1. BASIC MATHEMATICS

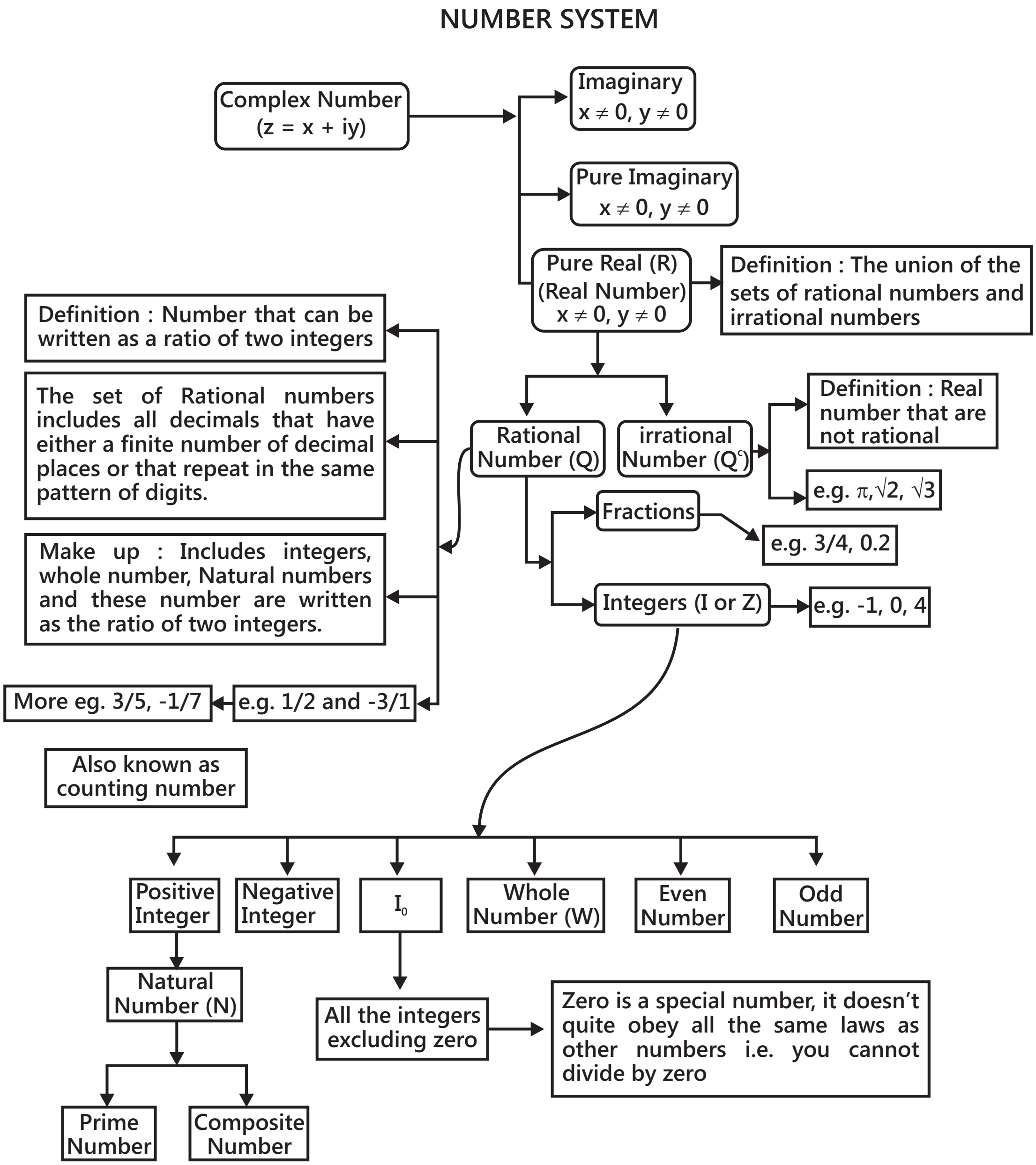


Figure 1.1: The number system

1. NUMBER SYSTEM

(a) Natural Numbers: The counting numbers 1, 2, 3, 4, ... are called natural numbers. The set of natural numbers is denoted by N.

