curves. ***(1995)***

Q.4 A and B are two separate reservoir of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at the time. One hour after the water is released the quantity of water in reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water? ***(1997)***

Q.5 Let u(x) and v(x) satisfy the differential equation $\frac{du}{dx} + p(x)u = f(x)$ and $\frac{dv}{dx} + p(x) = g(x)$, where P(x), f(x) and g(x) are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and f(x) > g(x) for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equations y = u(x) and y = v(x) ***(1997)***

Q.6 A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve. ***(1999)***

Q.7 A country has food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - (0.03)}$ ***(2000)***

Q.8 Let f: $R \to R$ be a continuous function, which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of f(log 5) is .... ***(2009)***

Q.9 If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the values of $g'(1)$ is .... ***(2009)***

Q.10 Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) =g(2) = 0. Then the value of y(2) is .... ***(2011)***

Q.11 A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point (1, 3). Then the solution curve dy dx ***(2016)***

(A) Intersects y = x + 2 exactly at one point.

(B) Intersects y = x + 2 exactly at two points

(C) Intersects $y = (x + 2)^2$

(D) Does NOT intersect $y = (x + 3)^2$

Q.12 Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation Py″ + Qy′ + 1 = 0, where P, Q are functions of x, y and y′ $\left(\text{here } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true ? ***(2015)***

(A) P = y + x

(B) P = y – x

(C) $Q = 1 + y_1 + y_1^2$

(D) $P - Q = x + y - y' - (y')^2$

Q.13 The function y = f (x) is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in (-1, 1) satisfying f(0) = 0. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is ***(2014)***

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

(B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

(D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Q.14 A curve passes through the point $\left(1, \frac{\pi}{6}\right)$, Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$, x > 0. Then the equation of the curve is ***(2013)***

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$

(B) $\text{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$

(D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Q.15 If y(x) satisfies the differential equation y′ - ytanx = 2x secx and y(0) = 0, then **(2012)**

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$

(B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$

(D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{2}}$

Q.16 Let f: $[1, \infty) \to [2, \infty)$ be a differentiable function such that f(1)= 2. If $6\int_1^x f(1)dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of f(2) is ***(2011)***

Q.17 Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to ***(2010)***

Q.18 Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$ ***(2009)***

Q.19 Let a solution y = y(x) of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$

Statement-I: y(x) = sec $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ and

Statement-II: y(x) is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

(2008)

(A) Statement-I is True, statement-II is True; statement-II is a correct explanation for statement-I

(B) Statement-I is True, statement-II is True; statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is True, statement-II is False

(D) Statement-I is False, statement-II is True

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.9 Q.14 Q.20 Q.21
Q.26

Exercise 2

Q.3 Q.4 Q.8 Q.10
Q.11 Q.16 Q.14

Previous Years' Questions

Q.3 Q.5 Q.8

JEE Advanced/Boards

Exercise 1

Q.3 Q.6 Q.14 Q.18
Q.19

Exercise 2

Q.1 Q.4 Q.5 Q.9
Q.11 Q.13

Previous Years' Questions

Q.2 Q.4 Q.7
Q.10

Answer Key

JEE Main/Boards

Exercise 1

Q.1 Order = 1; Degree = 1

Q.2 $\sin^{-1}y + \sin^{-1}x = c$

Q.3 Order = 1, degree is not defined

Q.4 Separate the variables after factorizing

Q.5 $e^{\tan^{-1}y}$

Q.6 $r = \sin\theta + c$

Q.7 $(\text{I.F.})y = \int(\text{I.F.}).Q$

Q.9 $y'' = -4A\cos 2x - 4B\sin 2x = -4y$

Q.12 $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

Q.13 $e^y = e^x + \dfrac{x^4}{4} + c$

Q.14 $\tan y - \cot x = c$

Q.15 $\sqrt{x^2+1} + \sqrt{y^2+1} + \dfrac{1}{2}\log\left(\dfrac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + c = 0$

Q.16 $\log|y| = \dfrac{x^3}{3y^3} + c$

Q.17 $y = \dfrac{(\cos x)^6}{6} - \dfrac{(\cos x)^4}{4} + e^x(x+1) + c$

Q.18 $-[\log y - \log(y-1)] = -\dfrac{1}{2}\log(1-x^2) + c$

Q.19 $y - 1 = xy$

Q.20 $x = \dfrac{y^3}{4} + \dfrac{2}{y}$

Q.21 $\dfrac{y}{y-\dfrac{1}{a}} = \left(\dfrac{x}{\left(x+\dfrac{1}{a}\right)}\right)\left(\dfrac{a^2+1}{a^2-1}\right)$

Q.22 $y\log|x| = \dfrac{-2\log|x|}{x} - \dfrac{2}{x} + c$

Q.23 $y = (x+1)\log|x+1| - x + 4$

Q.24 $\dfrac{1}{y} = 2x + \dfrac{2}{\pi}$

Q.25 $y = \sin x$

Q.26 $\dfrac{y}{x} = \log|y| + 1$

Q.27 $\tan^{-1}(\sin x) + \tan^{1}y = \dfrac{\pi}{4}$

Q.28 $y - 5\log|y+5| = \log|x| - 6\log 5$

Q.29 $y = \dfrac{1}{2}\log(x^2+4x+9) - \log 3$

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** A	**Q.3** C	**Q.4** B	**Q.5** D	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** A	**Q.10** C	**Q.11** D	**Q.12** C
Q.13 D	**Q.14** C	**Q.15** B	**Q.16** C		

Previous Years' Questions

Q.1 C	**Q.2** C	**Q.3** A	**Q.4** A	**Q.5** A	**Q.6** $T = \dfrac{dy}{dx}$
Q.7 $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = \pm x + c$		**Q.8** C	**Q.9** C	**Q.10** A	**Q.11** C
Q.12 D	**Q.13** D	**Q.14** D			

JEE Advanced/Boards

Exercise 1

Q.1 (i) $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6} = \exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$ where $\exp x = e^x$ (ii) $(x^2+y^2)^2 = (x^2-y^2)c$

Q.2 $\frac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \log\left|\left(y \pm \sqrt{y^2-x^2}\right).\frac{c^2}{x^3}\right|$, where same sign has to be taken

Q.4 $x^2 + y^2 - 2x = 0$

Q.5 $\frac{1}{2}\log|x^2+a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c$, where $a = x + y^2$

Q.6 $xy\cos\frac{y}{x} = c$

Q.7 $x^2 + y^2 = cx$

Q.8 $e^{c\tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right)}$

Q.9 $(x + y - 2) = c(y - x)^3$

Q.10 $\tan^{-1}\frac{y+3}{x+2} + \log c\sqrt{(y+3)^2+(x+2)^2} = 0$

Q.11 $x + y + \frac{4}{3} = ce^{3(x-2y)}$

Q. 12 $c = e^{-2\tan^{-1}\frac{y+2}{x-3}} = (y+2)$

Q.14 $\frac{1}{2}$

Q.15 $\frac{1}{2}$

Q.16 $y\sqrt{1+x^2} = c + \frac{1}{2}\log\left[\tan\frac{1}{2}\arctan x\right]$ another form is $y\sqrt{1+x^2} = c + \frac{1}{2}\log\frac{\sqrt{1+x^2}-1}{x}$

Q.17 $y = c(1-x^2) + \sqrt{1-x^2}$

Q.18 (i) $y = cx^2 + x$ (ii) $xy = 2$

Q.19 $y(x-1) = x^2(x^2 - x + c)$

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** B	**Q.3** C	**Q.4** A	**Q.5** A	**Q.6** C
Q.7 D	**Q.8** C	**Q.9** D	**Q.10** B	**Q.11** B	**Q.12** C

Multiple Correct Choice Type

Q.13 B, C

Previous Years' Questions

Q.1 A	**Q.2** C	**Q.3** $x + y = 2$, $xy = 1$		**Q.4** $\log_{3/4}\left(\frac{1}{2}\right)$	**Q.6** $x^2+y^2=2x$
Q.8 0	**Q.9** 2	**Q.10** 0	**Q.11** A, C	**Q.12** B, C	**Q.13** B
Q.14 A	**Q.15** A, D	**Q.16** 6	**Q.17** 9	**Q.19** C	

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $x = \cos\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = \cos^{-1} x$

$\therefore$ Degree = 1, order = 1

Sol 2: $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$

$\Rightarrow \int -\frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$

$\Rightarrow -\sin^{-1}y = \sin^{-1}x + c$ or $\sin^{-1}x + \sin^{-1}y = c$

Sol 3: $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

Highest order derivative = 1

Degree is not defined as differential coefficient is not free from radical and fraction.

Sol 4: $\frac{dy}{dx} = 1 + x + y(1 + x)$ or $\frac{dy}{dx} = (1 + x)(1 + y)$

(Separation of variables method)

$\Rightarrow \int \frac{dy}{(1+y)} = \int (1+x)dx$

$\Rightarrow \log(1 + y) = x + \frac{x^2}{2} + c$

Sol 5: $(1 + y^2)dx = (\tan^{-1}y - x)dy$

$\frac{dx}{dy} + \frac{1}{(1+y^2)}x = \frac{\tan^{-1} y}{1+y^2}$

$\therefore$ Integrating factor $= e^{\int\left(\frac{1}{1+y^2}\right)dy} = e^{\tan^{-1} y}$

Sol 6: $\frac{dr}{d\theta} = \cos\theta$

$\int dr = \int \cos\theta d\theta$

$r = \sin\theta + c$

Sol 7: $\frac{dy}{dx} + 2y = 6e^x$

This is a linear equation

$\therefore$ Integrating factor $= e^{\int 2\,dx} = e^{2x}$

$\therefore e^{2x}\frac{dy}{dx} + 2e^{2x}y = 6e^{3x}$

$\Rightarrow \int d(e^{2x}y) = \int 6e^{3x}dx \Rightarrow e^{2x}y = \frac{6}{3}e^{3x} + c$

$\therefore y = 2e^x + ce^{-2x}$

Sol 8: Ellipse with their axis coincide with x-axis

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$\therefore y' = -\frac{b^2}{a^2}\frac{x}{y} \Rightarrow y'' = \frac{-b^2}{a^2}\left[\frac{y - xy'}{y^2}\right]$

$\Rightarrow yy'' = \frac{-b^2}{a^2}\left[1 - \frac{x}{y}y'\right] = \frac{-b^2}{a^2} + \left(\frac{b^2}{a^2}\frac{x}{y}\right)y'$

$\therefore yy'' + (y')^2 = \frac{-b^2}{a^2}$

For parabola

Equation will be $y^2 = 4ax$

$\Rightarrow 2yy' = 4a$ or $y' = \frac{2a}{y}$

or $2a = yy'$

$\Rightarrow (y')^2 + yy'' = 0$

Sol 9: $y = A\cos 2x + B\sin 2x$

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x = -4y$

Sol 10: $y = Ax + \frac{B}{x}$

$y' = A - \frac{B}{x^2}$

$y'' = \frac{2B}{x^3}$

$\therefore x^2y'' + xy' - y$

$= \frac{x^2(2B)}{x^3} + x\left(A - \frac{B}{x^2}\right) - \left(Ax + \frac{B}{x}\right)$

$= \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$

Hence proved.

