Solved Examples

JEE Main/Boards

Example 1: Find the centre, the eccentricity, the foci, the directrices and the lengths and the equations of the axes of the ellipse

$$5x^2 + 9y^2 + 10x - 36y - 4 = 0$$

Sol: Rewrite the equation in the standard form and compare them to get the centre, eccentricity etc.

$5x^2 + 9y^2 + 10x - 36y - 4 = 0$, the given equation can be written as

$$5(x^2 + 2x) + 9(y^2 - 4y) = 4$$

$$5(x+1)^2 + 9(y-2)^2 = 45$$

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{5} = 1$$

Shift the origin to $O' \equiv (-1, 2)$

$\therefore \quad X = x + 1; \; Y = y - 2$

$\therefore \quad \frac{X^2}{9} + \frac{Y^2}{5} = 1$ ...(i)

This is in standard form

$\therefore \quad a = 3, b = \sqrt{5}$

$\therefore \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4}{9} \Rightarrow e = \frac{2}{3}$

Also $ae = 3 \cdot \frac{2}{3} = 2$ and $\frac{a}{e} = \frac{9}{2}$.

Now for an ellipse in the standard form we have Centre $\equiv$ (0, 0); foci $\equiv$ ($\pm$ae, 0); directrices $x = \pm\frac{a}{e}$; axes x = 0, y = 0, length of major axis = 2a, length of minor axis = 2b.

Now for (i) the centre is given by X = 0, Y = 0

$\Rightarrow \quad x + 1 = 0, y - 2 = 0$

i.e. Centre $\frac{ax}{3} + \frac{by}{4} = c$

Foci are given by $X = \pm ae, Y = 0$

i.e. $x + 1 = \pm 2$ and $y - 2 = 0$

i.e. $x = 1, y = 2$ and $x = -3, y = 2$

$\therefore$ Foci $\equiv (1, 2); (-3, 2)$

The equation of directrices are given by $X = \pm\frac{a}{e}$

i.e. $x + 1 = \pm\frac{9}{2}$

i.e. $x = \frac{7}{2}, x = -\frac{11}{2}$

The equation of the axes are given by

X = 0, Y = 0

i.e. $x + 1 = 0, y - 2 = 0$

i.e. $x = -1, y = 2$

Length of the axes being 2a, 2b

i.e, 6, $2\sqrt{5}$.

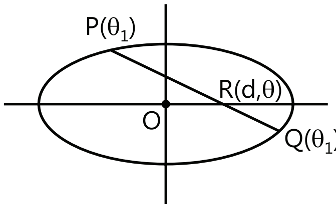
Example 2: If the chord through point θ_1 and θ_2 on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersects the major axis at (d, 0) prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{d-a}{d+a}$.

Sol: Substitute the point (d , 0) in the equation of the chord to prove the given result.

Equation of the chord joining the points θ_1 and θ_2 is

$$\frac{x}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

Since (d, 0) lies on it



$$\therefore \quad \frac{d}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

$$\frac{\cos\left((\theta_1 - \theta_2)/2\right)}{\cos\left((\theta_1 + \theta_2)/2\right)} = \frac{d}{a}$$

Applying componendo and dividendo, we get

$$\frac{d-a}{d+a} = \frac{\cos\left((\theta_1 - \theta_2)/2\right) - \cos\left((\theta_1 + \theta_2)/2\right)}{\cos\left((\theta_1 - \theta_2)/2\right) + \cos\left((\theta_1 + \theta_2)/2\right)}$$

$$=\frac{2\sin(\theta_1/2)\sin(\theta_2/2)}{2\cos(\theta_1/2)\cos(\theta_2/2)}=\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$$

Example 3: A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches it at the point P in the first quadrant and meets the x and y axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent at P.

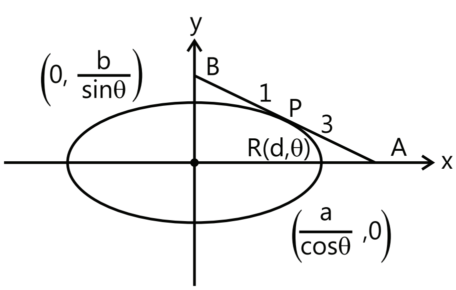
Sol: Consider a point in the parametric form and obtain the points A and B. Now use the condition that the point P divides AB in the ratio 3:1.

Let $P\equiv(a\cos\theta, b\sin\theta)$:

$$0<\theta<\frac{\pi}{2} \quad \text{...(i)}$$

Equation of the tangent at $P(\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$



$$\therefore A\equiv\left(\frac{a}{\cos\theta},0\right) \quad \text{and } B\equiv\left(0,\frac{b}{\sin\theta}\right)$$

Now P divides segment AB in the ratio 3 : 1

$$\therefore \quad P\equiv\left(\frac{a}{4\cos\theta},\frac{3b}{4\sin\theta}\right) \quad \text{...(ii)}$$

By (i) and (ii), we have

$$\therefore \quad \cos\theta=\frac{1}{2};\ \sin\theta=\frac{\sqrt{3}}{2}$$

$\therefore$ Equation of tangent at P is $bx+a\sqrt{3}y=2ab$.

Example 4: If the tangent drawn at a point $(t^2,2t); t\neq 0$ on the parabola $y^2=4x$ is the same as the normal drawn at a point $\left(\sqrt{5}\cos\phi, 2\sin\phi\right)$ on the ellipse $4x^2+5y^2=20$, find the value of t and ϕ.

Sol: Write the equation of the tangent and the normal using 't' and 'ϕ' and compare.

Equation of the tangent at $P(t^2,2t)$ to $y^2=4x$ is $yt=x+t^2$ ...(i)

Equation of normal at $Q\left(\sqrt{5}\cos\phi, 2\sin\phi\right)$ is

$$2y\cos\phi=x\sqrt{5}\sin\phi-\sin\phi\cos\phi \quad \text{...(ii)}$$

Equation (i) and (ii) represent the same line. Comparing the coefficients in equations (i) and (ii).

$$\frac{t}{2\cos\phi}=\frac{1}{\sqrt{5}\sin\phi}=\frac{t^2}{-\sin\phi.\cos\phi}$$

$$\Rightarrow \quad t=\frac{2}{\sqrt{5}}\cot\phi,\ t^2=-\frac{\cos\phi}{\sqrt{5}}$$

$$\frac{4}{5}\cot^2\phi=-\frac{\cos\phi}{\sqrt{5}}$$

$$\Rightarrow \quad \cos\phi\left(\frac{4\cos\phi}{\sin^2\phi}+\sqrt{5}\right)=0$$

$(\cos\phi\neq 0 \because t\neq 0)$

$$\Rightarrow \quad \frac{(x-5)^2}{9}+\frac{y^2}{25}=1$$

$$\Rightarrow \quad \sqrt{5}\left(1-\cos^2\phi\right)+4\cos\phi=0$$

$$\Rightarrow \quad \cos\phi=-\frac{1}{\sqrt{5}}$$

$$\therefore \phi=2n\pi\pm\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \text{ where } 0\le\phi\le 2\pi.$$

$\therefore$ Corresponding values of t are given by

$$t^2=-\frac{\cos\phi}{\sqrt{5}}=\frac{1}{5}.$$

$$\therefore \quad t=\pm\frac{1}{\sqrt{5}}$$

Example 5: Show that the sum of the squares of the perpendiculars on any tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from 2 points on the minor axis, each of which is at a distance $\sqrt{a^2-b^2}$ from the centre, is $2a^2$.

Sol: Use the standard equation of a tangent in terms of m and then proceed accordingly,

The general equation of a tangent to the ellipse is

$$y=mx\pm\sqrt{a^2m^2+b^2} \quad \text{...(i)}$$

Let the points on the minor axis be $P(0,ae)$ and $Q(0,-ae)$ as $b^2=a^2(1-e^2)$

Length of the perpendicular from P on (i) is

$$P_1 = \frac{\left|ae \pm \sqrt{a^2m^2 + b^2}\right|}{\sqrt{1+m^2}}$$

Similarly, $P_2 = \frac{\left|-ae \pm \sqrt{a^2m^2 + b^2}\right|}{\sqrt{1+m^2}}$

Hence, $P_1^2 + P_2^2 = \frac{2}{1+m^2}\{a^2e^2 + (a^2m^2 + b^2)\}$

$$= \frac{2}{1+m^2}\{(a^2 - b^2) + a^2m^2 + b^2\} = 2a^2$$

Example 6: Find the equation of the ellipse having its centre at the point (2,–3), one focus at (3,–3) and one vertex at (4, –3).

Sol: Use the basic knowledge of the major axis, centre and focus to get the equation of the ellipse.

$C \equiv (2,-3)$, $S \equiv (3,-3)$ and $A \equiv (4,-3)$

Now, CA = 2

$\therefore \quad a = 2$

Again CS = 1

$\therefore \quad ae = 1 \qquad \Rightarrow e = \frac{1}{a} = \frac{1}{2}$

We know that

$b^2 = a^2 - a^2e^2$

$\Rightarrow \quad b = \sqrt{3}$

$\therefore$ Equation of ellipse is

$$\frac{(x-2)^2}{2^2} + \frac{(y+3)^2}{(\sqrt{3})^2} = 1$$

$\Rightarrow \quad 3(x-2)^2 + 4(y+3)^2 = 12$

$\Rightarrow \quad 3x^2 + 4y^2 - 12x + 24y + 36 = 0$

Example 7: Show that the angle between pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2) is $\tan^{-1}\left(-\frac{12}{\sqrt{5}}\right)$.

Sol: Starting from the standard equation of a tangent in terms of m, satisfy the point (1 ,2) and get the values of m. Using the value of m, find the angle between the two tangents. Let the equation of the tangents be

$y = mx \pm \sqrt{a^2m^2 + b^2}$. It passes through (1, 2)

$\therefore \quad (2-m)^2 = \frac{5}{3}m^2 + \frac{5}{2}$

$\Rightarrow \quad 4m^2 + 24m - 9 = 0$

Angle between the tangents is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{4\sqrt{36+9}}{-5} = -\frac{12}{\sqrt{5}}$$

$\therefore \quad \theta = \tan^{-1}\left(-\frac{12}{\sqrt{5}}\right)$

Example 8: The locus of the foot of the perpendicular drawn from the centre to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(A) A circle (B) An ellipse

(C) A hyperbola (D) None of these.

Sol: Find the foot of the perpendicular from the centre to any tangent and eliminate the parameter.

Equation of a tangent to the ellipse is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \qquad \text{...(i)}$$

Equation of the line through the centre (0, 0) perpendicular to (i) is

$$y = \left(\frac{-1}{m}\right)x \qquad \text{...(ii)}$$

Eliminating m from (i) and (ii) we get the required locus of the foot of the perpendicular as

$$y = -\frac{x^2}{y} \pm \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$

which does not represent a circle, an ellipse or a hyperbola.

Example 9: The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). The equation of the ellipse is

(A) $4x^2 + 48y^2 = 48$ (B) $4x^2 + 6y^2 = 48$

(C) $x^2 + 16y^2 = 16$ (D) $x^2 + 12y^2 = 16$

Sol: Consider the standard equation of the ellipse. Use the two points given in the question to find the value of ‘a’ and ‘b’.

Let the equation of the required ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Given that it passes through (4, 0)

$\Rightarrow$ a = 4

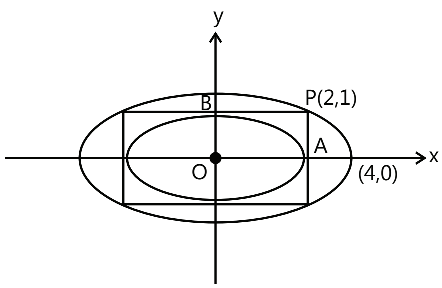
It also passes through (2, 1), one of the vertex of rectangle.

$\Rightarrow \quad \frac{4}{4^2}+\frac{1}{b^2}=1$

$\Rightarrow \quad b^2=\frac{4}{3}$ and the required equation is

$\frac{x^2}{16}+\frac{3y^2}{4}=1$

$\Rightarrow \quad x^2+12y^2=16$



Example 10: Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at point A and B. The equation of the locus of the point whose distances from the point P and the line AB are equal is

(A) $9x^2+y^2-6xy-54x-62y+241=0$

(B) $x^2+9y^2+6xy-54x+62y-241=0$

(C) $9x^2+9y^2-6xy-54x-62y-241=0$

(D) $x^2+y^2-2xy+27x+31y-120=0$

Sol: Write the equation of the chord of contact w.r.t. point P. Then follow the standard procedure to find the locus.

AB being the chord of contact of the ellipse from P(3, 4) has its equation

$\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow x+3y=3$

If (h, k) is any point on the locus, then

$\sqrt{(h-3)^2+(k-4)^2}=\left|\frac{h+3k-3}{\sqrt{1+9}}\right|$

$\Rightarrow 10(h^2+k^2-6h-8k+25)=(h+3k-3)^2$

Locus of (h, k) is

$9x^2+y^2-6xy-54x-62y+241=0$.

Example 11: If an ellipse slides between two perpendicular straight lines, then the locus of its centre is

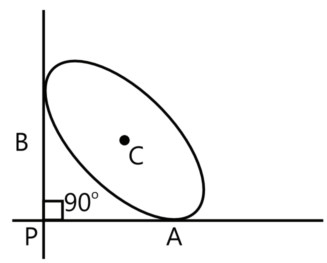
(A) A parabola (B) An ellipse

(C) A hyperbola (D) A circle

Sol: Use the concept of a Director Circle.

Let 2a, 2b be the length of the major and minor axes respectively of the ellipse. If the ellipse slides between two perpendicular lines, the point of intersection P of these lines being the point of intersection of perpendicular tangents lies on the director circle of the ellipse. This means that the centre C of the ellipse is always at a constant distance $\sqrt{a^2+b^2}$ from P. Hence the locus of C is a circle

$\frac{x^2}{9}+\frac{y^2}{4}=1$



Example 12: If α, β are the eccentric angles of the extremities of a focal chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, then $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=$

Sol: Equate the slope of the line joining the focus and the two points.

The eccentricity $e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$.

Let P $(4\cos\alpha, 3\sin\alpha)$ and $Q(4\cos\beta, 3\sin\beta)$ be a focal chord of the ellipse passing through the focus at $(\sqrt{7},0)$.

Then $\frac{3\sin\beta}{4\cos\beta-\sqrt{7}}=\frac{3\sin\alpha}{4\cos\alpha-\sqrt{7}}$

$\Rightarrow \quad \frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}=\frac{\sqrt{7}}{4}$

$$\Rightarrow \quad \frac{\cos[(\alpha-\beta)/2)]}{\cos[(\alpha+\beta)/2]} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{\sqrt{7}-4}{\sqrt{7}+4} = \frac{8\sqrt{7}-23}{9}.$$

JEE Advanced/Boards

Example 1: Common tangents are drawn to the parabola $y^2 = 4x$ and the ellipse $3x^2 + 8y^2 = 48$ touching the parabola A and B and the ellipse at C and D. Find the area of the quadrilateral ABCD.

Sol: Write the standard equation of the parabola in the slope. Use the condition for the line to be a tangent and obtain the value of m. We then find the points of contact with the ellipse and parabola and then find the area.

Let $y = mx + \frac{1}{m}$ be a tangent to the parabola $y^2 = 4x$.

It will touch the ellipse $\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{6})^2} = 1$, if $\frac{1}{m^2} = 16m^2 + 6$

[Using: $c^2 = a^2m^2 + b^2$]

$$\Rightarrow \quad 16m^4 + 6m^2 - 1 = 0$$

$$\Rightarrow \quad (8m^2-1)(2m^2+1) = 0 \;\Rightarrow\; m = \pm\frac{1}{\sqrt{8}}$$

We know that a tangent of slope m touches the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. So, the coordinates of the points of contact of the common tangents of slope $m = \pm\frac{1}{2\sqrt{2}}$ to the parabola $y^2 = 4x$ are A(8, $4\sqrt{2}$) and B(8, $-4\sqrt{2}$).

We also know that a tangent of slope m touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Therefore, the coordinates of the points of contact of common tangents of slope $m \pm \frac{1}{2\sqrt{2}}$ to the ellipse are $C\left(-2, \frac{3}{\sqrt{2}}\right)$ and $D\left(-2, -\frac{3}{\sqrt{2}}\right)$.

Clearly AB || CD. So, the quadrilateral ABCD is a trapezium.

We have, AB $= 8\sqrt{2}$, CD $= 3\sqrt{2}$ and the distance between AB and CD is

PQ = 8 + 2 = 10

$\therefore$ Area of quadrilateral ABCD

$$= \frac{1}{2}(AB + CD)PQ = \frac{1}{2}\left(8\sqrt{2} + 3\sqrt{2}\right)10 = 55\sqrt{2} \text{ sq. units.}$$

Example 2: Show that the angle between the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b$), and the circle $x^2 + y^2 = ab$ at their points of intersection in the first quadrant is $\tan^{-1}\left(\frac{(a-b)}{\sqrt{ab}}\right)$.

Sol: We find the point of intersection of the ellipse and the circle. Then we find the slope of the tangents to the circle and the ellipse and hence the angle.

At the points of intersection of ellipse and circle,

$$\frac{ab-y^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 1 - \frac{b}{a} \;\Rightarrow\; y^2 = \frac{ab^2}{a+b}$$

$$\therefore \quad y = \pm b\sqrt{\frac{a}{a+b}} \text{ and } x = \pm a\sqrt{\frac{b}{a+b}}$$

$$P\left(\frac{a\sqrt{b}}{\sqrt{a+b}}, \frac{b\sqrt{a}}{\sqrt{a+b}}\right)$$

lies in first quadrant

Equation of tangent at P to the circle is

$$\frac{xa\sqrt{b}}{\sqrt{a+b}} + \frac{yb\sqrt{a}}{\sqrt{a+b}} = ab$$

Its slope is: $m_1 = -\frac{\sqrt{a}}{\sqrt{b}}$

Equation of the tangent at P to the ellipse is

$$\frac{xa\sqrt{b}}{a^2\sqrt{a+b}} + \frac{yb\sqrt{a}}{b^2\sqrt{a+b}} = 1$$

Its slope in $m_2 = -\frac{b^{3/2}}{a^{3/2}}$

If α is the angle between these tangents, then

$$\tan\alpha = \frac{|m_2 - m_1|}{|1 + m_1m_2|} = \left|\frac{-(b^{3/2}/a^{3/2}) + (a^{1/2}/b^{1/2})}{1 + (b^{3/2}/a^{3/2})(a^{1/2}/b^{1/2})}\right|$$

$$\frac{a^2-b^2}{a^{1/2}b^{1/2}(a+b)} = \frac{a-b}{\sqrt{ab}}.$$

Example 3: Any tangent to an ellipse is cut by the tangents at the extremities of the major axis at T and T′. Prove that the circle on TT′ as the diameter passes through the foci.

Sol: We find out the point of intersection of the tangent with the axis and then use these points to find the equation of the circle.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The extremities A and A′ of the major axis are A (a, 0), A′ (–a, 0). Equations of tangents A and A′ are x = a and x = –a. Any tangent to the ellipse is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

The points of intersection are

$$T\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right), T'\left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

The equation of the circle on TT′ as diameter is

$$\left(x^2-a^2\right)+\left(\left(y-\frac{b}{\sin\theta}\right)^2-\left(\frac{b\cos\theta}{\sin\theta}\right)^2\right)=0$$

$$\Rightarrow x^2-a^2+y^2-\frac{b}{\sin\theta}2y+\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}=0$$

$$\Rightarrow \quad x^2+y^2-\frac{2by}{\sin\theta}+b^2-a^2=0$$

$$\Rightarrow \quad x^2+y^2-\frac{2by}{\sin\theta}=a^2e^2$$

Foci S (ae, 0) and S′ (–ae, 0) lie on this circle.

Example 4: Let ABC be an equilateral triangle inscribed in the circle $x^2+y^2=a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a > b), meets the ellipse at P, Q, R respectively so that P, Q, R lie on the same side of the major axis as are the corresponding points A, B, C. Prove that the normals to the ellipse drawn at the points P, Q, R are concurrent.

Sol: Find the points of intersection of the perpendicular and the ellipse. Then apply the condition for the normals at these three points to be concurrent.