N = {1, 2, 3, 4, ...} N is also denoted by I′ or Z′

(b) Whole Numbers: Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W.

Thus W = {0, 1, 2, ...}

(c) Integers: The numbers ..... -3, -2, -1, 0, 1, 2, 3 .... are called integers and the set of integers is denoted by I or Z.

Thus I (or Z) = {......-3, -2, -1, 0, 1, 2, 3........}

(a) Set of negative integers is denoted by I^- and consists of {......., -3, -2, -1}

(b) Set of non-negative integers is denoted by W.

(c) Set of non-positive integers {......., -3, -2, -1, 0}

(d) Even integers: Integers which are divisible by 2 are called even integers. e.g. 0, ±2, ± 4, ...

(e) Odd integers: Integers which are not divisible by 2 are called odd integers. e.g. ±1, ±3, ±5, ±7

(f) Prime numbers: A natural number (except unity) is said to be a prime number if it is exactly divisible by unity and itself only. e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(g) Composite numbers: Natural numbers which are not prime (except unity) are called composite numbers.

(h) Co-prime numbers: Two natural numbers (not necessarily prime) are said to be co-prime, if their H.C.F. (Highest common factor) is one. e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (4, 9), (5, 6), (7, 8) etc. These numbers are also called as relatively prime numbers.

(i) Twin prime numbers: If the difference between two prime numbers is two, then the numbers are called twin prime numbers. e.g. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

(j) Rational numbers: All the numbers that can be represented in the form p/q, where p and q are integers and q ≠ 0, are called rational numbers and their set is denoted by Q. e.g. $\frac{1}{2}$, 2, 0, -5, $\frac{22}{7}$, 2.5, 0.3333 .... etc. Thus $Q = \left\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\right\}$. It may be noted that every integer is a rational number since it can be written as p/1. The decimal part of rational numbers is either terminating or recurring.

(k) Irrational numbers: There are real numbers which cannot be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^C or $\overline{Q}$ (i.e. complementary set of Q). The decimal part of irrational numbers is neither terminating nor recurring e.g. $\sqrt{2}$, $1 + \sqrt{3}$, π etc.

(l) Real numbers: The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^C$.

(m) Complex numbers: A number of the form a + ib is called a complex number, where $a, b \in R$ and $i = \sqrt{-1}$. A complex number is usually denoted by 'z' and a set of complex numbers is denoted by C.

MASTERJEE CONCEPTS

- Zero is neither positive nor negative but zero is non-negative and non-positive.
- '1' is neither prime nor composite
- '2' is the only even prime number
- '4' is the smallest composite number

- Two distinct prime numbers are always co-prime but the converse need not be true.
- Consecutive natural numbers are always co-prime numbers.

  $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$. And both are irrational.

Vaibhav Gupta (JEE 2009, AIR 22)

2. RATIO AND PROPORTION

2.1 Ratio

(a) If A and B are two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (this may be an integer or fraction)

(b) A ratio may be represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb}$ = ... where m, n, ... are non-zero numbers.

(c) To compare two or more ratios, reduce them to their common denominator.

2.2 Proportion

When two ratios are equal, then the four quantities composing them are said to be proportional. If $\frac{a}{b} = \frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

(a) 'a' and 'd' are known as extremes and 'b' and 'c' are known as means.

(b) An important property of proportion; product of extremes = product of means.

(c) If a: b = c : d, then b : a = d : c (invertendo)

(d) If a: b = c : d, then a : c = b : d (alternando)

(e) If a: b = c : d, then $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo)

(f) If a: b = c : d, then $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo)

(g) If a: b = c : d, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo)

3. DEFINITION OF INDICES

If 'a' is any non-zero real or imaginary number and 'm' is a positive integer, then $a^m = a.a.a......a$ (m times). Here 'a' is called the base and m is the index, power, or exponent.