Sol 11: $1 + 8y^2\tan x = cy^2$

$y^2(c - 8\tan x) = 1$

$\therefore 2y(c - 8\tan x)\frac{dy}{dx} + y^2(-8\sec^2 x) = 0$

$2y\frac{1}{y^2}\frac{dy}{dx} = 8y^2\sec^2 x$

$\therefore \frac{dy}{dx} = 4y^3\sec^2 x$ or $\cos^2 x\frac{dy}{dx} = 4y^3$

Sol 12: $y = Ae^{Bx}$

$\Rightarrow y' = ABe^{Bx}$

$\Rightarrow y'' = AB^2e^{Bx}$

$\Rightarrow yy'' = A^2B^2e^{2Bx} = (y')^2$

$\therefore y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Sol 13: $\frac{dy}{dx} = e^{x-y} + x^3e^{-y}$

$\Rightarrow \int e^y dy = \int (e^x + x^3)\ dx$

$\Rightarrow e^y = e^x + \frac{x^4}{4} + c$

Sol 14: $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

$\Rightarrow \int \frac{dy}{1+\cos 2y} = \int \frac{dx}{\cos 2x - 1}$

$\Rightarrow \int \frac{dy}{2\cos^2 y} = -\int \frac{dx}{2\sin^2 x}$

$\Rightarrow \int \sec^2 y\ dy = -\int \operatorname{cosec}^2 x\ dx$

$\Rightarrow \tan y = \cot x + c$

$\therefore \tan y - \cot x = c$

Sol 15: $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$

$\sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$

$\Rightarrow xy\frac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)}$

$\Rightarrow \int \frac{y}{\sqrt{1+y^2}}dy = \int \frac{\sqrt{(1+x^2)}}{x}dx$

$\Rightarrow$ take $1 + y^2 = t$, differentiating both sides

$2ydy = dt$

$I_1 = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}2t^{1/2} = \sqrt{t} = \sqrt{1+y^2}$

$\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{(1+x^2)}{x\sqrt{1+x^2}}dx$

$= \int \frac{1}{x\sqrt{1+x^2}}dx + \int \frac{x}{\sqrt{1+x^2}}dx$

$I_3 = \int \frac{x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}$

$I_2 = \int \frac{1}{x\sqrt{1+x^2}}dx$

Put $x = \tan\theta$; $dx = \sec^2\theta\, d\theta$

$= \int \frac{\sec^2\theta d\theta}{\tan\theta\sec\theta} = \int \operatorname{cosec}\theta d\theta = \log|\operatorname{cosec}\theta - \cot\theta|$

$= \log\left|\frac{1-\cos\theta}{\sin\theta}\right| = \frac{1}{2}\log\left(\frac{1-\cos\theta}{\sin\theta}\right)^2$

$= \frac{1}{2}\log\frac{(1-\cos\theta)^2}{(1-\cos^2\theta)} = \frac{1}{2}\log\left(\frac{1-\cos\theta}{1+\cos\theta}\right)$

or $I_2 = \frac{1}{2}\log\left(\frac{\sec\theta - 1}{\sec\theta + 1}\right) = \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)$

$\therefore \sqrt{y^2+1} = -\left[\left(\sqrt{x^2+1}\right) + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + c\right]$

or $\sqrt{x^2+1} + \sqrt{y^2+1} - \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + c = 0$

Sol 16: $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$

$$\frac{dy}{dx} = \left(\frac{\left(\frac{x}{y}\right)^2}{\left(\frac{x}{y}\right)^3+1}\right)$$

Put x = vy ⇒ 1 = $v\frac{dy}{dx}+y\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{v}\left(1-\frac{x}{v}\frac{dv}{dx}\right) = \frac{v^2}{v^3+1}$$

$$\frac{-x}{v}\frac{dv}{dx} = \frac{v^3}{v^3+1} - 1 = \frac{-1}{v^3+1}$$

$$\Rightarrow \int\frac{(v^3+1)}{v}dv = \int\frac{dx}{x} \Rightarrow \frac{v^3}{3} + \log v = \log x + c$$

$$\Rightarrow \frac{1}{3}\left(\frac{x}{y}\right)^3 + \log x - \log y = \log x + c$$

$$\therefore \log y + c = \frac{x^3}{3y^3} \Rightarrow y = Ce^{\frac{x^3}{3y^3}}$$

Sol 17: $\frac{dy}{dx}$ = sin³x cos³x + xeˣ

$$dy = \int(\sin^3 x\cos^3 x + xe^x)dx$$

$$= \int\sin x(1-\cos^2 x)\cos^3 x dx + xe^x - e^x$$

Put cosx = t ⇒ –sinxdx = dt

$$= -\int(1-t^2)(t^3)dt + x(e^x) - e^x$$

$$= \int(t^5-t^3)dt + e^x(x-1)$$

$$= \frac{t^6}{6}-\frac{t^4}{4} + e^x(x-1) + c$$

$$= \frac{(\cos x)^6}{6}-\frac{(\cos x)^4}{4} + e^x(x+1) + c$$

Sol 18: $(1-x^2)dy + xydx = xy^2dx$

$(1-x^2)dy = (xy^2-xy)dx = x(y^2-y)dx$

$$\int\frac{dy}{(y^2-y)} = \int\frac{x}{1-x^2}dx$$

$$-\int\left[\frac{1}{y}-\frac{1}{(y-1)}\right]dy = -\frac{1}{2}\int\frac{-2x}{1-x^2}dx$$

$$\Rightarrow -[\log y - \log(y-1)] = -\frac{1}{2}\log(1-x^2) + c$$

Sol 19: $x\frac{dy}{dx} + y = y^2$

$$\Rightarrow x\frac{dy}{dx} = (y^2-y)$$

$$\Rightarrow \int\frac{dy}{y^2-y} = \int\frac{dx}{x} \Rightarrow -\int\left[\frac{1}{y}-\frac{1}{(y-1)}\right]dy = \log x + c$$

⇒ log(y – 1) – log(y) = logx + logc

⇒ (x – 1) = xy

Sol 20: $(x-y^3)dy + ydx = 0$

$\Rightarrow xdy + ydx = y^3dy$

$$\Rightarrow \int d(xy) = \int y^3dy$$

$$\Rightarrow xy = \frac{y^4}{4}+c \text{ or } x = \frac{y^3}{4}+\frac{c}{y}$$

Sol 21: $y - x\frac{dy}{dx} = a\left(y^2+x^2\frac{dy}{dx}\right)$

$$\Rightarrow \frac{dy}{dx}(ax^2+x) = y - ay^2$$

$$\therefore \int\frac{dy}{y-ay^2} = \int\frac{dx}{ax^2+x}$$

$$\Rightarrow \int\frac{1}{y(1-ay)}dy = \int\frac{1}{x(ax+1)}dx$$

$$\Rightarrow \int\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy = \int\left(\frac{1}{x}-\frac{a}{ax+1}\right)dx$$

$$\Rightarrow \int\left[\frac{1}{y}-\frac{1}{\left(y-\frac{1}{a}\right)}\right]dy = \int\left(\frac{1}{x}-\frac{1}{\left(x+\frac{1}{a}\right)}\right)dx$$

$$\log y - \log\left(y-\frac{1}{a}\right) = \log x - \log\left(x+\frac{1}{a}\right) + \log c$$

$$\log a - \log\left(a-\frac{1}{a}\right) = \log a \log.\left(a+\frac{1}{a}\right) + \log c$$

$$\log c = \log\frac{a^2+1}{a^2-1}$$

$$\Rightarrow \frac{y}{y-\frac{1}{a}} = \left(\frac{x}{\left(x+\frac{1}{a}\right)}\right)\left(\frac{a^2+1}{a^2-1}\right)$$

Sol 22: $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x^2}$

I. F. = $e^{\int \frac{1}{x\log x}dx} = e^{\ell n \ell nx} = \log x$

$\therefore (\log x)y = \int \frac{2}{x^2}\log x dx$

Put logx = t $\Rightarrow \frac{1}{x}dx = dt$

$\therefore (\log x)y = 2\int e^{-t} t dt$

u = –t

dt = – du =+ $2\int e^u u du$ = +2[e^u(u – 1)] + c

$\therefore (\log x)y = +2[e^{-\log x}(-\log x - 1)] + c$

$\therefore (\log x)y = -\frac{2}{x}(\log|x|+1) + c$

Sol 23: $e^{dy/dx} = x + 1$, y(0) = 4

$\frac{dy}{dx} = \log(x+1) \Rightarrow \int dy = \int \ell n(x+1)dx$

$\int dy = \log(x+1)\int 1dx - \int\left(\frac{d\ell n(x+1)}{dx}\int 1.dx\right)dx$

$y = x\log(x+1) - \int \frac{x}{x+1}dx + c$

= xlog(x + 1) – x + log(x+1) + c

Or y = (x + 1) log(x+1) – x + c

y(0) = 4

$\therefore$ 4 = c

$\therefore$ y = (x + 1) log(x + 1) – x + 4

Sol 24: $y' + 2y^2 = 0$

$\frac{dy}{dx} = -2y^2$

$\therefore \int -\frac{1}{2y^2}dy = \int dx$

$\Rightarrow -\frac{1}{2}\left(-\frac{1}{y}\right) = x + c$ or $\frac{1}{2y} = x + c$

$y(0) = \frac{\pi}{2}$

$\therefore c = \frac{1}{2\times\frac{\pi}{2}} = \frac{1}{\pi}$ $\therefore y = \frac{1}{2\left(x + \frac{1}{\pi}\right)}$

or $\frac{1}{y} = 2x + \frac{2}{\pi}$

Sol 25: $x\frac{dy}{dx} + y = x\cos x + \sin x$

Or $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \cos x + \frac{\sin x}{x}$

I. F. = $e^{\int \frac{1}{x}dx} = x$

$\therefore \frac{d(xy)}{dx} = \left(\cos x + \frac{\sin x}{x}\right)x$

$\therefore \int d(xy) = \int (x\cos x + \sin x)dx$

$xy = \int x\cos x dx + \int \sin x dx$

$= x\int \cos x dx - \int\left(\frac{dx}{dx}\int \cos x dx\right)dx + \int \sin x dx$

$= x\sin x - \int \sin dx + \int \sin x dx + c$

$\therefore$ xy = xsinx + c

$y\left(\frac{\pi}{2}\right) = 1$

$\frac{\pi}{2} = \frac{\pi}{2} + c \quad \Rightarrow c = 0$

$\therefore$ y = sinx + 0

$\therefore$ y = sinx

Sol 26: $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{y^2}{xy - x^2} = \frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right) - 1}$

Let y = vx

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2}{v-1}$ or $x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v}{v-1}$

$\therefore \int \frac{v-1}{v}dv = \int \frac{dx}{x}$

$\Rightarrow v - \log v = \log x + c$ or $\frac{y}{x} - \log\left(\frac{y}{x}\right) = \log x + c$

when x = 1, y = 1

1 – log1 = log1 + c or c = 1

$\therefore \frac{y}{x} = \log|y| + 1$

or $y = x\log(ey)$

Sol 27: $(1 + \sin^2 x)dy + (1 + y^2)\cos x dx = 0$

$\Rightarrow \int \frac{dy}{1+y^2} = -\int \frac{\cos x}{1+\sin^2 x}dx$

$\tan^{-1}y = -\int \frac{dt}{1+t^2}$ (Putting sinx = t)

$\therefore \cos x dx = dt$

$\therefore \tan^{-1}y = -\tan^{-1}\sin x + c$

At $x = \frac{\pi}{4}$, y = 0

$\therefore c = \tan^{-1}1 = \frac{\pi}{4}$

$\therefore \tan^{-1}\sin x + \tan^{-1}y = \frac{\pi}{4}$

Sol 28: $xydy = (y + 5)dx$

$\Rightarrow \left(\frac{y}{y+5}\right)dy = \frac{1}{x}dx$

$\Rightarrow \int\left(1-\frac{5}{y+5}\right)dy = \log x + c$

$\Rightarrow y - 5\log(y + 5) = \log x + c$

since for x = 5, y = 0

$\Rightarrow 0 - 5\log 5 = \log 5 + c \Rightarrow c = -6\log 5$

$\therefore y = 5\log(y + 5) + \log x - 6\log 5$

$y = 5\log\left(\frac{y+5}{5}\right) + \log\left(\frac{x}{5}\right)$

or $y - 5\log|y + 5| = \log|x| - 6\log 5$

Sol 29: $(x + 2)dx = (x^2 + 4x + 9)dy$

$\therefore dy = \int \frac{(x+2)}{(x^2+4x+9)}dx \Rightarrow y = \frac{1}{2}\int \frac{2x+4}{(x^2+4x+9)}dx$

Put $x^2 + 4x + 9 = t$

$\therefore (2x + 4)dx = dt$

$\therefore y = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t + c$ or $y = \frac{1}{2}\log(x^2+4x+9)+c$

for x = 0, y = 0

$\therefore c = -\frac{1}{2}\log 9$

$\therefore y = \frac{1}{2}\log\frac{(x^2+4x+9)}{9}$

Or $y = \frac{1}{2}\log(x^2 + 4x + 9) - \log 3$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $y' + y\,\phi'(x) - \phi(x)\,\phi'(x) = 0$

This is a linear equation

I. F. $= e^{\int \phi'(x)dx} = e^{\phi(x)}$

$\therefore \int d(e^{\phi(x)}.y) = \int e^{\phi(x)}\phi(x)\phi'(x)dx$

$\therefore e^{\phi(x)}y = te^t - \int e^t t dx$

Let $\phi(x) = t$

$\phi'(x)dx = dt$

$e^{\phi(x)}y = te^t - \int e^t dt + c = te^t - e^t + c = (\phi(x) - 1)e^{\phi(x)} + c$

$\therefore y = (\phi(x) - 1) + ce^{-\phi(x)}$

or $y = ce^{-\phi(x)} + \phi(x) - 1$

Sol 2: (A) $y\frac{dy}{dx} + x = c \Rightarrow \int ydy = \int(c-x)dx$

$\frac{y^2}{2} = cx - \frac{x^2}{2} \Rightarrow \frac{x^2}{2} - cx + \frac{y^2}{2} = 0$

$\Rightarrow x^2 - 2cx + y^2 = 0$ or $(x - c)^2 + y^2 = c^2$

$\therefore$ Circle with centre at (c, 0) and radius c.

