11. ELLIPSE

1. INTRODUCTION

An ellipse is defined as the locus of a point which moves such that the ratio of its distance from a fixed point (called focus) to its distance from a fixed straight line (called directrix, not passing through fixed point) is always constant and less than unity. The constant ratio is denoted by e and is known as the eccentricity of the ellipse.

Ellipse can also be defined as the locus of a point such that the sum of distances from two fixed points (foci) is constant. i.e. SP + S'P = constant where $S_1S^{'}$ are foci (two fixed points), P being a point on it. It has a lot of applications in various fields. One of the most commonly known applications is Kepler's first law of planetary motion, which says that the path of each planet is an ellipse with the sun at one focus.

Illustration 1: Find the equation of the ellipse whose focus is (1, 0) and the directrix x + y + 1 = 0 and eccentricity is equal to $\frac{1}{\sqrt{2}}$. **(JEE MAIN)**

Sol: Using the definition of ellipse we can easily get the equation of ellipse.

Let S (1, 0) be the focus and ZZ' be the directrix. Let P(x, y) be any point on the ellipse and PM be the perpendicular drawn from P on the directrix. Then by definition

SP = e. PM, where e = $\frac{1}{\sqrt{2}}$.

$\Rightarrow SP^2 = e^2PM^2 \qquad \Rightarrow (x-1)^2 + (y-0)^2 = \frac{1}{2}\left\{\frac{x+y+1}{\sqrt{1+1}}\right\}^2$

$\Rightarrow 4\{(x-1)^2 + y^2\} = (x+y+1)^2$

$\Rightarrow 4x^2 + 4y^2 - 8x + 4 = x^2 + y^2 + 1 + 2xy + 2x + 2y$

$\Rightarrow 3x^2 + 3y^2 - 2xy - 10x - 2y + 3 = 0$

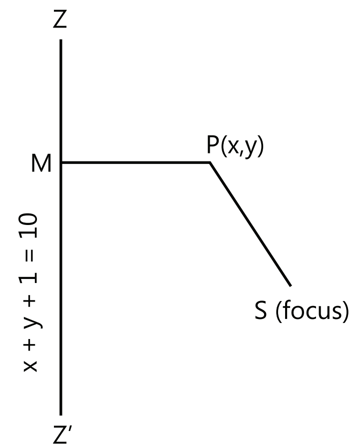


Figure 11.1

Note: The general equation of a conic can be taken as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

This equation represents ellipse if it is non degenerate (i.e. eq. cannot be written into two linear factors)

Condition: $\Delta \neq 0$, $h^2 < ab$. Where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

MASTERJEE CONCEPTS

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

  Degeneracy condition depends on determinant of 3x3 matrix and the type of conic depends on determinant of 2x2 matrix.

- Also the equation can be taken as the intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

Vaibhav Gupta (JEE 2009, AIR 54)

2. STANDARD EQUATION OF ELLIPSE

Let the origin be the centre of the ellipse and the major and minor axis be on the x-axis and y-axis respectively. It means foci lies on x-axis and the coordinates of F_1 are (-c, o) and F_2 be (c, o). Let P be any point (x, y) on the ellipse. By the definition of the ellipse, the sum of the distances from any point P(x, y) to foci F_1 and F_2 = constant.

Let us consider this constant to be 2a for the sake of simplicity.

$$PF_1 + PF_2 = 2a \qquad ...(i)$$

$$PF_1^{\,2} = (x + c)^2 + (y - 0)^2$$

$$\Rightarrow \quad PF_1 = \sqrt{(x+c)^2 + y^2} \qquad ...(ii)$$

Similarly, $PF_2 = \sqrt{(x-c)^2 + y^2}$ ...(iii)

Putting the value of PF_1 and PF_2 in (i) from (ii) and (iii), we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \quad \Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

On squaring, we get

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow 4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$$

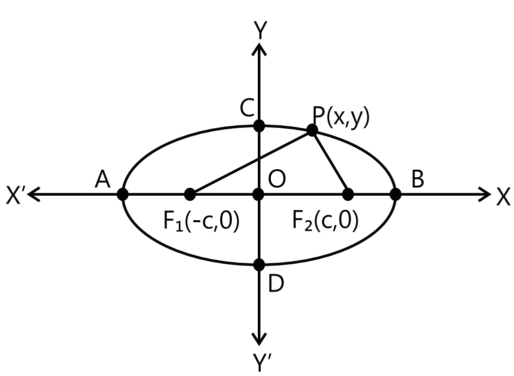


Figure 11.2

Squaring both sides, we get

$$(x-c)^2 + y^2 = a^2 - 2cx + \left(\frac{c}{a}x\right)^2$$

$$\Rightarrow \quad x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$$

$$\Rightarrow \left(1-\frac{c^2}{a^2}\right)x^2+y^2=a^2-c^2$$

$$\Rightarrow \left(\frac{a^2-c^2}{a^2}\right)x^2+y^2=a^2-c^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(iv)}$$

[taking $b^2=a^2-c^2$]

This is the standard form of the equation of an ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Where $b^2=a^2(1-e^2)$ i.e. b > a

MASTERJEE CONCEPTS

Domain and range of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are [–a, a] and [–b, b] respectively.

Vaibhav Krishnan (JEE 2009, AIR 22)

3. TERMS RELATED TO AN ELLIPSE

Vertices: The points A and A′, in the figure where the curve meets the line joining the foci S and S′, are called the vertices of the ellipse. The coordinates of A and A′ are (a, 0) and (–a, 0) respectively.

Major and Minor Axes: In the figure, the distance AA′= 2a and BB′= 2b are called the major and minor axes of the ellipse. Since $e<1$ and $b^2=a^2(1-e^2)$. Therefore a > b $\Rightarrow$ AA′ > BB′.

Foci: In figure, the points S (ae, 0) and S′ (–ae, 0) are the foci of the ellipse.

Directrix: ZK and Z′K′ are two directrix of the ellipse and their equations are x = a/e and x=– a/e respectively.

Centre: Since the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ every chord is bisected at C (0, 0). Therefore, C is the centre of the ellipse in the figure and C is the mid-point of AA′.

Eccentricity of the Ellipse: The eccentricity of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b is $e=\sqrt{1-\left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}=\sqrt{1-\left(\frac{b}{a}\right)^2}$

Ordinate and Double Ordinate: Let P be a point on the ellipse and let PN be perpendicular to the major axis AA′ such that PN produced meets the ellipse at P′. Then PN is called the ordinate of P and PNP′ the double ordinate of P.

Latus Rectum: It is a double ordinate passing through the focus. In Fig. 3, LL′ is the latus rectum and LS is called the semi-latus rectum. MSM′ is also a latus rectum. The length of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is $\frac{2b^2}{a}=2a(1-e^2)$.

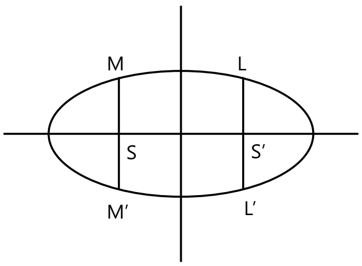


Figure 11.3

Focal Distances of a Point on the Ellipse: Let P(x, y) be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in Fig. 11.4. Then, by definition, we have

$SP = ePN$ and $S'P = e\,PN'$

$\Rightarrow$ $SP = eP'Q$ and $S'P = e\,(P'Q')$

$\Rightarrow$ $SP = e\,(CQ - CP)$ and $S'P = e\,(CQ' + CP')$

$\Rightarrow$ $SP = e\left(\frac{a}{e} - x\right)$ $\quad$ $S'P = e\left(\frac{a}{e} + x\right]$

$\Rightarrow$ $SP = a - ex$ and $S'P = a + ex$

Thus, the focal distances of a point P(x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are a –ex and a + ex.

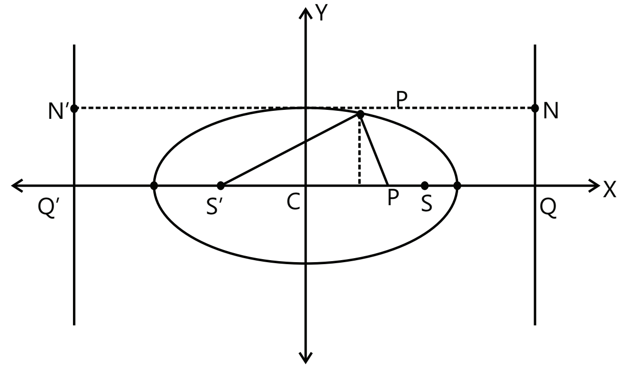


Figure 11.4

Also, SP + S'P= a – ex + a + ex = 2a = Major axis (constant)

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

MASTERJEE CONCEPTS

The above property of an ellipse gives us a mechanical method of tracing an ellipse as explained below:

Take an inextensible string of a certain length and fasten its ends to two fixed knobs. Now put a pencil on the string and turn it round in such a way that the two portions of the string between it and the fixed knobs are always tight. The curve so traced will be an ellipse having its foci at the fixed knobs.

Shrikant Nagori (JEE 2009, AIR 30)

4. PROPERTIES OF ELLIPSE

Important Terms \ **Ellipse**	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	For b > a
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	(±ae, 0)	(0, ± be)
Equation of directrices	x = ±a/e	y = ±b/e
Relation in a, b and e	$b^2 = a^2\,(1 - e^2)$	$a^2=b^2\,(1 - e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations **(Discussed later)**	(a cos ϕ, b sinϕ) ($0 \le \phi < 2\pi$)	(a cos ϕ, b sin ϕ) ($0 \le \phi < 2\pi$)
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii **SP + S′P =**	2a	2b
Distance between foci	2ae	2be
Distance between directrices	2a/e	2b/e
Tangents at the vertices	x = –a, x = a	y = b, y = –b

MASTERJEE CONCEPTS

The vertex divides the join of the focus and the point of intersection of directrix with the axis internally and externally in the ratio e: 1

Misconceptions: If a>b it is a horizontal ellipse, if b > a it is a vertical ellipse unlike hyperbola.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 2: Find the equation of the ellipse whose foci are (4, 0) and (–4, 0) and whose eccentricity is 1/3. **(JEE MAIN)**

Sol: Use the property of the centre of an ellipse and the foci to find the equation.

Clearly, the foci are on the x-axis and the centre is (0, 0), being midway between the foci. So the equation will be in the standard form.

Let it be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Foci are $(a\cos\theta, b\sin\theta)$. Here they are $(\pm 4, 0)$.