Law of indices:

(a) $a^0 = 1$, $(a \neq 0)$

(b) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(c) $a^{m+n} = a^m.a^n$, where m and n are real numbers

(d) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are real numbers

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

4. SOME IMPORTANT IDENTITIES

(a) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(b) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(c) $a^2 - b^2 = (a + b)(a - b)$

(d) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(e) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(f) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(g) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(h) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

(i) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(j) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(k) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(l) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

5. SURDS

Any root of an arithmetical number which cannot be completely found is called surd. E.g. $\sqrt[3]{2}$, $\sqrt[4]{5}$, $\sqrt[3]{7}$ etc. are all surds.

(a) **Pure Surd:** A surd which consists of purely an irrational number expressed as $\sqrt{n}$ where $a \neq x^n$ ($x \in I$) is called a pure surd. e.g. $\sqrt[3]{7}$, $\sqrt[5]{5}$ etc.

(b) **Mixed surd:** A pure surd when multiplied with a rational number becomes a mixed surd. e.g. $2\sqrt[3]{3}$, $4\sqrt[5]{5}$, $2\sqrt{3}$ etc.

A mixed surd can be written as a pure surd. e.g. $2 \times \sqrt[3]{3} = \sqrt[3]{3 \times 8} = \sqrt[3]{24}$, $2\sqrt{5} = \sqrt{20}$

(c) **Order of Surd:** The order of a surd is indicated by the number denoting the roots i.e. $\sqrt[4]{2}$, $\sqrt[3]{5}$, $\sqrt[6]{7}$ are surds of the 4th, 3rd and 6th order respectively.

(d) **Simple Surd:** Surds consisting of one term only are called simple surds. E.g. $\sqrt[5]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{a^2bc}$ etc. are simple surds or Monomial surds.

(e) **Compound Surd:** An expression consisting of two or more simple surds connected by (+) or (-) sign is called a compound surd. E.g. $5\sqrt{2} + 4\sqrt{3}$, $\sqrt{3} + \sqrt{2}$, $\sqrt{3} - \sqrt{5}$.

6 LOGARITHM

6.1 Introduction

It is very lengthy and time consuming to find the value of $5\sqrt[2]{0.0000165}$, $\sqrt{\dfrac{(45.5)^2}{(3.2)^2(6.5)^2}}$ or finding number of digits in 3^{12}, 2^8. John Napier (1550-1617 AD) invented logarithm (in 1614 AD) to solve such problems. The word "Logarithm" was formed by two Greek words, 'logos' which means 'ratio', and 'arithmos' meaning 'number'. Henry Briggs (1556-1630 AD) introduced common logarithm. He published logarithm in 1624 AD.

In its simplest form, a logarithm answers the question, "How many of one number do we multiply to get another number?"

Illustration 1: How many 2s do we multiply to get 8? **(JEE MAIN)**

Sol: 2 × 2 × 2 = 8, So we needed to multiply 3 of the 2s to get 8. So the logarithm of 8 to the base 2, written as $\log_2(8)$ is 3.

6.1.1 How to Write it

We would write "the number of 2s you need to multiply to get 8 is 3" as

$$\underbrace{2\times2\times2}_{3}=8 \;\leftrightarrow\; \underset{\substack{\uparrow\\ \text{Base}}}{\log_2}(8) = 3$$

So these two things are the same.

The number we are multiplying is called the "base", so we would say "The logarithm of 8 with the base 2 is 3".

Or "log base 2 of 8 is 3" or "the base-2 log of 8 is 3"

6.1.2 Exponents

Exponents and Logarithms are related, let's find out how...