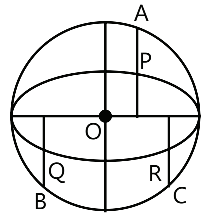
Let A, B, C have coordinates $(a\cos\theta, a\sin\theta)$,

$$\left[a\cos\left(\theta+\frac{2\pi}{3}\right), a\sin\left(\theta+\frac{2\pi}{3}\right)\right],$$

$$\left[a\cos\left(\theta+\frac{4\pi}{3}\right), a\sin\left(\theta+\frac{4\pi}{3}\right)\right]$$ respectively.

Then P, Q, and R have coordinates given by:

$P(a\cos\theta, b\sin\theta)$ $Q\left[a\cos\left(\theta+\frac{2\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right]$ and



$R\left[a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{4\pi}{3}\right)\right]$ respectively.

Normals at P, Q, R to ellipse are concurrent, if the determinants of the coefficients is zero. i.e., if

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$$

$$\therefore \sin\left(2\theta+\frac{2\pi}{3}\right)+\sin\left(2\theta+\frac{6\pi}{3}\right)+\sin\left(2\theta+\frac{4\pi}{3}\right)$$

$$=\sin(2\theta)+\sin\left(2\theta+\frac{2\pi}{3}\right)+\sin\left(2\theta+\frac{4\pi}{3}\right)$$

= 0 for all values of θ

$\therefore$ The normals are concurrent.

Example 5: Prove that the sum of the eccentric angles of the extremities of a chord which is drawn in a given direction is constant and equal to twice the eccentric angle of the point, at which the tangent is parallel to the given direction.

Sol: Consider two points on the ellipse and evaluate the slope of the chord. If the slope is constant prove that the sum of the angles is constant.

Slope of chord AB = m

$$=-\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$$

$$=\frac{2b\cos((\alpha+\beta)/2).\sin((\alpha-\beta)/2)}{2a\sin((\alpha+\beta)/2).\sin((\beta-\alpha)/2)}=-\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right).$$

$\therefore \quad \frac{\alpha+\beta}{2}=$ constant if m is constant

Eq. of a tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Slope of this tangent is $-\frac{b}{a}\cot\theta$.

Now if $m=-\frac{b}{a}\cot\theta$, then $\theta=\frac{\alpha+\beta}{2}$

So, the slopes are equal. They are parallel to each other. Hence proved.

Example 6: P and Q are two points of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ such that sum of their ordinates is 3. Prove that the locus of the intersection of the tangents at P and Q is $9x^2+25y^2=150y$.

Sol: Find the relation between the ordinate and use it to find the locus.

If (h, k) is the point of intersection of tangents at θ and ϕ, then

$$\frac{h}{a}=\frac{\cos\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)};\ \frac{k}{b}=\frac{\sin\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)}$$

$$\therefore\quad \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{1}{\cos^2\left((\theta-\phi)/2\right)} \quad ...(i)$$

We are given that sum of ordinates is 3.

$$\therefore\quad b(\sin\theta+\sin\phi)=3$$

$$\Rightarrow\quad 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=1 \quad ...(ii)$$

$$\text{Now, } \frac{k}{b}=\frac{\sin\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)}=\frac{1}{2\cos^2\left((\theta-\phi)/2\right)}$$

$$\therefore\quad \frac{2k}{b}=\frac{1}{\cos^2\left((\theta-\phi)/2\right)} \quad ...(iii)$$

Hence from (i) and (iii) we get $\frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{2k}{b}$

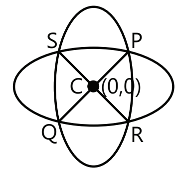
$\therefore$ Locus of (h, k) is $\frac{x^2}{25}+\frac{y^2}{9}=\frac{2y}{3}$

$$\Rightarrow\quad 9x^2+25y^2=150y$$

Example 7: If the points of intersection of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ are the extremities of the conjugate diameters of the first ellipse, then prove that $\frac{a^2}{p^2}+\frac{b^2}{q^2}=2$.

Sol: Use the condition for the pair of lines to represent conjugate diameters.

Subtracting in order to find points of intersection, we get $x^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$



Above equation will represent a pair of conjugate diameters of the first ellipse if

$$m_1m_2=-\frac{b^2}{a^2}$$

$$\therefore\quad \frac{\left((1/a^2)-(1/p^2)\right)}{\left((1/b^2)-(1/q^2)\right)}=-\frac{b^2}{a^2}$$

$$\Rightarrow\quad a^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+b^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$$

$$\Rightarrow\quad \frac{a^2}{p^2}+\frac{b^2}{q^2}=2$$

Example 8: The points of intersection of the two ellipses $x^2+2y^2-6x-12y+23=0$ and $4x^2+2y^2-20x-12y+35=0$.

(A) Lie on a circle centred at $\left(\frac{8}{3},3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(B) Lie on a circle centred at $\left(-\frac{8}{3},3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(C) Lie on a circle centred at (8, 9) and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(D) Are not cyclic.

Sol: Use the concept of the curve passing through the intersection of two ellipses.

Equation of any curve passing through the intersection of the given ellipse is

$$4x^2+2y^2-20x-12y+35+$$
$$\lambda(x^2+2y^2-6x-12y+23)=0$$

Which represents a circle is

$$4+\lambda=2+2\lambda\Rightarrow\lambda=2$$

and the equation of the circle is thus,

$$6x^2+6y^2-32x-36y+81=0$$

$$\Rightarrow\quad x^2+y^2-\left(\frac{16}{3}\right)x-6y+\frac{81}{6}=0$$

Centre of the circle is $\left(\frac{8}{3}, 3\right)$ and the radius is

$$\sqrt{\left(\frac{8}{3}\right)^2 + (3)^2 - \frac{81}{6}}$$

$$= \sqrt{\frac{128 + 162 - 243}{18}} = \frac{1}{3}\sqrt{\frac{47}{2}}.$$

Paragraph for Questions 9 to 12

$C: x^2 + y^2 = 9$, $E: \frac{x^2}{9} + \frac{y^2}{4} = 1$, $L: y = 2x$

Example 9: P is a point on the circle C, the perpendicular PQ to the major axis of the ellipse E meets the ellipse at M, then $\frac{MQ}{PQ}$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{2}$ (D) None of these

Sol: Proceed accordingly using parametric coordinates..

Let the coordinates of P be $(3\cos\theta, 3\sin\theta)$ then the eccentric angle of M, the point where the ordinate PQ through P meets the ellipse is θ and the coordinates of M are $(3\cos\theta, 2\sin\theta)$, $\frac{MQ}{PQ} = \frac{2\sin\theta}{3\sin\theta} = \frac{2}{3}$.

Example 10: If L represents the line joining the point P and C to its centre O and intersects E at M, then the equation of the tangent at M to the ellipse E is

(A) $x + 3y = 3\sqrt{5}$ (B) $4x + 3y = \sqrt{5}$

(C) $x + 3y + 3\sqrt{5} = 0$ (D) $4x + 3 + \sqrt{5} = 0$

Sol: Find the point of intersection of the line L and E. Write the equation of the tangent at M.

Line L: y = 2x meets the circle $C: x^2 + y^2 = 9$ at points for which $x^2 + 4x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{5}}$.

Coordinates of P are $(\pm\frac{3}{\sqrt{5}}, \pm\frac{6}{\sqrt{5}})$

$\Rightarrow$ Coordinates of M are $(\pm\frac{3}{\sqrt{5}}, \pm\frac{4}{\sqrt{5}})$

Equation of the tangent at M to the ellipse E is

$\frac{x(\pm 3)}{9\sqrt{5}} + \frac{y(\pm 4)}{4\sqrt{5}} = 1$; $x + 3y = \pm 3\sqrt{5}$.

Example 11: Equation of the diameter of the ellipse E conjugate to the diameter represented by L is

(A) $9x + 2y = 0$ (B) $2x + 9y = 0$

(C) $4x + 9y = 0$ (D) $4x - 9y = 0$

Sol: Use the condition of conjugate diameters to find the slope and hence write the equation of the line.

Let y = mx be the diameter conjugate to the diameter $L: y = 2x$ of the ellipse E, then

$$2m = -\frac{4}{9}\left(mm' = -\frac{b^2}{a^2}\right)$$

$\Rightarrow$ $m = -\frac{2}{9}$ and the equation of the conjugate diameter is $y = \left(-\frac{2}{9}\right)x$ or $2x + 9y = 0$.

Example 12: If R is the point of intersection of the line L with the line x = 1, then

(A) R lies inside both C and E

(B) R lies outside both C and E

(C) R lies on both C and E

(D) R lies inside C but outside E

Sol: Use the position of a point w.r.t a circle.

Coordinates of R are (1, 2)

C (1, 2) = $1 + 2^2 - 9 < 0$

$\Rightarrow$ R lies inside C; E (1, 2) = $\frac{1}{9} + 1 - 1 > 0$

$\Rightarrow$ R lies outside E.

Example 13: If CF is perpendicular from the centre C of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the tangent at any point P, and G is the point where the normal at P meets the minor axis, then $(CF \times PG)^2$ is equal to

Sol: Consider a parametric point on the ellipse and proceed to find CF and PG.

Equation of the tangent at $P(7\cos\theta, 5\sin\theta)$ on the ellipse is $\frac{x}{7}\cos\theta + \frac{y}{5}\sin\theta = 1$, then

$$(CF)^2 = \frac{7^2 \times 5^2}{5^2\cos^2\theta + 7^2\sin^2\theta} = \frac{25 \times 49}{25\cos^2\theta + 49\sin^2\theta}$$

Equation of the normal at P is

$$\frac{7x}{\cos\theta} - \frac{5y}{\sin\theta} = 7^2 - 5^2$$

Coordinates of G are $\left(0, \frac{-24\sin\theta}{5}\right)$

$$(PG)^2 = (7\cos\theta)^2 + \left(5\sin\theta + \frac{24\sin\theta}{5}\right)^2$$

$$= \frac{49}{25}(25\cos^2\theta + 49\sin^2\theta)$$

So, $(CF.PG)^2 = (49)^2 = 2401$.

JEE Main/Boards

Exercise 1

Q.1 Find the equation of the ellipse whose vertices are (5, 0) and (–5, 0) and foci are (4, 0) and (–4, 0).

Q.2 Find the eccentricity of the ellipse $9x^2 + 4y^2 - 30y = 0$.

Q.3 Find the equations of the tangents drawn from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.

Q.4 Find the eccentric angle of a point on the ellipse $\frac{x^2}{5} + \frac{y^2}{4} = 2$ at a distance 3 from the centre.

Q.5 Obtain equation of chord of the ellipse $4x^2 + 6y^2 = 24$ which has (0, 0) as its midpoint.

Q.6 Find the foci of the ellipse

$25(x+1)^2 + 9(y+2)^2 = 225$.

Q.7 Find the eccentricity of the ellipse if

(a) Length of latus rectum = half of major axis

(b) Length of latus rectum = half of minor axis.

Q.8 Find the condition so that the line $\ell x + my + n = 0$ may be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q.9 If the normal at the point P(θ) to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point Q(2θ), show that $\cos\theta = -\frac{2}{3}$.

Q.10 The common tangent of $\frac{x^2}{25} + \frac{y^2}{4} = 1$ and α lies in 1st quadrant. Find the slope of the common tangent and length of the tangent intercepted between the axis.

Q.11 Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

Q.12 Find the equations to the normals at the ends of the latus recta and prove that each passes through an end of the minor axis if $e^4 + e^2 = 1$.

Q.13 Find the co-ordinates of those points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, tangent at which make equal angles with the axes. Also prove that the length of the perpendicular from the centre on either of these is $\sqrt{\frac{1}{2}(a^2 + b^2)}$.

Q.14 Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.

Q.15 The tangent and normal at any point A of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut its major axis in points P and Q respectively. If PQ = a, prove that the eccentric angle of the point P is given by $e^2\cos^2\phi + \cos\phi - 1 = 0$.

Q.16 A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$.

Q.17 Show that the locus of the middle points of those chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are drawn through the positive end of the minor axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$.

Q.18 Tangents are drawn from a point P to the circle $x^2+y^2=r^2$ so that the chords of contact are tangent to the ellipse $a^2x^2+b^2y^2=r^2$. Find the locus of P.

Q.19 Show that the tangents at the extremities of all chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which subtend a right angle at the centre intersect on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.

Q.20 Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose middle point is $\left(\frac{1}{2},\frac{2}{5}\right)$.

Q.21 Prove that the circle on any focal distance as diameter touches the auxiliary circle.

Q.22 Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0<b<a$. Let the line parallel to y-axis passing thorugh P meet the circle $x^2+y^2=a^2$ at the point Q such that P and Q are on the same side of the x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse.

Q.23 Consider the family of circles $x^2+y^2=r^2$, $2<r<5$. In the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2+25y^2=100$ meets the coordinate axes at A and B, then find the equation of the locus of the mid-point of AB.

Q.24 A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=a+b$ in the points P and Q. Prove that the tangents at P and Q are at right angles.

Q.25 The co-ordinates of the mid-point of the variable chord $y=\frac{1}{2}(x+c)$ of the ellipse $4x^2+9y^2=36$ are

Q.26 A triangle ABC right angled at 'A' moves so that it walsys circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The locus of the point 'A' is

Exercise 2

Single Correct Choice Type

Q.1 The equation $\frac{x^2}{2-r}+\frac{y^2}{r-5}+1=0$ represents an ellipse, if

(A) r > 2 (B) $2<r<5$ (C) r > 5 (D) $r\in\{2,5\}$

Q.2 The eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(A) $\frac{5}{6}$ (B) $\frac{3}{5}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{5}}{3}$

Q.3 If $\tan\theta_1 . \tan\theta_2=-\frac{a^2}{b^2}$ then the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at:

(A) Focus (B) Centre

(C) End of the major axis (D) End of the minor axis

Q.4 If the line $y=2x+c$ be a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then c is equal to

(A) ±4 (B) ±6 (C) ±1 (D) ±8

Q.5 If the line $3x+4y=-\sqrt{7}$ touches the ellipse $3x^2+4y^2=1$ then, the point of contact is

(A) $\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}}\right)$ (B) $\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$

(C) $\left(\frac{1}{\sqrt{7}},\frac{-1}{\sqrt{7}}\right)$ (D) $\left(\frac{-1}{\sqrt{7}},\frac{-1}{\sqrt{7}}\right)$

Q.6 The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxiliary circle meet on the line:

(A) $x = \frac{a}{e}$ (B) $x = 0$ (C) $y = 0$ (D) None of these

Q.7 The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive and of latus rectum

(A) $x + ey + e^2a = 0$ (B) $x - ey - e^3a = 0$

(C) $x - ey - e^2a = 0$ (D) None of these

Q.8 The normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through an end of the minor axis, if:

(A) $e^4 + e^2 = 1$ (B) $e^3 + e^2 = 1$

(C) $e^2 + e = 1$ (D) $e^3 + e = 1$

Q.9 If CF is perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at P and G is the point where the normal at P meets the major axis, then the product CF.PG is:

(A) a^2 (B) $2b^2$ (C) b^2 (D) $a^2 - b^2$

Q.10 $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ and $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then the value of b and the other common tangent are given by:

(A) $b = \sqrt{3}$; $x + 2y + a = 0$ (B) $b = 3$; $x + 2y + 4 = 0$

(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$ (D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

Q.11 An ellipse is such that the length of the latus rectum is equal to the sum of the lengths of its semi principal axes. Then:

(A) Ellipse bulges to a circle

(B) Ellipse becomes a line segment between the two foci

(C) Ellipse becomes a parabola

(D) None of these

Q.12 Which of the following is the common tangent to the ellipses, $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1$?

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$

(B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(D) $by = ax + \sqrt{a^4 + a^2b^2 + b^4}$

Q.13 In the ellipse the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is

(A) $\frac{4}{5}$ (B) $\frac{1}{52}$ (C) $\frac{3}{5}$ (D) None of these

Q.14 Equation of a tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which cuts off equal intercepts on the axes is-

(A) $x + y - \sqrt{41} = 0$ (B) $x - y + 9 = 0$

(C) $x + y - 9 = 0$ (D) None of these

Q.15 An ellipse has OB as a semi minor axis. FBF' are its foci, and the angle FPF' is a right angle. Then the eccentricity of the ellipse, is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{2}}$

(C) $\frac{1}{2}$ (D) None of these

Q.16 The length of the latus rectum of the ellipse $9x^2 + 4y^2 = 1$, is

(A) $\frac{3}{2}$ (B) $\frac{8}{3}$ (C) $\frac{4}{9}$ (D) $\frac{8}{9}$

Q.17 If the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$, then length of the minor axis is

(A) 3 (B) $4\sqrt{2}$ (C) 6 (D) $\frac{16}{\sqrt{3}}$

Q.18 Let 'E' be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ & 'C' be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then:

(A) Q lies inside C but outside E

(B) Q lies outside both C and E

(C) P lies inside both C and E

(D) P lies inside C but outside E

Q.19 The line, $\ell x+my+n=0$ will cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in poins whose eccentric angle differe by $\frac{\pi}{2}$ if:

(A) $x^2\ell^2+b^2n^2=2m^2$ (B) $a^2m^2+b^2\ell^2=2n^2$

(C) $a^2\ell^2+b^2m^2=2n^2$ (D) $a^2n^2+b^2m^2=2\ell^2$

Q.20 The locus of point of intersection of tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at two points the sum of whose eccentric angles is constant is:

(A) A hyperbola (B) An ellipse

(C) A circle (D) A straight line

Q.21 Q is a point on the auxiliary circle of an ellipse. P is the corresponding point on ellipse. N is the foot of perpendicular from focus S, to the tangent of auxiliary circle at Q. Then

(A) SP = SN (B) SP = PQ

(C) PN = SP (D) NQ = SP

Q.22 A tangent to the ellipse $4x^2+9y^2=36$ is cut by the tangent at the extremities of the major axis at T and T′. The circle on TT′ as diameter passes thorugh the point

(A) (0, 0) (B) $(\pm 5,0)$ (C) $(\pm\sqrt{5},0)$ (D) $(\pm 3,0)$

Q.23 Q is a point on the auxiliary circle corresponding to the point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If T is the foot of the perpendicular dropped from the focus S onto the tangent to the auxiliary circle at Q then the Δ SPT is:

(A) Isosceles (B) Equilateral

(C) Right angled (D) Right isosceles

Q.24 $y=mx+c$ is a normal to the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if c^2 is equal to

(A) $\frac{(a^2-b^2)^2}{a^2m^2+b^2}$ (B) $\frac{(a^2-b^2)^2}{a^2m^2}$

(C) $\frac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$ (D) $\frac{(a^2-b^2)^2m^2}{a^2m^2+b^2}$

Q.25 The equation $2x^2 + 3y^2 - 8x - 18y + 35 = K$ represents:

(A) A point if K = 0

(B) An ellipse if K < 0

(C) A hyperbola if K < 0

(D) A hyperbola if K > 0

Previous Years' Questions

Q.1 If P = (x, y), $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals ***(1998)***

(A) 8 (B) 6 (C) 10 (D) 12

Q.2 The number of values of c such that the straight line y = 4x + c touches the curve $\frac{x^2}{4}+y^2=1$ is ***(1998)***

(A) 0 (B) 2 (C) 1 (D) ∞

Q.3 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is ***(2009)***

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Q.4 Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is................ ***(1994)***

Q.5 An ellipse has OB as a semi minor axis. F and F′ are its foci and the angle FBF′ is a right angle. Then, the eccentricity of the ellipse is............ ***(1997)***

Q.6 Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1, 0<b<a. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r ans s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse. ***(2001)***

Q.7 Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. ***(2005)***

Q.8 A focus of an ellipse is at the origin. The directrix is the line x = 4 and the eccentricity is 1/2. Then the length of the semi−major axis is ***(2008)***

(A) $\frac{8}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

Q.9 The ellipse $x^2 + 16y^2 = 16$ is inscribed in a rectangle aligned with the coordinate axes, which in turn in inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is ***(2009)***

(A) $x^2 + 16y^2 = 16$ (B) $x^2 + 12y^2 = 16$

(C) $4x^2 + 48y^2 = 48$ (D) $4x^2 + 64y^2 = 48$

Q.10 Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (-3, 1) and has eccentricity is ***(2011)***

(A) $5x^2 + 3y^2 - 48 = 0$ (B) $3x^2 + 5y^2 - 15 = 0$

(C) $5x^2 + 3y^2 - 32 = 0$ (D) $3x^2 + 5y^2 - 32 = 0$

Q.11 An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semiminor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$as its semi-major axis. If the centre of the ellipse is the origin and its axes are the coordinate axes, then the equation of the ellipse is ***(2012)***

(A) $4x^2 + y^2 = 4$ (B) $x^2 + 4y^2 = 8$

(C) $4x^2 + y^2 = 8$ (D) $x^2 + 4y^2 = 16$

Q.12 The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having centre at (0, 3) is ***(2013)***

(A) $x^2 + y^2 - 6y - 7 = 0$ (B) $x^2 + y^2 - 6y + 7 = 0$

(C) $x^2 + y^2 - 6y - 5 = 0$ (D) $x^2 + y^2 - 6y + 5 = 0$

Q.13 The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is ***(2014)***

(A) $\left(x^2 + y^2\right)^2 = 6x^2 + 2y^2$

(B) $\left(x^2 + y^2\right)^2 = 6x^2 - 2y^2$

(C) $\left(x^2 - y^2\right)^2 = 6x^2 + 2y^2$

(D) $\left(x^2 - y^2\right)^2 = 6x^2 - 2y^2$

Q.14 The area (in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the Ellipse ***(2015)***

(A) $\frac{27}{4}$ (B) 18 (C) $\frac{27}{2}$ (D) 27

JEE Advanced/Boards

Exercise 1

Q.1 Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and containing the point (4, 6).