Sol 3: (C) Parabola equation $y^2 = 4ax$

$\therefore 2y\frac{dy}{dx} = 4a$

or $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

Sol 4: (B) $xy\frac{dy}{dx} = \frac{(1+y^2)}{(1+x^2)}(1+x+x^2)$

$\Rightarrow \int \frac{y}{1+y^2}dy = \int \frac{(1+x^2)+x}{(1+x^2)x}dx$

$\Rightarrow \frac{1}{2}\log(1+y^2) = \int \frac{1}{x}dx + \int \frac{1}{1+x^2}dx$

$\Rightarrow \frac{1}{2}\log(1+y^2) = \log x + \tan^{-1} x + c$

For x = 1, y = 0

$\therefore$ c = $-\tan^{-1}1 = -\frac{\pi}{4}$

or $\log(1 + y^2) = 2\log x + 2\tan^{-1}x - \frac{\pi}{2}$

or $\log\frac{(1+y^2)}{x^2} = 2\tan^{-1}x - \frac{\pi}{2}$

Sol 5: (D) y = 1 + cosx

y = 1 + sinx

using option we can see that

y = 1 + cosx + sinx is satisfying the equation

$\frac{dy}{dx} = \cos x - \sin x$

$\frac{d^2y}{dx^2} = -\sin x - \cos x \therefore \frac{d^2y}{dx^2} + y = 1$

Sol 6: (B) $(x + 2y^3)\frac{dy}{dx} = y$

$x\frac{dy}{dx} + 2y^3\frac{dy}{dx} = y$

$\Rightarrow ydx - xdy = 2y^3dy$

$\Rightarrow \frac{ydx - xdy}{y^2} = 2ydy$

$\therefore \frac{x}{y} = y^2 + c$

Sol 7: (B) Equation of normal at P(x, y)

$(Y - y) = -\frac{dx}{dy}(X - x)$

OA = x-intercept = $x + y\frac{dy}{dx}$

OB = y-intercept = $y + x\frac{dx}{dy}$

$\therefore \frac{1}{x + y\frac{dy}{dx}} + \frac{1}{y + x\frac{dx}{dy}} = 1$ or $\frac{1 + \frac{dy}{dx}}{y\frac{dy}{dx} + x} = 1$

$\Rightarrow (y - 1)\frac{dy}{dx} = (1 - x)$ or $\int (y-1)dy = \int (1-x)dx$

$\Rightarrow \frac{y^2}{2} - y = x - \frac{x^2}{2} + c$

$\Rightarrow (y - 1)^2 + (x - 1)^2 - 2 = 2c$

$\therefore (y - 1)^2 + (x - 1)^2 = 2 + 2c$

at x = 5, y = 4

$\therefore 4^2 + 3^2 = 2 + 2c$ or $c = \frac{23}{2}$

$\therefore (y - 1)^2 + (x - 1)^2 = (5)^2$

This is circle with centre (1, 1) and radius 5.

Sol 8: (B) X-intercept of normal = $y\frac{dy}{dx} + x = x + 1$

$\therefore y\frac{dy}{dx} = 1$; $\Rightarrow \frac{y^2}{2} = x + c \Rightarrow y^2 = 2(x + c)$

$\therefore$ This curve pass through origin

So c = 0

$\therefore y^2 = 2x$

$\therefore$ Latus rectum = 2

Sol 9: (A) $2x^2y\frac{dy}{dx} + 2xy^2 = \tan[(xy)^2]$

Put xy = t

$\therefore x\frac{dy}{dx} + y = \frac{dt}{dx}$

$\therefore 2xy\left(x\frac{dy}{dx} + y\right) = \tan x^2y^2$ or $2t\frac{dt}{dx} = \tan t^2$

$\therefore \int \frac{2t}{\tan t^2}dt = \int dx$

Put $t^2 = u$

2tdt = du

$\therefore \int \frac{du}{\tan u} = x + c$

or logsinu = x + c or logsinxy = x + c

for x = 1, y = $\sqrt{\frac{\pi}{2}}$

$\therefore \log\sin\frac{\pi}{2} = c + 1$;   c = –1

$\therefore \sin(xy)^2 = e^{x-1}$

Sol 10: (C) $\frac{dm}{dt} = cm$

$\therefore$ logm = $c_1t + c_2$

at t = 0, m = maximum = M

$\therefore$ logm = c_2

at t = 1 hr, moisture content remains $\frac{M}{2}$

$\therefore \log\frac{M}{2} = c_1 + c_2$

$\therefore \log\frac{M}{2} - \log M = c_1$

$\therefore c_1 = \log\frac{1}{2}$

After t = x hr moisture content remains

100 – 9Sol.9 99 = 0. 1% = $\frac{0.1M}{100}$

$\therefore \log\frac{0.1M}{100} = xc_1 + c_2$

$\therefore \log\frac{0.1M}{100} - \log m = xc_1 = x\ell n\frac{1}{2}$

$\therefore x = \frac{\log\frac{0.1}{100}}{\log\frac{1}{2}} \approx 96 \approx 10$ hr

Sol 11: (D) Considering option taking $\phi\left(\frac{x}{y}\right) = \frac{-y^2}{x^2}$

$\frac{dy}{dx} = \frac{y}{x} - \frac{-y^2}{x^2}$

$y = vx \Rightarrow \ell v + x\frac{dv}{dx} = v - v^2$

$\Rightarrow x\frac{dv}{dx} = -v^2 \Rightarrow \int -\frac{1}{v^2}dv = \int\frac{1}{x}dx$

$\Rightarrow \frac{1}{v} = \log x + c$

or $\frac{x}{y} = \log cx$ or $y = \frac{x}{\log cx}$

Sol 12: (C) In 10 min. total litres run into tank = 50 × 10 = 500 lt

In 100 litres there is 20 gm salt

$\therefore$ in 500 litres we have $\frac{20\times 500}{100}$ = 100 gm = 0. 1 kg

Initially we have 10 kg of salt

$\therefore$ Total salt after 10 minutes = 10 + 0. 1 = 10.1 kg

Sol 13: (D) (A) order 2 degree 1

(B) $y_2 = (x - xy - (\sin x)y_1)^2$

$\therefore$ order 2 degree 1

(C) $y_1 + y = (x + 1)^2$

$\therefore$ Order 1 degree 1

Sol 14: (C) $\frac{dy}{dx} = \frac{xy + y}{xy + x}$

$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\frac{x+1}{y+1}\right)$

$\int\frac{(y+1)}{y}dy = \int\frac{(x+1)}{x}dx$

y + logy = x + logx + c

$\Rightarrow \log ye^y = \log xe^yA$ [$\because$ c = logA]

$\therefore ye^y = Axe^x$ or $y = Axe^{x-y}$

Sol 15: (B) Degree is 2

Sol 16: (C) $y^2 = 4xa$

$\therefore 2y\frac{dy}{dx} = 4a$

$\therefore y^2 = 2y\frac{dy}{dx}x$ or $y = 2x\frac{dy}{dx}$

$\therefore$ Order 1 degree 1

or $\frac{dx}{dy} = \frac{1}{2a}y$ or $\frac{d^2x}{dy^2} = \frac{1}{2a}$

Previous Years' Questions

Sol 1: (C) Given,$y = (c_1 + c_2)\cos(x + c_1) -$

$y\frac{dy}{dx} = \left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\}x$ ... (i)

$\Rightarrow y=(c_1 + c_2)\cos(x + c_3) - c\,e^x$. $y\frac{dy}{dx} = xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2$

Now, let $c_1 + c_2 = A, c_3 = B, c_4$ $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0 = c$

$\Rightarrow y = A\cos(x + B) - ce^x$ ... (ii)

On differential w.r.t. x, we get $x\frac{dy}{dx}$ ... (iii)

Again differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+1.\frac{dy}{dx}+1.\frac{dy}{dx} = -A\cos(x + B) - ce^x \quad \text{... (iv)}$$

$$\Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx} \quad \text{... (v)}$$

$$\Rightarrow \begin{bmatrix} \because xy = Ae^x + Be^{-x} + x^2 \\ \Rightarrow Ae^x + Be^{-x} = xy - x^2 \end{bmatrix}$$

Again differenting w.r.t. x, we get

$$\frac{dy}{dx} = c.2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) \quad \text{... (vi)}$$

$$\Rightarrow \frac{d^3y}{dx^3}+\frac{dy}{dx} = \frac{d^2y}{dx^2}+y \text{ [from Eq. (v)]}$$

Which is a differential equation of order 3

Sol 2: (C) (a) y = 2

$$\Rightarrow \left(x-y\frac{dy}{dx}\right)$$

On putting in equation, (i),

$0^2 - x(0) + y = 0$

$\Rightarrow$ y = 0 which is not satisfied.

(b) y = 2x $\Rightarrow \left(2x+2y\frac{dy}{dx}\right)$

on putting equation (i),

$(2)^2 - x . 2 + y = 0$

$\Rightarrow 4 - 2x + y = 0$

$\Rightarrow$ y = 2x – 4 which is not satisfied.

(c) $y = 2x^2 - 4$

$\frac{dy}{dx}$

On putting in equation (i),

$(4x)^2 - x . 4x + y = 0$

$\Rightarrow$ y = 0 which is not satisfied.

Therefore, C is the answer.

Sol 3: (A) Given, $\frac{dy}{dx}$ and y(0) = –1

Which represents linear differential equation of first order

$\therefore$ IF = $\frac{dy}{dx} = e^{-t + \log(1 + t)}$

$= e^{-t}.(1 + t)$

Required solution is

$ye^{-1}(1 + t) = \frac{dy}{dx}(1 + t)dt + c$

$= \frac{d^2y}{dx^2}-\frac{dy}{dx}=\frac{dy}{dx}$

$\Rightarrow ye^{-t}(1 + t) = -e^{-t} + c$

Since, y(0) = –1

$\Rightarrow -1\ e^0(1 + 0) = -e^0 + c$

c = 0;

Putting t = 1, we get y (1) = $\frac{-1}{2}$

Sol 4: (A) Given, $\frac{dy}{dx} = \frac{-\cos x(y+1)}{2+\sin x}$

$$\Rightarrow \frac{dy}{y+1} = \frac{-\cos x}{2+\sin x} - dx$$

On integrating both sides

$\Rightarrow \log(y+1) = -\log(2 + \sin x) + \log c,$

When, x = 0, $y = 1 \Rightarrow c = 4$

$$\Rightarrow \quad y+1 = \frac{4}{2+\sin x}$$

$$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{4}{3} - 1$$

$$\Rightarrow \quad y\left(\frac{\pi}{2}\right) = \frac{1}{3}$$

Sol 5: (A) $x\frac{dy}{dx} = y(\log y - \log x + 1)$

$$\therefore \frac{dy}{dx} = \left(\frac{y}{x}\right)\left(\log\frac{y}{x}+1\right)$$

Put $\frac{y}{x} = t \Rightarrow y = xt \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$\therefore \quad$ t log dx = x dt

$$\Rightarrow \quad \frac{dt}{t\log t} = \frac{dx}{x}$$

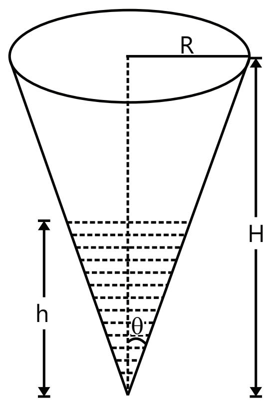
$$\Rightarrow \quad \log\log t = \log x + \log c$$

$$\Rightarrow \quad \log\left(\frac{y}{x}\right) = cx$$

Sol 6: Given, liquid evaporates at a rate proportional to its surface are.