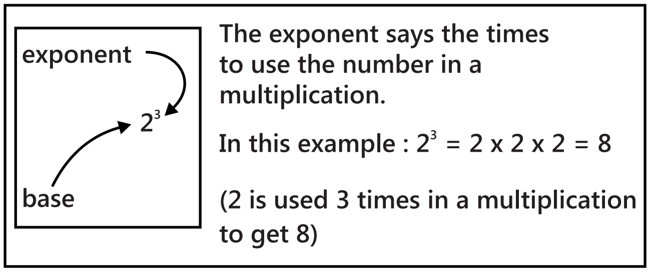


Figure 1.2

So a logarithm answers a question like this: $2^? = 8$

In this way

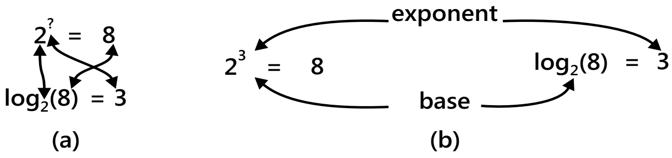


Figure 1.3

So the logarithm answer the question: The general case is

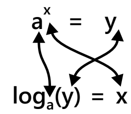


Figure 1.4

6.2 Definition of Logrithm

If $a^x = N$, then x is called the logarithm of N to the base a. It is also designated as $\log_a N$.

So $\log_a N = x$; $a^x = N$, $a > 0$, $a \neq 1$ and $N > 0$

Note:

(a) The logarithm of a number is unique i.e. No number can have two different log to a given base.

(b) From the definition of the logarithm of the number to a given base 'a'. $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ and $N > 0$ is known as the fundamental logarithmic identity.

(c) The base of log can be any positive number other than 1, but basically two bases are mostly used. They are 10 and e (= 2.718 approx.)

Logarithm of a number to the base 10 are named as common logarithm, whereas the logarithm of numbers to the base e are called as Natural or Napierian logarithm.

6.2.1 Common Logarithm: Base 10

Many a times, the logarithm is written without a base, like this, log (100)

This usually means that the base is really 10.

It is called "common logarithm". Engineers love to use it.

6.2.2 Natural Logarithms: Base "e"

Another base that is often used is e (Euler's Number) which is approximately 2.71828.

This is called a "natural logarithm". Mathematicians use this one quite often.

But There is Some Confusion ....!

Mathematicians use "log" (instead of "ln") to mean the natural logarithm. This can lead to confusion:

Example	Engineer Thinks	Mathematician Thinks	
log(50)	$\log_{10}(50)$	$\log_e(50)$	Confusion
ln (50)	$\log_e(50)$	$\log_e(50)$	No confusion
$\log_{10}$ (50)	$\log_{10}$ (50	$\log_{10}$ (50)	No confusion

So make sure that when you read "log" that you know what base they mean.

Note: Since NCERT assumed log x to be $\log_e x$, for JEE Main and Advanced this convention is to be used.

6.3 Properties of Logarithm

Let M and N be arbitrary positive numbers such that $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and x, y are real numbers. Then,

(a) $\log_a (M \times N) = \log_a M + \log_a N$ (Product rule)

Proof: Let $\log_a M = x$ and $\log_a N = y$

Then from the basic definition of logarithm, $M = a^x$ and $N = a^y$

$\Rightarrow (M \times N) = a^{x+y} \Rightarrow \log_a (M \times N) = x + y$

$\Rightarrow \log_a(M \times N) = \log_a M + \log_a N$

(b) $\log_a (M/N) = \log_a M - \log_a N$ (Division rule)

Proof: Let $\log_a M = x$ and $\log_a N = y$

$M = a^x$ and $N = a^y$

$\Rightarrow M/N = a^{x-y} \Rightarrow \log_a (M/N) = x - y = \log_a M - \log_a N$

(c) $\log_a M^x = x \log_a.M$ (Power rule)

Proof: Let $\log_a M^x = y$ ... (i)