Q.2 The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point A (a, 0) in T and T is joined to B, the other end of the diameter through A, prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$.

Q.3 The tangent at the point α on a standard ellipse meets the auxiliary circle in two points which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2\alpha)^{-1/2}$.

Q.4 An ellipse passes through the points (−3, 1) and (2, −2) and its principal axis are along the coordinate axes in order. Find its equation.

Q.5 If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}.\tan\frac{\delta}{2} = 1$ where $\alpha, \beta, \gamma, \delta$ are the eccentric angles of the extremities of the chords.

Q.6 (a) Obtain the equations of the tangents to the ellipse $5x^2 + 9y^2 = 45$, perpendicular to $3x + 4y = 11$.

(b) Prove that the straight line $\frac{ax}{3} + \frac{by}{4} = c$ will be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $5c = a^2e^2$.

Q.7 Prove that the equation to the circle having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum, is $x^2+y^2-2ae^3x=a^2(1-e^2-e^4)$.

Q.8 Find the equations of the lines with equal intercepts on the axis & which touch the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Q.9 The tangent at $P\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x-15=0$. Find the θ. Find also the equation to the common tangent.

Q.10 A tangent having slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$, intersects the axis of x and y in points A and B respectively. If O is the origin, find the area of triangle OAB.

Q.11 'O' is the origin & also the centre of two concentric circles having radii of the inner & the outer circle as 'a' and 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the foci of this ellipse lie on the inner circle, find the ratio of inner outer radii & find also the eccentricity of the ellipse.

Q.12 ABC is an isosceles triangle with its base BC twice its altitude. A point P moves within the triangle such that the square of its distance from BC is half the rectangle contained by its distances from the two sides. Show that the locus of P is an ellipse with eccentricity $\sqrt{\frac{2}{3}}$ passing through B & C.

Q.13 Find the equations of the tangents drawn from the point (2, 3) to the ellipse, $9x^2+16y^2=144$.

Q.14 Common tangents are drawn to the parabola $y^2=4x$ & the ellipse $3x^2+8y^2=48$ touching the parabola at A and B and the ellipse at C & D. Find the area of the quadrilateral.

Q.15 If the normal at a point P on the ellipse of semi axes a, b & centre C cuts the major & minor axes at G and g, show that $a^2\,(CG)^2+b^2.(Cg)^2=(a^2-b^2)^2$. Also prove that $CG=e^2CN$, where PN is the ordinate of P.

Q.16 Prove that the length of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is inclined to the major axis at angle θ is $\frac{2ab^2}{a^2+\sin^2\theta+b^2\cos^2\theta}$.

Q.17 The tangent at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersects the major axis in T & N is the foot of the perpendicular from P to the same axis. Show that the circle on NT as diameter intersects the auxiliary circle orthogonally.

Q.18 The tangents from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles. Show that the normals at the points of contact meet on the line $\frac{y}{y_1}=\frac{x}{x_1}$.

Q.19 Find the locus of the point the chord of contact of the tangent drawn from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=C^2$, where $c<b<a$.

Q.20 Find the equation of the common tangents to the ellipse $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$.

Q.21 P and Q are the corresponding point on a standard ellipse and its auxiliary circle. The tangent at P to the ellipse meets the major axis in T. prove that QT touches the auxiliary circle.

Q.22 If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, intersects it again at the point $Q(2\theta)$, show that $\cos\theta=-\left(\frac{2}{3}\right)$.

Q.23 A straight line AB touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the circle $x^2+y^2=r^2$; where $a>r>b$. PQ is a focal chord of the ellipse. If PQ be parallel to AB and cuts the circle in P & Q, find the length of the perpendicular drawn from the centre of the ellipse to PQ. Hence show that PQ = 2b.

Q.24 If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity of the ellipse is given by: $e=\frac{\cos\beta}{\cos\alpha}$.

Q.25 An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. The radius of the circle is________.

Q.26 Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. 1 f OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) = ________________________.

Exercise 2

Single Correct Choice Type

Q.1 The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ included between the co-ordinate axes is the curve.

(A) $9x^2+16y^2=4x^2y^2$ (B) $16x^2+9y^2=4x^2y^2$

(C) $3x^2+4y^2=4x^2y^2$ (D) $9x^2+16y^2=x^2y^2$

Q.2 P & Q are corresponding points on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R where C is centre of the ellipse. Then $\ell(CR)$ is

(A) 5 units (B) 6 units (C) 7 units (D) 8 units

Q.3 The equation of the ellipse with its centre at (1, 2), focus at (6, 2) and passing through the point (4, 6) is

(A) $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$ (B) $\frac{(x-1)^2}{20}+\frac{(y-2)^2}{45}=1$

(C) $\frac{(x-1)^2}{25}+\frac{(y-2)^2}{16}=1$ (D) $\frac{(x-1)^2}{16}+\frac{(y-2)^2}{25}=1$

Q.4 A line of fixed length (a+b) moves so that its ends are always on two fixed perpendicular straight lines. The locus of the point which divided this line into portions of lengths a and b is:

(A) An ellipse (B) An hyperbola

(C) A circle (D) None of these

Q.5 An ellipse is described by using an endless string which passes over two pins. If the axes are 6 cm and 4 cm, the necessary length of the string and the distance between the pins respectively in cm, are

(A) $6, 2\sqrt{5}$ (B) $6, \sqrt{5}$

(C) $4, 2\sqrt{5}$ (D) None of these

Q.6 If F_1 & F_2 are the feet of the perpendiculars from the foci S_1 and S_2 of an ellipse $\frac{x^2}{5}+\frac{y^2}{3}=1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to:

(A) 2 (B) 3 (C) 4 (D) 5

Q.7 a & b are positive real numbers, such that $a>b$. If the area of the ellipse $ax^2+by^2=3$ equals area of the ellipse $(a+b)x^2+(a-b)y^2=3$, then a/b is equal to

(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}+1}{2}$ (C) $\frac{\sqrt{6}-1}{2}$ (D) $\frac{\sqrt{5}-1}{4}$

Q.8 The locus of image of the focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ with respect to any of the tangent to the ellipse is

(A) $(x\pm ae)^2=y^2+4a^2$ (B) $(x\pm ae)^2=4a^2-y^2$

(C) $(x\pm ae)^2=y^2-4a^2$ (D) $(x\pm ae)^2=4a^2$

Q.9 The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that:

(A) e' is independent of e

(B) e ' = 1

(C) e ' = e

(D) e ' = 1/e

Q.10 A circle has the same centre as an ellipse & passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of the triangle PF_1F_2 is 30, then the distance between the foci is:

(A) 11 (B) 12

(C) 13 (D) None of these.

Q.11 The arc of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is:

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$ (B) $\frac{(a^2+b^2)ab}{a^2-b^2}$

(C) $\frac{(a^2-b^2)}{ab(a^2+b^2)}$ (D) $\frac{(a^2+b^2)}{(a^2-b^2)ab}$

Q.12 Co-ordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0) respectively. The vertex A is varying in such a way that $4\tan\frac{B}{2}\tan\frac{C}{2}=1$. Then locus of A is

(A) $\frac{(x-5)^2}{25}+\frac{y^2}{16}=1$ (B) $\frac{(x-5)^2}{16}+\frac{y^2}{25}=1$

(C) $\frac{(x-5)^2}{25}+\frac{y^2}{9}=1$ (D) $\frac{(x-5)^2}{9}+\frac{y^2}{25}=1$

Multiple Correct Choice Type

Q.13 Identify the statement which are True

(A) The equation of the director circle of the ellipse, $5x^2+9y^2=45$ is $x^2+y^2=14$

(B) The sum of the focal distances of the point (0, 6) on the ellipse $\frac{x^2}{25}+\frac{y^2}{36}=1$ is 10

(C) The point of intersection of anytangent to a parabola and the perpendicular to it from the focus lies on the tangent at the vertex

(D) The line through focus and $(at_1^2, 2at_1)$ $y^2=4ax$, meets it again in on the point $(at_2^2, 2at_2)$, if $t_1t_2=-1$.

Q.14 The angle between pair of tangents drawn to the ellipse $3x^2+2y^2=5$ from the point (1, 2) is

(A) $\tan^{-1}\frac{12}{5}$ (B) $\tan^{-1}\frac{6}{\sqrt{5}}$

(C) $\tan^{-1}\frac{12}{\sqrt{5}}$ (D) $\pi-\cot^{-1}\left(-\frac{\sqrt{5}}{12}\right)$

Q.15 If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S′. Let $\angle PSS'=\alpha$ and $\angle PS'S=\beta$, then

(A) PS + PS′ = 2a, if $a>b$

(B) PS + PS′ = 2b, if $a<b$

(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$

(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{\sqrt{a^2-b^2}}{b^2}[a-\sqrt{a^2-b^2}]$ when $a>b$

Q.16 The equation of the common tangents to the ellipse $x^2+4y^2=8$ and the parabola $y^2=4x$ are

(A) $2y-x=4$ (B) $2y+x=4$

(C) $2y+x+4=0$ (D) $2y+x=0$

Q.17 The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$, from its centre is 2. Then the eccentric angle is:

(A) $\pi/4$ (B) $3\pi/4$ (C) $5\pi/4$ (D) $7\pi/4$

Q.18 The tangents at any point F on the standard ellipse with foci as S and S′ meets the tangents at the vartices A and A′ in the points V and V′, then:

(A) $\ell(AV).\ell(AV')=b^2$

(B) $\ell(AV).\ell(A'V')=a^2$

(C) $\angle V'SV=90^0$

(D) V′S′ VS is a cyclic quadrilateral

Previous Years' Questions

Q.1 If $a>2b>0$, then positive value of m for which $y=mx-b\sqrt{a+m^2}$ is a common tangent to $x^2+y^2=b^2$ and $(x-a)^2+y^2=b^2$ is ***(2002)***

(A) $\frac{2b}{\sqrt{a^2-4b^2}}$ (B) $\frac{\sqrt{a^2-4b^2}}{2b}$

(C) $\frac{2b}{a-2b}$ (D) $\frac{b}{a-2b}$

Q.2 Tangent is drawn to ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta\in(0,\pi/2)$). Then the value of θ such that the sum of intercepts on axes made by this tangent is minimum, is ***(2003)***

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

Q.3 The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. if M is the mid point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points. ***(2009)***

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Q.4 An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse, in the standard form is............... ***(1996)***

Paragraph Based Questions 5 to 7

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B. ***(2010)***

Q.5 The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3,0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

Q.6 The orthocenter of the triangle PAB is

(A) $\left(5, \frac{8}{7}\right)$ (B) $\left(\frac{7}{5}, \frac{25}{8}\right)$ (C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

Q.7 The equation of the locus of the point whose distances from the point P and the line AB are equal is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Q.8 Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that ***(1995)***

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

Q.9 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipses $x^2 + 2y^2 = 6$ at P and Q. Prove that tangents at P and Q of ellipse $x^2 + 2y^2 = 6$ are at right angles. ***(1997)***

Q.10 Find the coordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area fo the triangle PON is maximum, where O denotes the origin and N be the foot of the perpendicular from O to the tangent at P. ***(1999)***

Q.11 Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ meets the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that, the normals to the ellipse drawn at the points P, Q and R are concurrent. ***(2000)***

Q.12 Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse of the point of contact meet on the corresponding directrix. ***(2002)***

Q.13 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0, y_2<0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are ***(2008)***

(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Q.14 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then ***(2009)***

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $\left(\pm\sqrt{2},0\right)$

Q.15 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2+9y^2=9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is ***(2009)***

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Q.16 Match the conics in column I with the statements/ expressions in column II. ***(2009)***

Column I	**Column II**
(A) Circle	(p) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2+y^2=4$
(B) Parabola	(q) Points z in the complex plane satisfying $\lvert z+2\rvert-\lvert z-2\rvert=\pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x=\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y=\frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1\le x<\infty$
	(t) Points z in the complex plane satisfying $\operatorname{Re}(z+1)^2=\lvert z\rvert^2+1$

Q.17 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is ***(2010)***

(A) $2x-\sqrt{5}y-20=0$ (B) $2x-\sqrt{5}y+4=0$

(C) $3x-4y+8=0$ (D) $4x - 3y + 4 = 0$

Q.18 The ellipse $E_1:\frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes therectangle R. The eccentricity of the ellipse E_2 is ***(2012)***

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Q.19 A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)=$ area of the triangle PQR, $\Delta_1=\max_{1/2\le h\le 1}\Delta(h)$ and $\Delta_2=\min_{1/2\le h\le 1}\Delta(h)$, then

$\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=$ ____ ***(2013)***

Q.20 Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where, $f_1 > 0$ and $f_1 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. Then m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m^2}+m_2^2\right)$ is ***(2015)***

Q.21 Consider the hyperbola $H: x^2-y^2=1$ and a circle S with center $N(x_2,0)$. Suppose that H and S touch each other at a point $P(x_1,y_1)$ with $x_1>1$ and $y_1>0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle Δ PMN, then the correct expression(s) is(are) ***(2015)***

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1>1$

(B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1>1$

(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1>1$

(D) $\frac{dm}{dx_1}=\frac{1}{3}$ for $y_1>0$

Q.22 If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is ***(2016)***

(A) 3 : 1 (B) 4 : 5

(C) 5 : 8 (D) 2 : 3

MASTERJEE Essential Questions

JEE Main/Boards

Exercise 1

Q.6 Q.13 Q.16
Q.19 Q.21 Q.22
Q.23 Q.25 Q.27

Exercise 2

Q.6 Q.8 Q.11
Q.14 Q.15 Q.21
Q.23 Q.25

Previous Years' Questions

Q.2 Q.5 Q.6

JEE Advanced/Boards

Exercise 1

Q.3 Q.5 Q.7
Q.9 Q.12 Q.15
Q.17 Q.26

Exercise 2

Q.1 Q.4 Q.6
Q.10 Q.13

Previous Years' Questions

Q.1 Q.4 Q.11
Q.12

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $9x^2+25y^2=225$

Q.2 $\sqrt{5}/3$

Q.3 $y-3=0,\ x+y=5$

Q.4 $\pi/4,\ 3\pi/4$

Q.5 All lines passing through origin

Q.6 (–1, 2) and (–1, –6)

Q.7 (a) $e=\frac{1}{\sqrt{2}}$ (b) $e=\frac{\sqrt{3}}{2}$

Q.8 $\frac{a^2}{\ell^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$

Q.10 slope = $\pm\frac{2}{\sqrt{3}}$; length = $\frac{14}{\sqrt{3}}$

Q.11 (2, 1)

Q.12 $\frac{a^2x}{ae}-\frac{b^2y}{\pm\left(\frac{b^2}{a}\right)}=a^2-b^2$ or $\frac{a^2x}{-ae}-\frac{b^2y}{\pm\left(\frac{b^2}{a}\right)}=a^2-b^2$

Q. 13 $\left(\pm\frac{a^2}{a^2+b^2},\pm\frac{b^2}{a^2+b^2}\right)$

Q.18 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2$

Q.20 $\frac{7}{5}\sqrt{41}$

Q.22 $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(ra+sb)^2}=1$

Q.23 $4x^2y^2=25y^2+4x^2$

Q.25 $-9c/25, 8c/25$

Q.26 $x^2+y^2=a^2+b^2$, a director circle

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** D **Q.3** B **Q.4** B **Q.5** D **Q.6** C

Q.7 B **Q.8** A **Q.9** C **Q.10** A **Q.11** A **Q.12** B

Q.13 C **Q.14** A **Q.15** B **Q.16** C **Q.17** D **Q.18** D

Q.19 C **Q.20** B **Q.21** A **Q.22** C **Q.23** A **Q.24** C

Q.25 A

Previous Years' Questions

Q.1 C **Q.2** B **Q.3** D **Q.4** $b\sqrt{a^2-b^2}$ **Q.5** $\frac{1}{\sqrt{2}}$

Q.6 $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(ar+bs)^2}=1$ **Q.7** $\frac{14}{\sqrt{3}}$ **Q.8** A **Q.9** B

Q.10 D **Q.11** D **Q.12** A **Q.13** A **Q.14** D

JEE Advanced/Boards

Exercise 1

Q.1 $4x^2+9y^2-8x-36y-175=0$

Q.4 $3x^2+5y^2=32$

Q.6 (a) $4x-3y+3\sqrt{21}=0$; $4x-3y-3\sqrt{21}=0$

Q.8 $x+y-5=0, x+y+5=0$

Q.9 $\theta=\frac{\pi}{3}$ or $\frac{5\pi}{3}$; $4x\pm\sqrt{33}y-32=0$

Q.10 24 sq. units

Q.11 $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Q.13 $y-3=0$ & $x+y=5$

Q.14 $55\sqrt{2}$ sq. units

Q.19 $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$

Q.20 $by=\pm ax\sqrt{a^4+a^2b^2+b^4}$

Q.23 $\sqrt{r^2-b^2}$

Q.25 2 units

Q.26 65

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** A	**Q.5** A	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** C	**Q.10** C	**Q.11** A	**Q.12** A

Multiple Correct Choice Type

Q.13 A, C, D	**Q.14** C, D	**Q.15** A, B, C	**Q.16** A, C	**Q.17** A, B, C, D	**Q.18** A, C, D

Previous Years' Questions

Q.1 A **Q.2** B **Q.3** C **Q.4** $\frac{(x-(1/3))^2}{1/9}+\frac{(y-1)^2}{1/12}=1$

Q.5 D **Q.6** C **Q.7** A **Q.10** $\left(\frac{\pm a^2}{\sqrt{a^2+b^2}},\frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$

Q.13 B, C **Q.14** A, B **Q.15** D **Q.16** A $\rightarrow$ p; B $\rightarrow$ s, t; C $\rightarrow$ r; D $\rightarrow$ q, s

Q.17 B **Q.18** C **Q.20** D **Q.21** A, B, D **Q.22** C

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $2a = 10$

$\Rightarrow a = 5$

$ae = 4 \Rightarrow e = \frac{4}{a} = \frac{4}{5}$

$\therefore b^2 = a^2(1 - e^2)$

$b^2 = 25\left(1-\frac{16}{25}\right) = 9$

$\therefore b = \pm 3$

Equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$ or $9x^2 + 25y^2 = 225$

Sol 2: $9x^2 + 4y^2 - 30y = 0$

$9x^2 + 4\left(y^2-\frac{15}{2}y\right) = 0$

$9x^2 + 4\left(y-\frac{15}{4}\right)^2-\frac{225}{4}=0$

or $9x^2 + 4\left(y-\frac{15}{4}\right)^2=\left(\frac{15}{2}\right)^2$

or $\frac{x^2}{\left(\frac{5}{2}\right)^2}+\frac{\left(y-\frac{15}{4}\right)^2}{\left(\frac{15}{4}\right)^2}=1$

$\therefore a = \frac{15}{4}, b = \frac{5}{2}$

$e^2 = 1-\frac{b^2}{a^2} = 1 - \frac{25}{4} \times \frac{16}{225} = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore e = \frac{\sqrt{5}}{3}$

Sol 3: $\frac{x^2}{16}+\frac{y^2}{9}=1$

P(2, 3)

$\therefore a^2 = 16$ or $a = \pm 4$

$b^2 = 9$ or $b = \pm 3$

$\therefore$ Equation is $\frac{x^2}{(4)^2}+\frac{y^2}{(3)^2}=1$

$\therefore$ Equation tangent will be

$y = mx + \sqrt{a^2m^2+b^2}$

$y = mx + \sqrt{16m^2+9}$

As this line passes through (2, 3)

$\therefore 3 - 2m = \sqrt{16m^2+9}$

$\Rightarrow 9 + 4m^2 - 12m = 16m^2 + 9$

or $12m^2 + 12m = 0$

$12(m + 1)m = 0$

$\Rightarrow m = 0, -1$

$\therefore$ y = 3 or y = – x + 5

i.e. y – 3 = 0 or x + y = 5

Sol 4: $\frac{x^2}{5}+\frac{y^2}{4}=2 \Rightarrow \frac{x^2}{10}+\frac{y^2}{8}=1$

let ϕ be eccentric angle

$\therefore$ Any point on ellipse will be (a cosϕ, b sinϕ)

$\therefore P = \left(\sqrt{10}\cos\phi, \sqrt{8}\sin\phi\right)$

$\therefore \sqrt{(\sqrt{10}\cos\phi)^2+(\sqrt{8}\sin\phi)^2} = 3$

$\Rightarrow 10\cos^2\phi + 8\sin^2\phi = 9$

or $2\cos^2\phi = 1$

or $\cos^2\phi = \frac{1}{2}$

$\therefore \cos\phi = \pm\frac{1}{\sqrt{2}}$ or $\phi = \frac{\pi}{4},\frac{3\pi}{4}$

Sol 5: $\frac{x^2}{4}+\frac{y^2}{6}=1$

Since midpoint of chord is (0, 0)

$\therefore$ Take one point as (a cosα, b sinα)

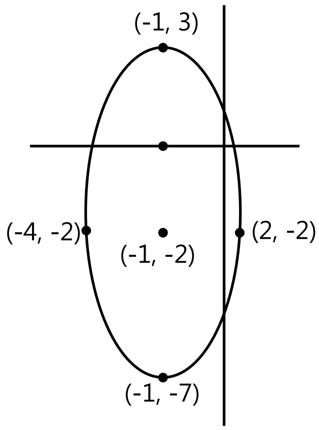
and another point as(–acosα, – bsinα)

$\therefore -a\cos\alpha = a\cos\beta \Rightarrow \beta = \pi +$

So there are infinite value of α which will satisfy this condition therefore all line passing through origin will be chord to the given ellipse.