$\Rightarrow \frac{B}{x} \propto -S$ ... (i)

We know, volume of cone = $\frac{d^2y}{dx^2}+x\frac{dy}{dx}$



and surface area = πr^2

or V = $\frac{1}{3}\pi r^2 h$ and S = πr^2 ... (ii)

where $\tan\theta = \frac{dy}{dx}$ and $x\frac{d^2y}{dx^2}+1.\frac{dy}{dx}+\frac{dy}{dx} = \tan\theta$ ... (iii)

From equation (ii) and (iii), we get

V = $\frac{1}{3}\pi r^3 \cot\theta$ and S = πr^2 ... (iv)

On substituting equation (iv) in equation (i), we get

$\frac{1}{3}\cot\theta 3r^2 \frac{dr}{dt} = -k\pi r^2$

$\Rightarrow \cot\theta \; x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 = -k\int_0^T dt$

$\Rightarrow \cot\theta(0 - R) = -k(T - 0)$

$\Rightarrow R\cos\theta = kT$

$\Rightarrow H = kT$ [from Equation (iii)]

$\Rightarrow T = y.\sqrt{x^2+1} = \log\left[\sqrt{x^2+1}-x\right]$

$\therefore$ Required time after which the cone is empty, $T = \frac{dy}{dx}$

Sol 7: Since, the length of tangent = $\frac{dy}{dx} = 1$

$\Rightarrow \frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1-x}} = 1 \quad \therefore \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \pm \frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

$\Rightarrow \frac{dy}{dx} = \frac{a\sin(\log x)}{x}+\frac{b\cos(\log x)}{x} = \pm x + c$

Put y = $\sin\theta$

$\Rightarrow dy = \cos\theta d\theta \qquad \therefore x\frac{dy}{dx} = \pm x + c$

$\Rightarrow x\frac{d^2y}{dx^2}+1.\frac{dy}{dx} = -\frac{a\cos(\log x)}{x}-\frac{b\sin(\log x)}{x} = \pm x + c$

Again put $\cos\theta = t \Rightarrow -\sin\theta d\theta = dt$

$-\int\frac{t^2}{1+t^2}dt = \pm x + c$

$\therefore - x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = \pm x + c$

$\Rightarrow t - \log x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = \pm x + c$

$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = \pm x + c$

Sol 8: (C) $y(1 + xy)\,dx = xdy \Rightarrow ydx - xdy + xy^2dx = 0$

$y^2 d\left(\frac{x}{y}\right)+xy^2dx = 0 \Rightarrow \frac{x}{y}+\frac{x^2}{y} = c$ ... (i)

Since, (1,-1) satisfies the above equation

$-1+\frac{1}{2} = c \Rightarrow c = -\frac{1}{2}$

Put in (i) x = $-\frac{1}{2}$

$\frac{-\frac{1}{2}}{y}+\frac{\frac{1}{4}}{2} = -\frac{1}{2} \Rightarrow \frac{-1}{2y} = \frac{-1}{2}-\frac{1}{8}$

$\Rightarrow \frac{1}{2y} = \frac{5}{8}; \quad \Rightarrow \quad y = \frac{4}{5}$

Sol 9: (C)

$\frac{dy}{dx}+\frac{y}{x\log x} = 2$ at $x = 1$; $y = 0$

$I.F = \int_e \frac{1}{x\log x}dx = e^{\log(\log x)} = \log x$

$\Rightarrow y(\log x) = \int 2(\log x)dx$

$\Rightarrow y(\log x) = 2[x\log - x] + c$

At $x=1$, $c=2$ $x=e$

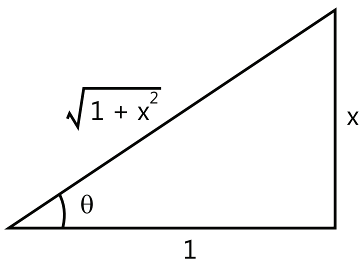
$y = 2(e - e) + 2 \Rightarrow y = 2$

Sol 10: (A)

$y = \sec(\tan^{-1} x)$

Let $\tan^{-1} x = \theta$

$x = \tan\theta$



$\Rightarrow y = \sec\theta$

$\Rightarrow\ y = \sqrt{1+x^2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1+x^2}}.2x$

At x = 1

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2}}$

Sol 11: (C)

$$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d}{dy}\left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{dy}{dx}\right)^{-2}\frac{1}{\left(\frac{dy}{dx}\right)}\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

Sol 12: (D) $\cos x\, dy = y(\sin x - y)\, dx$

$\dfrac{dy}{dx} = y\tan x - y^2(\sec x)$

$\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y}\tan x = -\sec x$

Let $\dfrac{1}{y} = t \ \Rightarrow\ -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$-\dfrac{dy}{dx} - t\,(\tan x) = -\sec x \Rightarrow \dfrac{dt}{dx} + (\tan x)t = \sec x$

$\text{I.F.} = e^{\int \tan x\, dx} = \sec x$

Solution is $t(\text{I.F}) = \int(\text{I.F})\ \sec x\, dx$

$\dfrac{1}{y}\sec x = \tan x + c$

Sol 13: (D) $y = c_1 e^{c_2 x}$ ... (i)

$y' = c_2 c_1 e^{c_2 x}$

$y' = c_2 y$ ... (ii)

$y'' = c_2 y'$

From (ii)

$c_2 = \dfrac{y'}{y}$

So, $y'' = \dfrac{(y')^2}{y} \Rightarrow yy'' = (y')^2$

Sol 14: (D) Y = vx

$\dfrac{dy}{dx} = v + x\dfrac{dy}{dx}$

$v + x\dfrac{dv}{dx} = 1 + v$

$\Rightarrow dv = \dfrac{dx}{x}$

$\therefore v = \log x + c$

$\Rightarrow \dfrac{y}{x} = \log x + c$

Since, y (i) = 1, we have

$y = x\log x + x$

JEE Advanced/Boards

Exercise 1

Sol 1: (i) $\dfrac{dy}{dx} = \dfrac{x^2 + xy}{x^2 + y^2}$

$\dfrac{dy}{dx} = \dfrac{1 + \frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2}$

Put $\dfrac{y}{x} = v$

$\therefore v + x\dfrac{dv}{dx} = \dfrac{1+v}{1+v^2}$

$\therefore x\dfrac{dv}{dx} = \dfrac{1-v^3}{1+v^2} \ \Rightarrow \int \dfrac{(1+v^2)}{(1-v^3)}dv = \int \dfrac{dx}{x}$

$\Rightarrow \int \dfrac{1}{(1-v^3)}dv + \left(-\dfrac{1}{3}\right)\int \dfrac{-3v^2}{(1-v^3)}dv = \log x$

$$\Rightarrow \frac{1}{3}\int\left(\frac{1}{(1-v)}+\frac{v+2}{(v^2+v+1)}\right)dv$$

$$-\frac{1}{3}\log(1-v^3)=\log x+c$$

$$=-\frac{1}{3}\log(1-v)+\frac{1}{6}\int\left(\frac{2v+1+3}{v^2+v+1}\right)dv$$

$$=\frac{1}{3}\log(x^3-y^3)-\log x+\log x$$

$$-\frac{1}{3}\log(1-v)\frac{3}{6}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}\log(x^3-y^3)\ +\ \frac{1}{3}\log(x-y)\ -\ \frac{1}{3}\log x+c$$

$$\Rightarrow \frac{1}{6}\log(y^2+xy+x^2)-\frac{1}{3}\log x+\frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$=\frac{1}{3}\log(x^3-y^3)+\frac{1}{3}\log(x-y)-\frac{1}{3}\log x+c$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\frac{(2y+x)}{\sqrt{3}x}$$

$$=\log(x^3-y^3)^{1/3}(x-y)^{1/3}(y^2+xy+x^2)^{-1/6}+c$$

$$\therefore (x-y)^{2/3}(y^2+xy+x^2)^{1/6}=e^{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)}$$

(ii) $$\frac{dy}{dx}=\frac{1-3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3-3\left(\frac{y}{x}\right)}$$

Put $\frac{y}{x}=v$

$$v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

or $$x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

$$\therefore \int\frac{v^3-3v}{1-v^4}dv=\int\frac{dx}{x}$$

$$\Rightarrow\left(-\frac{1}{4}\right)\int\frac{-4v^3}{1-v^4}dv-3\int\frac{v}{1-v^4}dv=\log x+c$$

$$\Rightarrow -\frac{1}{4}\log(1-v^4)-\frac{3}{2}\int\frac{dt}{(1-t^2)}\log x+c$$

$$\Rightarrow -\frac{1}{4}\log(1-v^4)-\frac{3}{2}\times\frac{1}{2}\log\left(\frac{1+t}{1-t}\right)=\log x+c$$

$$-\frac{1}{4}\log(x^4-y^4)+\log x-\frac{3}{4}\ell n\left(\frac{x^2+y^2}{x^2-y^2}\right)=\log x+c$$

$$\therefore \log\left(\frac{x^2+y^2}{x^2-y^2}\right)^{3/4}\log(x^4-y^4)^{1/4}=\log c$$

Or $(x^2+y^2)(x^2-y^2)^{-1/2}=c$

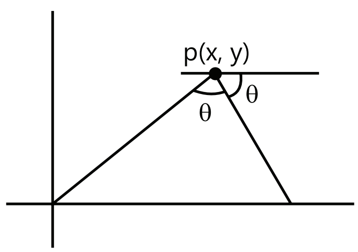
Or $(x^2+y^2)^2=(x^2-y^2)c$

Sol 2: Projection of ordinate on normal

$y\cos\theta$

$y\cos\theta=x$

$\cos\theta=\frac{x}{y}$



$$1-\sin^2\theta=\frac{x^2}{y^2}\Rightarrow\sin\theta=\sqrt{\frac{y^2-x^2}{y^2}}$$

$$\therefore \tan\theta=\frac{dy}{dx}=\frac{\sqrt{y^2-x^2}}{x}=\sqrt{\left(\frac{y}{x}\right)^2-1}$$

$y=vx$

$$v+x\frac{dv}{dx}=\sqrt{v^2-1}$$

$$\therefore \int\frac{1}{\sqrt{v^2-1}-v}dv=\log x+c$$

$$\Rightarrow -\int\left(\sqrt{v^2-1}+v\right)dv=\log x+c$$

$$\Rightarrow -\frac{v^2}{2}-\int\sqrt{v^2-1}dv=\log x+c$$

$$\Rightarrow -\frac{v^2}{2}-\left[\frac{v}{2}\sqrt{v^2-1}-\frac{1}{2}\log[v+\sqrt{v^2-1}\right]$$

$=\log x+c$

$$\Rightarrow \frac{y^2\pm y\sqrt{y^2-x^2}}{x^2}=\log\left[\left[y\pm\sqrt{y^2-x^2}\right].\frac{c^2}{x^3}\right]+c$$

Sol 3: $\therefore\ 2\tan\theta = -\dfrac{0+\frac{y}{x}}{1+0\times\frac{y}{x}}$

or $\tan\theta = \dfrac{dy}{dx} = +\dfrac{y}{2x}$

$\therefore$ 2logy = logx + c

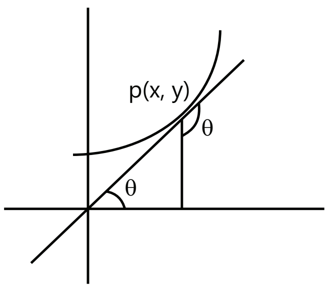
$\therefore\ y^2 = cx$

$\therefore$ This is a parabola

Sol 4: Equation of tangent

(Y – y)

$= \dfrac{dy}{dx}(X - x)$



$\Rightarrow X\dfrac{dy}{dx} - Y + y - x\dfrac{dy}{dx} = 0$

Distance from origin

$\Rightarrow \dfrac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = x$ or $y^2 - 2xy\dfrac{dy}{dx}$

$= \left[1+\left(\dfrac{dy}{dx}\right)^2\right]x^2 - x^2\left(\dfrac{dy}{dx}\right)^2$

or $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy} = \dfrac{1}{2}\left(\dfrac{y}{x} - \dfrac{x}{y}\right)$

$\dfrac{y}{x} = v$ or $v + x\dfrac{dv}{dx} = \dfrac{1}{2}\left(v - \dfrac{1}{v}\right)$

$x\dfrac{dv}{dx} = -\dfrac{v}{2} - \dfrac{1}{2v}$ or $\displaystyle\int \frac{1}{-\frac{1}{2}\left(\frac{v^2+1}{v}\right)}dv = \int\frac{dx}{x}$

$= -\displaystyle\int\left(\frac{2v}{v^2+1}\right)dv = \log x + c$

Or $-\log(v^2 + 1) = \log x + c$ or $\log(x^2 + y^2) + 2\log x$

$= \log x + c$ or $\log x - \log(x^2 + y^2) = c$

for (1, 1)

c = –log2

$\therefore \log 2x = \log(x^2 + y^2)$

or $x^2 + y^2 - 2x = 0$

Sol 5: $y^3\dfrac{dy}{dx} + x + y^2 = 0$

$y^2 + x = a\ \therefore\ \dfrac{da}{dx} = 2y\dfrac{dy}{dx} + 1$

$\Rightarrow 2y^3\dfrac{dy}{dx} + y^2 = y^2\dfrac{da}{dx} = (a - x)\dfrac{da}{dx}$