$\therefore$ ae = 4

Given $e=\frac{1}{3}$

$\therefore$ $a.\frac{1}{3}=4$, i.e., a = 12

Again, $b^2=a^2(1-e^2)$

$\Rightarrow b^2=12^2.\left(1-\frac{1}{3^2}\right)=12^2.\frac{8}{3^2}=128$

$P(a\cos\theta, b\sin\theta)$. The equation of the ellipse is $\frac{x^2}{144}+\frac{y^2}{128}=1$

Illustration 3: From a point Q on the circle $x^2 + y^2 = a^2$ perpendicular QM is drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1. **(JEE MAIN)**

Sol: Starting from a point on the circle find the foot of the perpendicular on the X-axis and hence find the locus.

Let $by\sec\phi + ax\,\text{cosec}\,\phi + (a^2 + b^2) = 0$, $M \equiv (a\cos\theta, 0)$ and $P \equiv (h, k)$

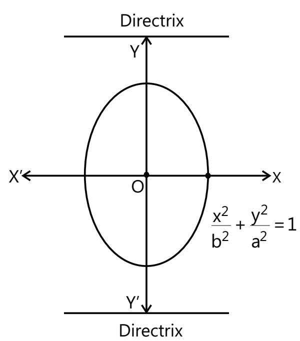
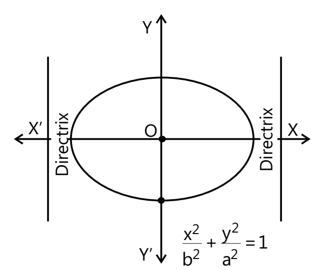
$$\therefore\ h = a\cos\theta,\ k = \frac{a\sin\theta}{3} \Rightarrow \left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$$

$$\Rightarrow \text{ Locus of P is } \frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1.$$

Illustration 4: Draw the shape of the given ellipse and find their major axis, minor axis, value of c, vertices, directrix, foci, eccentricity and the length of the latus rectum. **(JEE MAIN)**

(i) $36x^2 + 4y^2 = 144$ (ii) $4x^2 + 9y^2 = 36$

Sol: Using the standard form and basic concepts of curve tracing, sketch the two ellipses.

1.	Ellipse	$36x^2 + 4y^2 = 144$ or $\frac{x^2}{4} + \frac{y^2}{36} = 1$	$4x^2 + 9y^2 = 36$ or $\frac{x^2}{9} + \frac{y^2}{4} = 1$
2.	Shape	Since the denominator of $\frac{y^2}{36}$ is larger then the denominator of $\frac{x^2}{4}$, so the major axis lies along y-axis	Since the denominator of $\frac{x^2}{9}$ is greater than the denominator of $\frac{y^2}{4}$, so the major axis lies along x-axis
		 **Figure 11.5**	 **Figure 11.6**
3.	Major axis	$2a = 2 \times 6 = 12$	$2a = 2 \times 3 = 6$
4.	Minor axis	$2b = 2 \times 2 = 4$	$2b = 2 \times 2 = 4$
5.	Value of c	$a^2 = 36,\ b^2 = 4$ $c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = 4\sqrt{2}$	$a^2 = 9,\ b^2 = 4$ $c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$
6.	Vertices	(0, –a) and (0, a) (0, –6) and (0, 6)	(–a, 0) and (a, 0) (–3, 0) and (3, 0)
7.	Directrices	$y = \pm\frac{a^2}{c} = \pm\frac{36}{4\sqrt{2}} = \pm\frac{9}{\sqrt{2}}$	$x = \pm\frac{a^2}{c} = \pm\frac{9}{\sqrt{5}}$

8.	Foci	(0, –c), (0, c) $(0, -4\sqrt{2}), (0, 4\sqrt{2})$	(–c, 0) and (c, 0) $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$
9.	Eccentricity	$e = \frac{c}{a} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$	$e = \frac{c}{a} = \frac{\sqrt{5}}{3}$
10.	Length of latus rectum	$2l = \frac{2b^2}{a} = \frac{2\times 4}{6} = \frac{4}{3}$	$2l = \frac{2b^2}{a} = \frac{2\times 4}{3} = \frac{8}{3}$

MASTERJEE CONCEPTS

The semi-latus rectum of an ellipse is the harmonic mean of the segments of its focal chord.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 5: Show that $x^2 + 4y^2 + 2x + 16y + 13 = 0$ is the equation of an ellipse. Find its eccentricity, vertices, foci, directrices, length of the latus rectum and the equation of the latus rectum. **(JEE ADVANCED)**

Sol: Represent the equation given in the standard form and compare it with the standard form to get the eccentricity, vertices etc.

We have,

$x^2 + 4y^2 + 2x + 16y + 13 = 0 \qquad \Rightarrow (x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$

$\Rightarrow (x+1)^2 + 4(y+2)^2 = 4 \qquad \Rightarrow \frac{(x+1)^2}{2^2} + \frac{(y+2)^2}{1^2} = 1$ ... (i)

Shifting the origin at (–1, –2) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y,

we have $x = X - 1$ and $y = Y - 2$ ... (ii)

Using these relations, equation (i) reduces to

$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$, where ... (iii)

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where a = 2 and b = 1.

Thus, the given equation represents an ellipse.

Clearly a > b, so, the given equation represents an ellipse whose major and minor axes are along the X and Y axes respectively.

Eccentricity: The eccentricity e is given by $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

Vertices: The vertices of the ellipse with respect to the new axes are $(X = \pm a, Y = 0)$ i.e. $(X = \pm 2, Y = 0)$.

So, the vertices with respect to the old axes are given by

(± 2–1, –2) i.e., (–3, –2) and (1, –2) [Using (ii)]

Foci: The coordinates of the foci with respect to the axes are given by

$(X = \pm ae, Y = 0)$ i.e. $(X = \pm\sqrt{3}, Y = 0)$.

So, the coordinates of the foci with respect to the old axes are given by

$(\pm\sqrt{3}-1, -2)$ [Putting $X = \pm\sqrt{3}, Y = 0$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are

$$X = \pm\frac{a}{e} \text{ i.e. } ea^2\left(1-\frac{b^2}{d^2}\right) = 4a^2e^2\cos^2\theta$$

So, the equations of the directrices with respect to the old axes are

$$x = +\frac{4}{\sqrt{3}}-1 \text{ i.e. } x = \frac{4}{\sqrt{3}}-1 \text{ and } x = -\frac{4}{\sqrt{3}}-1 \quad \left[\text{Putting } X = \pm\frac{4}{\sqrt{3}} \text{ in (ii)}\right]$$

Length of the latus rectum: The length of the latus rectum = $\frac{2b^2}{a} = \frac{2}{2} = 1$.

Equation of latus rectum: The equations of the latus rectum with respect to the new axes are

$X = \pm ae$ i.e. $X = \pm\sqrt{3}$

So, the equations of the latus rectum with respect to the old axes are

$x = \pm\sqrt{3}-1$ [Putting $X = \pm\sqrt{3}$ in (ii)]

i.e., $x = \sqrt{3}-1$ and $x = -\sqrt{3}-1$.

Illustration 6: A straight rod of given length slides between two fixed bars which include an angle of 90º. Show that the locus of a point on the rod which divides it in a given ratio is an ellipse. If this ratio is 1/2, show that the eccentricity of the ellipse is $\sqrt{3}/2$. **(JEE ADVANCED)**

Sol: Consider a rod of particular length and write the coordinates of the point in terms of the parameter. Eliminate the parameters to get eccentricity equal to $\sqrt{3}/2$.

Let the two lines be along the coordinate axes. Let PQ be the rod of length a such that $\angle OPQ = \theta$. Then, the coordinates of P and Q are $(a\cos\theta, 0)$ and $(0, a\sin\theta)$ respectively. Let R(h,k) be the point dividing PQ in the ratio

$\lambda : 1$. Then, $h = \frac{a\cos\theta}{\lambda+1}$ and $k = \frac{\lambda a\sin\theta}{\lambda+1}$.

$$\Rightarrow \quad \cos\theta = \frac{h}{a}(\lambda+1) \text{ and } \sin\theta = \frac{k}{a\lambda}(\lambda+1)$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = \frac{h^2}{a^2}(\lambda+1)^2 + \frac{k^2}{a^2\lambda^2}(\lambda+1)^2$$

$$\Rightarrow \quad \frac{h^2}{\left((a/(\lambda+1)\right)^2} + \frac{k^2}{\left(a\lambda/(\lambda+1)\right)^2} = 1.$$

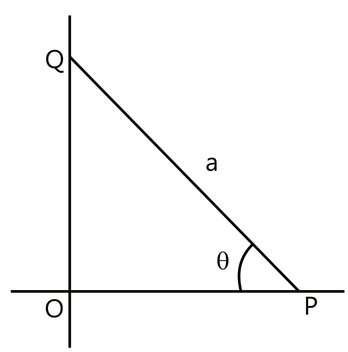


Figure 11.7

Hence, the locus of R (h, k) is $\frac{x^2}{\left(a/(\lambda+1)\right)^2} + \frac{y^2}{\left(a\lambda/(\lambda+1)\right)^2} = 1$

which is an ellipse of eccentricity given by $e = \begin{cases} \sqrt{1-\lambda^2} & \text{if } \lambda < 1 \\ \sqrt{1-(1/\lambda^2)} & \text{if } \lambda > 1 \end{cases}$

When $\lambda = \frac{1}{2}$, we have $e = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$.

Illustration 7: A man running a race course notes that the sum of the distances from the two flag posts from him is always 10 metres and the distances between the flag posts is 8 metres. Find the equation of the path traced by the man. **(JEE ADVANCED)**

Sol: Use the basic definition of an ellipse. Clearly, the path traced by the man is an ellipse having its foci at two flag posts. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } b^2=a^2(1-e^2)$$

It is given that the sum of the distances of the man from the two flag posts is 10 metres. This means that the sum of the focal distances of a point on the ellipse is 10 m.

$\Rightarrow$ PS + PS′ = 2a = 10 $\Rightarrow$ a = 5 ...(i)

It is also given that the distance between the flag posts is 8 metres.

$\therefore$ 2ae = 8 $\Rightarrow$ ae = 4 ...(ii)

Now, $b^2=a^2(1-e^2)=a^2-a^2e^2=25-16$

$\Rightarrow$ $b^2=9$ $\Rightarrow$ b = 3 [Using (i) and (ii)]

Hence, the equation of the path is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

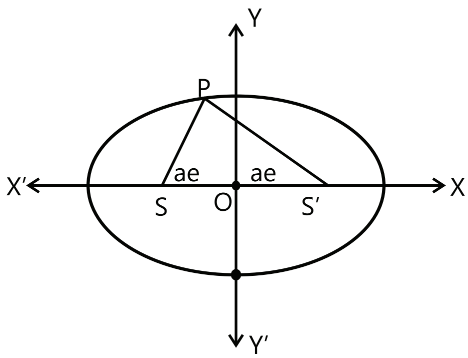


Figure 11.8

5. AUXILIARY CIRCLE

A circle with its centre on the major axis, passing through the vertices of the ellipse is called an auxiliary circle.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse, then its auxiliary circle is $x^2+y^2=a^2$.