Sol 6:

$\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1$



$a^2 = 25$

$b^2 = 9$

$b^2 = a^2(1 - e^2)$

$9 = 25(1 - e^2)$

$\Rightarrow e = \frac{4}{5}$

y coordinate of foci

$= -2 \pm \left(5\times\frac{4}{5}\right)$

$= -2 \pm 4 = (2, 6)$

$\therefore$ foci is at (–1, 2), (–1, –6)

Sol 7: Length of latus rectum $= \frac{2b^2}{a}$

(a) If $\frac{2b^2}{a}=a$

$\Rightarrow 2b^2 = a^2$

$\therefore \frac{b^2}{a^2}=\frac{1}{2}$ or $e^2 = 1-\frac{1}{2}=\frac{1}{2}$

$\therefore e=\frac{1}{\sqrt{2}}$

(b) if $\frac{2b^2}{a}=b$

$\Rightarrow 2b = a$

$\therefore 4b^2 = a^2$

or $\frac{b^2}{a^2}-\frac{1}{4}$

$\therefore e^2 = 1-\frac{1}{4}=\frac{3}{4}$

$\therefore e=\frac{\sqrt{3}}{2}$

Sol 8: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\lambda x + my + n = 0$

$y = -\frac{\ell}{m}x - \frac{n}{m}$

Equation of normal

$\Rightarrow y = m'x \mp \frac{(a^2-b^2)m'}{\sqrt{a^2+b^2m'^2}}$

$\therefore \frac{-l}{m} = m'$; $\mp \frac{(a^2-b^2)m'}{\sqrt{a^2+b^2(m')^2}} = -\frac{n}{m}$

$\therefore \frac{n}{m} = \mp \frac{(a^2-b^2)\frac{\ell}{m}}{\sqrt{a^2+b^2\times\frac{\ell^2}{m^2}}}$

$\Rightarrow \frac{\sqrt{a^2m^2+b^2\ell^2}}{m} = \mp (a^2-b^2)\frac{\ell}{n}$

$\therefore a^2m^2 + b^2\lambda^2 = (a^2-b^2)^2\frac{\ell^2m^2}{n^2}$

$\Rightarrow \frac{a^2}{\ell^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$

Sol 9: Normal of P(a cosθ, b sinθ)

$\frac{x^2}{14} + \frac{y^2}{5} = 1$

$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2 - b^2$

$\frac{\sqrt{14}x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 14 - 5 = 9$

As this passes through (a cos2θ, b sin2θ)

$\therefore \frac{14\cos 2\theta}{\cos\theta} - \frac{5\sin 2\theta}{\sin\theta} = 9$

$\therefore 14[2\cos^2\theta - 1] - 5 \times 2\cos^2\theta = 9\cos\theta$

$\therefore 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\cos\theta = \frac{9 \pm \sqrt{81+4\times14\times18}}{36}$

$= \frac{9\pm.33}{36} = -\frac{24}{36}, \frac{42}{36}$

$\therefore \cos\theta = -\frac{2}{3}$ is only possible solution

Sol 10: Equation of tangent for $\frac{x^2}{25} + \frac{y^2}{4} = 1$

$y = mx \pm \sqrt{a^2m^2+b^2}$

$y = mx \pm \sqrt{25m^2+4}$

equation of tangent for $\frac{x^2}{16} + \frac{y^2}{16} = 1$ is

$y = mx \pm 4\sqrt{m^2+1}$

$\therefore 25m^2 + 4 = 16m^2 + 16$

or $9m^2 = 12$ or $m = \pm\frac{2}{\sqrt{3}}$

equation tangent is $y = \pm\frac{2}{\sqrt{3}}x \pm 4\sqrt{\frac{4}{3}+1}$

$y = \pm\frac{2}{\sqrt{3}}x \pm \frac{4\sqrt{7}}{\sqrt{3}}$

y intercept = $\frac{4\sqrt{7}}{\sqrt{3}}$

x intercept = $2\sqrt{7}$

$\therefore$ Length = $\sqrt{4\times7+\frac{16}{3}\times7}$

$= \frac{\sqrt{7}\times\sqrt{28}}{\sqrt{3}} = \frac{7\times2}{\sqrt{3}} = \frac{14}{\sqrt{3}}$

Sol 11: Point on curve $\frac{x^2}{6} + \frac{y^2}{3} = 1$

is $(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$

$\therefore$ Distance of point from line x + y – 7 = 0 is

$\frac{\sqrt{6}\cos\theta + \sqrt{3}\sin\theta - 7}{\sqrt{2}} = f(\theta)$

$\therefore f'(\theta) = -\sqrt{3}\sin\theta + \frac{\sqrt{3}}{\sqrt{2}}\cos\theta = 0$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$

or $\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$

$\therefore$ Point is $\left(\frac{\sqrt{2}.\sqrt{6}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}\right)$

P = (2, 1)

Sol 12: End of latus rectum is $\left(ae, \pm\frac{b^2}{a}\right)$

or $\left(-ae, \pm\frac{b^2}{a}\right)$

∴ Equation of normal is

$$\frac{a^2x}{-ae} - \frac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$$

or $\frac{a^2x}{-ae} - \frac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$

if they passed through (0, b)

$\Rightarrow \mp ab = a^2 - b^2$

$\Rightarrow \left(\frac{b^2}{a^2} - 1\right)^2 = \frac{b^2}{a^2}$

$\Rightarrow (-e^2) = 1 - e^2$

$\Rightarrow e^4 + e^2 = 1$

Sol 13: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let point be (a cosϕ, b sinϕ)

i.e. (a cosϕ, b sin ϕ)

∴ Equation of tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Tangent is inclined at equal angle to axis

∵ m = ± 1 or $\left|-\frac{b}{a}\cot\theta\right| = |\pm 1|$

or $\tan\theta = \frac{b}{a}$

∴ $\cos\theta = \pm\frac{a}{\sqrt{a^2+b^2}}$,

$\sin\theta = \pm\frac{b}{\sqrt{a^2+b^2}}$

∴ Points are $\left(\frac{\pm a^2}{a^2+b^2}, \pm\frac{b^2}{a^2+b^2}\right)$

And equation of tangent is

$$\frac{x}{a^2}\left(\pm\frac{a^2}{a^2+b^2}\right) + \frac{y}{b^2}\left(\pm\frac{b^2}{a^2+b^2}\right) = 1$$

or $\frac{x}{a^2+b^2} + \frac{y}{a^2+b^2} \pm 1 = 0$

∴ Distance of tangent from origin is

$$\left|\frac{\pm 1}{\sqrt{\left(\frac{1}{a^2+b^2}\right) + \left(\frac{1}{a^2+b^2}\right)}}\right|$$

$= \sqrt{\frac{1}{2}(a^2+b^2)}$

Hence proved.

Sol 14: Let P = (acosθ, bsinθ)

Slope of tangent = $-\frac{b}{a\tan\theta}$

∴ Slope of normal to tangent

$= \frac{a\tan\theta}{b}$

∴ Equation of line

FN = (y) = $\frac{a}{b}\tan\theta(x - ae)$ ...(i)

And equation of CP

$y = \frac{b}{a}\tan\theta x$ ...(ii)

∴ $\frac{bx}{a} = \frac{a}{b}(x - ae)$

∴ $\left(\frac{b^2 - a^2}{ab}\right)x = -\frac{a^2e}{b}$

∴ $x = \frac{a}{e}$

∴ The two lines intersect on directrix

Sol 15: Normal : axsecθ – by cosecθ = $a^2 - b^2$

Tangent : $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

for y = 0

point of intersection of normal is

$x = \frac{a^2 - b^2}{a\sec\theta} = \frac{ae^2}{\sec\theta} = ae^2\cos\theta$

Point of intersection of tangent is

$x = a\sec\theta = \frac{a}{\cos\theta}$

$-\frac{ae^2}{\sec\theta} + a\sec\theta = a$

or $-e^2 + \sec^2\theta = \sec\theta$

or $e^2\cos^2\theta - 1 = -\cos\theta$

$\Rightarrow e^2\cos^2\theta + \cos\theta - 1 = 0$

Sol 16: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x^2 + y^2 = r^2$

equation of tangent at circle $\Rightarrow x\cos\theta + y\sin\theta = r$

or equation of tangent at ellipse is $y = mx + \sqrt{a^2m^2 + b^2}$

if it is a tangent to circle, then perpendicular from (0, 0) is equal to r.

$$\frac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}} = |r|$$

or $a^2m^2 + b^2 = m^2r^2 + r^2$

or $(a^2 - r^2)m^2 = r^2 - b^2$

$$\therefore m = \sqrt{\frac{r^2-b^2}{a^2-r^2}}$$

$$\therefore \tan\theta = \sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)} \text{ or } \theta = \tan^{-1}\sqrt{\frac{(r^2-b^2)}{(a^2-r^2)}}$$

Sol 17: Let (h, k) be midpoints of chords,

$\therefore$ Equation of chord with midpoint (h. , k) is

$$\frac{xh}{a^2}+\frac{4k}{b^2}-1 = \frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

It passes through (0, b)

$$\therefore \text{Equation is: } \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{k}{b} \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$$

Sol 18: Let P be (h, k)

$\therefore$ Chord of contact is

$xh + yk - r^2 = 0$

or $y = -\frac{hx}{k}+\frac{r^2}{k}$

$c^2 = a^2m^2 + b^2$

$$\text{or } \frac{r^4}{k^2}=\frac{r^2}{a^2}\times\frac{b^2}{k^2}+\frac{r^2}{b^2}$$

or $r^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is locus of P

Sol 19: We have to find the locus of P(h, k) such that the chord to contact subtends 90° at centre the equation of chord of contact is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \qquad \text{...(i)}$$

the equation of the straight line joining the centre of ellipse,. To the points of intersection of ellipse and (i) is obtained be making homogenous equation of (i) and then ellipse & is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{hx}{a^2}+\frac{by}{b^2}\right)^2 = 0$$

$$\therefore x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+y^2\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)\frac{-2hk}{a^2b^2}xy = 0 \qquad \text{...(ii)}$$

It chord of contact subtends 90° at origin then the lines separated by (ii) should be $\perp$

$$\Rightarrow \frac{1}{a^2}-\frac{h^2}{a^4}+\frac{1}{b^2}-\frac{k^2}{b^4}=0$$

$$\text{or } \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Sol 20: Equation of chord whose middle point is $\left(\frac{1}{2},\frac{2}{5}\right)$ is

$$\frac{x\times\frac{1}{2}}{25}+\frac{y\times\frac{2}{5}}{16}=\frac{\left(\frac{1}{2}\right)^2}{25}+\frac{\left(\frac{2}{5}\right)^2}{16}$$

$$\therefore \frac{x}{50}+\frac{y}{40}=\frac{1}{50}$$

$$y = -\frac{4}{5}(x-1)$$

$$\therefore \frac{x^2}{25}+\frac{\left(\frac{4}{5}\right)^2(x-1)^2}{16}=1$$

$\Rightarrow x^2 + x - 12 = 0$

$\therefore x_1 = -4$ and $x_2 = 3$

$\therefore$ Length of chord

$$L = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$= \left|(x_1-x_2)\sqrt{1+m^2}\right| = 7\sqrt{1+\frac{16}{25}} = \frac{7}{5}\sqrt{41}$$

Sol 21: Let F = (ae, 0) & p = $(a\cos\theta, b\sin\theta)$

Radius of circle

$$= \frac{1}{2}\sqrt{b^2\sin^2\theta+a^2(e-\cos\theta)^2}$$

$$= \frac{1}{2}\sqrt{b^2\sin^2\theta+a^2e^2-2ae\cos\theta+a^2\cos^2\theta}$$

$$= \frac{1}{2}\sqrt{a^2-b^2\cos^2\theta+a^2\cos^2\theta-2ae\cos\theta}$$

$= \frac{1}{2}\sqrt{a^2e^2\cos^2\theta - 2ae\cos\theta + a^2}$

$= \frac{1}{2}\sqrt{(a - ae\cos\theta)^2} = \frac{1}{2}(a - ae\cos\theta)$

Radius of auxillary circle = a

$\therefore r_1 - r_2 = \frac{1}{2}$ (a + ae cosθ)

Centre of circle with FP as diameter

$= C = \left(\frac{ae + \cos\theta}{2}, \frac{b\sin\theta}{2}\right)$

Distance between centre

$= \sqrt{\frac{a(e + \cos\theta)^2}{4} + \frac{b^2\sin b^2\theta}{4}}$

$= \frac{1}{2}\sqrt{a^2(e + \cos\theta)^2 + b^2\sin^2\theta}$

$= \frac{1}{2}\sqrt{(a + ae\cos\theta)^2}$

$= \frac{1}{2}(a + ae\cos\theta)$

∴ The two circle touch each other internally.

Sol 22: Let the co-ordinate of P be (a cos θ, bsinθ) the coordinates of Q are (acosθ, bsinθ)

Let R(h, k) be a point on PQ such that PR : RQ = r : s

then h = acosθ and

$R = \frac{ra\sin\theta + sb\sin\theta}{r + s}$

$\Rightarrow \cos\theta = \frac{h}{a}$ & $\sin\theta = \frac{(r + s)k}{ra + sb}$

$\Rightarrow \left(\frac{h}{a}\right)^2 + \frac{(r + s)^2k^2}{(ra + sb)^2} = 1$

or $\frac{x^2}{a^2} + \frac{(r + s)^2y^2}{(ra + sb)^2} = 1$

Sol 23: Equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{4} = 1$

Tangent to ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

or $mx - y \pm \sqrt{a^2m^2 + b^2} = 0$

it is tangent to circle

$\pm\frac{\sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + 1}} = r$

or $a^2m^2 + b^2 = r^2(m^2 + 1)$

$m^2(25 - r^2) = r^2 - 4$

$m = -\sqrt{\frac{r^2 - 4}{2r - r^2}}$

since tangent lies in first quadrant m < 0

$\therefore m^2 = \frac{(r^2 - 4)}{25 - r^2}$

equation of tangent is

$y = -\sqrt{\frac{(r^2 - 4)}{25 - r^2}}x + \sqrt{\frac{25(r^2 - 4)}{25 - r^2} + 4}$

$y = -\sqrt{\frac{r^2 - 4}{25 - r^2}}x + \sqrt{\frac{2/r^2}{25 - r^2}}$

∴ Midpoint is

$\left(\frac{1}{2}\sqrt{\frac{21r^2}{r^2 - 4}}, \frac{1}{2}\sqrt{\frac{21r^2}{25 - r^2}}\right)$

$\therefore 2x = \sqrt{\frac{21r^2}{r^2 - 4}}; 2y = \sqrt{\frac{21r^2}{25 - r^2}}$

$\therefore 4x^2 = \frac{21r^2}{r^2 - 4}; 4y^2 = \frac{21r^2}{25 - r^2}$

$\therefore \frac{25}{4x^2} + \frac{4}{4y^2} = 1$

or $25y^2 + 4x^2 = 4x^2y^2$

Sol 24: The given ellipses are

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a(a + b)} + \frac{y^2}{b(a + b)} = 1$

chord of contact of $P(x_1, y_1)$ w. r. t. ellipse is

$\frac{xx_1}{a(a + b)} + \frac{yy_1}{b(a + b)} = 1$

or λx + my = n

It is touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2\lambda^2 + b^2m^2 = n^2$

$$\frac{a^2 \times x_1^2}{a^2(a+b)^2} + \frac{b^2 x y_1 2}{b^2(a+b)^2} = 1$$

$x_1^2 + y_1^2 = (a + b)^2$

Locus of P is

$x^2 + y^2 = (a + b)^2$

$= a(a + b) + b(a + b)$

Which is the director circle.