$\Rightarrow M^x = a^y \quad \Rightarrow (M^x)^{1/x} = (a^y)^{1/x}$

$\Rightarrow M = a^{y/x} \quad \Rightarrow \log_a M = y/x$

$\Rightarrow x \log_a M = y$ ... (ii)

From (i) and (ii), we can say that $\log_a M^x = x \log_a M$

(d) $\log_{a^x} M = \frac{1}{x} \log_a M \ (x \neq 0)$ (Power rule for base)

Proof: Let $\log_{a^x} M = y$ ... (i)

$\Rightarrow M = a^{xy} \Rightarrow M^{1/x} = a^y \quad \Rightarrow \log_a M^{1/x} = \log_a a^y$

$\frac{1}{x} \log_a M = y$ ... (ii)

Using (i) and (ii) $\quad \log_{a^x} M = \frac{1}{x} \log_a M$

(e) $\log_b a = \frac{\log_c a}{\log_c b}$ $(c > 0, c \neq 1) = \frac{\log a}{\log b}$ (Base changing theorem)

Proof: Let $\log_c a = x$ and $\log_c b = y$

$\Rightarrow a = c^x$ and $b = c^y \Rightarrow a^{1/x} = c$ and $b^{1/y} = c \Rightarrow a^{1/x} = b^{1/y} \Rightarrow (a^{1/x})^x = (b^{1/y})^x \Rightarrow a = b^{x/y}$

$$\Rightarrow \log_b a = \frac{x}{y} = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b}$$

This is the most important property of logarithms and applies to most of the problems. Here, the base can be taken as any positive real number except unity.

E.g. $\log_3 5 = \frac{\log_2 5}{\log_2 3} = \frac{\log_{10} 5}{\log_{10} 3} = \frac{\log_{1/2} 5}{\log_{1/2} 3}$

Note: $\log_3 \pi$ and $\log_\pi 3$ are reciprocals of each other.

The following properties can be deduced using base changing theorem.

(a) $\log_b a = \frac{1}{\log_a b}$; $\quad$ Proof: $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$

(b) $\log_b a \,.\, \log_c b \,.\, \log_d c = \log_d a$ $\quad$ Proof: $\log_b a \,.\, \log_c b \,.\, \log_d c = \frac{\log a}{\log b} \cdot \frac{\log b}{\log c} \cdot \frac{\log c}{\log d} = \frac{\log a}{\log d} = \log_d a$

(c) $a^{\log_e c} = c^{\log_e a}$ $\quad$ Proof: $a^{\log_e c} = (a^{\log_e c})^{\log_e a} = c^{\log_e a}$ $\quad (\because \ a^{\log_e N} = N)$

(i) $(\log_b a \,.\, \log_a b = 1 \quad \Rightarrow \log_b a = \frac{1}{\log_a b}$ $\quad$ (ii) $e^{x \ln a} = a^x$

Illustration 2: What is logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ **(JEE MAIN)**

Sol: Here we can write $32\sqrt[5]{4}$ as $2^5\,4^{1/5} = (2)^{27/5}$ and $2\sqrt{2}$ as $2^{\frac{3}{2}}$ and then by using the formulae $\log_a M^x = x\log_a .M$ and $\log_{a^x} M = \frac{1}{x}\log_a M$ we can solve it.

$$\log_{2\sqrt{2}} 32\sqrt[5]{4} = \log_{\left(2^{3/2}\right)}\left(2^5\,4^{1/5}\right) = \log_{\left(2^{3/2}\right)}(2)^{27/5} = \frac{2}{3}\frac{27}{5}\log_2 2 = \frac{18}{5} = 3.6$$

Illustration 3: Prove that, $\log_{4/3}(1.\overline{3}) = 1$ **(JEE MAIN)**

Sol: By solving we get $1.\overline{3} = \frac{4}{3}$, and use the formula $\log_a a = 1$.