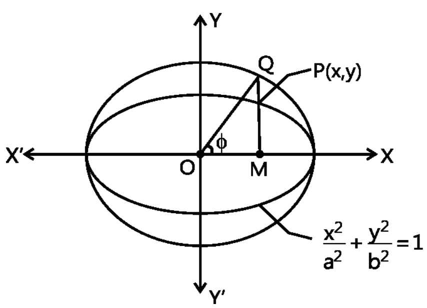


Figure 11.9

Eccentric angle of a point: Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q. Join OQ. The angle $\angle QOM=\phi$ is called the eccentric angle of the point P on the ellipse.

Note that the angle $\angle XOP$ is not the eccentric angle of point P.

MASTERJEE CONCEPTS

A circle defined on the minor axis of an ellipse as diameter $x^2+y^2=b^2$ is called a minor auxiliary circle.

Ravi Vooda (JEE 2009, AIR 71)

6. PARAMETRIC FORM

6.1 Parametric Co-Ordinates of a Point on an Ellipse

Let P(x, y) be a point on an ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and Q be the corresponding point on the auxiliary circle $x^2+y^2=a^2$.

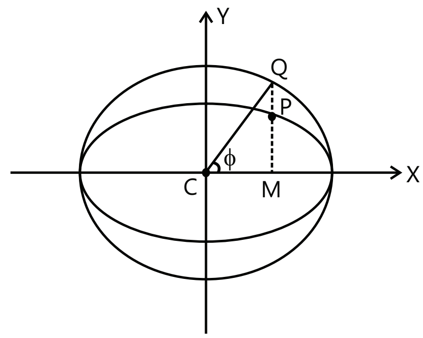


Figure 11.10

Let the eccentric angle of P be ϕ. Then $\angle XCQ = \phi$.

Now, x = CM

$\Rightarrow$ x = CQ cos ϕ [$\because$ CQ = radius of $x^2+y^2=a^2$]

Since P(x, y) lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\Rightarrow$ $\frac{a^2\cos^2\phi}{a^2}+\frac{y^2}{b^2}=1$.

$\Rightarrow$ $y^2=b^2(1-\cos^2\phi)=b^2\sin^2\phi$ $\Rightarrow$ $y=b\sin\phi$.

Thus, the coordinates of point P having eccentric angle ϕ can be written as (a cosϕ, b sin ϕ) and are known as the parametric coordinates of an ellipse.

6.2 Parametric Equation of an Ellipse

The equations x = acosϕ, y = bsinϕ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.

MASTERJEE CONCEPTS

Always remember that θ is not the angle of P with x-axis. It is the angle of corresponding point Q.

Rohit Kumar (JEE 2012, AIR 79)

Illustration 8: Find the distance from the centre to the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which makes an angle α with x-axis. **(JEE MAIN)**

Sol: Establish a relation between the angle α and the eccentric angle. Use parametric coordinates of an ellipse and the distance formula to find the distance.

Let $P\equiv(a\cos\theta, b\sin\theta)$ $\therefore$ $(b/a)\tan\theta=\tan\alpha \Rightarrow$ $\tan\theta=(a/b)\tan\alpha$

$$OP = \sqrt{a^2\cos^2\theta+b^2\sin^2\theta} = \sqrt{\frac{a^2\cos^2\theta+b^2\sin^2\theta}{\sin^2\theta+\cos^2\theta}} \qquad \because \sin^2\theta+\cos^2\theta=1$$

$$=\sqrt{\frac{a^2+b^2\tan^2\theta}{1+\tan^2\theta}}=\sqrt{\frac{a^2+b^2\times(a^2/b^2)\tan^2\alpha}{1+(a^2/b^2)\tan^2\alpha}}$$

7. SPECIAL FORMS OF AN ELLIPSE

(a) If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(b) If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $x^2+2y^2-6x-12y+23=0$ and $mx-ly+p=0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$, $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

Illustration 9: Find the equation to the ellipse whose axes are of lengths 6 and $e^2\cos^2\phi+\cos\phi-1=0$ and their equation are $x-3y+3=0$ and $3x+y-1=0$ respectively. **(JEE MAIN)**

Sol: Given the equation of the axis, we can find the centre. Use the length of the axes of the ellipse to find the required equation of the ellipse. Let P (x, y) be any point on the ellipse and let p_1 and p_2 be the lengths of perpendiculars drawn from P on the major and minor axes of the ellipse.

Then, $p_1=\frac{x-3y+3}{\sqrt{1+9}}$ and $p_2=\frac{3x+y-1}{\sqrt{9+1}}$.

Let 2a and 2b be the lengths of major and minor axes of the ellipse respectively. We have, 2a = 6 and 2b= $2\sqrt{6}$.

$\Rightarrow$ a = 3 and b = $\sqrt{6}$. The equation of the ellipse is $\frac{p_1^2}{b^2}+\frac{p_2^2}{a^2}=1$

$\Rightarrow \frac{(x-3y+3)^2}{60}+\frac{(3x+y-1)^2}{90}=1 \Rightarrow (x-3y+3)^2+2(3x+y-1)^2$

$\Rightarrow 21x^2-6xy+29y^2+6x-58y-151=0$

8. EQUATION OF A CHORD

Let P ($a\cos\alpha$, $b\sin\alpha$), Q ($a\cos\beta$, $b\sin\beta$) be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then, the equation of the chord joining these two points is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

Illustration 10: Find the angle between two diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose extremities have eccentric angles α and $\beta=\alpha+\frac{\pi}{2}$. **(JEE MAIN)**

Sol: Find the slope of the two diameters and then use the relation between the given angles.

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Slope of OP = $m_1=\frac{b\sin\alpha}{a\cos\alpha}=\frac{b}{a}\tan\alpha$; Slope of OQ = $m_2=\frac{b\sin\beta}{a\cos\beta}=\frac{-b}{a}\cot\alpha$ given $\beta=\alpha+\frac{\pi}{2}$

$\therefore \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{(b/a)(\tan\alpha+\cot\alpha)}{1-(b^2/a^2)}\right|=\left|\frac{2ab}{(a^2-b^2)\sin 2\alpha}\right|$

Illustration 11: If the chord joining the two points whose eccentric angles are α and β, cut the major axis of an ellipse at a distance c from the centre, show that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$. **(JEE ADVANCED)**

Sol: Use the fact that the point (c, 0) lies on the chord joining points whose eccentric angles are α and β. The equation of the chord joining points whose eccentric angles are α and β on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{, is }\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

This will cut the major axis at the point (c, 0) if

$$\frac{c}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)\Rightarrow\frac{\cos\left((\alpha+\beta)/2\right)}{\cos\left((\alpha-\beta)/2\right)}=\frac{a}{c}\Rightarrow\frac{\cos\left((\alpha+\beta)/2\right)+\cos\left((\alpha-\beta)/2\right)}{\cos\left((\alpha+\beta)/2\right)-\cos\left((\alpha-\beta)/2\right)}=\frac{a+c}{a-c}$$

$$\Rightarrow\frac{2\cos(\alpha/2)\cos(\beta/2)}{-2\sin(\alpha/2)\sin(\beta/2)}=\frac{a+c}{a-c}\Rightarrow\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}.$$

Illustration 12: The eccentric angle of any point P on the ellipse is ϕ. If S is the focus nearest to the end A of the major axis A′A such that $\angle ASP=\theta$. Prove that $\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}$. **(JEE ADVANCED)**

Sol: Find the distance of the point P from the X-axis and the horizontal distance of the point from nearest focus. Use trigonometry to get the desired result.

In ΔPSL, we have

PL = $b\sin\phi$ and SL = $a\cos\phi - ae$

$$\therefore\tan\theta=\frac{b\sin\phi}{a\cos\phi-ae}\Rightarrow\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}=\frac{2\sqrt{1-e^2}\tan(\phi/2)}{(1-e)-(1+e)\tan^2(\phi/2)}$$

$$\Rightarrow\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}=\frac{2\sqrt{(1+e)/(1-e)}\tan(\phi/2)}{1-\left(\sqrt{(1+e)/(1-e)}\right)}\Rightarrow\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}$$

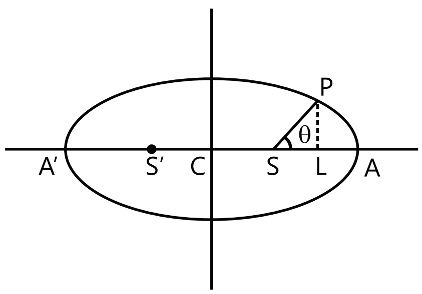


Figure 11.11

9. POSITION OF A POINT W.R.T. AN ELLIPSE

The point $P(x_1,y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to $\frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}-1>0$, = 0 or < 0 respectively. $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

Illustration 13: Find the set of value(s) of 'α' for which the point $P(\alpha,-\alpha)$ lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. **(JEE MAIN)**

Sol: Apply the concept of position of a point w.r.t. the ellipse.

If $P(\alpha,-\alpha)$ lies inside the ellipse $2a^2$ $S_1<0$

$$\Rightarrow\frac{\alpha^2}{16}+\frac{\alpha^2}{9}-1<0\Rightarrow\frac{25}{144},\alpha^2<1\Rightarrow\alpha^2<\frac{144}{25}\qquad\therefore\alpha\in\left(-\frac{12}{5},\frac{12}{5}\right).$$

10. LINE AND AN ELLIPSE

Consider a straight line of the form $y = mx + c$ and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

By solving these two equations we get, $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow \quad (b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$

For this equation

$\Rightarrow \quad D = 4\left(a^4m^2c^2 - (b^2 + a^2m^2)a^2(c^2 - b^2)\right)$

$\Rightarrow \quad D = 4a^2b^2(b^2 - c^2 + a^2m^2)$

$\therefore$ The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

Illustration 14: Find the condition for the line $lx + my + n = 0$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the theory of equations or the standard form of the tangent. The equation of the line is $lx + my + n = 0$

$\Rightarrow \quad y = \left(-\frac{l}{m}\right)x + \left(-\frac{n}{m}\right)$. We know that the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

$\Rightarrow \quad \left(\frac{-n}{m}\right)^2 = a^2\left(-\frac{l}{m}\right)^2 + b^2 \Rightarrow n^2 = a^2l^2 + b^2m^2$.

Illustration 15: Find the condition for the line $x\cos\alpha + y\sin\alpha = p$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the theory of equations or the standard form of the tangent.