Sol 25: Let mid-point = (h, k)

$\therefore$ Equation of ellipse is

$4xh + 9ky - 36 = 4h^2 + 9k^2 - 36$

And equation of chord is

$x - 2y + c = 0$

$$\therefore \frac{4h}{1} = -\frac{ak}{2} = -\left(\frac{4h^2 + 9k^2}{c}\right) \Rightarrow h = -\frac{9}{8}k$$

$$-\frac{9k}{2} = -\frac{4 \times \frac{8}{64}k^2 + 9k^2}{c} \Rightarrow +\frac{1}{2} = \frac{\frac{9k}{16} + k}{c}$$

$\therefore 25K = 8c$

$$\therefore k = \frac{8c}{25};\ h = -\frac{9c}{25}$$

Sol 26: $\therefore$ From A two $\perp$ tangents can be drawn to ellipse

$\therefore$ A is the director circle

i.e. $x^2 + y^2 = a^2 + b^2$

Exercise 2

Single Correct Choice Type

Sol 1: (B) $\frac{x}{r-2} + \frac{y^2}{5-r} = 1$

$\therefore r - 2 > 0$ and $5 - r > 0$

$\therefore r \in (2, 5)$

Sol 2: (D) $4x^2 + 8x + 9y^2 + 36y + 4 = 0$

$\Rightarrow (2x + 2)^2 + (3y + 6)^2 = 36$

$$\Rightarrow \frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

$$\therefore e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Sol 3: (B) Let P be (θ_1) and Q = (θ_2)

Slope OP = $\frac{b}{a}\tan\theta_1$

and slope OQ = $\frac{b}{a}\tan\theta_2$

$$\therefore M_{OP} \times M_{OQ} = \frac{b^2}{a^2} \times \left(\frac{-a^2}{b^2}\right) = -1$$

$\therefore$ It subtends 90° at centre

Sol 4: (B) $c^2 = a^2m^2 + b^2$

$\therefore c^2 = 8 \times 4 + 4$

$\therefore c = \pm\sqrt{36} = \pm 6$

Sol 5: (D) Let (x_1, y_1) be point of contact to ellipse

$\therefore 3xx_1 + 4yy_1 = 1$ is equation of tangent at (x_1, y_1)

$$\therefore \frac{3x_1}{3} = \frac{4y_1}{4} = -\frac{1}{\sqrt{7}}$$

$$\therefore (x_1, y_1) = \left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$$

Sol 6: (C) x-axis

Sol 7: (B) The positive end of latus rectum is $\left(ae, \frac{b^2}{a}\right)$

$a\cos\theta = ae \therefore \cos\theta = e$

$b\sin\theta = \frac{b^2}{a^2}$ & $\sin\theta = \frac{b}{a}$

equation of normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\therefore \frac{ax}{e} - \frac{by}{b} \times a = a^2(e^2)$

or $x - ey = ae^3$

$\therefore x - ey - e^3a = 0$

Sol 8: (A) Consider normal at positive end of latus rectum from above equation of normal is $x - ey - e^3a = 0$

It passes through (0, –b)

$\therefore be - e^3a = 0$

$\Rightarrow b - e^2a = 0$

$\therefore \dfrac{b}{a} = e^2$

$\Rightarrow 1 - e^2 = e^4$ or $e^2 + e^4 = 1$

Sol 9: (C) Tangent at P is

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} - 7 = 0$

$\therefore$ CF $= \dfrac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}$

P = (acosθ, bsinθ)

Equation of normal at

P = $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

$\therefore\ G = \left(\dfrac{(a^2 - b^2)\cos\theta}{a}, 0\right)$

$\therefore PG = \sqrt{b^2\sin^2\theta + \dfrac{b^4}{a^2}\cos^2\theta} = b^2\sqrt{\dfrac{\sin^2\theta}{b^2} + \dfrac{\cos^2\theta}{a^2}}$

$\therefore$ PG. CF $= b^2$

Sol 10: (A) $y = \dfrac{x}{2} + 2$

$\therefore c^2 = a^2m^2 + b^2$

$\Rightarrow 4 = 4 \times \dfrac{1}{4} + b^2 \Rightarrow b = \sqrt{3}$

$\therefore$ The other common tangent has slope – m

$\therefore c = \dfrac{1}{-\dfrac{1}{2}} = -2$

$\therefore$ Equation is $y = -\dfrac{1}{2}x - 2$

or $x + 2y + 2 = 0$

Sol 11: (A) $\dfrac{2b^2}{a} = a + b$

or $2b^2 = a^2 + ab$

$a^2 + ab - 2b^2 = 0$

$a = -\dfrac{b \pm \sqrt{b^2 + 8b^2}}{2}$

$a = -\dfrac{b + 3b}{2} = b$

$\therefore$ Ellipse bulges to circle

Sol 12: (B) $\Rightarrow y = \pm \dfrac{a}{b}x \pm \sqrt{a^2 \cdot \dfrac{a^2}{b^2} + a^2 + b^2}$

$\Rightarrow y = \pm \dfrac{a}{b}x \pm \sqrt{\dfrac{a^4 + a^2b^2 + b^4}{b^2}}$

$\Rightarrow y = \pm \dfrac{a}{b}x \pm \dfrac{1}{b}\sqrt{a^4 + a^2b^2 + b^4}$

$\Rightarrow yb = \pm ax \pm \sqrt{a^4 + a^2b^2 + b^4}$

Sol 13: (C) 2ae = 6

2b = 8

$\therefore$ ae = 3 & b = 4

$\therefore a^2 - b^2 = 9$ & $b^2 = 16$

$\therefore a^2 = 25$

$e = \sqrt{1 - \dfrac{16}{25}} = \dfrac{3}{5}$

Sol 14: (A) Slope of ellipse = – 1

$\therefore$ Equation is $y = -x \pm \sqrt{25 + 16}$

$\therefore x + y = \pm\sqrt{41}$

Sol 15: (B) FBF′ is 90°

We know that BF = BF′ = a

$\therefore 2a^2 = 4a^2e^2$

$\therefore e^2 = \dfrac{1}{2} \Rightarrow e = \dfrac{1}{\sqrt{2}}$

Sol 16: (C) Ellipse is

$\dfrac{x^2}{\left(\dfrac{1}{3}\right)^2} + \dfrac{y^2}{\left(\dfrac{1}{2}\right)^2} = 1$

since a < 0

$\therefore$ Latus rectum is $\dfrac{2a^2}{b} = \dfrac{2 \times 1}{9 \times \dfrac{1}{2}} = \dfrac{4}{9}$

Sol 17: (D) $\dfrac{a}{e} - ae = 8$

$e = \dfrac{1}{2}$

$\therefore \dfrac{3a}{2} = 8\ ; a = \dfrac{16}{3}$

$\frac{1}{4} = 1 - \frac{b^2}{a^2}$

$\therefore b^2 = a^2 (1 - l^2) = \frac{16}{3} \times \frac{16}{3} \times \frac{3}{4}$

$b = \frac{8}{\sqrt{3}}$

$\therefore$ Length of minor axis = 2b = $\frac{16}{\sqrt{3}}$

Sol 18: (D) E(P) > 0 , E(Q) < 0

C(P) < 0 and C(Q) < 0

$\therefore$ P lies inside C but outside E

Sol 19: (C) Let P = (acosθ, bsinθ)

$Q = \left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$

= (– asinθ, bcosθ)

aλcosθ +mbsinθ + n = 0

And ⇒ aλcosθ + mbsinθ = – n → 1 – aλsinθ + mbcosθ + n = 0 & –aλsinθ + mbcosθ = – n(2)

squaring and adding 1 and 2

we get $a^2l^2 + m^2b^2 = 2n^2$

Sol 20: (B) $Q_1 + Q_2 = C$

Point of intersection of tangent at (θ_1) and (θ_2) is

$$(x, y) = \left(\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

$$\therefore \frac{x}{a\cos\left(\frac{c}{2}\right)} = \frac{y}{b\sin\left(\frac{C}{2}\right)}$$

$\therefore$ Locus of P is a straight line

Sol 21: (A) P = (acosθ, bsinθ)

Q = (acosθ, asinθ)

equation of tangent at Q

is (y – asinθ) = – 1tanθ (x – acosθ)

$x + y\tan\theta - \frac{a}{\cos\theta} = 0$ & (ae, 0)

$$\therefore SN = \frac{ae - \frac{a}{\cos\theta}}{\sqrt{1+\tan^2\theta}} = |ae\cos\theta - a|$$

$SP = \sqrt{(ae - a\cos\theta)^2 + b^2\sin^2\theta}$

$= \sqrt{a^2(\cos^2\theta + a^2 - b^2\cos^2\theta - 2a^2e\cos\theta)}$

$= \sqrt{(a\ell\cos\theta)^2 + a^2 - 2a^2e\cos\theta} = |ae\cos\theta - a|$

$\therefore$ SP = SN

Sol 22: (C) Equation of tangent is $\frac{x}{3}\cos\theta + \frac{y}{2}\sin\theta = 1$

T is x = 3 and T′ x = – 3

$\therefore$ Point of intersection of tangent & T let say

$P = \left(3, -\frac{2(1-\cos\theta)}{\sin\theta}\right) = \left(3, 2\tan\frac{\theta}{2}\right)$

$P' = \left(-3, \frac{2(1+\cos\theta)}{\sin\theta}\right) = \left(-3, 2\cot\frac{\theta}{2}\right)$

$\therefore$ Equation of circle is

$(x + 3)(x - 3) + \left(y - 2\tan\frac{\theta}{2}\right)\left(y - 2\cot\frac{\theta}{2}\right) = 0$

$\therefore$ When y = 0

$x^2 - 5 = 0$

$\therefore x = \pm\sqrt{5}$

$\therefore$ It always passes through $(\pm\sqrt{5}, 0)$

i.e. it always passes through focus.

Sol 23: (A) Q = (acosθ, asinθ)

P = (acosθ, bsinθ)

Δ SPT is an isosceles triangle.

Sol 24: (C) Equation of normal in slope form is

$$y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$$

$$\therefore c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2m^2}$$

Sol 25: (A) $(\sqrt{2}x - 2\sqrt{2})^2 + (\sqrt{3}y - 3\sqrt{3})^2 = k$

$$\therefore \frac{2(x-2)^2}{k} + \frac{3(y-3)^2}{k} = 1$$

For ellipse k > 0

For a point k = 0

Previous Years' Questions

Sol 1: (C) Given, $16x^2 + 25y^2 = 400$

$\Rightarrow \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

Here, $a^2 = 25,\ b^2 = 16$

But $a^2(1 - e^2)$

$\Rightarrow 16 = 25(1 - e^2)$

$\Rightarrow \dfrac{16}{25} = 1 - e^2$

$\Rightarrow e^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25} \Rightarrow e = \dfrac{3}{5}$

Now, foci of the ellipse are $(\pm ae,\ 0) \equiv (\pm 3, 0)$

We have $3 = a.\dfrac{3}{5}$

$\Rightarrow a = 5$

Now, $PF_1 + PF_2 =$ major axis = 2a

$= 2 \times 5 = 10$

Sol 2: (B) For ellipse, condition of tangency is $c^2 = a^2m + b^2$

Given line is $y = 4x + c$ and curve $\dfrac{x^2}{4} + y^2 = 1$

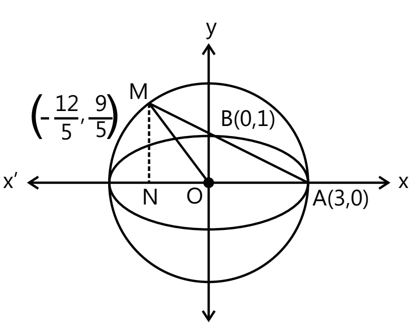
$\Rightarrow \left| ma - b\sqrt{1+m^2} \right| = \left| 1 - b\sqrt{1+m^2} \right|$

$\Rightarrow c = \pm\sqrt{65} = \sqrt{65}$ or $-\sqrt{65}$

So, number of values are 2.

Sol 3: (D) Equation of auxiliary circle is $x^2 + y^2 = 9$ ...(i)

Equation of AM is $\dfrac{x}{3} + \dfrac{y}{1} = 1$ ...(ii)

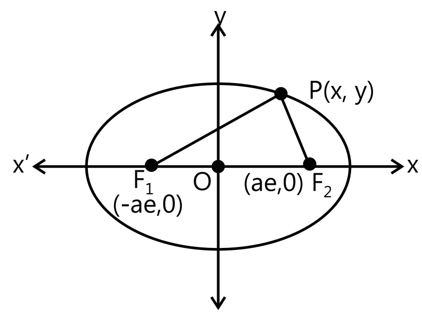


On solving equation (i) and (ii), we get $M\left(-\dfrac{12}{5}, \dfrac{9}{5}\right)$

Now, area of $\Delta AOM = \dfrac{1}{2}.OA \times MN = \dfrac{27}{10}$ sq. unit

Sol 4: Given, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Foci F_1 and F_2 are (—ae, 0) and (ae, 0) respectively. Let P(x, y) be any variable point on the ellipse. The area A of the triangle PF_1F_2 is given by



$\theta = \dfrac{\pi}{6}$

$= \dfrac{1}{2}(-y)(-ae \times 1 - ae \times 1)$

$= -\dfrac{1}{2}y(-2ae) = aey$

$= ae.b\sqrt{1 - \dfrac{x^2}{a^2}}$

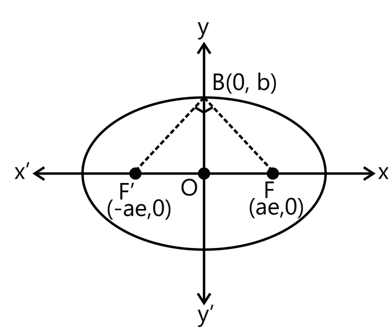
So, A is maximum when x = 0

$\Rightarrow$ Maximum of A

$= abe = ab\sqrt{1 - \dfrac{b^2}{a^2}} = ab\sqrt{\dfrac{a^2 - b^2}{a^2}} = b\sqrt{a^2 - b^2}$

Sol 5: Since, angle FBF' is right angled

$\therefore$ (slope of FB). (slope of F'B) = –1



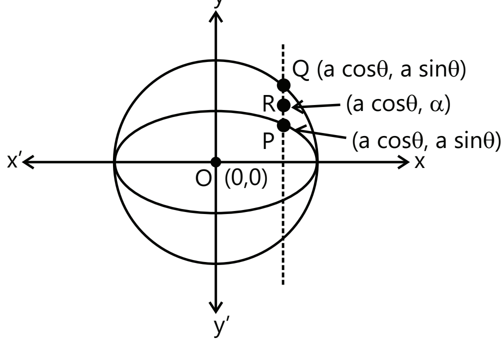
$\Rightarrow \left(\dfrac{0-b}{ae-0}\right)\left(\dfrac{0-b}{-ae-0}\right) = -1$

$$\Rightarrow \frac{b^2}{-a^2e^2} = -1 \Rightarrow b^2 = a^2e^2$$

$$\Rightarrow a^2(1-e^2) = a^2e^2 \Rightarrow e^2 = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

Sol 6: Given, $\frac{PR}{RQ} = \frac{r}{s}$



$$\Rightarrow \frac{\alpha - b\sin\theta}{a\sin\theta - \alpha} = \frac{r}{s}$$

$$\Rightarrow \alpha s - b\sin\theta.s = ra\sin\theta - \alpha r$$

$$\Rightarrow \alpha s + \alpha r = ra\sin\theta + b\sin\theta.s$$

$$\Rightarrow \alpha(s+r) = \sin\theta(ra+bs)$$

$$\Rightarrow \alpha = \frac{\sin\theta(ra+bs)}{r+s}$$

Let the coordinate of R be (h, k)

$$\Rightarrow h = a\cos\theta$$

and $k = \alpha = \frac{(ar+bs)\sin\theta}{r+s}$

$$\Rightarrow \cos\theta = \frac{h}{a}, \sin\theta = \frac{k(r+s)}{ar+bs}$$

On squaring and adding, we get

$$\sin^2\theta + \cos^2\theta = \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(ar+bs)^2}$$

$$\Rightarrow 1 = \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(ar+bs)^2}$$

Hence, locus of R is $\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(ar+bs)^2} = 1$.

Sol 7: In 1st quadrat eq. of target will be of fly from

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \begin{cases} \text{if x intercept} = q \\ \text{y intercept} = b \end{cases}$$

$\because$ (1) is tangent to circle $x^2 + y^2 = 16$

$$\Rightarrow x^2 + \frac{(ab-bx)^2}{a} = 16$$

$$\Rightarrow x^2 + \frac{a^2b^2 + b^2x^2 - 2ab^2x}{a^2} = 16$$

$$\Rightarrow x^2 + b^2 + \frac{b^2}{a^2}x^2 - \frac{2b^2}{a}.x = 16$$

$$\Rightarrow x^2\left(1 + \frac{b^2}{a^2}\right) - \frac{2b^2}{a}x + b^2 - 16 = 0$$

For unique solution

$$\Rightarrow \frac{4b^2}{a^2} - 4\left(1 + \frac{b^2}{a^2}\right)\left(1^2 - 16\right) = 0$$

$$\Rightarrow \frac{b^4}{a^2} = b^2 - 16 + \frac{b^4}{a^2} - \frac{16b^2}{a^2}$$

$$\Rightarrow b^2 - 16 = 16\frac{b^2}{a^2}$$

$$\Rightarrow a^2b^2 - 16a^2 = 16b^2$$

$$\Rightarrow a^2b^2 = 16\left(a^2 + b^2\right) \qquad \text{... (ii)}$$

Similarly (i) is tamest to ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ are will get the relatiesn

$$a^2b^2 = 4a^2 + 25b^2 \qquad \text{... (iii)}$$

Solving (i) (ii) we get $a = 2\sqrt{7}$

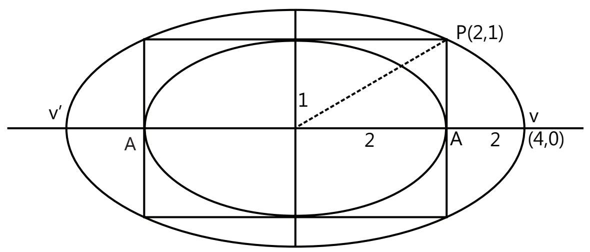
$$b = 4\sqrt{\frac{7}{3}}$$

$\Rightarrow$ Eq. of tangent $\frac{x}{2\sqrt{7}} + \frac{y}{4\sqrt{\frac{7}{3}}} = 1$

Distances $= \sqrt{a^2 + b^2} = \frac{14}{\sqrt{3}}$

Focus (S = 6, 2)

Sol 8: (A) Major axis is along x-axis.



$$\frac{a}{e} - ae = 4$$

$$a\left(2 - \frac{1}{2}\right) = 4$$

$$a = \frac{8}{3}.$$

Sol 9:

$$x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1 \Rightarrow a = 2, b = 1 \Rightarrow P = (2,1)$$

Required Ellipse is $\frac{x}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1$

(2, 1) lies on it

$$\Rightarrow \frac{4}{16} + \frac{1}{b^2} = 1 \Rightarrow \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$$

$$\therefore \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1 \Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1 \Rightarrow x^2 + 12y^2 = 16$$

Sol 10: (D) $b^2 = a^2\left(1 - e^2\right) = a^2\left(1 - \frac{2}{5}\right) = a^2\,\frac{3}{5} = \frac{3a^2}{5}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

$$a^2 = \frac{32}{3}$$

$$b^2 = \frac{32}{5}$$

$\therefore$ Required equation of ellipse $3x^2 + 5y^2 - 32 = 0$

Sol 11: (D) Semi minor axis b = 2

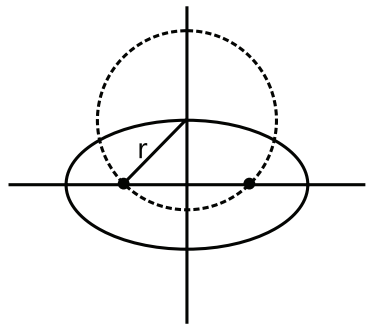
Semi major axis a = 4

Equation of ellipse $= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\Rightarrow x^2 + 4y^2 = 16.$$

Sol 12: (A)



$$a = 4, b = 3, e = \sqrt{1 - \frac{9}{16}} \quad \Rightarrow \frac{\sqrt{7}}{4}$$

Focii is $(\pm\, ae,\, 0) \qquad \Rightarrow \left(\pm\sqrt{7}, 0\right)$

$$r = \sqrt{(ae)^2 + b^2}$$

$$\sqrt{7 + 9} = 4$$

Now equation of circle is $(x - 0)^2 + (y - 3)^2 = 16$

$$x^2 + y^2 - 6y - 7 = 0$$

Sol 13: (A)

Here ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 6, b^2 = 2$

Now, equation of any variable tangent is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \qquad \text{....(i)}$$

where m is slope of the tangent

So, equation of perpendicular line drawn from centre to tangent is $y = \frac{-x}{m}$ ....(ii)

Eliminating m, we get

$$\left(x^2 + y^2\right)^2 = a^2x^2 + b^2y^2$$

$$\Rightarrow \boxed{\left(x^2 + y^2\right)^2 = 6x^2 + 2y^2}$$

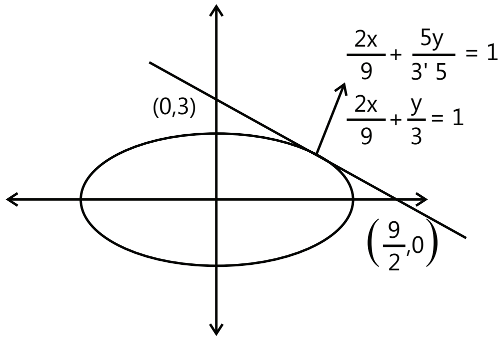
Sol 14: (D) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$$a = 3,\ b = \sqrt{5} \qquad \left(ae, \frac{b^2}{a}\right)$$

$$\frac{b^2}{a} = \frac{5}{3}, \qquad \left(2, \frac{5}{3}\right)$$

$$e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

Area = $4\left(\frac{1}{2}\times\frac{9}{2}\times 3\right) = 27$



JEE Advanced/Boards

Exercise 1

Sol 1: Let X = x – 1

And Y = y – 2

$\therefore$ Centre = (0, 0)

Focus (F_1) (5, 0)

F_2 = (–5, 0) and point on ellipse

= (3, 4)