$\log_{4/3} 1.\overline{3} = 1$

Let $x = 1.333\ ...$ ... (i)

$10x = 13.3333\ ...$ ... (ii)

From Equation (i) and (ii), we get

So $9x = 12 \Rightarrow x = 12/9,\ x = 4/3;$

Now $\log_{4/3} 1/\overline{3} = \log_{4/3}(4/3) = 1$

Illustration 4: If $N = n!\ (n \in N,\ n \ge 2)$ then $\lim\limits_{N\to\infty} [(\log_2 N)^{-1} + (\log_3 N)^{-1} + \ldots + (\log_n N)^{-1}]$ is **(JEE ADVANCED)**

Sol: Here by using $\log_a b = \frac{1}{\log_b a}$ we can write given expansion as $\log_N 2 + \log_N 3 + \ldots\ldots + \log_N n$ and then by using $\log_a(M.N) = \log_a M + \log_a N$ and $N = n!$ we can solve this.

$(\log_2 N)^{-1} + (\log_3 N)^{-1} + \ldots\ldots + (\log_n N)^{-1} = \log_N 2 + \log_N 3 + \ldots\ldots + \log_N n = \log_n(2.3\ \ldots.\ N) = \log_N N = 1.$

Illustration 5: If $\log x^2 - \log 2x = 3\log 3 - \log 6$ then x equals **(JEE ADVANCED)**

Sol: By using $\log_a(M.N) = \log_a M + \log_a N$ and $\log_a M^x = x\log_a .M$ we can easily solve above problem.
Clearly $x > 0$. Then the given equation can be written as $2\log x - \log 2 - \log x = 3\log 3 - \log 2 - \log 3$

$\Rightarrow \log x = 2\log 3 \Rightarrow x = 9$

Illustration 6: Prove that, $\log_{2-\sqrt{3}}(2+\sqrt{3}) = -1$ **(JEE ADVANCED)**

Sol: By multiplying and dividing by $2+\sqrt{3}$ to $2-\sqrt{3}$ we will get $2+\sqrt{3} = \frac{1}{2-\sqrt{3}}$. Therefore by using $\log_{1/N} N = -1$ we can easily prove this.

$$\Rightarrow \log_{2-\sqrt{3}}\frac{1}{2-\sqrt{3}} \quad \Rightarrow \log_{2-\sqrt{3}}\left(2-\sqrt{3}\right)^{-1} \quad \Rightarrow -1.\log_{2-\sqrt{3}}\left(2-\sqrt{3}\right) = -1$$

Illustration 7: Prove that, $\log_5\sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}} = 1$ **(JEE ADVANCED)**

Sol: Here $\sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$ can be represented as $y = \sqrt{5y}$ where $y = \sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$. Hence, by obtaining the value of y we can prove this.

Let $y = \sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$

$y = \sqrt{5y} \quad \Rightarrow y^2 = 5y$ or $y^2 - 5y = 0$

$y(y-5) = 0 \quad \Rightarrow y = 0, y = 5$

y = 0 is not possible because log is not defined for zero.

$\therefore \quad \log_5 5 = 1$

Illustration 8: Prove that, $\log_{2.25}\left(0.\overline{4}\right) = -1$ **(JEE MAIN)**

Sol: As similar to illustration 3 we can solve it by using $\log_{1/N} N = -1$.

$x = 0.4444.....$ ... (i)

$10x = 4.4444.....$ ... (ii)

Equ (ii) – Equ (i)

So $9x = 4 \Rightarrow x = 4/9$

Also, $2.25 = \frac{225}{100} = \frac{9}{4}$; $\quad \log_{2.25}\left(0.\overline{4}\right) = \log_{\left(\frac{9}{4}\right)}\left(\frac{4}{9}\right) = -1$

Illustration 9: Find the value of $2^{\log_6 18} . 3^{\log_6 3}$ **(JEE MAIN)**

Sol: We can solve above problem by using $\log_a(M.N) = \log_a M + \log_a N$ and $a^{\log_e c} = c^{\log_e a}$ step by step.