The equation of the given line ℓ is $x\cos\alpha + y\sin\alpha = p \Rightarrow y = (-\cot\alpha)x - p\,\mathrm{cosec}\,\alpha$

This will touch $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, If $(-p\,\mathrm{cosec}^2\alpha)^2 = a^2\cot^2\alpha + b^2$ [Using: $c^2 = a^2m^2 + b^2$]

$\Rightarrow \quad p^2\mathrm{cosec}^2\alpha = \frac{a^2\cos^2\alpha + b^2\sin^2\alpha}{\sin^2\alpha} \Rightarrow p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

Illustration 16: Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{16} = 1$ at two distinct points. **(JEE ADVANCED)**

Sol: Same as previous illustration.

Solving the given line with ellipse, we get $\frac{(4y-\lambda)^2}{9\times 16} + \frac{y^2}{9} = 1 \Rightarrow 32y^2 - 8\lambda + (\lambda^2 - 144) = 0$

Since the line intersects the parabola at two distinct points:

$\therefore$ Roots of above equation are real & distinct $\therefore$ D > 0

$\Rightarrow \quad (8\lambda)^2 - 4.32(\lambda^2 - 144) > 0 \Rightarrow -12\sqrt{2} < \lambda < 12\sqrt{2}$

11. TANGENT TO AN ELLIPSE

11.1 Equation of Tangent

(a) Point form: The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(b) Slope form: If the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=a^2m^2+b^2$.

Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to the ellipse.

(i) Point of contact: Line $\left(\frac{0-b}{ae-0}\right)\left(\frac{0-b}{-ae-0}\right)=-1$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right).$$

(c) Parametric form: The equation of tangent at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Remark: The equation of the tangents to the ellipse at points $p(a\cos\theta_1, b\sin\theta_1)$ and $Q(a\cos\theta_2, b\sin\theta_2)$ are

$$\frac{x}{a}\cos\theta_1+\frac{y}{b}\sin\theta_1=1 \text{ and } \frac{x}{a}\cos\theta_2+\frac{y}{b}\sin\theta_2=1$$

And these two intersect at the point $\left(\frac{a\cos\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}, \frac{b\sin\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}\right)$

11.2 Equation of Pair of Tangents

Pair of tangents: The equation of a pair of tangents PA and PB is $SS_1 = T^2$.

Where $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

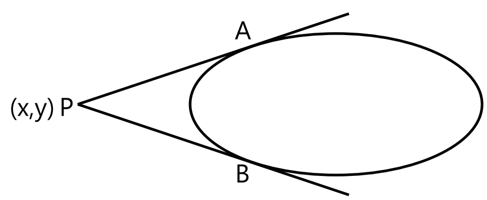


Figure 11.12

MASTERJEE CONCEPTS

The portion of the tangent to an ellipse intercepted between the curve and the directrix subtends a right angle at the corresponding focus.

B Rajiv Reddy (JEE 2012, AIR 11)

11.3 Director Circle

Definition The locus of the point of intersection of the perpendicular tangents to an ellipse is known as its director circle.

Equation of the director circle the equation of the director circle, is $(x \pm ae)^2 = y^2 - 4a^2$. Clearly, it is a circle concentric to the ellipse and radius equal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

It follows from the definition of the director circle that the tangents drawn from any point on the director circle of a given ellipse to the ellipse are always at right angles.

MASTERJEE CONCEPTS

Director circle is the circumcircle of ellipse's circumrectangle whose sides are parallel to the major and minor axis.

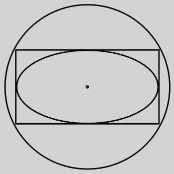


Figure 11.13

Anvit Tanwar (JEE 2009, AIR 9)

Illustration 17: A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **(JEE ADVANCED)**

Sol: Use the condition of tangency and the standard equation of tangent. The equations of the two ellipses are

$$\frac{x^2}{4^2} + \frac{y^2}{1^2} = 1 \quad \text{...(i)}$$

and $$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \text{...(ii) respectively.}$$

Suppose the tangents P and Q to ellipse (ii) intersect at R(h, k). PQ is the chord of contact of tangents drawn from R(h, k) to ellipse (ii). So, the equation of PQ is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \quad \text{...(ii)}$$

$\Rightarrow \frac{ky}{3} = \frac{-hx}{6} + 1 \Rightarrow y = -\frac{hx}{2k} + \frac{3}{k}$. This touches the ellipse given in (i). Therefore,

$$\frac{9}{k^2} = 4\left(\frac{-h}{2k}\right)^2 + 1 \qquad [\text{Using}: c^2 = a^2m^2 + b^2]$$

$\Rightarrow h^2 + k^2 = 9 \Rightarrow (h, k)$ lies on the circle $x^2 + y^2 = 9$.

Clearly, $x^2 + y^2 = 9$ is the director circle of the ellipse (ii). Hence, the angle between the tangents at P and Q to the ellipse is a right angle.

11.4 Chord of Contact

If PQ and PR are the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact QR is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or T = 0 at (x_1, y_1).

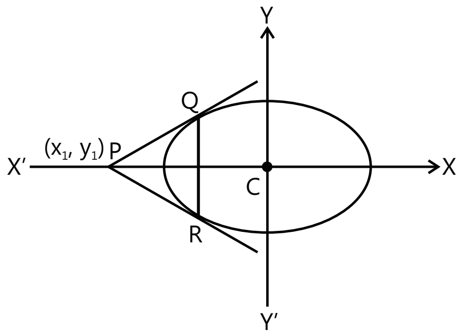


Figure 11.14

Illustration 18: Prove that the chord of contact of tangents drawn from the point (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at the centre, if $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$. **(JEE ADVANCED)**

Sol: Make the equation of the ellipse homogeneous using the chord and then apply the condition for the pair of straight lines to be perpendicular.

The equation of the chord of contact of tangents drawn from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \qquad \text{...(i)}$$

The equation of the straight lines joining the centre of the ellipse i.e. the origin, to the points of intersection of the ellipse and (i) is obtained by making a homogeneous equation with the help of (i) and the ellipse and is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2=0 \text{ or } x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+y^2\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)-\frac{2hk}{a^2b^2}xy=0 \qquad \text{...(ii)}$$

If the chord of contact of tangents subtends a right angle at the centre, then the lines represented by (ii) should be at right angles.

$$\Rightarrow \left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)=0 \quad \Rightarrow \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Illustration 19: Find the equations of the tangents to the ellipse $3x^2+4y^2=12$ which are perpendicular to the line $y+2x=4$. **(JEE MAIN)**

Sol: Use the slope form of the tangent. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $y+2x=4$

$\therefore\ m(-2)=-1 \Rightarrow m=\frac{1}{2}$; Now, $3x^2+4y^2=12 \Rightarrow \frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a^2=4$ and $b^2=3$.

So, the equations of the tangents are $y=mx\pm\sqrt{a^2m^2+b^2}$

i.e. $y=\frac{1}{2}x\pm\sqrt{4(1/4)+3} \Rightarrow y=\frac{x}{2}\pm 2 \Rightarrow 2y=x\pm 4$.

Illustration 20: Find the equations of the tangents to the ellipse $9x^2+16y^2=144$ which pass through the point (2, 3). **(JEE MAIN)**

Sol: Put the given point in the standard equation of the tangent and find the value of m.

The equation of the ellipse is $9x^2+16y^2=144 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

This if of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=4^2$ and $b^2=3^2$.

The equation of any tangents to this ellipse is $y=mx\pm\sqrt{a^2m^2+b^2}$ i.e. $y=mx\pm\sqrt{16m^2+9}$ ...(i)

If it passes through (2, 3) then $3=2m+\sqrt{16m^2+9}$

$\Rightarrow (3-2m)^2=16m^2+9\ x^2+y^2=4 \Rightarrow m=0,-1$

Substituting these values of m in (i), we obtain y = 3 and y = –x + 5 as the equations of the required tangents.

Note: If the question was asked to find combined eq. of a pair of tangents then use $SS_1 = T^2$.

Illustration 21: The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse $x^2+2y^2=6$ which touch the ellipse $x^2+4y^2=4$ is. **(JEE MAIN)**

(A) $x^2+y^2=4$ (B) $x^2+y^2=6$ (C) $x^2+y^2=9$ (D) None of these.

Sol: Find the equation of the tangents for the two ellipses and compare the two equations.

We can write $x^2+4y^2=4$ as $\frac{x^2}{4}+\frac{y^2}{1}=1$ ...(i)

Equation of a tangent to the ellipse (i) is $\frac{x}{2}\cos\theta+y\sin\theta=1$ ...(ii)

Equation of the ellipse $x^2+2y^2=6$ can be written as $\frac{x^2}{6}+\frac{y^2}{3}=1$ ...(iii)

Suppose (ii) meets the ellipse (iii) at P and Q and the tangents at P and Q to the ellipse (iii) intersect at (h, k), then (ii) is the chord of contact of (h, k) with respect to the ellipse (iii) and thus its equation is $\frac{hx}{6}+\frac{ky}{3}=1$ ...(iv)

Since (ii) and (iv) represent the same line

$DA=CA-CD=\frac{a^2}{x_1}-x_1 \Rightarrow$ h = 3 cos θ, k = 3 sin θ and the locus of (h, k) is $x^2+y^2=9$.

Illustration 22: Show that the locus point of intersection of the tangents at two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angles differ by a right angle is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$. **(JEE MAIN)**

Sol: Solve the equation of the tangents at the two points whose eccentric angles differ by $\frac{\pi}{2}$.

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be two points on the ellipse such that $\theta-\phi=\frac{\pi}{2}$. The equations of tangents at P and Q are

$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ ...(i)

and, $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$ ...(ii) respectively.

Since $\frac{\sqrt{2}}{3}$, so (i) can be written as $-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1$ ...(iii)

Let (h, k) be the point of intersection of (i) and (ii). Then,

$\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1$ and $-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta=1$

$$\Rightarrow \left(\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta\right)^2+\left(-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta\right)^2=1+1 \qquad \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=2$$

Hence, the locus of (h, k) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

Illustration 23: Prove that the locus of the mid-points of the portion of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes $a^2y^2+b^2x^2=4x^2y^2$. **(JEE ADVANCED)**

Sol: Starting from the equation of the tangent, find the mid point of the tangent intercepted between the axes. Eliminate the parameter to get the locus.

The equation of the tangent at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

This cuts the coordinates axes at $A\left(\frac{a}{\cos\theta}, 0\right)$ and $B\left(0, \frac{b}{\sin\theta}\right)$

Let P(h, k) be the mid-point of AB. Then, $\frac{a}{2\cos\theta}=h$ and $\frac{b}{2\sin\theta}=k$

$$\Rightarrow \cos\theta=\frac{a}{2h} \text{ and } \sin\theta=\frac{b}{2k} \Rightarrow \cos^2\theta+\sin^2\theta=\frac{a^2}{4h^2}+\frac{b^2}{4k^2} \Rightarrow \frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

Hence, the locus of P (h, k) is $\frac{1}{52}$, $a^2y^2+b^2x^2=4x^2y^2$.