$F_1P + F_2P = 2a$

$= \sqrt{80}+\sqrt{20} = 2a$

$3\sqrt{20} = 2a$

$\therefore \boxed{a = 3\sqrt{5}}$

ae = 5

$\therefore a^2 - b^2 = 25$

$\therefore b^2 = 20$

$b = 2\sqrt{5}$

$\therefore$ Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{9}+\frac{y^2}{4}=5$

$\Rightarrow \frac{(x-1)^2}{9}+\frac{(y-2)^2}{4} = 5$

$4x^2 + 9y^2 - 8x - 36y - 175 = 0$

Sol 2: Let T = (h, k)

$\therefore$ AP is

$xh + yk = a^2$

It passes through (a, 0)

$\therefore$ h = a

$\therefore$ T = (a, k)

TB is

$y = \frac{(k-0)}{2a} \times (x + a)$

$\therefore 2ay = kx + ka \Rightarrow kx - 2ay + ka = 0$

and $ax + ky = a^2$

Let point of intersection be (x_1, y_1)

$y_1 = \frac{2a^2k}{k^2+2a^2}$ and $x_1 = \frac{2a^3-ak^2}{2a^2-k^2}$

$x_1 = \frac{a(2a^2-k^2)}{2a^2+k^2}$

$(x_1)^2 + 2(y_1)^2 = a^2$

$\because \left(\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{\left(\frac{a}{\sqrt{2}}\right)^2} = a^2\right)$

$\therefore$ Eccentricity is = $\sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Sol 3: Equation of auxiliary circle is

$x^2 + y^2 = a^2$ .... (i)

Equation of tangent at P(α) is

$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$ ...(ii)

Equation of pair of lines OA, OB is obtained by making homogenous equation of i w. r. t. (ii)

$\therefore x^2 + y^2 = a^2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2$

$\therefore (1 - \cos^2\alpha)x^2 - \frac{2xya\sin\alpha\cos\alpha}{b}$

$+ y^2\left(1-\frac{a^2}{b^2}\sin^2\alpha\right) = 0$

But $\angle AOB = 90^o$

$\therefore$ coeff of x^2 + coeff of y^2 = 0

$\therefore 1 - \cos^2\alpha + 1 - \frac{a^2}{b^2}\sin^2\alpha = 0$

$1 = \frac{a^2-b^2}{b^2}\sin^2\alpha$

$1 = \frac{a^2e^2}{a^2(1-e^2)}\sin^2\alpha$

$\Rightarrow e^2 = \frac{1}{(1+\sin^2\alpha)}$ or $e = (1 + \sin^2\alpha)^{-1/2}$

Sol 4: $(-3, 1) = (a\cos a_1, b\sin a_1)$

$(a_1 - \alpha) = (a\cos a_2, b\sin a_2)$

$\therefore \frac{9}{a^2}+\frac{1}{b^2}=1$ & $\frac{4}{a^2}+\frac{4}{b^2}=1$

$\therefore \frac{32}{a^2} = 3$

$\therefore a^2 = \frac{32}{3}$ & $b^2 = \frac{32}{5}$

$\therefore$ Equation of ellipse is $\frac{x^2}{\frac{32}{3}}+\frac{y^2}{\frac{32}{5}} = 1$

$\therefore 3x^2 + 5y^2 = 32$

Sol 5: Let α and β form a chord which interests the major axis at (c, 0)

$\therefore$ Equation of chord is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\frac{(\alpha+\beta)}{2} = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\frac{c}{a}\cos\frac{(\alpha+\beta)}{2}=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{c}$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a+c}{a-c}$$

$$\Rightarrow \frac{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}=\frac{a+c}{a-c}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a}$$

Similarly let γ and δ intersect major axis at (– c, 0)

$$\therefore \tan\frac{\gamma}{2}\tan\frac{\delta}{2}=\frac{-c-a}{a-c}$$

$$\tan\frac{\gamma}{2}\tan\frac{\delta}{2}\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a}\times\frac{-(c+a)}{-(c-a)} = 1$$

Sol 6: (a) m of line = $-\frac{3}{4}$

$\therefore$ Slope of line $\perp$ to given line = $\frac{4}{3}$

Equation of ellipse is $\frac{x^2}{(3)^2}+\frac{y^2}{5}=0$

$\therefore$ Equation of tangent is $y = \frac{4}{3}x\pm\sqrt{9\times\left(\frac{4}{3}\right)^2+5}$

$y = \frac{4x}{3}\pm\sqrt{21}$

$\therefore 3y = 4x \pm 3\sqrt{21}$

(b) Equation of normal to the ellipse is $ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$

or $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

the normal given is $\frac{ax}{3}+\frac{by}{4}=c$

$\therefore \frac{3}{\cos\theta}=\frac{4}{-\sin\theta}=\frac{a^2-b^2}{c}$

$\therefore \frac{(3c)^2}{(a^2-b^2)^2}+\frac{(4c)^2}{(a^2-b^2)^2}=1$

$\therefore 5c = a^2 - b^2$ or $5x = a^2e^2$

Sol 7: Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the ends of latus rectum are

$L_1\left(ae,\frac{b^2}{9}\right)$ & $L_2\left(ae,-\frac{b^2}{9}\right)$

For double contact the centre of circle should lie on normal of L_1 & L_2. By symmetry y-coordinates of centre = 0

Equation of normal at L_l is

$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a} = a^2 - b^2$

For centre y = 0

$\therefore$ x centre = $\frac{\ell a^2}{a}\left(1-\frac{b^2}{a^2}\right) = ae^3$

$\therefore$ Equation of circle is

$(x - ae^3)^2 + y^2 = (ae - ae^3)^2 + \left(\frac{b^2}{9}\right)^2$

$(x - ae^3)^2 + y^2 = (ae - ae^3)^2 + \left(\frac{b^2}{9}\right)^2$

$x^2 - 2ae^3 + a^2e^6 + y^2 = (ae - ae^3)^2 + a^2(1 - e^2)^2$

$\therefore x^2 - 2ae^3 + y^2 = a^2(1 - e^2 - e^4)$

Sol 8: Since lines have equal intercept on axis

$\therefore$ Slope = – 1

$\therefore$ Equation is $y = -x \pm \sqrt{a^2m^2 + b^2}$

$y = -x \pm \sqrt{25}$

or $x + y \pm 5 = 0$ are the equation of tangents.

Sol 9: $P = \left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right) = (a\cos\theta, b\sin\theta)$

$\therefore$ Equation of tangent is

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{\frac{16}{\sqrt{11}}} = 1$$

It is also tangent to circle

$x^2 + y^2 - 2x - 15 = 0$

$c = (1, 0)$ and $r = 4$

$\therefore$ Distances from center = radius

$$\therefore \frac{\frac{\cos\theta}{4} - 1}{\sqrt{\frac{\cos^2\theta}{16} + \frac{11\sin^2\theta}{256}}} = 4$$

$$\Rightarrow \frac{(\cos\theta - 4)^2}{16} = \cos^2\theta + \frac{11\sin^2\theta}{16}$$

$(\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

$\cos^2\theta - 8\cos\theta + 16 = 11 + 5\cos^2\theta$

$4\cos^2\theta + 8\cos\theta - 5 = 0$

$4\cos^2\theta + 10\cos\theta - 2\cos\theta - 5 = 0$

$\therefore \cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

Sol 10: $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$y = -\frac{4}{3}x \pm \sqrt{18 \times \left(\frac{4}{3}\right)^2 + 32}$$

$y = -\frac{4}{3}x \pm 8$

$\therefore A = \pm \frac{3}{4} \times 8 = \pm 6$ & $B = \pm 8$

$\therefore$ Area of A = $\left|\frac{1}{2}A \times B\right| = \frac{1}{2} \times 6 \times 8 = 24$

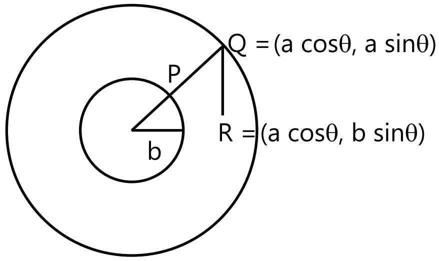
Sol 11: $P = (b\cos\theta, b\sin\theta)$

$Q = (a\cos\theta, a\sin\theta)$

$R = (a\cos\theta, b\sin\theta)$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is locus of R which is

An ellipse since focus lie on inner circle



$\therefore b = ae$

$\Rightarrow b^2 = a^2\left(1 - \frac{b^2}{a^2}\right)$

$\therefore \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow \frac{b}{a} = \frac{1}{\sqrt{2}}$

And $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$

Sol 12: Let B = (a, 0) C = (–a, 0) & A = (a, 0)

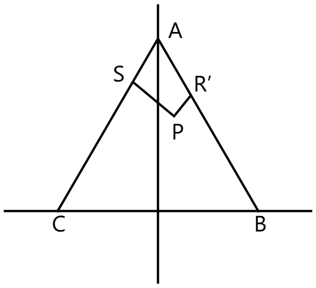
Equation of line AB is

$x + y = a$ and that of AC is $y = x + a$

Let P= (x, y) distance from BC = y

And area of PRAS = PR x PS

$$= \frac{(x + y - a)}{\sqrt{2}} \times \frac{(x - y + a)}{\sqrt{2}}$$



according to Q.

$y^2 = \pm \frac{1}{2}\frac{(x^2-(y-a)^2)}{2}$

$\therefore \pm 4y^2 = x^2 - y^2 - 2ay - a^2$

When it is $+4y^2$ it forms a hyperbola.

When it is $-4y^2$ it forms an ellipse

$\therefore x^2 + 3y^2 - 2ay - a^2 = 0$

$x^2 + 3\left(y-\frac{1}{3}\right)^2 = \frac{4a^2}{3}$

$\therefore \frac{x^2}{\frac{4a^2}{3}} + \frac{\left(y-\frac{1}{3}\right)^2}{\frac{4a^2}{9}} = 1$

$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow e = \sqrt{\frac{2}{3}}$

Sol 13: Let equation of tangent be

$y = mx \pm \sqrt{16m^2+9}$ is passes through (2, 3)

$\therefore (3-2m) = \pm \sqrt{16m^2+9}$

$\Rightarrow 4m^2 - 12m + 9 = 16m^2 + 9$

$\Rightarrow 12m^2 + 12m = 0$

$m = 0$ or $m = -1$

Rechecking we get when $m = 0$ $C > 0$ & when $m = -1$ $C > 0$

$\therefore$ Equation of tangent is

$y = 3$ and $y = -x + 5$ or $x + y = 5$.

Sol 14: Let $y = mx + \frac{1}{m}$ be tangent to parabola $y^2 = 4x$.

It will touch ellipse $\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{6})^2} = 1$ if

$\frac{1}{m^2} = 16m^2 + 6$

$\Rightarrow 16m^4 + 6m^2 - 1 = 0$

$\Rightarrow (8m^2 - 1)(2m^2 + 1) = 0$

$m = \pm \frac{1}{2\sqrt{2}}$

we know that a tangent at slope m touches parabola at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

so coordinate of point of contact are $A(8, 4\sqrt{2})$ and $B(8, -4\sqrt{2})$ we also know that tangent of slope m touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

$\left(\frac{\mp a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

$\therefore C = \left(-2, \frac{3}{\sqrt{2}}\right)$ & $D\left(-2, \frac{-3}{\sqrt{2}}\right)$

AB || CD $\therefore$ Quadrilateral is trapezium

Area $= \frac{1}{2} \times h\,(AB + CD)$

$= \frac{1}{2} \times 10 \times \left(8\sqrt{2} + \frac{6}{\sqrt{2}}\right) = 55\sqrt{2}$ sq. units

Sol 15: Equation of normal at $P(a\cos\theta, b\sin\theta)$ is

$\frac{ax}{\cos\theta} - \frac{by}{mn\theta} = a^2 - b^2$

$G = \left(\frac{\cos\theta}{a}(a^2-b^2), 0\right)$

And $g = \left(0, \frac{-\sin\theta}{b}(a^2-b^2)\right)$

$\therefore a^2CG^2 + b^2(Cg)^2$

$= a^2 \times \frac{\cos^2\theta}{a^2}(a^2-b^2)^2 + b^2\frac{\sin^2\theta}{b^2}(a^2-b^2)^2$

$= (a^2 - b^2)^2$

$CG = a\cos\theta\left(1 - \frac{b^2}{a^2}\right)$

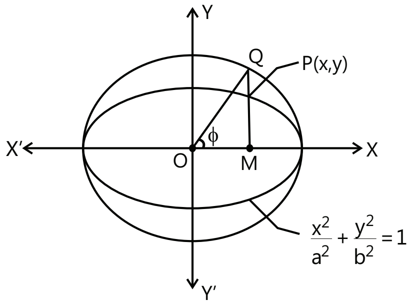
$= a\cos q e^2 = e^2 \times$ abscissa of P

Sol 16: $\therefore (ae + r\cos\theta, r\sin\theta)$ lies on ellipse

$\therefore \frac{(ae+r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$

$\therefore \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)r^2 + \frac{2e\cos\theta}{a}r + e^2 - 1 = 0$

To find the ℓ chord we have to find $(r_1 - r_2)$ as r_1 +ve and $r_2 < 0$



$\therefore (r_1 - r_2)^2$

$$= \frac{4e^2\cos^2\theta}{a^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^2} - \frac{4(e^2-1)}{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}$$

$$= \frac{4e^2a^2b^4\cos^2\theta}{(b^2\cos^2\theta+a^2\sin^2\theta)^2}$$

$$-\frac{4a^2b^2(b^2\cos^2\theta+a^2\sin^2\theta)(e^2-1)}{(b^2\cos^2\theta+a^2\sin^2\theta)^2}$$

$$= \frac{4\left(1-\frac{b^2}{a^2}\right)a^2b^4\cos^2\theta+4b^4(b^2\cos^2\theta+a^2\sin^2\theta)}{(b^2\cos^2\theta+a^2\sin^2\theta)^2}$$

$$= \frac{4a^2b^4}{(b^2\cos^2\theta+a^2\sin^2\theta)^2}$$

$$\therefore \quad (r_1 - r_2) = \frac{2ab^2}{(b^2\cos^2\theta+a^2\sin^2\theta)}$$

Sol 17: The tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

where p = $(a\cos\theta, b\sin\theta)$

$\therefore T = \left(\frac{a}{\cos\theta}, 0\right)$ & N = $(a\cos\theta, 0)$

equation of circle with TN as diameter is

$\left(x-\frac{a}{\cos\theta}\right)(x - a\cos\theta) + y^2 = 0$

$\Rightarrow x^2 - a\left(\frac{1}{\cos\theta}+\cos\theta\right)x + y^2 + a^2 = 0$

Equation of auxiliary circle is $x^2 + y^2 - a^2 = 0$

$2gg_1 + 2ff_1 = 2\,\frac{a}{2}\left(\frac{1}{\cos\theta}+\cos\theta\right)\times 0+2\times 0\times 0 = 0$

$c_1 + c_2 = a^2 - a^2 = 0$

The two circle as orthogonal

Sol 18: Let P = $(a\cos\alpha), b\sin\alpha)$ and Q = $(a\cos\beta, b\sin\beta)$

Since tangents at P & Q are $\perp$ is

$\therefore \frac{-b}{a\tan}\times\frac{-b}{a\tan\beta} = -1$

$\therefore$ tanatan$\beta = -\frac{b^2}{a^2}$

the point of intersection to tangents is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Find the point of interaction of normal from their equations.

You can easily show that

slop ON = slope OT

$\therefore$ N lies on $\frac{y}{y_1} = \frac{x}{x_1}$

Sol 19: Let (h, k) be the point the chord of contact is $\frac{xh}{a^2}+\frac{ky}{b^2} - 1 = 0$

It touches circle $x^2 + y^2 = c^2$

$\therefore \frac{|-1|}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}} = c$

$\therefore 1 = c^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)$

on $\frac{x^2}{a^4}+\frac{y^2}{b^4} = \frac{1}{c^2}$ is locus of P.

Sol 20: Equation of tangent to ellipse

$\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$

$y = mx \pm \sqrt{(a^2+b^2)m^2+b^2}$

It is also tangent to the ellipse,

$\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1$

$\therefore c^2 = a^2 m^2 + (a^2 + b^2)$

$\therefore (a^2 + b^2)m^2 + b^2 = a^2m^2 + (a^2 + b^2)$

$\therefore\ m^2 = \frac{a^2}{b^2}$ $m = \pm\frac{a}{b}$

$\therefore$ Tangents are

by = ± ax ± $\sqrt{a^4 + a^2b^2 + b^4}$

Sol 21: P = (acosθ, bsinθ) &

Q = (acosθ, asinθ)

tangent at P is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$\therefore$ T = $\left(\frac{a}{\cos\theta}, 0\right)$

Slope QT = $\frac{a\sin\theta}{a\cos\theta - \frac{a}{\cos\theta}} = -\frac{1}{\tan\theta}$

$\therefore$ QT ⊥ OQ $\therefore$ QT is tangent to the auxiliary circle

Sol 22: Normal at P(θ) is

ax secθ – bycosecθ = $a^2 - b^2$

It passes through Q(acos2θ, bsin2θ)

$$\therefore\ \frac{a^2 \times \cos^2\theta}{\cos\theta} - \frac{b^2 \times 2\sin\theta\cos\theta}{\cos\theta\sin\theta} = a^2 - b^2$$

$\therefore$ $a^2(2\cos^2\theta - 1) - 2b^2\cos^2\theta = (a^2 - b^2)\cos\theta$.

$\therefore$ $18\cos^2\theta - 9\cos\theta - 14 = 0$

$18\cos^2\theta - 21\cos\theta - 12\cos\theta - 14 = 0$

$\therefore 3\cos\theta(6\cos\theta - 7) + 2(6\cos\theta - 7) = 0$

$\therefore\ \cos\theta = -\frac{2}{3}$

Sol 23: Let equation of tangent to ellipse be

y = mx + $\sqrt{a^2m^2 + b^2}$

Now it touches circle

$\therefore$ $c^2 = r^2(m^2 + 1)$

$\therefore$ $a^2m^2 + b^2 = r^2m^2 + 1$

$$\boxed{m^2 = \frac{r^2 - b^2}{a^2 - r^2}}$$

Equation of PQ is y = m(x – ae)

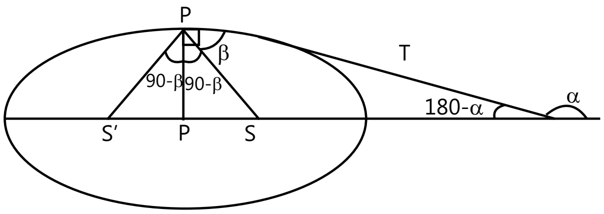
or mx – y – mae = 0

Length of ⊥ from centre to PQ (LP)

$$= \frac{+mae}{\sqrt{m^2+1}} = \frac{+\sqrt{\frac{r^2-b^2}{a^2-r^2}} \times ae}{\sqrt{\frac{a^2-b^2}{a^2-r^2}}}$$

$$= +\sqrt{\frac{r^2-b^2}{a^2-b^2}} \times a \times \frac{1}{a}\sqrt{a^2-b^2} = \sqrt{r^2-b^2}$$

PQ = $2\sqrt{r^2 - LP^2} = 2\sqrt{b^2} = 2b$



Sol 24:

∠SPN = 90 – b

∠S′PN = ∠SPN. As normal bisects angle between S′P and SP

∠SPS′ = 180 – 2β

∠PSS′ = 180 – α + b

∠PS′S = α + β – 180

Applying sine rule on DSPS′

$$\frac{\sin\angle PS'S}{PS} = \frac{\sin\angle PSS}{PS'} = \frac{\sin\angle S'PS}{SS'}$$

$$\therefore\ \frac{\sin(\alpha+\beta-180°)}{PS}$$

$$= \frac{\sin(180° - (\alpha-\beta))}{PS'} = \frac{\sin(180° - 2\beta)}{2ae}$$

PS + PS′ = 2a

$$\Rightarrow \frac{\sin(\alpha-\beta) - \sin(\alpha+\beta)}{2a} = \frac{\sin 2\beta}{2ae}$$

$$\therefore\ e = \left|\frac{2\sin\beta\cos\beta}{2\sin\beta\cos\alpha}\right| = \left|\frac{\cos\beta}{\cos\alpha}\right|$$

Sol 25: Let P = (acosθ, bsinθ)

normal is $\frac{5x}{\cos\theta} - \frac{4y}{\sin\theta} = 9$

It passes through (ae, 0) = (3, 0)

when sinθ ≠ 0

$\frac{15}{\cos\theta} = 9 \therefore \cos\theta = \frac{5}{3}$ × not possible

When $\sin\theta = 0$ equation of normal is y = 0 which passes through (3, 0)

$\therefore$ Radius = a – ae = 5 – 3 = 2

Sol 26: PF = QF and PC = RC

$\therefore$ Ae – 1 + b – 1 = $\sqrt{a^2e^2 + b^2}$

$(ae + b - 2)^2 = (a^2e^2 + b^2)$

$\therefore$ 4 – 4ae – 4b + 2aeb = 0

2 = 2ae + 2b – aeb

ae = 6

$\therefore$ 2 = 12 + 2b – 6b

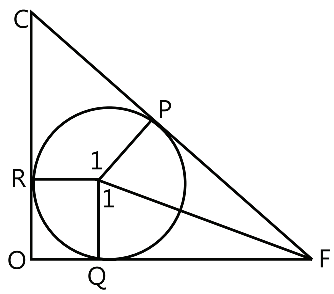
$\therefore b = \frac{5}{2}$

$a^2l^2 = a^2 - b^2$

$36 = a^2 - \frac{25}{4}$

$a^2 = \frac{16a}{4} \therefore a = \frac{13}{2}$

AB. CD = 4ab = 65



Exercise 2

Single Correct Choice Type

Sol 1: (A) Tangent to ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is

$\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$

Let P be point of intersection of x-axis & Q be Let Q be the point on tangent and y-axis

$\therefore P = \left(\frac{4}{\cos\theta}, 0\right)$ & $Q = \left(0, \frac{3}{\sin\theta}\right)$

Let M = (x, y)

$x = \frac{2}{\cos\theta}$ & $y = \frac{3}{2\sin\theta}$

$\therefore \frac{4}{x^2} + \frac{9}{4y^2} = 1$

$\therefore 16y^2 + 9x^2 = 4x^2y^2$

Sol 2: (C) P = (acosθ, bsinθ)

Q = (acosθ, asinθ)

normal at P is

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\Rightarrow \frac{4x}{\cos\theta} - \frac{3y}{\sin\theta} = 7$

Equation of CQ is y = tanθx

$\therefore \frac{x}{\cos\theta} = 7 \Rightarrow x = 7\cos\theta$ & $y = 7\sin\theta$ s

$\therefore$ R = (7cosθ, 7sinθ)

$\therefore \ell(CR) = 7$

Sol 3: (A) Lines that C = (1, 2)

Patting through P = (4, 6)

$\because$ Centre and focus have same y coordie

$\therefore$ This will be ellipse where major axis is horizontal so.