$2^{\log_6 18}(3)^{\log_6 3} = 2^{\log_6 (6\times 3)}.3^{\log_6 3} = 2^{1+\log_6 3}.3^{\log_6 3} = 2 \cdot 2^{\log_6 3}.3^{\log_6 3} \qquad (\because a^{\log_e c} = c^{\log_e a})$

$= 2\cdot(3)^{\log_6 2}\cdot(3)^{\log_6 3} = 2(3)^{\log_6 2 + \log_6 3} = 2(3)^{\log_6 (6)} = 2\cdot(3) = 6$

Illustration 10: Find the value of, $\log_{\sec\alpha}\left(\cos^3\alpha\right)$ where $\alpha \in (0, \pi/2)$ **(JEE MAIN)**

Sol: Consider $\log_{\sec\alpha}\left(\cos^3\alpha\right) = x$. Therefore by using formula $y = \log_a x \Leftrightarrow a^y = x$ we can write $\cos^3\alpha = (\sec\alpha)^x$. Hence by solving this we will get the value of x.

Let $\log_{\sec\alpha}\cos^3\alpha = x$

$\cos^3\alpha = (\sec\alpha)^x \Rightarrow (\cos\alpha)^3 = \left(\frac{1}{\cos\alpha}\right)^x \Rightarrow (\cos\alpha)^3 = (\cos\alpha)^{-x} \Rightarrow x = -3$

Illustration 11: If $k \in N$, such that $\log_2 x + \log_4 x + \log_8 x = \log_k x$ and $\forall\, x \in R'$ **(JEE ADVANCED)**

If $k = (a)^{1/b}$ then find the value of a + b; $a \in N$, $b \in N$ and b is a prime number.

Sol: By using $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b}$ we can obtain the value of k and then by comparing it to $k = (a)^{1/b}$ we can obtain value of a + b.

Given, $\frac{\log x}{\log 2} + \frac{\log x}{2\log 2} + \frac{\log x}{3\log 2} = \frac{\log x}{\log k} \Rightarrow \frac{\log x}{\log 2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] = \frac{\log x}{\log k} \Rightarrow \frac{\log x}{\log 2}\left(\frac{11}{6}\right) = \frac{\log x}{\log k} \Rightarrow \log x\left[\frac{11}{6}\frac{1}{\log 2} - \frac{1}{\log k}\right] = 0$

Also, $\frac{11}{6}\frac{1}{\log 2} - \frac{1}{\log k} = 0 \Rightarrow \frac{11}{6} = \frac{\log 2}{\log k} \Rightarrow \frac{11}{6} = \log_k 2$

So $2 = k^{\frac{11}{6}}$; $\quad 2^{6/11} = k \Rightarrow \left(2^6\right)^{\frac{1}{11}} = k \Rightarrow (64)^{\frac{1}{11}} = k$

Comparing by $k = (a)^{1/b} \Rightarrow a = 64$ and $b = 11 \Rightarrow a + b = 64 + 11 = 75$

6.4 Logarithmic Equation

While solving logarithmic equation, we tend to simplify the equation. Solving the equation after simplification may give some roots which do not define all the terms in the initial equation. Thus, while solving an equation involving logarithmic function, we must take care of all the terms involving logarithm.

Let a = log (x) and b = log (x + 2)

In general, a + b = log (x) + log (x + 2) = log [x (x + 2)]

If we take, x = –3, a and b both are not defined, but a + b will be defined.

as a + b = log [(–3) (–3 + 2)] = log (3)

Here, the problem lies in the definition of a and b. a and b is not defined here, so addition of a and b i.e. a + b will not be defined.

Note: A similar situation might arise while solving logarithmic equations. To avoid or to reject extraneous roots we have to define the logarithm.