Illustration 24: Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse, then show that $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

(JEE ADVANCED)

Sol: Use the fact that focal distances of a point (x, y) on the ellipse are a+ex and a–ex.

Let the coordinates of P be $(a\cos\theta, b\sin\theta)$, where θ is a parameter. The coordinates of F_1 and F_2 are (ae, 0) and (–ae, 0) respectively. We know that.

Therefore, $PF_1 = a + ae\cos\theta$ and $PF_2 = a - ae\cos\theta$

i.e., $PF_1 = a(1+e\cos\theta)$ and $PF_2 = a(1-e\cos\theta)$

$\therefore$ $(PF_1-PF_2)^2=\{a(1+e\cos\theta)-a(1-e\cos\theta)\}^2=4a^2e^2\cos^2\theta$ ...(i)

The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ ...(ii)

$\therefore$ d = Length of the perpendicular from (0, 0) on (ii)

$$\Rightarrow \quad d=\left|\frac{(0/a)\cos\theta+(0/b)\sin\theta-1}{\sqrt{\cos^2\theta/a^2+\sin^2\theta/b^2}}\right|$$

$$\Rightarrow \quad \frac{1}{d^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\Rightarrow\frac{b^2}{d^2}=\frac{b^2}{a^2}\cos^2\theta+\sin^2\theta \;\Rightarrow\quad 1-\frac{b^2}{d^2}=1-\frac{b^2}{a^2}\cos^2\theta-\sin^2\theta$$

$$\Rightarrow \quad 1-\frac{b^2}{d^2}=\cos^2\theta-\frac{b^2}{a^2}\cos^2\theta=\cos^2\theta\left(1-\frac{b^2}{a^2}\right)=e^2\cos^2\theta$$

$$\Rightarrow \quad 4a^2\left(1-\frac{b^2}{d^2}\right)=4a^2e^2\cos^2\theta \qquad \text{...(iii)}$$

Hence, from (i) and (iii), we have $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

Illustration 25: The tangent at point P(cosθ, bsinθ) of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, meets its auxiliary circle on two points, the chord joining which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1+\sin^2\theta)^{-1/2}$. **(JEE ADVANCED)**

Sol: Homogenize the equation of the ellipse using the equation of the tangent and then use the condition for the pair of straight lines to be perpendicular.

The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \text{...(i)}$$

The equation of the auxiliary circle is $x^2+y^2=a^2$ ...(ii)

The combined equation of the lines joining the origin with the points of intersection of (i) and (ii) is obtained by making (ii) homogeneous w.r.to (i)

$$\therefore \quad x^2+y^2=a^2\left(\frac{x}{a}\cos\theta+\frac{y}{b}\cos\theta\right)^2$$

$$\Rightarrow \quad x^2(1-\cos^2\theta)+y^2\left(1-\frac{a^2}{b^2}\sin^2\theta\right)-2xy\frac{a}{b}\sin\theta\cos\theta=0$$

These two lines are mutually perpendicular. Therefore, coefficient of x^2 + Coefficient of y^2 = 0

$$\Rightarrow \quad \sin^2\theta+1-\frac{a^2}{b^2}\sin^2\theta\Rightarrow\sin^2\theta\left(1-\frac{a^2}{b^2}\right)+1=0\Rightarrow\frac{a^2-b^2}{b^2}\sin^2\theta=1$$

$$\Rightarrow \quad \frac{a^2e^2\sin^2\theta}{a^2(1-e^2)}=1\Rightarrow e^2\sin^2\theta=1-e^2\Rightarrow e=(1+\sin^2\theta)^{-1/2}.$$

Illustration 26: If the tangent at (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the auxiliary circle $x^2+y^2=r^2$ at points whose ordinates are y_1 and y_2, show that $\frac{1}{y_1}+\frac{1}{y_2}=\frac{2}{k}$. **(JEE ADVANCED)**

Sol: Form a quadratic in y using the equation of the tangent and the ellipse and then use the sum and product of the roots to prove the above result.

The equation of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and point (h, k) is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$. The ordinates of the points of intersection of (i) and the auxiliary circle are the roots of the equation

$$\frac{a^4}{h^2}\left(\frac{b^2-ky}{b^2}\right)^2+y^2=a^2$$

$$\Rightarrow \quad y^2\left(a^4k^2+b^4h^2\right)-2a^4b^2ky+a^4b^4-a^2b^4h^2=0$$

Since y_1 and y_2 are the roots of this equation.

Therefore, $y_1+y_2=\frac{2a^4b^2k}{a^4k^2+b^4h^2}$ and $y_1y_2=\frac{a^4b^4-a^2b^4h^2}{a^4k^2+b^4h^2}$

$$\Rightarrow \quad \frac{1}{y_1}+\frac{1}{y_2}=\frac{2a^4b^2k}{a^4b^4-a^2b^4h^2} \Rightarrow \frac{1}{y_1}+\frac{1}{y_2}=\frac{2ka^2}{(a^2-h^2)b^2} \quad \Rightarrow \frac{1}{y_1}+\frac{1}{y_2}=\frac{2a^2k}{a^2k^2} \quad \Rightarrow \quad \frac{1}{y_1}+\frac{1}{y_2}=\frac{2}{k}$$

Illustration 27: Find the locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse. **(JEE ADVANCED)**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

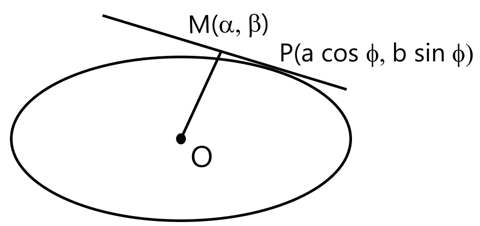


Figure 11.15

Sol: Follow the procedure for finding the locus starting from the parametric equation of the tangent.

The equation of the tangent at any point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1 \qquad ....(i)$$

Let $M(\alpha,\beta)$ be the foot of the perpendicular drawn from the centre (0, 0) to the tangent (i).

$x^2+y^2=C^2$ M is on the tangent, $\frac{\alpha\cos\phi}{a}+\frac{\beta\sin\phi}{b}=1$ ...(ii)

$x^2+y^2=C^2$ CM $\perp$ PM, $\frac{\beta}{\alpha}\left(-\frac{b\cos\phi}{a\sin\phi}\right)=-1$

or $\quad b\beta\cos\phi=a\sin\phi\,\alpha \qquad \therefore \quad \frac{\cos\phi}{a\alpha}=\frac{\sin\phi}{b\beta}=\frac{1}{\sqrt{a^2\alpha^2+b^2\beta^2}}$.

Putting in (ii), $\frac{\alpha}{a}.\frac{a\alpha}{\sqrt{a^2\alpha^2+b^2\beta^2}}+\frac{\beta}{b}.\frac{b\beta}{\sqrt{a^2\alpha^2+b^2\beta^2}}=1$

or $\quad \alpha^2+\beta^2=\sqrt{a^2\alpha^2+b^2\beta^2} \qquad \therefore \ (\alpha^2+\beta^2)^2=a^2\alpha^2+b^2\beta^2$

$\therefore$ The equation of the required locus is $(x^2+y^2)^2=a^2x^2+b^2y^2$.

12. NORMAL TO AN ELLIPSE

12.1 Equation of Normal in Different Forms

Following are the various forms of equations of the normal to an ellipse.

(a) Point form: The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

(b) Parametric form: The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\phi, b\sin\phi)$ is $x-2y+4=0$.

(c) Slope form: If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of normal is $x+2y+a=0$.

The co-ordinates of the point of contact are. $\left(\frac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \frac{\mp mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

12.2 Number of Normal and Co-normal Points

On a given ellipse exactly one normal can be drawn from a point lying on ellipse. If the point is not lying on the given ellipse, at most 4 lines which are normal to the ellipse at the points where they cut the ellipse. Such points on the ellipse are called co-normal points. In this section, we shall learn about the co-normal points and various relations between their eccentric angles.

Conormal points are the points on ellipse, whose normals to the ellipse pass through a given point are called co-normal points.

12.3 Properties of Eccentric Angles of Conormal Points

Property 1: The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

Property 2: If θ_1, θ_2 and θ_3 are eccentric angles of three co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$

Property 3: Co-normal points lie on a fixed curve called Apollonian Rectangular Hyperbola $\left(a^2-b^2\right)xy+b^2kx-a^2hy=0$

Property 4: If the normal at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

Illustration 28: If the normal at an end of the latus rectum of an ellipse passes through one extremity of the minor, show that the eccentricity of the ellipse is given by $e^4+e^2-1=0$. **JEE MAIN)**

Sol: Subtitute the point (0, ±b) in the equation of the normal and simplify it.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse. The coordinates of an end of the latus rectum are $(ae, b^2/a)$.

The equation of normal at $(ae, b^2/a)$ is $\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2$

It passes through one extremity of the minor axis whose coordinates are (0, ±b).

$\therefore \quad \pm ab=a^2-b^2$

$\Rightarrow \quad a^2b^2=(a^2-b^2)^2 \Rightarrow a^2.a^2(1-e^2)=(a^2e^2)^2 \Rightarrow 1-e^2=e^4 \Rightarrow e^4+e^2-1=0$

Illustration 29: Any ordinate MP of an ellipse meets the auxiliary circle in Q. Prove that the locus of the point of intersection of the normal P and Q is the circle $x^2+y^2=(a+b)^2$. **(JEE MAIN)**

Sol: Consider a point on the ellipse and find the intersection of the ordinate with the circle. Next find the intersection of the normal at P and Q and eliminate the parameter θ.

Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let $Q(a\cos\theta, a\sin\theta)$ be the corresponding point on the auxiliary circle $x^2+y^2=a^2$. The equation of the normal at $P(a\cos\theta, b\sin\theta)$ to the ellipse is

$ax\sec\theta-by\,\mathrm{cosec}\,\theta=a^2-b^2$ ...(i)

The equation of the normal at $Q(a\cos\theta, a\sin\theta)$ to the circle $x^2+y^2=a^2$ is

$y=x\tan\theta$ ...(ii)

Let (h , k) be the point of intersection of (i) and (ii). Then,

$ah\sec\theta-bk\,\mathrm{cosec}\,\theta=a^2-b^2$ ...(iii)

and, $P(a\cos\theta, b\sin\theta)$ ...(iv)

Eliminating θ from (iii) and (iv), we get

$$ah\sqrt{1+\frac{k^2}{h^2}}-bk\sqrt{1+\frac{h^2}{k^2}}=a^2-b^2$$

$\Rightarrow \quad (a-b)\sqrt{h^2+k^2}=a^2-b^2 \Rightarrow h^2+k^2=(a+b)^2$

Hence, the locus of (h, k) is $x^2+y^2=(a+b)^2$.