Eqn. will be $\frac{(x-n)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Centre is (h, k)

The left focus (h – c, k)

Right focus (h + c, k)

Where $c^2 = a^2 - b^2$

On putting values

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1 \quad \text{(i)}$$

Patting through (9, 6)

$$\Rightarrow \frac{3^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1 \quad \text{(ii)}$$

The given frees (6, 2) must be right focuses

$\Rightarrow (h + c, k) = (6, 2)$

$\Rightarrow h + c = 6 \quad k = 2$

$\Rightarrow c = 5 \quad (\because h = 1)$

Now, $C^2 = a^2 - b^2 \Rightarrow a^2 - b^2 = 25$ (iii)

Solving (ii) and (iii)

Substituting in (i)

$$\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$$

Sol 4: (A) Let the lines be x-axis & y-axis

Let P = (h, 0) and Q = (0, k)

$h^2 + k^2 = (a + b)^2$

$(x, y) = \left(\frac{bh}{(a+b)}, \frac{ak}{(a+b)}\right)$

$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$

∴ Equation of P is an ellipse

Sol 5: (A) Now sum of distance of points from two foci = constant = 2a for an ellipse

∴ Necessary length of string = 2a = 6

a = 3 and b = 2

Distance between pins = 2ae

$= 6\sqrt{1-\frac{b^2}{a^2}} = \frac{6\times\sqrt{5}}{3} = 2\sqrt{5}$

Sol 6: (B) It is a known property that

$SF_1 \cdot SF_2 = b^2 = 3$

Sol 7: (B) Ellipse 1 is

$$\frac{x^2}{\left(\sqrt{\frac{3}{a}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{3}{b}}\right)^2} = 1$$

And E_2 is

$$\frac{x^2}{\left(\sqrt{\frac{3}{a+b}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{3}{a-b}}\right)^2} = 1$$

are of ellipse = pab

$\therefore pa_1b_1 = pa_2b_2$

$\Rightarrow \frac{3}{\sqrt{ab}} = \frac{3}{\sqrt{a^2-b^2}}$

$\Rightarrow a^2 - b^2 = ab$

$\Rightarrow \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0$

$\because \frac{a}{b} = \frac{1+\sqrt{5}}{2}$

Sol 8: (B) Let P = (h, k)

Foot of perpendicular from focus to any tangent of the ellipse lies on its auxiliary circle.

∴ Midpoint of P & S lies on auxiliary circle

$\therefore M = \left(\frac{h\pm ae}{2}, \frac{k}{2}\right)$

$\Rightarrow \left(\frac{h\pm ae}{2}\right)^2 + \left(\frac{k}{2}\right)^2 = a^2.$

$\Rightarrow \left(\frac{x\pm ae}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = a^2.$

Sol 9: (C) Equation of normal is

$ax\sec\theta - by\,\text{cosec}\theta = a^2 - b^2$

$Q = \left(\frac{a^2-b^2}{a}\right)\cos\theta$

$R = -\left(\frac{a^2-b^2}{b}\right)\sin\theta$

$M = (x, y) = \left(\frac{a^2-b^2}{2a}\times\cos\theta, -\frac{(a^2-b^2)}{2b}\sin\theta\right)$

∴ Locus of M is

$(ax)^2 + (by)^2 = \frac{(a^2-b^2)^2}{4}$

$\therefore \frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 0$

coeff of y is > coeff of x.

$$e' = 1 - \frac{\left(\frac{a^2-b^2}{2a}\right)^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 1 - \frac{b^2}{a^2} = 0$$

Sol 10: (C) Equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and of a circle is

$x^2 + y^2 = a^2 - b^2$

$\therefore \frac{a^2-b^2-y^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow (a^2 - b^2)y^2 = a^2b^2 - a^2b^2 + b^4$

$\therefore y^2 = \dfrac{b^4}{a^2 - b^2}$

$y = \dfrac{b^2}{\sqrt{a^2 - b^2}}$

$2a = 17 \therefore\ a = \dfrac{17}{2}$

And aexy = 30

$\therefore\ \sqrt{a^2 - b^2} \times \dfrac{b^2}{\sqrt{a^2 - b^2}} = 30$

$\therefore$ Distance between foci

$= 2ae = 2\sqrt{a^2 - b^2} = 2\sqrt{\dfrac{289}{4} - 30}$

$= 2 \times \dfrac{13}{2} = 13$

Sol 11: (A) Equation of tangent is

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} - 1 = 0$

$\Rightarrow \dfrac{x}{\sqrt{2}a} + \dfrac{y}{\sqrt{2}b} - 1 = 0$

$\therefore P' = \dfrac{1}{\sqrt{\dfrac{1}{2a^2} + \dfrac{1}{2b^2}}}$

Equation of normal is $\sqrt{2}ax - \sqrt{2}by = a^2 - b^2$

$\therefore P^2 = \dfrac{a^2 - b^2}{\sqrt{2(a^2 + b^2)}}$

$\therefore$ Area of rectangle = P_1P_2

$= \dfrac{\sqrt{2}ab}{\sqrt{a^2 + b^2}} + \dfrac{(a^2 - b^2)}{\sqrt{2(a^2 + b^2)}} = \dfrac{(a^2 - b^2)ab}{(a^2 + b^2)}$

Sol 12: (A) If for an ellipse S & S′ are focus, then

$\tan\left(\dfrac{PSS'}{2}\right) \times \tan\left(\dfrac{PS'S}{2}\right) = \dfrac{1-e}{1+e}$

$\therefore$ Centre of ellipse = (5, 0)

$\dfrac{1-e}{1+e} = \dfrac{1}{4}$

$\therefore 5e = 3e = \dfrac{3}{5}$

2ae = 6

$\therefore a = 5$

$\therefore 1 - \dfrac{b^2}{a^2} = \dfrac{a}{25}$

$\therefore b = 4$

$\therefore$ Equation of ellipse can be $\dfrac{(x-5)^2}{25} + \dfrac{y^2}{16} = 1$

Multiple Correct Choice Type

Sol 13: (A, C, D) (A) ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

Equation of director circle is $x^2 + y^2 = a^2 + b^2$

$\Rightarrow x^2 + y^2 = 14$

(B) Sum of focal distances = 2b(when b > 0)

and 2a when a > 0

$\therefore S = 2 \times 6 = 12$

(C) Free (a known property of parabola)

(D) Line passes through focus.

$\dfrac{2at_2 - 0}{at_2^2 - a} = \dfrac{2at_1}{at_1^2 - a}$

(slope of PF = QF)

$\therefore t_2(t_1^2 - 1) = t_1(t_2^2 - 1)$

$t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$

$\therefore t_1 = t_2$ or $t_1t_2 = -1$

But points are distinct

$\therefore t_1t_2 = -1$

Sol 14: (C, D) Equation of tangent is

$y = mx \pm \sqrt{\dfrac{5}{3}m^2 + \dfrac{5}{2}}$

It passes through (1, 2)

$(m - 2)^2 = \dfrac{5}{3}m^2 + \dfrac{5}{2}$

$m^2 - 4m + 4 = \dfrac{5}{3}m^2 + \dfrac{5}{2}$

$\dfrac{2}{3}m^2 + 4m - \dfrac{3}{2} = 0$

$4m^2 + 24m - 9 = 0$

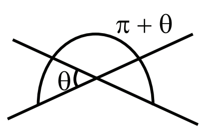
angle between tangents

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\right|$

$= \left|\frac{\sqrt{6^2 + \frac{9}{4} \times 4}}{1 - \frac{9}{4}}\right| = \left|\frac{3\sqrt{5}}{5} \times 4\right| = \frac{12}{\sqrt{5}}$

$\therefore \theta = \tan^{-1}\frac{12}{\sqrt{5}}$

other angle $= \pi - \tan^{-1}\frac{12}{\sqrt{5}} = \pi - \cot^{-1}\left(\frac{\sqrt{5}}{12}\right)$



If angle between lines is E_0

$\therefore$ $\pi + \theta$ can also be considered angle between lines,

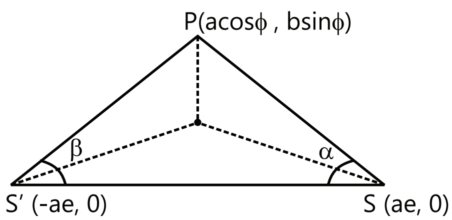
$\therefore$ D is also correct

Sol 15: (A, B, C) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(A) and (B) are true

By sine rule in DPSS' then

$\frac{SP}{\sin(\beta)} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin(\pi - (\alpha + \beta))}$



$\frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin(\alpha + \beta)}$

$\frac{SP + S'P}{\sin\alpha + \sin\beta} = \frac{SS'}{\sin(\alpha + \beta)}$

$\therefore \frac{2a}{\sin\alpha + m\beta} = \frac{2ae}{\sin(\alpha + \beta)}$

$\therefore \frac{1}{e} = \frac{\sin\beta + \sin\alpha}{\sin(\alpha + \beta)}$

$\frac{1}{e} = \frac{2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)}{2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)}$

$\frac{1}{e} = \frac{\cos\left(\frac{\alpha - \beta}{2}\right)}{\cos\left(\frac{\alpha + \beta}{2}\right)}$

$\frac{1 - e}{1 + e} = \frac{\cos\left(\frac{\alpha - \beta}{2}\right) - \cos\left(\frac{\alpha + \beta}{2}\right)}{\cos\left(\frac{\alpha - \beta}{2}\right) + \cos\left(\frac{\alpha + \beta}{2}\right)}$

$\therefore \tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{1 - e}{1 + e} = \frac{(1 - e)^2}{1 - e^2}$

$= \frac{1 + 1 - \frac{b^2}{a^2} - 2\sqrt{1 - \frac{b^2}{a^2}}}{\frac{b^2}{a^2}} = \frac{2a^2 - b^2 - 2a\sqrt{a^2 - b^2}}{b^2}$

Sol 16: (A, C) Equation of tangent to parabola is

$y = mx + \frac{1}{m}$

for ellipse $c^2 = a^2 m^2 + b^2$

$\therefore \frac{1}{m^2} = 8m^2 + 2$

$8m^4 + 2m^2 - 1 = 0$

$8m^4 + 4m^2 - 2m^2 - 1 = 0$

$\therefore m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

$\therefore$ Equation of tangents

are $y = \frac{1}{2}x + 2$ or $x - 2y + 4 = 0$

and $y = -\frac{1}{2}x - 2$ or $2y + x + 4 = 0$

Sol 17: (A, B, C, D) $a^2\cos^2\theta + b^2\sin^2\theta = 4$

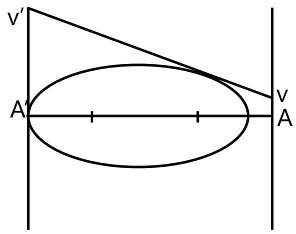
$6\cos^2\theta + 2\sin^2\theta = 4$

$4\cos^2\theta = 2$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore$ Eccentric angle is $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Sol 18: (A, C, D) Equation of tangent is



$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

A =(a, 0)

$$\therefore\ V=\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right)$$

A′ = (–a, 0)

$$\therefore\ V'=\left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$AV\times A'V=\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}$$

We know that V′V subtend a right angle at each of the foci

∴ VV′ SS′ lie on a circle with VV′ as diameter.

Previous Years' Questions

Sol 1: (A) Given, $y=mx-b\sqrt{1+m^2}$ touches both the circles, so distance from centre = radius of both the circles.

$$\frac{\left|ma-0-b\sqrt{1+m^2}\right|}{\sqrt{m^2+1}}=b$$

and $$\frac{\left|-b\sqrt{1+m^2}\right|}{\sqrt{m^2+1}}=b$$

$$\Rightarrow \left|ma-b\sqrt{1+m^2}\right|=\left|-b\sqrt{1+m^2}\right|$$

$$\Rightarrow m^2a^2-2abm\sqrt{1+m^2}+b^2=b^2(1+m^2)$$

$$\Rightarrow ma-2b\sqrt{1+m^2}=0$$

$$\Rightarrow m^2a^2=4b^2(1+m^2)$$

$$\Rightarrow m=\frac{2b}{\sqrt{a^2-4b^2}}$$

Sol 2: **(B)** Given tangent is drawn at $(3\sqrt{3}\cos\theta,\sin\theta)$ to $\frac{x^2}{27}+\frac{y^2}{1}=1$.

$$\therefore\ \text{Equation of tangent is } \frac{x\cos\theta}{3\sqrt{3}}+\frac{y\sin\theta}{1}=1$$

Thus, sum of intercepts

$$=\left(\frac{3\sqrt{3}}{\cos\theta}+\frac{1}{\sin\theta}\right)=f(\theta)\ \text{(say)}$$

$$\Rightarrow f'(\theta)=\frac{3\sqrt{3}\sin^3\theta-\cos^3\theta}{\sin^2\theta\cos^2\theta}\ \text{Put } f'(\theta)=0$$

$$\Rightarrow \sin^3\theta=\frac{1}{3^{3/2}}\cos^3\theta$$

$$\Rightarrow \tan\theta=\frac{1}{\sqrt{3}}\ \text{, i.e., } \theta=\frac{\pi}{6}$$

and at $\theta=\frac{\pi}{6}$, $f''(0)>0$

∴ Hence, tangent is minimum at $\theta=\frac{\pi}{6}$.

Sol 3: (C) There are two common tangents to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. These are x = 1 and x = –1. But x = 1 is nearer to the point P(1/2, 1). Therefore, directrix of the required ellipse is x = 1.

$$\therefore\ e=\frac{\sqrt{3}}{2}$$

$$\therefore\ x=\pm4\times\frac{\sqrt{3}}{2}=\pm2\sqrt{3}\ (\because x=\pm ae) \qquad \text{...(ii)}$$

On solving equation (i) and (ii), we get $\frac{4}{49}\times12+y^2=1$

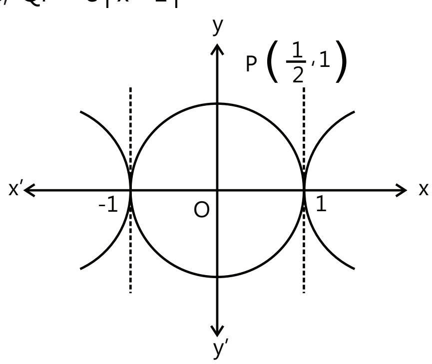
$$\Rightarrow y^2=1-\frac{48}{49}=\frac{1}{49}\ \Rightarrow y=\pm\frac{1}{7}$$

$$\therefore\ \text{Required points } \left(\pm2\sqrt{3},\pm\frac{1}{7}\right).$$

Sol 4: Now, If Q(x, y) is any point on the ellipse, then its distance from the focus is

$$QP=\sqrt{(x-1/2)^2+(y-1)^2}$$

and its distance from the directrix is $|x-1|$ by definition of ellipse, $QP=e|x-1|$



$$\Rightarrow \sqrt{\left(x-\frac{1}{2}\right)^2+(y-1)^2}=\frac{1}{2}|x-1|$$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2+(y-1)^2=\frac{1}{4}(x-1)^2$$

$$\Rightarrow x^2-x+\frac{1}{4}+y^2-2y+1=\frac{1}{4}(x^2-2x+1)$$

$$\Rightarrow 4x^2-4x+1+4y^2-8y+4=x^2-2x+1$$

$$\Rightarrow 3x^2-2x+4y^2-8y+4=0$$

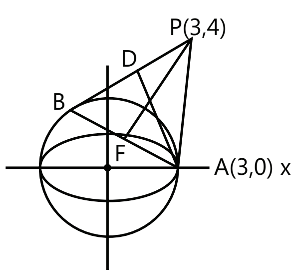
$$\Rightarrow 3\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}\right]+4(y-1)^2=0$$

$$\Rightarrow 3\left(x-\frac{1}{3}\right)^2+4(y-1)^2=\frac{1}{3}$$

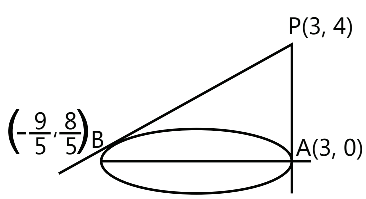
$$\Rightarrow \frac{\left(x-\frac{1}{3}\right)^2}{1/9}+\frac{(y-1)^2}{1/12}=1$$

Comprehenesion Type

Sol 5: (D) Figure is self explanatory.



Sol 6: (C) Equation of AB is



$$y-0=\frac{\frac{8}{5}}{-\frac{9}{5}-3}(x-3)$$

$$=\frac{8}{-24}(x-3)$$

$$\Rightarrow y=-\frac{1}{3}(x-3)$$

$\Rightarrow$ x + 3y = 3 ...(i)

Equation of the straight line perpendicular to AB through P is $3x-y=5$.

Equation of PA is $x-3=0$.

The equation of straight line perpendicular to PA through $B\left(-\frac{9}{5},\frac{8}{5}\right)$ is $y=\frac{8}{5}$.

Hence, the orthocenter is $\left(\frac{11}{5},\frac{8}{5}\right)$.

Sol 7: (A) Equation of AB is $y-0=-\frac{1}{3}(x-3)$

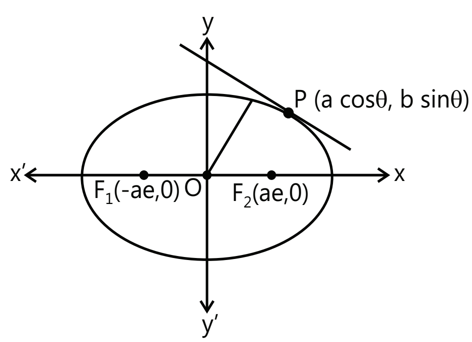
$x+3y-3=0$ $\quad |x+3y-3|^2=10[(x-3)^2+(y-4)^2]$

(Look at coefficient of x^2 and y^2 in the answers).