$\therefore\ x + (a - x)\dfrac{da}{dx} = 0$

$\dfrac{da}{dx} = \dfrac{x}{x-a} = \dfrac{1}{1-\frac{a}{x}}$ $\qquad \because a = vx$

$v + x\dfrac{dv}{dx} = \dfrac{1}{1-v} = \displaystyle\int\frac{(1-v)}{(v^2 - v + 1)}dv = \log x + c$

$= \displaystyle\int\frac{1}{\left(v-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dv + \int\frac{(-v)}{(v^2 - v + 1)}dv = \log x + c$

$= \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{\left(v-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} - \dfrac{1}{2}\log(v^2 - v + 1)$

$-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{v-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \log x + c$

$\Rightarrow \dfrac{1}{2}\log|x^2 + a^2| - \tan^{-1}\left(\dfrac{a}{x}\right) = c$

Where $a = x + y^2$

Sol 6: $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{\cos\frac{y}{x} + \frac{y}{x}\sin\frac{y}{x}}{\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}}\right]$

Put $\dfrac{y}{x} = v$

$v + x\dfrac{dv}{dx} = \dfrac{v\cos v + v^3\sin v}{v\sin v - \cos v}$

or $x\dfrac{dv}{dx} = \dfrac{2v\cos v}{v\sin v - \cos v}$

or $\displaystyle\int\frac{v\sin v - \cos v}{2v\cos v}dv = \int\frac{dx}{x}$

$\dfrac{1}{2}\left[\displaystyle\int\tan v\,dv - \int\frac{1}{v}dv\right] = \log x + c$

$\frac{1}{2}\log|\sec v| - \frac{1}{2}\log v = \log x + c$

$\log\left|\frac{\sec v}{v}\right| = \log x^2 + 2\log c$

or $\log\left|\frac{\sec\frac{y}{x}}{\frac{y}{x}\times x^2}\right| = \log c^2$ or $xy\cos\frac{y}{x} = c$

Sol 7: Equation of tangent

$y - \frac{dy}{dx}x + x\frac{dy}{dx} - y = 0$

Intercept at y-axis $\Rightarrow Y = y - x\frac{dy}{dx}$

$\therefore y - x\frac{dy}{dx} = \sqrt{x^2 + x^2\left(\frac{dy}{dx}\right)^2}$

or $y^2 - 2xy\frac{dy}{dx} = x^2$ or $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$\Rightarrow x^2 + y^2 = cx$

Sol 8: $(x - y)dy = (x + y + 1)dx$

$\frac{dy}{dx} = \frac{x+y+1}{x-y}$

Put $x = X + h$

$y = Y + k$

$X + Y + h + k + 1$

$X - Y + h - k$

$\therefore h + k + 1 = 0$

$h - k = 0$

$\Rightarrow h = k = -\frac{1}{2}$ $\quad\therefore \frac{dy}{dx} = \frac{X+Y}{X-Y}$

Put $\frac{Y}{X} = v$

$\therefore v + X\frac{dv}{dx} = \frac{1+v}{1-v}$

or $X\frac{dv}{dX} = \frac{1+v^2}{1-v}$ or $\int\left(\frac{1-v}{1+v^2}\right)dv = \log X + c$

$\Rightarrow \tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log X + c$

$\Rightarrow \tan^{-1}\frac{Y}{X} - \frac{1}{2}\log\left(1 + \frac{Y^2}{X^2}\right) = \log X + c$

or $\tan^{-1}\frac{Y}{X} - \log\sqrt{X^2 + Y^2} = c$

or $\sqrt{x^2 + y^2} = e^{c\tan^{-1}\frac{y}{x}}$

or $\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2} = e^{c\tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right)}$

Sol 9: $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

$x = X + h$

$y = Y + k$

$\therefore h + 2k - 3 = 0$

$2h + k - 3 = 0$

$\therefore h = 1, k = 1$

$\therefore x = X + 1, y = Y + 1$

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$

$Y = vX$

$V + X\frac{dv}{dx} = \frac{1+2v}{2+v}$

or $X\frac{dv}{dx} = \frac{1-v^2}{2+v}$ or $\int\left(\frac{2+v}{1-v^2}\right)dv = \log X + c$

$\log\frac{1+v}{1-v} + \left(-\frac{1}{2}\right)\log(1 - v^2) = \log cX$

or $\log\frac{X+Y}{X-Y} - \frac{1}{2}\log(X^2 - Y^2) + \log X = \log X + c$

or $\log\left(\frac{X+Y}{X-Y}\times\frac{1}{\sqrt{X^2 - Y^2}}\right) = c$

$\therefore \sqrt{X+Y} = (X - Y)^{3/2}C$

or $X + Y = (X - Y)^3 C$

$(X + Y - 2) = c(X - Y)^3$

or $(X + Y - 2) = c(Y - X)^3$

Sol 10: $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

$x = X + h, y = Y + k$

$h + k + 5 = 0$

$k - h + 1 = 0$

$\therefore$ k = –3

h = –2

$\therefore \frac{dy}{dx} = \frac{Y-X}{Y+X}$

Put y = vX

$\therefore v + X\frac{dv}{dX} = \frac{v-1}{v+1}$

$X\frac{dv}{dx} = \frac{-1-v^2}{1+v} = -\frac{(1+v^2)}{1+v}$

or $-\int \frac{(1+v)}{1+v^2}dv = \int \frac{dX}{X}$

$\Rightarrow -\tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log X + c$

$\Rightarrow -\tan^{-1}\frac{Y}{X} - \log\sqrt{(X^2+Y^2)} = c$

$\therefore \tan^{-1}\frac{y+3}{x+2} + \log c\sqrt{(x+2)^2+(y+3)^2} = 0$

Sol 11: $\frac{dy}{dx} = \frac{x+y+1}{2(x+y)+3}$

x + y = v

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$

$\frac{dv}{dx} = \frac{3v+y}{2v+3} \Rightarrow \int \frac{2v+3}{3v+4}dv = \int dx$

or $\int \left(\frac{\frac{2}{3}(3v+4)+\frac{1}{3}}{3v+4} \right) dv = x + c$

$\frac{2}{3}v + \frac{1}{9}\log\left(v+\frac{4}{3}\right) = x + c$

or $\frac{2}{3}(x+y) + \frac{1}{9}\log(3(x+y)+4) = x + c$

or $\frac{1}{9}\log\left(x+y+\frac{4}{3}\right) = \frac{x}{3} - \frac{2}{3}y + c$

$\log\left(x+y+\frac{4}{3}\right) = 3(x-2y) + \log c$

or $x + y + \frac{4}{3} = ce^{3(x-2y)}$

Sol 12: $\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y-1)^2}$

x = X + h   y = Y + k

k + 2 = 0 $\therefore$ k = –2

h + k – 1 = 0 and h = 3

$\therefore \frac{dY}{dX} = \frac{2Y^2}{(X+Y)^2} = \frac{2\left(\frac{Y}{X}\right)^2}{\left(1+\frac{Y}{X}\right)^2}$

Putting $\frac{Y}{X} = v$

$v + X\frac{dv}{dx} = \frac{2v^2}{(1+v)^2}$

$\therefore X\frac{dv}{dx} = \frac{2v^2 - v(1+v^2+2v)}{(1+v)^2} = -\frac{v(1+v^2)}{(1+v)^2}$

or $\int -\frac{(1+v)^2}{v(1+v^2)}dv = \log X + c$

$-\int \frac{1+v^2+2v}{v(1+v^2)}dv = \log X + c$

$\Rightarrow -\int\left(\frac{1}{v}+\frac{2}{(1+v^2)}\right)dv = \log X + c$

$-\int\left(\frac{1}{v}+\frac{2}{(1+v^2)}\right)dv = \log X + c$

$\Rightarrow -\log v - 2\tan^{-1}v = \log X + c$

$\therefore \log Y + 2\tan^{-1}\frac{Y}{X} = C$

or $Y = ce^{-2\tan^{-1}\frac{Y}{X}}$ or $(y+2) = ce^{-2\tan^{-1}\frac{y+2}{x-3}}$

Sol 13: Equation of tangent

$Y - y = X\frac{dy}{dx} - x\frac{dy}{dx}$

Equation of normal

$Y - y = -\frac{dx}{dy}(X-x)$

Distance of tangent from origin = $\frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$

Distance of normal from origin = $\dfrac{x\dfrac{dx}{dy}+y}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}$

$\therefore y - x\dfrac{dy}{dx} = \pm\left(x\dfrac{dx}{dy}+y\right)\dfrac{dy}{dx}$

$\therefore y - x\dfrac{dy}{dx} = \pm\left(x + y\dfrac{dy}{dx}\right)$

or $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ or $\dfrac{dy}{dx} = -\left(\dfrac{y+x}{y-x}\right) = \dfrac{y+x}{x-y}$

Put y = vx

$v + x\dfrac{dv}{dx} = \dfrac{v-1}{v+1}$ or $v + x\dfrac{dv}{dx} = -\left(\dfrac{v+1}{v-1}\right)$

$x\dfrac{dv}{dx} = -\dfrac{1-v^2}{v+1}$ $x\dfrac{dv}{dx} = -\dfrac{(v^2+1)}{v-1}$

or $\int -\dfrac{(v+1)}{(1+v^2)}dv = \log x + c$

or $\int -\dfrac{(v-1)}{(v^2+1)}dv = \log x + c$

$\Rightarrow -\dfrac{1}{2}\log(1+v^2) \pm \tan^{-1}v = \log x + c$

$-\log\sqrt{(x^2+y^2)} \pm \tan^{-1}\dfrac{y}{x} = \log c$

or $\sqrt{x^2+y^2} = ce^{\pm\tan^{-1}\frac{y}{x}}$

Sol 14: $\dfrac{dy}{dx} - y = 1 - e^{-x}$

If = $e^{\int -1dx} = e^{-x}$

$(e^{-x}y) = \int\left(e^{x} - e^{-2x}\right)dx$

$e^{-x}y = -e^{-x} + \dfrac{1}{2}e^{-2x} + c$

for x = 0

$y_0 = -1 + \dfrac{1}{2} + c$

$\therefore c = y_0 + \dfrac{1}{2}$

$\therefore y = -1 + \dfrac{1}{2}e^{-x} + ce^{x}$

For $x \to \infty$ and y to be finite

c = 0

$\therefore y_0 + \dfrac{1}{2} = 0$

$y_0 = -\dfrac{1}{2}$

Sol 15: $y' + 2ty = t^2$

I. F. = $e^{\int 2tdt} = e^{t^2}$

$\therefore e^{t^2}y = \int t^2e^{t^2}dt$

$\therefore y = \dfrac{1}{e^{t^2}}\int t^2e^{t^2}dt$

$\lim\limits_{t\to\infty}\dfrac{y}{t} = \lim\limits_{t\to\infty}\dfrac{1}{te^{t^2}}\int t^2e^{t^2}dt = \lim\limits_{t\to\infty}\dfrac{t^2e^{t^2}}{e^{t^2}+2t^2e^{t^2}}$

$= \lim\limits_{t\to\infty}\dfrac{t^2}{1+2t^2} = \lim\limits_{t\to\infty}\dfrac{1}{\dfrac{1}{t^2}+2} = \dfrac{1}{2}$

Sol 16: $\dfrac{dy}{dx} + \dfrac{x}{1+x^2}y = \dfrac{1}{2x(1+x^2)}$

I. F. = $e^{\int\frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ell n(1+x^2)} = \sqrt{1+x^2}$

$\therefore \left(\sqrt{1+x^2}\right)y = \int\dfrac{1}{2x\sqrt{1+x^2}}dx$

$\Rightarrow$ Put x = tanθ

$\Rightarrow dx = \sec^2\theta d\theta = \int\dfrac{\sec^2\theta d\theta}{2\tan\theta\sec\theta}$

$= \dfrac{1}{2}\int \text{cosec}\theta d\theta = \dfrac{1}{2}\log|\text{cosec}\theta - \cot\theta| = \dfrac{1}{2}\log\left|\dfrac{1-\cos\theta}{\sin\theta}\right|$