Illustration 30: If the length of the major axis intercepted between the tangent and normal at a point $\begin{vmatrix} \sec\theta & \mathrm{cosec}\,\theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \mathrm{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta-\frac{2\pi}{3}\right) & \mathrm{cosec}\left(\theta-\frac{2\pi}{3}\right) & 1 \end{vmatrix}$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to the semi-major axis, prove that the eccentricity of the ellipse is given by $e=\{\sec\theta(\sec\theta-1)\}^{1/2}$. **(JEE MAIN)**

Sol: Obtain the points of intersection of the tangent and the normal and then use the distance formula.

The equation of the tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point P

$$\begin{vmatrix} \sec\theta & \operatorname{cosec}\theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \operatorname{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta-\frac{2\pi}{3}\right) & \operatorname{cosec}\left(\theta-\frac{2\pi}{3}\right) & 1 \end{vmatrix}$$ are given by

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \text{...(i)}$$

and, $ax\sec\theta - by\operatorname{cosec}\theta=(a^2-b^2)$ ...(i) respectively.

Suppose (i) and (ii) meet the major axis i.e. y = 0 at Q and R respectively. Then, the coordinates of Q and R are given by

$Q(a\sec\theta,0)$ and $R\left(\frac{a^2-b^2}{a}\cos\theta,0\right)$ $\quad\therefore\quad$ QR = a [Given]

$$\Rightarrow \quad a\sec\theta-\frac{a^2-b^2}{a}\cos\theta=a \Rightarrow a^2-(a^2-b^2)\cos^2\theta=a^2\cos\theta \Rightarrow a^2-a^2e^2\cos^2\theta=a^2\cos\theta$$

$$\Rightarrow \quad 1-e^2\cos^2\theta=\cos\theta \Rightarrow e^2\cos^2\theta=1-\cos\theta \Rightarrow e^2=\sec\theta(\sec\theta-1) \Rightarrow e=\{\sec\theta(\sec\theta-1)\}^{1/2}$$

Illustration 31: If ω is one of the angles between the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$, then prove that $\frac{2\cot\omega}{\sin 2\theta}=\frac{e^2}{\sqrt{1-e^2}}$. **(JEE ADVANCED)**

Sol: Evaluate the equation of the normal at the two points and then use the formula of the angle between two lines.

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$ are $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$ and, $-ax\operatorname{cosec}\theta-by\sec\theta=a^2b^2$ respectively. Since ω is the angle between these two normals, therefore, $\tan\omega=\left|\frac{(a/b)\tan\theta+(a/b)\cot\theta}{1-(a^2/b^2)}\right|$

$$\Rightarrow \quad \tan\omega=\left|\frac{ab(\tan\theta+\cot\theta)}{b^2-a^2}\right| \Rightarrow \tan\omega=\left|\frac{2ab}{(\sin 2\theta)(b^2-a^2)}\right| \Rightarrow \tan\omega=\frac{2ab}{(a^2-b^2)\sin 2\theta}$$

$$\Rightarrow \quad \tan\omega=\frac{2a^2\sqrt{1-e^2}}{a^2e^2\sin 2\theta} \Rightarrow \frac{2\cot\omega}{\sin 2\theta}=-\frac{e^2}{\sqrt{1-e^2}}$$

Illustration 32: If the tangent drawn at point $(t^2,2t)$ on the parabola $y^2=4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2+5y^2=20$, find the values of t and θ. **(JEE ADVANCED)**

Sol: Write the equation for the tangent and normal in terms of the parameter. Compare the two equations to get the values of t and θ.

The equation of the tangent at (t^2, 2t) to the parabola $y^2 = 4x$ is

$$2ty = 2(x + t) \Rightarrow ty = x + t^2 \left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right) \quad \text{... (i)}$$

The equation of the normal at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$ is

$$\Rightarrow \quad (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y - 1 = 0 \quad \text{... (ii)}$$

It is given that (i) and (ii) represent the same line. Therefore, $\frac{\sqrt{5}\sec\theta}{1} = \frac{-2\operatorname{cosec}\theta}{-t} = \frac{-1}{t^2}$

$$\Rightarrow \quad t = \frac{2\operatorname{cosec}\theta}{\sqrt{5}\sec\theta} \text{ and } t = -\frac{1}{2\operatorname{cosec}\theta} \Rightarrow t = \frac{2}{\sqrt{5}}\cot\theta \text{ and } t = -\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad \frac{2}{\sqrt{5}}\cot\theta = -\frac{1}{2}\sin\theta \Rightarrow 4\cos\theta = -\sqrt{5}\sin^2\theta \Rightarrow 4\cos\theta = -\sqrt{5}(1-\cos^2\theta)$$

$$\Rightarrow \quad \sqrt{5}\cos^2\theta - 4\cos\theta - \sqrt{5} = 0 \Rightarrow \sqrt{5}\cos^2\theta - 5\cos\theta + \cos\theta - \sqrt{5} = 0$$

$$\Rightarrow \quad \sqrt{5}\cos\theta(\cos\theta - \sqrt{5}) + (\cos\theta - \sqrt{5}) = 0 \Rightarrow (\cos\theta - \sqrt{5})(\sqrt{5}\cos\theta + 1) = 0$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \qquad [\because \cos\theta \neq -\sqrt{5}]$$

Putting $\cos\theta = -\frac{1}{\sqrt{5}}$ in $t = -\frac{1}{2}\sin\theta$ we get $t = -\frac{1}{2}\sqrt{1-\frac{1}{5}} = -\frac{1}{\sqrt{5}}$

Hence, $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ and $t = -\frac{1}{\sqrt{5}}$.

Illustration 33: The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points.

(JEE ADVANCED)

(a) $\left(\pm\frac{3\sqrt{5}}{7}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$ (c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Sol: Put y = 0 in the equation of the normal to get the point Q in terms of θ. Get the locus of the mid-point as required. In the last step solve the equation of the locus and the latus rectum.

Equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Equation of the normal at $P(4\cos\theta, 2\sin\theta)$ to the ellipse is

$4x\sec\theta - 2y\operatorname{cosec}\theta = 4^2 - 2^2 \quad \Rightarrow 2x\sec\theta - y\operatorname{cosec}\theta = 6$

It meets x-axis at $Q(3\cos\theta, 0)$. If (h, k) are the coordinates of M, then

$$h = \frac{4\cos\theta + 3\cos\theta}{2}, k = \frac{2\sin\theta + 0}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{2h}{7}, \sin\theta = k$$

$$\Rightarrow \frac{4h^2}{49} + k^2 = 1 \text{ Locus of M is} \Rightarrow \frac{x^2}{(7/2)^2} + \frac{y^2}{1} = 1.$$

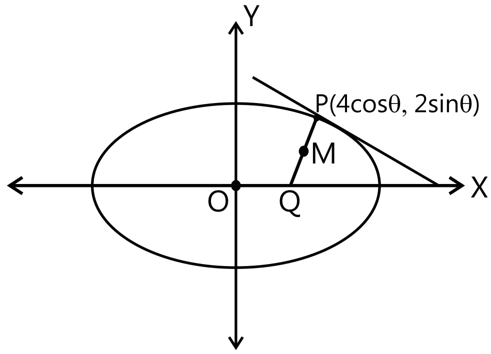


Figure 11.16

Latus rectum of the given ellipse is $x = \pm ae = \pm\sqrt{16-4} = \pm 2\sqrt{3}$

So locus of M meets the latus rectum at points for which $y^2 = 1 - \frac{12 \times 4}{49} = \frac{1}{49} \Rightarrow y = \pm\frac{1}{7}$

And hence the required points are $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$.

13. CHORD BISECTED AT A GIVEN POINT

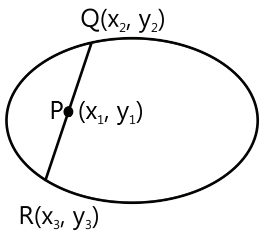


Figure 11.17

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

whose mid point is (x_1, y_1) is $T = S_1$

where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

$$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1.$$

Illustration 34: Find the locus of the midpoint of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: In the equation T = S_1, substitute x = ae and y = 0.

Let (h, k) be the midpoint of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the equation of the chord is

$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$ [Using : T = S_1]

or, $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ It passes through the focus (ae, 0) of the ellipse.

$\therefore \frac{hae}{a^2} + 0 = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. Hence, the locus of (h, k) is $\frac{xe}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Illustration 35: Find the locus of the mid-point of the normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(JEE ADVANCED)

Sol: Similar to the previous question.

Let (h, k) be the mid point of a normal chord of the given ellipse. Then, its equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \quad \text{[Using: T = } S_1\text{]}$$

or $$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \text{...(i)}$$

If (i) is a normal chord, then it must be of the form

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 \quad \text{...(ii)}$$

$$\therefore \frac{h}{a^3\sec\theta} = \frac{k}{-b^3\,\text{cosec}\,\theta} = \frac{\frac{h^2}{a^2} + \frac{k^2}{b^2}}{a^2 - b^2}$$

$$\Rightarrow \quad \cos\theta = \frac{a^3}{h(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right),\ \sin\theta = \frac{-b^3}{k(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Eliminating θ from the above relations, we get

$$\frac{a^6}{h^2(a^2-b^2)^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2 + \frac{b^6}{k^2(a^2-b^2)^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2 = 1 \qquad \Rightarrow \left(\frac{a^6}{h^2}+\frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2 = (a^2-b^2)$$

Hence, the locus of (h, k) is $\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = (a^2-b^2)^2$.

14. DIAMETERS

Definition: A chord through the centre of an ellipse is called a diameter of the ellipse.

The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = -\frac{b^2}{a^2m}x$, which is passing through (0, 0)

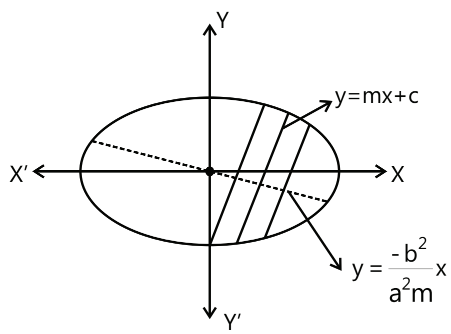


Figure 11.18

Conjugate diameter: Two diameters of an ellipse are said to be conjugate diameters if each bisects all chords parallel to the other. The coordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$; $P'(-a\cos\phi, -b\sin\phi)$

$Q(-a\sin\phi, b\cos\phi)$; $Q'(-a\cos\phi, -b\sin\phi)$

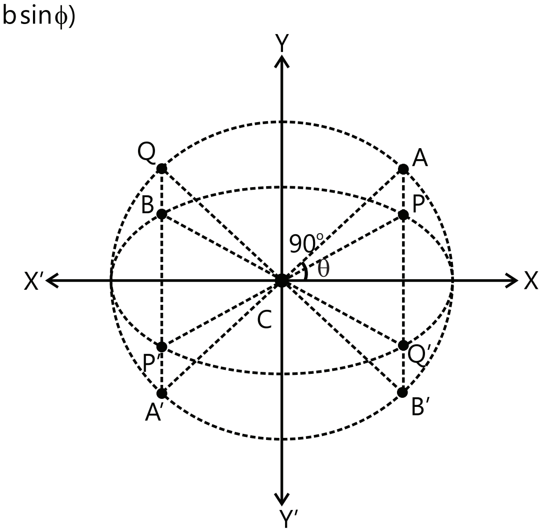


Figure 11.19

If $y = m_1 x$ and $y = m_2 x$ are two conjugate diameters of an ellipse, then $m_1 m_2 = \frac{-b^2}{a^2}$.