Sol 8: Let the coordinates of point P be (acosθ, bsinθ). Then equation fo tangent at P is

$$\frac{x}{a}\cos\theta+\frac{y}{b\sin\theta}=1 \quad ...(i)$$

We have , d = length of perpendicular from O to the tangent at P.



$$d=\frac{|0+0-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$\Rightarrow \frac{1}{d}=\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}$$

$$\Rightarrow \frac{1}{d^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

We have, to prove $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

Now, RHS = $4a^2\left(1-\frac{b^2}{d^2}\right)$

$$=4a^2-\frac{4a^2b^2}{d^2}$$

$= 4a^2 - 4a^2b^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$

$= 4a^2 - 4b^2\cos^2\theta - 4a^2\sin^2\theta$

$= 4a^2(1 - \sin^2\theta) - 4b^2\cos^2\theta$

$= 4a^2\cos^2\theta - 4b^2\cos^2\theta$

$= 4\cos^2\theta(a^2 - b^2)$

$= 4\cos^2\theta a^2 e^2\left(\because e = \sqrt{a - (b/a)^2}\right)$

Again, $PF_1 = e\left|a\cos\theta + a/e\right|$

$= a\left|e\cos\theta + 1\right|$

$= a(e\cos\theta + 1)$ $(\because -1 \le \cos\theta \le 1$ and $0 < e < 1)$

Similarly, $PF_2 = a(1 - e\cos\theta)$

Therefore, LHS = $(PF_1 - PF_2)^2$

$= [a(e\cos\theta + 1) - a(1 - e\cos\theta)]^2$

$= (ae\cos\theta + a - a - ae\cos\theta)^2$

$= (2ae\cos\theta)^2 = 4a^2e^2\cos^2\theta$

Hence, LHS = RHS.

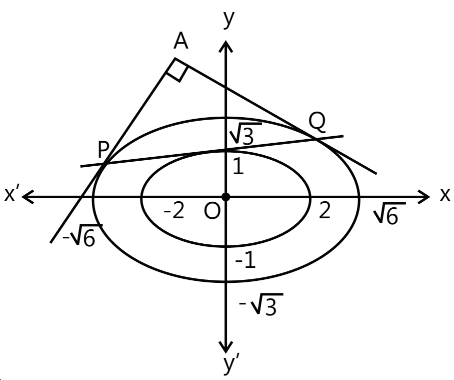
Sol 9: Given, $x^2 + 4y^2 = 4$

or $\frac{x^2}{4} + \frac{y^2}{1} = 1$ ...(i)

Equation of any tangent to the ellipse on (i) can be written as

$\frac{x}{2}\cos\theta + y\sin\theta = 1$ ...(ii)

Equation of second ellipse is



$x^2 + 2y^2 = 6$

$\Rightarrow \frac{x^2}{6} + \frac{y^2}{3} = 1$ ...(iii)

Suppose the tangents of P and Q meets in A(h,k). Equation of the chord of contact of the tangents through A(h,k) is

$\frac{hx}{6} + \frac{ky}{3} = 1$ ...(iv)

But Eqs. (iv) and (ii) represent the same straight line, so comparing Eqs. (iv) and (ii), we get

$\frac{h/6}{\cos\theta/2} = \frac{k/3}{\sin\theta} = \frac{1}{1} \Rightarrow h = 3\cos\theta$ and $k = 3\sin\theta$

Therefore, coordinates of A are $(3\cos\theta, 3\sin\theta)$.

Now, the joint equation of the tangents at A is given by $T^2 = SS_1$

i.e, $\left(\frac{hx}{6} + \frac{ky}{3} - 1\right)^2$

$= \left(\frac{x^2}{6} + \frac{y^2}{3} - 1\right)\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$ ...(v)

In equation (v).

Coefficient of $x^2 = \frac{h^2}{36} - \frac{1}{6}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$

$= \frac{h^2}{36} - \frac{h^2}{36} - \frac{k^2}{18} + \frac{1}{6} = \frac{1}{6} - \frac{k^2}{18}$

And coefficient of $y^2 = \frac{k^2}{9} - \frac{1}{3}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$

$= \frac{k^2}{9} - \frac{h^2}{18} - \frac{k^2}{9} + \frac{1}{3} = -\frac{h^2}{18} + \frac{1}{3}$

Again, coefficient of x^2 + coefficient of y^2

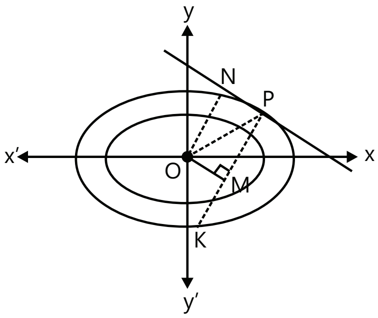
$= -\frac{1}{18}(h^2 + k^2) + \frac{1}{6} + \frac{1}{3}$

$= -\frac{1}{18}(9\cos^2\theta + 9\sin^2\theta) + \frac{1}{2}$

$= -\frac{9}{18} + \frac{1}{2} = 0$

Which shows that two lines represent by equation (v) are at right angles to each other.

Sol 10: Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Equations of tangents at P is



$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Again, equation of normal at point P is

$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

Let M be foot of perpendicular from O to PK, the normal at P.

Area of $\Delta OPN = \frac{1}{2}$ (Area of rectangle OMPN)

$= \frac{1}{2}$ ON. OM

Now,

$$ON = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

($\perp$ from O, to line NP)

$$\text{and } OM = \frac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\mathrm{cosec}^2\theta}}$$

$$= \frac{(a^2 - b^2).\cos\theta.\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

Thus area of

$$\Delta OPN = \frac{ab(a^2 - b^2).\cos\theta.\sin\theta}{2(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$$= \frac{ab(a^2 - b^2)\tan\theta}{2(a^2\tan^2\theta + b^2)}$$

Let $f(\theta) = \frac{\tan\theta}{a^2\tan^2\theta + b^2}$ $(0 < \theta < \pi/2)$

$$f'(\theta) = \frac{\sec^2\theta(a^2\tan^2\theta + b^2) - \tan\theta(2a^2\tan\theta\sec^2\theta)}{(a^2\tan^2\theta + b^2)^2}$$

$$= \frac{\sec^2\theta(a^2\tan^2\theta + b^2 - 2a^2\tan^2\theta)}{(a^2\tan^2\theta + b^2)^2}$$

$$= \frac{\sec^2\theta(a\tan\theta + b)(b - a\tan\theta)}{(a^2\tan^2\theta + b^2)^2}$$

For maximum or minimum, we put

$f'(\theta) = 0 \Rightarrow b - a\tan\theta = 0$

$[\sec^2\theta \neq 0.\ a\tan\theta + b \neq 0,\ 0 < \theta < \pi/2]$

$\Rightarrow \tan\theta = b/a$

Also, $f'(\theta)\begin{cases} > 0, & \text{if } 0 < \theta < \tan^{-1}(b/a) \\ < 0, & \text{if } \tan^{-1}(b/a) < \theta < \pi/2 \end{cases}$

Therefore, $f(\theta)$ has maximum, when

$\theta = \tan^{-1}\left(\frac{b}{a}\right) \Rightarrow \tan\theta = \frac{b}{a}$

Again $\sin\theta = \frac{b}{\sqrt{a^2 + b^2}} - \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$

By using symmetry, we get the required points

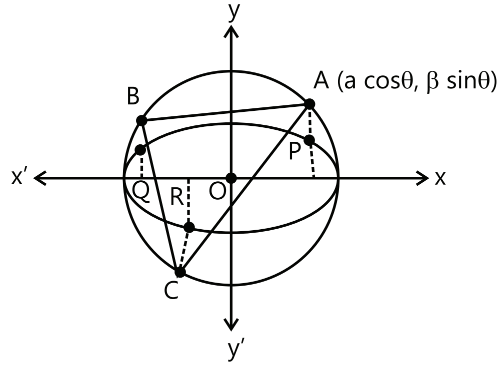
$\left(\frac{\pm a^2}{\sqrt{a^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2 + b^2}}\right)$.

Sol 11: Let the coordinates of $A \equiv (a\cos\theta, b\sin\theta)$, so that the coordinates of

$B = \{a\cos(\theta + 2\pi/3), a\sin(\theta + 2\pi/3)\}$

and $C = \{a\cos(\theta + 4\pi/3), a\sin(\theta + 4\pi/3)\}$

According to the given condition, coordinates of P are $(a\cos\theta, b\sin\theta)$ and that of Q are $\{a\cos(\theta + 2\pi/3), b\sin(\theta + 2\pi/3)\}$ and that of R are $\{a\cos(\theta + 4\pi/3), b\sin(\theta + 4\pi/3)\}$.



[It is given that P, Q, R are on the same side of x-axis as A, B and C].

Equation of the normal to the ellipse at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \text{ or } ax\sin\theta - by\cos\theta$$

$$= \frac{1}{2}(a^2 - b^2)\sin 2\theta \qquad \text{......(i)}$$

Equation of normal to the ellipse at Q is

$$ax\sin\left(\theta + \frac{2\pi}{3}\right) - by\cos\left(\theta + \frac{2\pi}{3}\right) =$$

$$\frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{4\pi}{3}\right) \qquad \text{...(ii)}$$

Equation of normal to the ellipse at R is

$$ax\sin\left(\theta + \frac{4\pi}{3}\right) - by\cos\left(\theta + \frac{4\pi}{3}\right) =$$

$$\frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{8\pi}{3}\right) \qquad \text{...(iii)}$$

$$\text{But } \sin\left(\theta + \frac{4\pi}{3}\right) = \sin\left(2\pi + \theta - \frac{2\pi}{3}\right) = \sin\left(\theta - \frac{2\pi}{3}\right)$$

$$\text{and } \cos\left(\theta + \frac{4\pi}{3}\right) = \cos\left(2\pi + \theta - \frac{2\pi}{3}\right) = \cos\left(\theta - \frac{2\pi}{3}\right)$$

$$\text{and } \sin\left(2\theta + \frac{8\pi}{3}\right) = \sin\left(4\pi + 2\theta - \frac{4\pi}{3}\right) = \sin\left(2\theta - \frac{4\pi}{3}\right)$$

Now, eq. (iii) can be written as

$$ax\sin(\theta - 2\pi/3) - by\cos(\theta - 2\pi/3) =$$

$$\frac{1}{2}(a^2 - b^2)\sin(2\theta - 4\pi/3) \qquad \text{...(iv)}$$

For the liens (i), (ii) and (iv) to be concurrent, we must have the determinant

$$\Delta_1 = \begin{vmatrix} a\sin\theta & -b\cos\theta & \frac{1}{2}(a^2 - b^2)\sin 2\theta \\ a\sin\left(\theta + \frac{2\pi}{3}\right) & -b\cos\left(\theta + \frac{2\pi}{3}\right) & \frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{4\pi}{3}\right) \\ a\sin\left(\theta - \frac{2\pi}{3}\right) & -b\cos\left(\theta - \frac{2\pi}{3}\right) & \frac{1}{2}(a^2 - b^2)\sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Thus, line (i), (ii) and (iv) are concurrent.

Sol 12: Any point on the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be P $(a\cos\theta, b\sin\theta)$.

The equation of tangent at point P is given by;

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

The equation of line perpendicular to tangent is,

$$\frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \lambda$$

Since, it passes through the focus (ae, 0), then

$$\frac{ae\sin\theta}{b} - 0 = \lambda$$

$$\Rightarrow \lambda = \frac{ae\sin\theta}{b}$$

$$\therefore \text{ Equation is } \frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \frac{ae\sin\theta}{b} \qquad \text{...(i)}$$

Equation of line joining centre and point of contact $P(a\cos\theta, b\sin\theta)$ is

$$y = \frac{b}{a}(\tan\theta)x \qquad \text{...(ii)}$$

Point of intersection Q of Eqs. (i) and (ii) has x coordinate $\frac{a}{e}$.

Hence, Q lies on the corresponding directrix $x = \frac{a}{e}$.

Sol 13: (B, C) $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$$b^2 = a^2\left(1 - e^2\right)$$

$$\Rightarrow e = \frac{\sqrt{3}}{2}$$

$$\Rightarrow P\left(\sqrt{3}, -\frac{1}{2}\right) \text{ and } Q\left(-\sqrt{3}, -\frac{1}{2}\right)$$

(given y_1 and y_2 less than 0).

Co-ordinates of mid-point of PQ are

$$R \equiv \left(0, -\frac{1}{2}\right).$$

$PQ = 2\sqrt{3}$ = length of latus rectum.

$\Rightarrow$ two parabola are possible whose vertices are

$$\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \text{ and } \left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right).$$

Hence the equations of the parabolas are

$$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$$

And $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$.

Sol 14: (A, B) Ellipse and hyperbola will be confocal

$$\Rightarrow (\pm ae, 0) \equiv (\pm 1, 0)$$

$\Rightarrow\left(\pm a\times\frac{1}{\sqrt{2}},0\right)\equiv(\pm 1, 0)$

$\Rightarrow a=\sqrt{2}$ and $e=\frac{1}{\sqrt{2}}$

$\Rightarrow b^2=a^2(1-e^2)\Rightarrow b^2=1$

$\therefore$ Equation of ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$

Sol 15: (D) Equation of line AM is x + 3y −3 = 0

Perpendicular distance of line from origin $=\frac{3}{\sqrt{10}}$

Length of AM $=2\sqrt{9-\frac{9}{10}}=2\times\frac{9}{\sqrt{10}}$

$\Rightarrow$ Area $=\frac{1}{2}\times 2\times\frac{9}{\sqrt{10}}\times\frac{3}{\sqrt{10}}=\frac{27}{10}$ sq. units

Sol 16: $A\to p$; $B\to s, t$; $C\to r$; $D\to q, s$

(p) $\frac{1}{k^2}=4\left(1+\frac{h^2}{k^2}\right)$

$\Rightarrow 1=4(k^2+h^2)$

$\therefore h^2+k^2=\left(\frac{1}{2}\right)^2$ which is a circle.

(q) If $|z-z_1|-|z-z_2|=k$ where $k<|z_1-z_2|$ the locus is a hyperbola.

(r) Let t = $\tan\alpha$

$\Rightarrow x=\sqrt{3}\cos 2\alpha$ and $\sin 2\alpha=y$

or $\cos 2\alpha=\frac{x}{\sqrt{3}}$ and $\sin 2\alpha=y$

$\therefore \frac{x^2}{3}+y^2=\sin^2 2\alpha+\cos^2 2\alpha=1$ which is an ellipse.

(s) If eccentricity is $[1,\infty)$, then the conic can be a parabola (if e = 1) and a hyperbola if $e\in(1,\infty)$.

(t) Let $z=x+iy$; $x,y\in R$

$\Rightarrow (x+1)^2-y^2=x^2+y^2+1$

$\Rightarrow y^2=x$; which is a parabola.

Sol 17: (B) Let equation of tangent to ellipse

$\frac{\sec\theta}{3}x-\frac{\tan\theta}{2}y=1$

$2\sec\theta x-3\tan\theta y=6$

It is also tangent to circle $x^2+y^2-8x=0$

$\Rightarrow\frac{|8\sec\theta-6|}{\sqrt{4\sec^2\theta+9\tan^2\theta}}=4$

$(8\sec\theta-6)^2=16(13\sec^2\theta-9)$

$\Rightarrow 12\sec^2\theta+8\sec\theta-15=0$

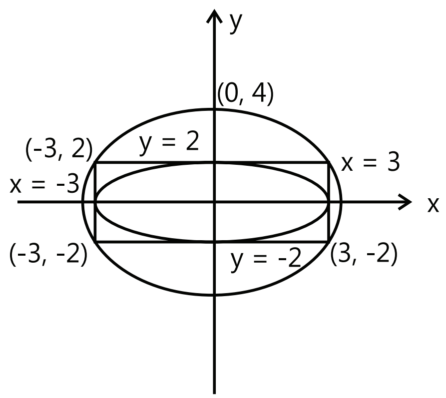
$\Rightarrow \sec\theta=\frac{5}{6}$ and $-\frac{3}{2}$ but $\sec\neq\frac{5}{6}$

$\Rightarrow \sec\theta=-\frac{3}{2}$ and $\Rightarrow \tan\theta=\frac{\sqrt{5}}{2}$

$\therefore$ Slope is positive

Equation of tangent $=2x-\sqrt{5}\,y+4=0$

Sol 18: (C)



Equation of ellipse is

$(y+2)(y-2)+\lambda(x+3)(x-3)=0$

It passes through (0, 4) $\Rightarrow\lambda=\frac{4}{3}$

Equation of ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$

$e=\frac{1}{2}$.

Sol 19: (9)

$\frac{x^2}{4}+\frac{y^2}{3}=1$

$y=\frac{\sqrt{3}}{2}\sqrt{4-h^2}$ at x = h

Let $R(x_1, 0)$

PQ is chord of contact, so $\frac{xx_1}{4}=1 \Rightarrow x=\frac{4}{x_1}$

which is equation of PQ, x = h

So $\frac{4}{x_1}=h \Rightarrow x_1=\frac{4}{h}$

$\Delta(h)$ = area of $\Delta PQR=\frac{1}{2}\times PQ\times RT$

$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2}\times(x_1-h)=\frac{\sqrt{3}}{2h}(4-h^2)^{3/2}$

$\Delta'(h)=\frac{-\sqrt{3}(4+2h^2)}{2h^2}\sqrt{4-h^2}$

which is always decreasing.

So Δ_1 = maximum of $\Delta(h)=\frac{45\sqrt{5}}{8}$ at $h=\frac{1}{2}$

Δ_2 = minimum of $\Delta(h)=\frac{9}{2}$ at h = 1

So $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=\frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8}-8.\frac{9}{2}=45-36=9$

Sol 20: (D)

The equation of P_1 is $y^2-8x=0$ and P_2 is $y^2+16x=0$

Tangent to $y^2-8x=0$ passes through (-4, 0)

$\Rightarrow 0=m_1(-4)+\frac{2}{m_1}\Rightarrow\frac{1}{m_1^2}=2$

Also tangent to $y^2+16x=0$ passes through

(2, 0)

$\Rightarrow 0=m_2\times 2-\frac{4}{m_2}\Rightarrow m_2^2=2$

$\Rightarrow\frac{1}{m_1^2}+m_2^2=4$

Sol 21: (A, B, D)

Tangent at P, $xx_1-yy_1=1$ intersects x axis at $M\left(\frac{1}{x_1},0\right)$

Slope of normal $=-\frac{y_1}{x_1}=\frac{y_1-0}{x_1-x_2}$

$\Rightarrow x_2=2x_1\Rightarrow N\equiv(2x_1,0)$

For centroid $\ell=\frac{3x_1+\frac{1}{x_1}}{3}$, $m=\frac{y_1}{3}$

$\frac{d\ell}{dx_1}=1-\frac{1}{3x_1^2}$

$\frac{dm}{dy_1}=\frac{1}{3},\frac{dm}{dx_1}=\frac{1}{3}\frac{dy_1}{dx_1}=\frac{x_1}{3\sqrt{x_1^2-1}}$

Sol 22: (C)

Equation of tangent at M is $\frac{x\times 3}{2\times 9}+\frac{y\sqrt{6}}{8}=1$

Put y = 0 as intersection will be on x-axis.

$\therefore R\equiv(6,0)$

Equation of normal at M is

$\sqrt{\frac{3}{2}}X+y=2\sqrt{\frac{3}{2}}+\left(\sqrt{\frac{3}{2}}\right)^3$

Put y = 0, $x=2+\frac{3}{2}=\frac{7}{2}$

$\therefore Q\equiv\left(\frac{7}{2},0\right)$

$\therefore$ Area $(\Delta MQR)=\frac{1}{2}\times\left(6-\frac{7}{2}\right)\times\sqrt{6}=\frac{5}{4}\sqrt{6}$ sq. units.

Area of quadrilateral

$(MF_1NF_2)=2\times$ Area (ΔF_1F_2M)

$=2\times\frac{1}{2}\times 2\times\sqrt{6}=2\sqrt{6}$

$\therefore$ Required Ratio = $\frac{5/4}{2}=\frac{5}{8}$