$= \dfrac{1}{2}\log\tan\left(\dfrac{\theta}{2}\right) + c$

$\therefore \left(\sqrt{1+x^2}\right)y = \dfrac{1}{2}\left\{\log\tan\left(\dfrac{1}{2}\tan^{-1}x\right) + c\right\}$

Sol 17: $\dfrac{dy}{dx} + \dfrac{2x}{(1-x^2)}y = \dfrac{x}{(1-x^2)^{1/2}}$

I. F. = $e^{\int\frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \dfrac{1}{(1-x^2)}$

$\therefore \left(\dfrac{1}{1-x^2}\right)y = \int\dfrac{x}{(1-x^2)^{3/2}}dx$

$= \left(-\dfrac{1}{2}\right)\int\dfrac{-2x}{(1-x^2)^{3/2}}dx = \left(-\dfrac{1}{2}\right)\int\dfrac{dt}{t^{3/2}}$

Put $1 - x^2 = t$

$-2x\,dx = dt$

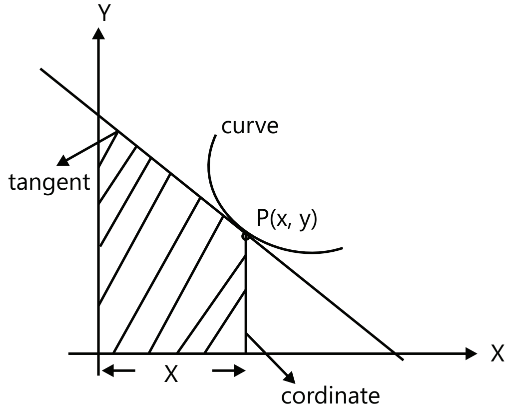
$$= \left(-\frac{1}{2}\right)\frac{1}{\left(-\frac{1}{2}\right)} \times \frac{1}{t^{1/2}} = \frac{1}{\sqrt{1-x^2}} + c$$

$$y = \sqrt{1-x^2} + c(1 - x^2)$$

Sol 18: (i) Equation of tangent

$$(Y - y) = \frac{dy}{dx}(X - x)$$

y-intercept



$$\Rightarrow Y = y - x\frac{dy}{dx}$$

$$\therefore \quad A = \frac{1}{2}x\left[y + y - x\frac{dy}{dx}\right] = \frac{1}{2}x^2$$

or $y + y - x\frac{dy}{dx} = x$

$$x\frac{dy}{dx} - 2y + x = 0$$

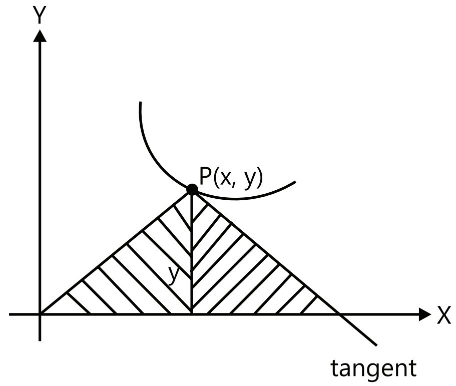
or $\frac{dy}{dx} - \frac{2}{x}y = -1$

$$\text{I. F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

$$\frac{1}{x^2}y = \int -\frac{1}{x^2}dx = \frac{1}{x} + c$$

$\therefore y = x + cx^2$

(ii) x-interupt because of tangent



$$= x - y\frac{dx}{dy}$$

$$\therefore A = \frac{1}{2} \times y \times \left(x - y\frac{dx}{dy}\right) = 2$$

$$xy - y^2\frac{dx}{dy} = 4$$

or $\frac{dx}{dy} - \frac{x}{y} + \frac{4}{y^2} = 0$

or $\frac{dx}{dy} - \frac{x}{y} = \frac{-4}{y^2}$

$$\text{I. F.} = e^{\int -\frac{1}{y}dy} = \frac{1}{y}$$

$$\left(\frac{1}{y}.x\right) = \int -\frac{4}{y^3}dy = \frac{-4y^{-2}}{-2} + c$$

$$\frac{x}{y} = \frac{2}{y^2} + c$$

$$x = \frac{2}{y} + cy$$

For $x = 2$, $y = 1$

$\therefore 2 = 2 + c \times 1 \Rightarrow c = 0$

$\therefore xy = 2$

Sol 19: $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

$$\frac{dy}{dx} - \frac{(x-2)}{(x-1)x}y = \frac{x^2(2x-1)}{(x-1)}$$

$$\text{I. F.} = e^{-\int \frac{(x-2)}{(x-1)x}dx} = e^{-\int\left(\frac{1}{x} - \frac{1}{x(x-1)}\right)dx} = e^{-\int\left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{x}\right)dx}$$

$$= e^{-\int\left(\frac{2}{x} - \frac{1}{x-1}\right)dx} = e^{-[2\log x - \log(x-1)]} = \frac{x-1}{x^2}$$

$\therefore \frac{(x-1)}{x^2} y = \int (2x-1)dx$

$\therefore \frac{(x-1)y}{x^2} = x^2 - x + c$

$y(x-1) = x^2(x^2 - x + c)$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$

Let y = vx

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = v - \cos^2 v$

or $\int \sec^2 v dv = \int -\frac{1}{x}dx \Rightarrow \tan v = -\log x + c$

or $\tan\frac{y}{x} = -\log x + c$

$\therefore$ This curve is passing through $\left(1, \frac{\pi}{4}\right)$

$\therefore \tan\frac{\pi}{4} = c \Rightarrow c = 1$

$\therefore y = x\tan^{-1}(1 - \log x)$

or $y = x\tan^{-1}\left(\log\frac{e}{x}\right)$

Sol 2: (B) $\frac{dy}{dx} - y = \cos x - \sin x$

I. F. $= e^{\int -1dx} = e^{-x}$

$\therefore \int d(e^{-x}y) = \int (\cos x - \sin x)e^{-x}dx$

$e^{-x}y = -\cos x e^{-x} - \int (-\sin x)(-e^{-x})dx - \int \sin x e^{-x}dx$

$= -\cos x e^{-x} - 2\int \sin x e^{-x}dx = -\cos x e^{-x} - 2$

$\left[-\sin x e^{-x} + \int \cos x e^{-x}dx\right]$

$= -\cos x e^{-x} + 2\sin x e^{-x} - 2\int \cos x e^{-x}dx$

Also

$e^{-x}y = \int \cos x e^{-x}dx - \int \sin x e^{-x}dx$

$= \int \cos x e^{-x}dx - \left[-\sin x e^{-x} + \int \cos x e^{-x}dx\right]$

$e^{-x}y = \sin x e^{-x} + c$

$\therefore y = \sin x + ce^{+x}$

As $x \to \infty$

$\therefore y \to \sin x$

$\therefore y = \sin x$

$\therefore y = f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$

Sol 3: (C) $2x^4y\frac{dy}{dx} + y^4 = 4x^6$

$\frac{dy}{dx} = \frac{4x^6 - y^4}{2x^4y}$

$y = u^m$

$\frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$

$\therefore mu^{m-1}\frac{du}{dx} = \frac{4x^6 - u^{4m}}{2x^4u^m}$

u = x

$\therefore mx^{m-1}\frac{dx}{dx} = \frac{4x^6 - x^{4m}}{2x^4x^m} = \frac{x^6(4 - x^{4m-6})}{2x^{4+m}}$

$\therefore 6 = 4 + 2m - 1$

$\therefore m = \frac{3}{2}$

Sol 4: (A) $x^2\frac{dy}{dx}\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$

$\therefore \frac{dy}{dx} - \frac{\tan\left(\frac{1}{x}\right)}{(x^2)}y = \frac{-\sec\frac{1}{x}}{(x^2)}$

$\therefore$ I. F. $= e^{\int -\frac{\tan\left(\frac{1}{x}\right)}{x^2}dx}$

Put $\frac{1}{x} = t$

$\Rightarrow -\frac{1}{x^2}dx = dt$

$\therefore$ I. F. $= e^{\int \tan t dt} = e^{\ell n \sec t} = \sec t = \sec\frac{1}{x}$

$\therefore \int d\left(\sec\frac{1}{x}y\right) = -\int \frac{\sec^2\frac{1}{x}}{x^2}dx$

$\Rightarrow \sec\frac{1}{x}y = +\int \sec^2 t dt$ Put $\frac{1}{x} = t$

$-\frac{1}{x^2} = dx = dt$

$\sec\frac{1}{x}y = \tan\frac{1}{x} + c\cos\frac{1}{x}$

at $x \to \infty, y \to -1$

$-1 = 0 + c \therefore = -1$

$\therefore y = \sin\frac{1}{x} - \cos\frac{1}{x}$

Sol 5: (A) P(x, y)

Equation of nor mal $(Y - y) = -\frac{dx}{dy}(X - x)$

$\therefore$ x-axis intercept $= x + y\frac{dy}{dx}$

$|r| = \sqrt{x^2 + y^2}$, $r = xi + yj$

$\therefore x\left(x + y\frac{dy}{dx}\right) = 2(x^2 + y^2)$ [given]

$\therefore xy\frac{dy}{dx} = x^2 + 2y^2$

or $\frac{dy}{dx} = \frac{x}{y} + \frac{2y}{x}$

Put y = vx

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{1}{v} + 2v$

or $x\frac{dv}{dx} = \left(v + \frac{1}{v}\right)$

$\therefore \int \frac{v}{v^2 + 1}dv = \int \frac{dx}{x}$ or $\frac{1}{2}\log(v^2 + 1)$

$= \ell nx + c$

or $\frac{1}{2}\log(x^2 + y^2) = 2\log x + c$

for x = 1, y = 0

$\therefore c = 0$

$\therefore x^2 + y^2 = (x^2)^2 = x^4$

Sol 6: (C) $y = c(x - c)^2$

$\frac{dy}{dx} = 2c(x - c)$

$\left(\frac{dy}{dx}\right)^2 = 4c^2(x - c)^2 = 4cy$

$\therefore c = \frac{(y')^2}{4y}$

$\therefore y = \frac{(y')^2}{4y}\left(x - \frac{(y')^2}{4y}\right)^2$

$4y^2 = \left(x(y') - \frac{(y')^3}{4y}\right)^2$

$\therefore$ Degree = 3

Order = 1

Sol 7: (D) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\log\left(\frac{d^2y}{dx^2}\right)$

This equation is not a polynomial equation in y′, y″ so degree of such a differential equation cannot be determined.

Sol 8: (C) $y^2 = 4a(x + a)$

$2y\frac{dy}{dx} = 4a$

Change $\frac{dy}{dx} \to -\frac{dx}{dy}$

$\therefore -2y\frac{dx}{dy} = 4a$

$\therefore \int \frac{dy}{y} = \int -\frac{1}{2a}dx$

$\Rightarrow \log y = -\frac{x}{2a} + c$

or $\log cy = -\frac{x}{2a}$ or $y = ce^{-\frac{x}{2a}}$

Sol 9: (D) $\frac{dy}{dx} = 4e^{4x} - 2e^{-x}$

$\frac{d^2y}{dx^2} = 16e^{4x} + 2e^{-x}$ and $\frac{d^3y}{dx^3} = 64e^{4x} - 2e^{-x}$

$\therefore \frac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = \frac{64e^{4x} - 2e^{-x} - 13(4e^{4x} - 2e^{-x})}{e^{4x} + 2e^{-x}}$

$= \frac{12e^{4x} + 24e^{-x}}{4x + 2e^{-x}} = 12$

Sol 10: (B) $\frac{dy}{dx}-\left(\frac{1}{x}\right)y = \frac{f\left(\frac{y}{x}\right)}{f'\left(\frac{y}{x}\right)}$

I. F. $e^{\int -\frac{1}{x}dx} = \frac{1}{x}$ $\therefore d\left(\frac{1}{x}y\right) = \frac{1}{x}\frac{f\left(\frac{y}{x}\right)}{f'\left(\frac{y}{x}\right)}$

or $\int \frac{f'\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \int \frac{1}{x}dx$

$\Rightarrow \log f\left(\frac{y}{x}\right) = \log x + c$

$\therefore f\left(\frac{y}{x}\right) = cx$

Sol 11: (B) $\frac{dy}{dx} = \frac{x^2+2xy+y^2}{x^2-2xy+2y^2}$

$\frac{dy}{dx} = \frac{1+2\frac{y}{x}+\left(\frac{y}{x}\right)^2}{1-\frac{2y}{x}+2\left(\frac{y}{x}\right)^2}$

Put y = vx

$\therefore v + x\frac{dv}{dx} = \frac{1+2v+v^2}{1-2v+2v^2}$

$x\frac{dv}{dx} = \frac{1+2v+v^2-v+2v^2-2v^3}{1-2v+2v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v+3v^2-2v^3}{1-2v+2v^2}$

$\therefore \int \frac{(1-2v+2v^2)}{1+v+3v^2-2v^3}dv = \int \frac{dx}{x}$

$\frac{1}{3}\int\left(\frac{6v^2-6v-1+4}{1+v+3v^2-2v^3}\right)dv = \log x$

$\frac{1}{3}\log(1 + v + 3v^2 - 2v^3) + \frac{4}{3}\int \frac{dv}{(1+v+3v^2-2v^3)} = \log x$

Rather than solving this integration we can solve this problem in another method

Line joining origin and A(1, 2)

$\Rightarrow (y - 0) = 2(x - 0)$

y = 2x

Let $\frac{dy}{dx}$ for y = 2x

$\frac{dy}{dx} = \frac{x^2+2x.2x+(2x)^2}{x^2-2x.2x+(2x)^2} = \frac{9}{5}$

$\therefore$ If y = 2x cuts c_1 or c_2 $\frac{dy}{dx} = \frac{9}{5}$

$\therefore$ If y = 2x cuts c_2 at b then also slope of tangent at B will be equal to $\frac{9}{5}$.