(a) Properties of diameters:

(i) The tangent at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter.

(ii) The tangents at the ends of any chord meets on the diameter which bisects the chord.

(b) Properties of conjugate diameters:

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle, i.e., $\phi - \phi' = \frac{\pi}{2}$

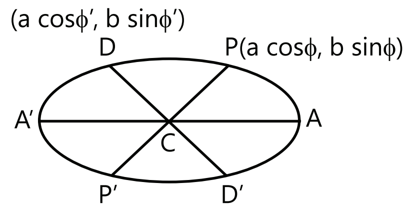


Figure 11.20

(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$.

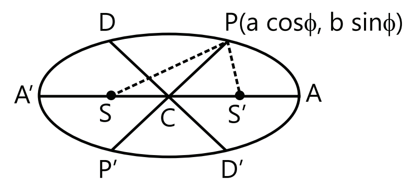


Figure 11.21

(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point i.e., $SP.S'P = CD^2$.

(iv) The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes i.e., Area of parallelogram = (2a)(2b) = Area of rectangle contained under major and minor axes.

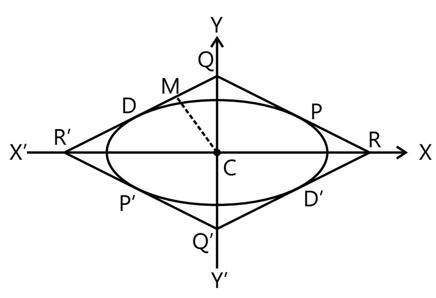


Figure 11.22

(v) The polar of any point with respect to an ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is (x_1, y_1), i.e., chord is $T = S_1$.

(vi) Major and minor axes of ellipse is also a pair of conjugate diameters.

(c) Equi-conjugate diameters: Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e., $(CP)^2 = (CD)^2$.

$\therefore (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

Illustration 36: If PCP′ and DCD′ form a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and R is any point on the circle $x^2+y^2=c^2$, then prove that $PR^2+DR^2+P'R^2+D'R^2=2(a^2+b^2+2c^2)$. **(JEE MAIN)**

Sol: Using the definition of conjugate diameters, get the coordinates of the point P, P′, Q and Q′. Starting from the L.H.S. prove the R.H.S.

Let R(h, k) be any point on the circle $x^2+y^2=c^2$. Then $h^2+k^2=c^2$ ...(i)

Since PCP′ and DCD′ form a pair of conjugate diameters, the coordinates of the extremities are:

$P(a\cos\theta, b\sin\theta)$, $P'(-a\cos\theta, -b\sin\theta)$ $D(-a\sin\theta, b\cos\theta)$, $D'(a\sin\theta, -b\cos\theta)$

$$\therefore \quad PR^2+DR^2+P'R^2+D'R^2 = (h-a\cos\theta)^2+(k-b\sin\theta)^2+(h+a\sin\theta)^2+(k-b\cos\theta)^2$$
$$+(h+a\cos\theta)^2+(k+b\sin\theta)^2+(h-a\sin\theta)^2+(k+b\cos\theta)^2$$

$= 4(h^2+k^2)+2a^2+2b^2$

$= 2a^2+2b^2+4c^2$ [Using (i)]

$= 2(a^2+b^2+2c^2)$

Illustration 37: CP and CD are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Show that the locus of the mid-point of PD is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$. **(JEE MAIN)**

Sol: Consider two points which lie on two conjugate diameters. Find the mid point of these two points and eliminate the parameter θ to get the locus of the mid point.

Let P $(a\cos\theta, b\sin\theta)$, D $(-a\sin\theta, b\cos\theta)$ and (h, k) be the mid-point of PD. Then,

$2h = a\cos\theta - a\sin\theta$ and $2k = b\sin\theta + b\cos\theta$

$$\Rightarrow \frac{2h}{a}=\cos\theta-\sin\theta \text{ and } \frac{2k}{b}=\sin\theta+\cos\theta \Rightarrow \frac{4h^2}{a^2}+\frac{4k^2}{b^2}=(\cos\theta-\sin\theta)^2+(\sin\theta+\cos\theta)^2$$

$\Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{1}{2}$. Hence, the locus of (h, k) is $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

Illustration 38: If $y=x$ and $3y+2x=0$ are the equations of a pair of conjugate diameters of an ellipse, then the eccentricity of the ellipse is **(JEE MAIN)**

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2}{\sqrt{5}}$

Sol: Use the condition of conjugacy of diameters in an ellipse to find the eccentricity.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Slope of the given diameters are $m_1=1$, $m_2=-\frac{2}{3}$.

$\Rightarrow m_1 m_2 = -\frac{2}{3} = -\frac{b^2}{a^2}$ [Using the condition of conjugacy of two diameters]

$3b^2 = 2a^2 \Rightarrow 3a^2(1-e^2) = 2a^2 \Rightarrow \quad 1-e^2 = \frac{2}{3} \quad \Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$

Illustration 39: Show that the locus of the point of intersection of tangents at the end-point of the conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is another ellipse of the same eccentricity. **(JEE ADVANCED)**

Sol: Using two points at the end points of the conjugate diameters of an ellipse, write the equation of the tangent. Solve the two equations to eliminate the parameter θ.

Let CP and CD be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the eccentric angles of P and D are θ and $\frac{\pi}{2}+\theta$ respectively. So, the coordinates of P and D are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively. The equation of the tangents at P and D are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad \text{...(i)}$$

and $$\frac{-x}{a}\sin\theta+\frac{y}{b}\cos\theta=1 \quad \text{...(ii)}$$

Let (h, k) be the point of intersection (i) and (ii). Then, $\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1$ and $\frac{-h}{a}\sin\theta+\frac{k}{b}\cos\theta=1$

$$\Rightarrow \left(\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta\right)^2+\left(-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta\right)^2=1+1 \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=2$$

Hence, the locus of (h, k) is $\frac{h^2}{a^2}+\frac{k^2}{b^2}=2$ which represents an ellipse of eccentricity e, given by

$$e_1 = \sqrt{1-\frac{2b^2}{2a^2}} = \sqrt{1-\frac{b^2}{a^2}}$$

Clearly, it is same as the eccentricity of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Illustration 40: If α and β are the angles subtended by the major axis of an ellipse at the extremities of a pair of conjugate diameters, prove that $\cot^2\alpha+\cot^2\beta$ = constant. **(JEE MAIN)**

Sol: Using the co-ordinates of the co-ordinates of the end points of a diameter, find the angle subtended by the major axis. Repeat the same process for the other end of the diameter. Then find the value of $\cot^2\alpha+\cot^2\beta$ and prove that it is independent of the parameter.

Let CP and CD be a pair of conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the coordinates of P and D are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively.

$$m_1 = \text{Slope of AP} = \frac{b\sin\theta}{a\cos\theta-a} = -\frac{b}{a}\cot\frac{\theta}{2}$$

$$m_2 = \text{Slope of A'P} = \frac{b\sin\theta}{a\cos\theta+a} = \frac{b}{a}\tan\frac{\theta}{2}$$

$$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad \Rightarrow \tan\alpha = \left|\frac{-(b/a)\cot(\theta/2) - (b/a)\tan(\theta/2)}{1 - (b^2/a^2)}\right|$$

$$\Rightarrow \tan\alpha = \frac{ab}{a^2 - b^2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) \Rightarrow \tan\alpha = \left(\frac{2ab}{a^2 - b^2}\right)\frac{1}{\sin\theta}$$

Replacing θ by $\left(\frac{\pi}{2} + \theta\right)$, we get $\tan\beta = \left(\frac{2ab}{a^2 - b^2}\right)\frac{1}{\cos\theta}$

$$\therefore \quad \cot^2\alpha + \cot^2\beta = \left(\frac{a^2 - b^2}{2ab}\right)^2 (\sin^2\theta + \cos^2\theta) = \left(\frac{a^2 - b^2}{2ab}\right)^2 = \text{Constant.}$$

Illustration 41: Find the locus of the points of intersection of normals at two points on an ellipse which are extremities of conjugate diameters. **(JEE MAIN)**

Sol: Solve the equation of the normal at the extremities of conjugate diameters.

Let PP′ and QQ′ be two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the eccentric angle of the point P be $'\phi'$. Then the eccentric angle of Q is $'\phi + \frac{\pi}{2}'$.

$$\therefore \quad P = (a\cos\phi, b\sin\phi)$$

$$Q = \left\{a\cos\left(\phi + \frac{\pi}{2}\right), b\sin\left(\phi + \frac{\pi}{2}\right)\right\}$$

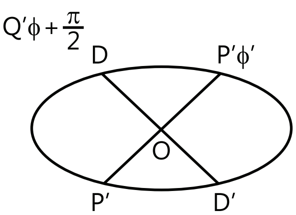


Figure 11.23

The equation of the normal at $P = (a\cos\phi, b\sin\phi)$ is $\frac{x - a\cos\phi}{(a\cos\phi)/a^2} = \frac{y - b\sin\phi}{(b\sin\phi)/b^2}$

or $\quad ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2$ ...(i)

Similarly, the equation of the normal at Q is

$$ax\sec\left(\phi + \frac{\pi}{2}\right) - by\,\text{cosec}\left(\phi + \frac{\pi}{2}\right) = a^2 - b^2$$

or $\quad -ax\,\text{cosec}\,\phi - by\sec\phi = a^2 - b^2$ ...(ii)

The locus of the point of intersection of (i) and (ii) is obtained by eliminating ϕ from them. Now we have

$$ax\sec\phi - by\,\text{cosec}\,\phi - (a^2 - b^2) = 0$$

$$by\sec\phi + ax\,\text{cosec}\,\phi + (a^2 - b^2) = 0$$

By cross multiplication,

$$\frac{\sec\phi}{-by + ax} = \frac{\text{cosec}\,\phi}{-by - ax} = \frac{a^2 - b^2}{a^2x^2 + b^2y^2}$$

$$\therefore \quad \cos\phi = \frac{a^2x^2 + b^2y^2}{ax - by}\cdot\frac{1}{a^2 - b^2}$$

$$\sin\phi = \frac{a^2x^2 + b^2y^2}{-(ax + by)}\cdot\frac{1}{a^2 - b^2}$$

Squaring and adding,

$$1=\frac{(a^2x^2+b^2y^2)^2}{(a^2-b^2)^2}\left\{\frac{1}{(ax-by)^2}+\frac{1}{(ax+by)^2}\right\}=\left(\frac{a^2x^2+b^2y^2}{a^2-b^2}\right)^2.\frac{2(a^2x^2+b^2y^2)}{(a^2x^2-b^2y^2)^2}$$

$\Rightarrow\quad 2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2.(a^2x^2-b^2y^2)^2.$

15. POLE AND POLAR

Let P(x_1, y_1) be any point inside or outside the ellipse. A chord through P intersects the ellipse at A and B respectively. If tangents to the ellipse at A and B meet at Q(h, k) then locus of Q is called polar of P with respect to the ellipse and point P is called the pole.