Sol 12: (C) $\ell n\left(\frac{dy}{dx}\right) = 4x - 2y - 2$

$\frac{dy}{dx} = e^{4x-2y-2} = \frac{e^{4x}}{e^{2y}e^2}$

$\therefore \int e^{2y}dy = \int \frac{e^{4x}}{e^2}dx \Rightarrow \frac{1}{2}e^{2y} = \frac{1}{4}\frac{e^{4x}}{e^2}dx$

For x = 1, y = 1

$\frac{e^2}{2} = \frac{1}{4}\frac{e^4}{e^2} + c \Rightarrow c = \frac{1}{4}e^2$

$\therefore \frac{1}{2}e^{2y} = \frac{1}{4}e^{4x-2} + \frac{1}{4}e^2$

or $2e^{2y} = \frac{e^{4x}}{e^2} + e^2$ or $2e^{2y+2} = e^{4x} + e^4$

Multiple Correct Choice Type

Sol 13: (B, C) $x\left(\frac{dy}{dx}\right) = y\log\left(\frac{y}{x}\right)$

$\frac{dy}{dx} = \frac{y}{x}\log\left(\frac{y}{x}\right)$

Put y = vx

$\therefore v + x\frac{dv}{dx} = v\log v$

$\therefore x\frac{dv}{dx} = v(\log v - 1)$

or $\int \frac{1dv}{V(\log v-1)} = \int \frac{1}{x}dx$

log(logv – 1) = logx + c y = xe. xe^{cx}

C = logc

or logv – 1 = $e^{\ell n(xc)}$

Previous Years' Questions

Sol 1: (A) Given that, $\lim_{t\to x} \frac{t^2f(x)-x^2f(t)}{t-x} = 1$

$\Rightarrow x2f'(x) - 2xf(x) + 1 = 0$

$\Rightarrow x2f'(x) - cx^2 + \frac{1}{3x}$

Since, $f(1) = 1, 1 = c + \frac{1}{3}$

$\Rightarrow c = \frac{2}{3}$

Hence, $f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

Sol 2: (C) Given that, $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$

$\Rightarrow \int \frac{y}{\sqrt{1-y^2}}\,dy = \int dx$

$\Rightarrow -\sqrt{1-y^2} = x + c$

$\Rightarrow (x+c)^2 + y^2 = 1$

Here, centre is (–c, 0); radius = $\sqrt{c^2 - c^2 + 1} = 1$

Sol 3: Equation of tangent to the curve y = f(x) at point (x, y) is

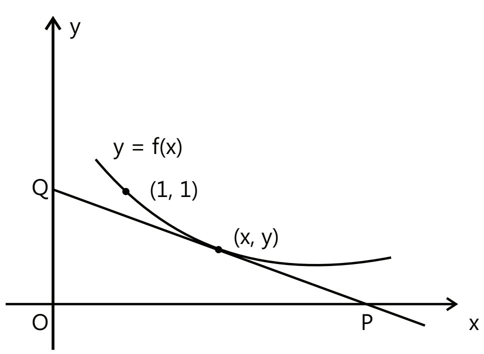
$Y - y = f'(x)(X - x)$ ... (i)

The line (i) meets the x-axis at $p\left(x - \frac{y}{f'(x)}, 0\right)$

And the y-axis at Q(0, y – xf′(x)).

Area of ΔOPQ is

$\frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(x - \frac{y}{f'(x)}\right)(y - xf'(x)) = -\frac{(y - xf'(x))^2}{2f'(x)}$



We are given that area of ΔOPQ = 2, therefore,

$-\frac{(y - xf'(x))^2}{2f'(x)} = 2$

$\Rightarrow (y - xf'(x))^2 + 4f'(x) = 0$

$\Rightarrow (y - px)^2 + 4p = 0$ ... (ii)

Where p = f′(x) = dy/dx.

Since, OQ > 0, y – xf′(x) > 0. Also, note that p = f′(x) < 0.

We can write (ii) as $y - px = 2\sqrt{-p}$

$\Rightarrow y = px + 2\sqrt{-p}$ ... (iii)

Differentiating (iii) with respect to x, we get

$p = \frac{dy}{dx} = p + \frac{dp}{dx}x + 2\left(\frac{1}{2}\right)(-p)^{\frac{-1}{2}}(-1)\frac{dp}{dx}$

$\Rightarrow \frac{dp}{dx}x - (-p)^{\frac{-1}{2}}\frac{dp}{dx} = 0$

$\Rightarrow \frac{dp}{dx}[x - (-p)^{\frac{-1}{2}}] = 0$

$\Rightarrow \frac{dp}{dx} = 0$ or $x = (-p)^{\frac{-1}{2}}$

If $\frac{dp}{dx} = 0$, then p = c where c < 0 [∵ p < 0]

Putting this value in (iii) we get

$y = cx + 2\sqrt{-c}$ ... (iv)

This curve will pass through (1, 1) if

$1 = c + 2\sqrt{-c}$

$\Rightarrow -c - 2\sqrt{-c} + 1 = 0$

$\Rightarrow (\sqrt{-c} - 1)^2 = 0$ or $\sqrt{-c} = 1$

$\Rightarrow -c = 1$ or $c = -1$

Putting the value of c in (iv) we get

y = -x + 2 or x + y = 2

Next, putting $x = (-p)^{\frac{-1}{2}}$ x or –p = x-2 in (iii) we get

$y = \frac{-x}{x^2} + 2\left(\frac{1}{x}\right) = \frac{1}{x}$

$\Rightarrow xy = 1$ (x > 0, y > 0)

Thus, the two required curves are x + y = 2 and xy = 1, (x > 0, y > 0)

Sol 4: $\frac{1}{e^y} + \frac{x^3}{3} \propto V$ for each reservoir

$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right) \propto -V_A \Rightarrow \frac{dv}{a + bf(v)} = -K_1V_A$

(K_1 is the proportional constant)

$$\Rightarrow \int_{V_A}^{V'_A} \frac{dV_A}{V_A} = -K_1 \int_0^1 dt$$

$$\Rightarrow \log \frac{dy}{dx} = a^2 = -K_1 t$$

$$\Rightarrow \left(\frac{dt}{dx} - 1\right) \quad \text{... (i)}$$

Similarly for B,

$$\frac{dt}{dx} = \frac{a^2}{t^2} + 1 \quad \text{... (ii)}$$

On dividing equation (i) by (ii), we get

$$\frac{a^2 + t^2}{t^2}$$

It is given that at t = 0, $V_A = 2V_B$ and at $t = \int \frac{t^2 dt}{t^2 + a^2}$,

$$V'_A = \frac{3}{2} V'_B$$

Thus, $\frac{dy}{dx}$

$$\Rightarrow \frac{x+y-1}{\sqrt{x+y+1}} \quad \text{... (iii)}$$

Now, let at t = t_0 both the reservoirs have some quantity of water. Then, $V'_A = V'_B$

From equation (iii),

$$\left(2t\frac{dt}{dx} - 1\right)$$

$$\Rightarrow \frac{t^2 - 2}{t}$$

$t_0 = \log_{3/4}(1/2)$

Sol 5: Let w(x) = u(x) – v(x) ... (i)

and h(x) = f(x) – g(x)

On differentiation equation (i) w.r.t. x

$$\frac{2tdt}{dx} = \frac{t^2 + t - 2}{t}$$

= {f(x) – p(x).u(x)} – {g(x)

– p(x)v(x)} (given)

= {f(x)–g(x)} – p(x)[u(x)–v(x)]

$$\Rightarrow \int \frac{2t^2 dt}{(t+2)(2t-2)} dt = h(x) - p(x).w(x) \quad \text{... (ii)}$$

$\frac{dy}{dx}$ + p(x)w(x) = h(x) which is linear differential equation.

The integrating factor is given by

$$IF = \frac{dt}{dx} = r(x)(\text{let})$$

On multiplying both sides of equation (ii) of r(x), we get

$$r(x). \int \frac{dt}{8\cos t + 10} = \int dx + p(x)(r(x))w(x) = r(x).h(x)$$

$$\Rightarrow \frac{y}{x} = r(r).h(x)$$

$$\frac{xdy - ydx}{x^2}$$

Now, $r(x) = \frac{y}{x} > 0, \forall x$

And h(x) = f(x) – g(x) > 0 for x > x_1

Thus,

$$\int \frac{dr}{r^2} = \int \cos\theta d\theta, \forall x > x_1$$

r(x)w(x) increases on the interval [x, ⇒]

Therefore, for all x > x_1

r(x)w(x) > r(x_1) w(x_1) > 0

[∵ r(x_1) > 0 and u(x_1) > v(x_1)]

⇒ w(x) > 0 ∀ x > x_1

⇒ u(x) > ∀ (x)A x >x_1

[∵ r(x) > 0]

Hence, there cannot exist a point (x, y) such the x > x_1 and y = u(x) and y = v(x)

Sol 6: Equation of normal at point (x, y) is

$$Y - y = -\frac{-1}{r} = \sin\theta + c \; (X - x)$$

Distance of perpendicular from the origin to Eq. (i)

$$= \frac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$$

Also, distance between P and x-axis is |y|

$$\therefore \frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1 - (x^2 + y^2)}{x^2 + y^2}}$$

$\Rightarrow y^2 + \frac{dx}{dy}.x^2 + 2xy\int \frac{dt}{\sqrt{1-r^2}}$

$= y^2\sqrt{x^2+y^2}$

$\Rightarrow \frac{y}{x}(x^2-y^2)+2xy\frac{xdx+ydy}{\sqrt{x^2+y^2}} = 0$

$\Rightarrow \frac{ydr - xdy}{x^2}$

$\Rightarrow \frac{rdr}{r^2 d\theta}$ or $\frac{r^2}{r\cos\theta}$

But $\int \sec\theta \, d\theta = \int \frac{dr}{r} \Rightarrow x = c,$

where c is a constant.

Since, curve passes through (1, 1) we get the equation of the curve as x = 1

The equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is a homogeneous equation

Put y = vx, $\Rightarrow \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

$v + x\frac{dy}{dx}$

$\Rightarrow \frac{dv}{dx}$

$= \frac{dv}{dx} = -\frac{dx}{x} = \frac{f(x,y)}{g(x,y)}$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dv}{f(v)-v}$

$\Rightarrow c_1 - \log(v^2 + 1) = \log|x|$

$\Rightarrow \log|x| (v^2 + 1) = c_1$

$\Rightarrow \int_c^r y\,dx = \frac{y^3}{x}$

$\Rightarrow x^2 + y^2 = \pm \frac{x.3y^2y' - y^3.1}{x^2}x$

or $x^2 + y^2 = \pm e^c x$

is passing through (1, 1)

$\therefore 1 + 1 = \pm e^c.1 \Rightarrow \pm e^c = 2$

Hence, required curve is $x^2 + y^2 = 2x$

Sol 7: Let X_0 be initial population of the country and Y_0 be its initial food production.

Let the average consumption be a unit. Therefore, food required initially aX_0. It is given

$y_p = aX_0\left(\frac{90}{100}\right) = 0.9aX_0$ ... (i)

Let X be the population of the country in year t.

They, $\frac{dX}{dt}$ = rate of change of population

$= \frac{3}{100}X = 0.03\,X$

$\Rightarrow \frac{dX}{X}dt$

$\Rightarrow \int \frac{dX}{X} = \int 0.03\,dt$

$\Rightarrow \log X = 0.03t + c$

$\Rightarrow X = A.e^{0.03t}$ when $A = e^c$

At t = 0, $X = X_0$, thus $X_0 = A$

$\therefore X = X_0 e^{0.03t}$

Let Y be the food productioninyear t.

Then $Y = Y_0\left(1+\frac{4}{100}\right)^t$

$= 0.9aX_0(1.04)^t$

$\because Y_0 = 0.9aX_0$ [from Eq. (i)]

Food consumption in the year t is $aX_0e^{0.03t}$

Again, $Y - X \geq 0$ (given)

$\Rightarrow 0.9X_0a(1.04)^t > aX_0e^{0.03t}$

$\Rightarrow \frac{(104)^t}{e^{0.03t}} > \frac{1}{0.9} = \frac{10}{9}$

Taking log on both sides, we get

t[log(1.04)–0.03]log10–log9

$\Rightarrow t \geq \frac{\log 10 - \log 9}{\log(1.04) - 0.03}$

Thus, the least integral values of the year n, when the country becomes self-sufficient, is the smallest integer greater than or equal to

$\frac{\log 10 - \log 9}{\log(1.04) - 0.03}$

Sol 8: from given integral equation f(0) = 0

Also, differentiation the given integral equation w.r.t. x

$f'(x) = f(x)$

If $f(x) \neq 0$ $\frac{f'(x)}{f(x)} = 1$

$\Rightarrow \log f(x) = x + C$

$\Rightarrow f(x) = e^c e^x$

$\because f(0) = 0 \Rightarrow e^C = 0,$

a contradiction

$\therefore f(x) = 0, \forall x \in R$

$\Rightarrow f(\log 5) = 0$

Alternate Solution

Given$f(x) = \int_0^x f(t)dt$

$\Rightarrow f(0) = 0$

And $f'(x) = f(x)$

It $f(x) \neq 0$

$\Rightarrow \frac{f'(x)}{f(x)} = 1$

$\Rightarrow \log f'(x) = x + C$

$\Rightarrow f(x) = e^c.e^x$

$\because f(0) = 0$

$\Rightarrow e^c = 0$, a contradiction

$\therefore f(x) = 0 \ \forall x \in R$

$\Rightarrow f(\log 5) = 0$

Sol 9: given, g{f(x)} = x

$\Rightarrow g'\{f(x)\}f'(x) = 1$ ... (i)

If $f(x) = 1 \Rightarrow x = 0, f(0) = 1$

Substitute x = 0 in eq. (1), we get

$g'(1) = \frac{1}{f'(0)}$

$\Rightarrow g'(1) = 2$

$$\left\{ \begin{aligned} &\because f'(x) = 3x^2 + \frac{1}{2}e^{x/2} \\ &\qquad\qquad \Rightarrow f'(0) = \frac{1}{2} \end{aligned} \right\}$$

Alternate solution

Given, $f(x) = x^3 + e^{x/2}$

$\Rightarrow f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

For x = 0, f(0) = 1, $f'(0) = \frac{1}{2}$ and $g(x) = f^{-1}(x)$

Replacing x by f(x), we have

$g(f(x)) = x$

$\Rightarrow g'(f(x)).f'(x) = 1$

Put x = 0, we get

$g'(1) = \frac{1}{f'(0)} = 2$

Sol 10: $\frac{dy}{dx} + y.g'(x) = g(x)g'(x)$

I.F. = $e^{\int g'(x)dx} = e^{g(x)}$

$\therefore$ Solution is $y(e^{g(x)})$

$= \int g(x).g'(x).e^{g(x)}dx + C$

Put g(x) = t, g′(x) = dx = dt

$y(e^{g(x)}) = \int t.e^t dt + C$

$= t.e^t - \int 1.e^t + C$

$= t.e^t - e^t + C$

$ye^{g(x)} = (g(x) - 1)e^{g(x)} + C$ ... (i)

Given, y(0) = 0, g(0) = g(2) = 0

$\therefore$ Equation (i) becomes

$y(0).e^{g(0)} = (g(0) - 1).e^{g(0)} + C$

$\Rightarrow 0 = (-1).1 + C \Rightarrow C = 1$

$\therefore y(x).e^{g(x)} = (g(x)-1)e^{g(x)} + 1$

$\Rightarrow y(2).e^{g(2)} = (g(2)-1)e^{g(2)} + 1,$

$\Rightarrow y(2).1 = (-1).1 + 1$

$y(2) = 0$

Sol 11: (A, C) $(x+2)^2 + y(x+2) = y^2.\frac{dx}{dy}$

$$\Rightarrow \frac{dx}{dy} = \frac{(x+2)^2}{y^2} + \frac{x+2}{y}$$

$$\Rightarrow \frac{1}{(x+2)^2}\frac{dx}{dy} = \frac{1}{y^2} + \frac{1}{y(x+2)}$$

$$\therefore \frac{1}{(x+2)^2}\frac{dx}{dy} - \frac{1}{(x+2)y} = \frac{1}{y^2}$$

$$-\frac{dt}{dy} - \frac{t}{y} = \frac{1}{y^2}$$

$$\therefore \text{ Put } \frac{1}{x+2} = t,\ -\frac{1}{(x+2)^2}\frac{dx}{dy} = \frac{dt}{dy}$$

$$\Rightarrow \frac{dt}{dy} + \frac{t}{y} = -\frac{1}{y^2} \qquad \text{I.F} = e^{\int \frac{1}{y}dy} = y$$

$$t.y = C + \int y\left(-\frac{1}{y^2}\right)dy$$

$$t.y = C - \log y$$

$$\therefore \frac{1}{x+2}.y = C - \log y$$

It passes (1, 3) $\Rightarrow 1 = C + \log 3 \Rightarrow C = 1 + \log(3)$

$$\frac{y}{x+2} = 1 + \log 3 - \log y$$

[A] option is correct.

For Option (C)

$$\frac{(x+2)^2}{(x+2)} = 1 + \log\left(\frac{y}{3}\right)$$

$$x + 1 - \log\left(\frac{3}{y}\right)$$

$$\therefore y = 3e^{-x-1}$$

$\Rightarrow$ Intersect

For Option (D)

$$\frac{(x+3)^2}{4+2} - 1 = -\log\left(\frac{(x+3)^2}{3}\right)$$

$$\therefore \frac{(x+3)^2 - 1}{x+2} = -\log\left\{\frac{(x+3)^2}{3}\right\}$$

$$3e\left(\frac{(x+3)^2 - 1}{-x-2}\right) = (x+3)^2$$

$\Rightarrow$ Will intersect.

$\Rightarrow$ (D) is not correct.

Sol 12: (B, C) Let centre of the circle is (a, a) and radius 'r'

Now equation of circle is $(x - a)^2 + (y - a)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2ax - 2ay + 2a^2 - r^2 = 0$ ... (i)

Differentiation w.r.t. x we get

$X + yy_1 - a - ay_1 = 0$ ... (ii)

$\Rightarrow a \dfrac{x + yy_1}{1 + y_1}$ ... (iii)

Differentiation once again equation (ii) w.r.t. x we get

$I + yy_2 + y_1^2 - ay_2 = 0$ ... (iv)

Using (iii) is (iv) we have

$$\left(1 + yy_2 + y_1^2\right) - \left(\frac{x + yy_1}{1 + y_1}\right)y_2 = 0$$

$$\Rightarrow 1 + (1 + y_1 + y_1^2)y_1 + (y - x)y_2 = 0$$

Hence, $p = y - x$ and $Q = 1 + y_1 + y_1^2$

Sol 13: (B) $\dfrac{dy}{dx} + \dfrac{x}{x^2 - 1}y = \dfrac{x^4 + 2x}{\sqrt{1 - x^2}}$

This is a linear differential equation

$$\text{I.F.} = e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\ln|x^2-1|} = \sqrt{1 - x^2}$$

$\Rightarrow$ solution is

$$y\sqrt{1 - x^2} = \int \frac{x(x^3 + 2)}{\sqrt{1 - x^2}}.\sqrt{1 - x^2}\,dx$$

$$\text{or } y\sqrt{1 - x^2} = \int\left(x^4 + 2x\right)dx = \frac{x^5}{5} + x^2 + c$$

$f(0) = 0 \Rightarrow C = 0$

$$\Rightarrow f(x)\sqrt{1 - x^2} = \frac{x^5}{5} + x^2$$

$$\text{Now, } \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx \text{ (Using property)}$$

$$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}dx = 2\int_0^{\pi/3} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \text{ (Taking } x = \sin\theta)$$

$$= 2\int_0^{\pi/3} \sin^2\theta\,d\theta = 2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/3}$$

$$= 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\sqrt{3}}{8}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Sol 14: (A) $\frac{dy}{dx}=\frac{y}{x}+\sec\frac{y}{x}$ Let y = vx

$\Rightarrow \frac{dv}{\sec v}=\frac{dx}{x}$

$\int \cos v dv=\int\frac{dx}{x}$

$\Rightarrow \sin v = 1nx+c$

$\Rightarrow \sin\left(\frac{y}{x}\right)=1nx+c$

The curve passes through $\left(1,\frac{\pi}{6}\right)$

$\Rightarrow \sin\left(\frac{y}{x}\right)=1nx+\frac{1}{2}$

Sol 15: (A, D) $\frac{dy}{dx}-y\tan x=2x\sec x$

$\Rightarrow \cos x\frac{dy}{dx}+(-\sin x)y=2x$

$\Rightarrow \frac{d}{dx}(y\cos x)=2x$

$\Rightarrow$ y (x) cos x = x^2 + c, where c = 0 since y (0) = 0

when $x=\frac{\pi}{4}, y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$, when $x=\frac{\pi}{3}, y\left(\frac{\pi}{3}\right)=\frac{2\pi^2}{9}$

when $x=\frac{\pi}{4}, y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}+\frac{\pi}{\sqrt{2}}$

when $x=\frac{\pi}{3}, y'\left(\frac{\pi}{3}\right)=\frac{2\pi^2}{3\sqrt{2}}+\frac{4\pi}{3}$

Sol 16:

$6\int_1^x f(t)dt=3xf(x)-x^3 \Rightarrow 6f(x)=3f(x)+3xf'(x)-3x^2$

$\Rightarrow 3f(x)=3xf'(x)-3x^2 \Rightarrow xf'(x)-f(x)=x^2$

$\Rightarrow x\frac{dy}{dx}-y=x^2 \Rightarrow \frac{dy}{dx}-\frac{1}{x}y=x$ ... (i)

$I.F=e\int^{-\frac{1}{x}x}=e-\log_e x$

Multiplying (i) both sides by $\frac{1}{x}$

$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=1 \Rightarrow \frac{d}{dx}\left(y.\frac{1}{x}\right)=1$

Integrating

$\frac{y}{x}=x+c$

Put x = 1, y = 2

$\Rightarrow$ 2 = 1 + c $\Rightarrow$ c = 1 $\Rightarrow$ y = x^2 + x

$\Rightarrow$ f(x) = x^2 + x $\Rightarrow$ f(2) = 6

Note: If we put x = 1 in the given equation we get f(1) = 1/3.

Sol 17: Y – y = m (X – x)

y-intercept (x = 0)

y = y – mXS

Given that y – mx = x^3 $\Rightarrow$ $x\frac{dy}{dx}-y=-x^3$

$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=-x^2$

Integrating factor $e^{-\int\frac{1}{x}dx}=\frac{1}{x}$

$\therefore$ Solution is $y.\frac{1}{x}=\int\frac{1}{x}.(-x^2)dx$

$\Rightarrow f(x)=y=-\frac{x^3}{2}+cx$

Given f(1) = 1 $\Rightarrow$ $c=\frac{3}{2}$

$\therefore f(x)=-\frac{x^3}{2}+\frac{3x}{2}$ $\Rightarrow$ f(-3)=9

Sol 18: $(x-3)^2\frac{dy}{dx}+y=0$

$\int\frac{dx}{(x-3)^2}=-\int\frac{dy}{y}$

$\Rightarrow \frac{1}{x-3}=1n|y|+c$

so domain is R − {3}.

Sol 19: (C) $\int\frac{dx}{x\sqrt{x^2-1}}=\int\frac{dy}{y\sqrt{y^2-1}}$

$\sec^{-1}2=\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)+c$

$\Rightarrow c=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$

$$\Rightarrow \sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$$

$$\Rightarrow y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$$

$$\Rightarrow \cos^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y} + \frac{\pi}{6}$$

$$\Rightarrow \cos^{-1}\frac{1}{y} = \cos^{-1}\frac{1}{x} - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$$