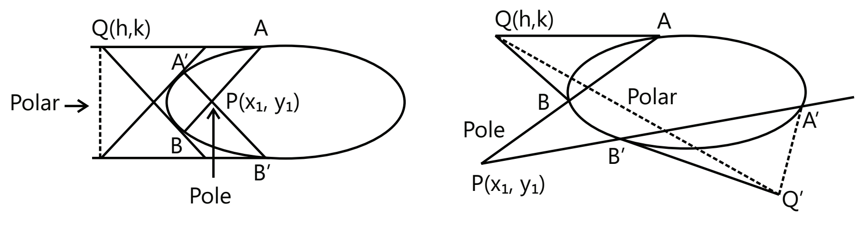


Figure 11.24

Note: If the pole lies outside the ellipse then the polar passes through the ellipse. If the pole lies inside the ellipse then the polar lies completely outside the ellipse. If the pole lies on the ellipse then the polar becomes the same as the tangent.

Equation of polar: Equation of polar of the point (x_1, y_1) with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, i.e., T = 0

Coordinates of Pole: The pole of the line $lx+my+n=0$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n},\frac{-b^2m}{n}\right)$.

Properties of pole and polar:

(a) If the polar of $P(x_1,y_1)$ passes through $Q(x_2,y_2)$, then the polar of $Q(x_2,y_2)$ goes through $P(x_1,y_1)$ and such points are said to be the conjugate points. Condition for conjugate points is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

(b) If the pole of line $l_1x+m_1y+n_1=0$ lies on another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(c) Pole of a given line is the same as the point of intersection of tangents at its extremities.

(d) Polar of focus is directrix.

Illustration 42: Obtain the locus of poles of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to concentric ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. **(JEE MAIN)**

Sol: Taking a point (h , k), write the equation of the polar w.r.t. the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. In the next step put the condition for polar to be the tangent to the other given ellipse.

The equation of the polar is $\frac{hx}{\alpha^2}+\frac{ky}{\beta^2}=1 \Rightarrow y=-\left(\frac{\beta^2 h}{\alpha^2 k}\right)x+\frac{\beta^2}{k}$

This touches $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Therefore, $\left(\frac{\beta^2}{k}\right)^2=a^2\left(\frac{-\beta^2 h}{\alpha^2 k}\right)^2+b^2$

$$\Rightarrow \quad \frac{\beta^4}{k^2}=a^2\frac{\beta^4 h^2}{\alpha^4 k^2}+b^2 \Rightarrow \frac{a^2h^2}{\alpha^4}+\frac{b^2k^2}{\beta^4}=1$$

Hence, the locus of (h, k) is $\Rightarrow \frac{a^2x^2}{\alpha^4}+\frac{b^2y^2}{\beta^4}=1$.

Illustration 43: Find the locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose poles are on the auxiliary circle or the tangents at the extremities of which intersect on the auxiliary circle. **(JEE ADVANCED)**

Sol: Compare the equation of the chord and the tangent to get the point which lies on the auxiliary circle. Substitute the point in the equation of the circle to get the required locus.

Let (h, k) be the mid-point of a chord of the ellipse. Then, its equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \text{...(i)}$$

Let (x_1, y_1) be its pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the equation of the polar is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{...(ii)}$$

Clearly, (i) and (ii) represent the same line. Therefore,

$$\frac{x_1}{h}=\frac{y_1}{k}=\frac{1}{(h^2/a^2)+(k^2/b^2)} \Rightarrow x_1=\frac{h}{(h^2/a^2)+(k^2/b^2)},\ y_1=\frac{k}{(h^2/a^2)+(k^2/b^2)}$$

It is given that (x_1, y_1) lies on auxiliary circle. Therefore $x_1^2+y_1^2=a^2 \Rightarrow h^2+k^2=a^2\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2$.

Hence the locus of (h, k) is $x^2+y^2=a^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$.

16. SUBTANGENT AND SUBNORMAL

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively. Length of subtangent at $P(x_1, y_1)$ to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $DA=CA-CD=\frac{a^2}{x_1}-x_1$

Length of sub-normal at $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$BD=CD-CB=x_1-\left(x_1-\frac{b^2}{a^2}x_1\right)=\frac{b^2}{a^2}x_1=(1-e^2)x_1.$$

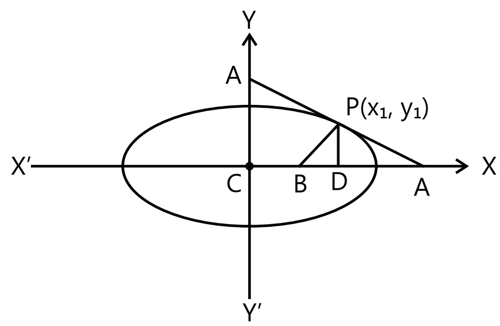


Figure 11.25

MASTERJEE CONCEPTS

Misconception: As there is no y_1 term involved in the above results, don't think that the lengths are independent of y_1. Always remember that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Vaibhav Krishnan (JEE 2009, AIR 22)

PROBLEM-SOLVING TACTICS

- If the line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c^2=\frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$ is the condition of normality of the line to the ellipse.
- The tangent and normal at any point of an ellipse bisect the external and angles between the focal radii to the point. It follows from the above property that if an incident light ray passing through the focus (S) strikes the concave side of the ellipse, then the reflected ray will pass through the other focus (S').
- If SM and S'M' are perpendicular from the foci upon the tangent at any point of the ellipse, then SM. S'M' = b^2 and M, M' lie on the auxiliary circle.
- If the tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major axis in T and minor axis in T', then CN. CT = a^2, CN'. CT' = b^2

  Where N and N' are the feet of the perpendicular from P on the respective axis.
- If SM and S' M' are perpendicular from the foci S and S' respectively upon a tangent to the ellipse, then CM and CM' are parallel to S'P and SP respectively.

FORMULAE SHEET

1. The general equation of second order $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents an ellipse

if $\Delta \neq 0$, $h^2 < ab$. where $\left(\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

2. The sum of the focal distance of any point on an ellipse is a constant and is equal to the length of the major axis of the ellipse i.e. SP + S'P = 2a.

3. Standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Where a = length of semi-major axis,

b = length of semi-minor axis

4.

Imp. Terms \ **Ellipse**	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	For b > a
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	$(\pm ae , 0)$	$(0 , \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2 (1 - e^2)$	$a^2 = b^2 (1 - e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	$(a \cos \phi, b \sin\phi)$	$(a \cos \phi, b \sin \phi)$ $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii (SP + S'P =)	2a	2b
Distance between foci	2ae	2be
Distance between directrices	2a/e	2b/e
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

5. The equations $x = a\cos\phi,\ y = b\sin\phi$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.

6. **(i)** If the centre of the ellipse is at (h, k) and the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(ii) If the equation of the ellipse is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $lx+my+n=0$ and $mx-ly+p=0$ are perpendicular lines. Substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$ and $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

7. If $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ are any two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of a chord joining these two points is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

8. The point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to $\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1>0, =0$ or <0 respectively.

9. The line $y=mx+c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on two distinct points if $a^2m^2+b^2>c^2$, on one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the auxiliary circle is $x^2+y^2=a^2$.

10. The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. The equation of tangent to the ellipse having its slope equal to m is $y=mx\pm\sqrt{a^2m^2+b^2}$ and the point of contact is $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$. The equation of the tangent at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $P(a\cos\theta_1, b\sin\theta_1)$, and $Q(a\cos\theta_2, b\sin\theta_2)$ is $\left(\frac{a\cos\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}, \frac{b\sin\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}\right)$.

11. Equation of pair of tangents drawn from an outside point $P(x_1, y_1)$ is $SS_1=T^2$.

12. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of director circle is $x^2+y^2=a^2+b^2$.

13. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point $(a\cos\phi, b\sin\phi)$ is $(ax\sec\phi-by\operatorname{cosec}\phi)=a^2-b^2$.

14. If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the normal is $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$. The co-ordinates of the point of contact are $\left(\frac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \frac{\pm mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

15. The properties of conormal points are

(i) **Property 1:** The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

(ii) **Property 2:** If θ_1, θ_2 and θ_3 are eccentric angles of three co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$.

(iii) **Property 3:** Co-normal points lie on a fixed curve called an Apollonian Rectangular Hyperbola $\left(a^2-b^2\right)xy+b^2kx-a^2hy=0$

(iv) **Property 4:** If the normal at four points $P(x_1y_1)$, $Q(x_2y_2)$, $R(x_3y_3)$ and $S(x_4y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

16. If SM and S′M′ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \times S'M' = b^2$ and M, M′ lie on the auxiliary circle.

17. If the tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major axis at T and minor axis at T′, then CN × CT = a^2, CN′ × CT′ = b^2. Where N and N′ are the feet of the perpendiculars from P on the respectively axis.

18. The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point is (x_1, y_1), is $T = S_1$.

19. The chord of contact from a point $P(x_1, y_1)$ to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0 is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

20. The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = -\frac{b^2}{a^2m}x$.

21. If m_1 and m_2 are the slopes of two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2 = \frac{-b^2}{a^2}$.

22. The eccentric angle of the ends of a pair of conjugate diameters of an ellipse differ by a right angle, i.e., $\phi - \phi' = \frac{\pi}{2}$.

23. The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and is equal to the sum of the squares of the semi axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$.

24. The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point i.e., $SP \times S'P = CD^2$.

25. The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes.

i.e. Area of the parallelogram = (2a)(2b) = Area of the rectangle contained under major and minor axes.

26. Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e., $(CP)^2 = (CD)^2$

$\therefore (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

27. Equation of the polar of the point (x_1, y_1) w.r.t. an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$.

28. The pole of the line $lx + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$.

29. Condition for a conjugate point is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

30. The length of a sub tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2}{x_1} - x_1$.

31. The length of a sub normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{b^2}{a^2}x_1 = (1-e^2)x_1$.