Illustration 12: Solve $\log_4 8 + \log_4 (x + 3) - \log_4 (x - 1) = 2$ **(JEE MAIN)**

Sol: As we know $\log_a(M.N) = \log_a M + \log_a N$, $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ and $y = \log_a x \Leftrightarrow a^y = x$. By using these formulae we can solve the problem above.

$\log_4 8 + \log_4 (x + 3) - \log_4(x - 1) = 2$

$\Rightarrow \log_4 \frac{8(x+3)}{x-1} = 2 \Rightarrow \frac{8(x+3)}{x-1} = 4^2 \Rightarrow x + 3 = 2x - 2 \Rightarrow x = 5$

Also for x = 5 all terms of the equation are defined.

Illustration 13: Solve log (–x) = 2 log (x + 1) **(JEE MAIN)**

Sol: Here it's given that log (–x) = 2 log (x + 1). Therefore by using the formula $\log_a M^x = x \log_a .M$. We can evaluate the value of x.

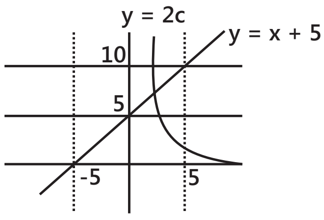
By definition, x < 0 and x + 1 > 0 ⇒ –1 < x < 0

Now $\log(-x) = 2\log(x + 1) \Rightarrow -x = (x + 1)^2 \Rightarrow x^2 + 3x + 1 = 0$

$\Rightarrow x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ (rejected). Hence, $x = \frac{-3+\sqrt{5}}{2}$ is the only solution.

Illustration 14: Find the number of solutions to the equation $\log_2 (x + 5) = 6 - x$. **(JEE MAIN)**

Sol: By using the formula $y = \log_a x \Leftrightarrow a^y = x$, we can write given the equation as $x + 5 = 2^{6-x}$. Hence, by checking the number of intersections made by the graph of $y = x + 5$ and $y = 2^{6-x}$ we will obtain the number of solutions.



Here, $x + 5 = 2^{6-x}$

Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersects only once. Hence, there is only one solution.

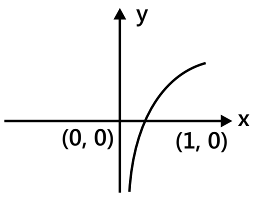
MASTERJEE CONCEPTS

Always check your answer by putting it back in the equation; sometimes answer might not be in the domain of logarithm.

Shrikant Nagori (JEE 2009, AIR 7)

6.5 Graph of Logarithmic Function

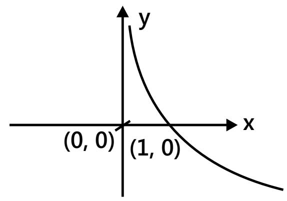
(i)



$y = \log_a x \ \forall\ a > 1$

Figure 1.6

(ii)



$y = \log_a x \ \forall\ 0 < a < 1$

Figure 1.7

If the number and the base are on the same side of unity, then the logarithm is positive, and if the number and the base are on different side of unity then the logarithm is negative.

Illustration 15: Which of the following numbers are positive/negative? **(JEE MAIN)**

(i) $\log_2 7$ (ii) $\log_{1/2} 3$ (iii) $\log_{1/3}(1/5)$ (iv) $\log_4 3$ (v) $\log_2 9$

Sol: By observing whether the Number and Base are on the same side of unity or not we can say whether the numbers are positive or negative.

(i) Let $\log_2 7 = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(ii) Let $\log_{1/2} 3 = x$ (number and base are on the same side of unity) $\Rightarrow x < 0$

(iii) Let $\log_{1/3}(1/5) = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(iv) Let $\log_4 3 = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(v) Let $(\log_2 9) = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

6.6 Characteristic and Mantissa

(a) Given a number N, Logarithm can be $\log_{10}N = \text{Integer} + \text{Fraction}$

$$\begin{array}{cc} \downarrow & \downarrow \\ \text{Characterstic} & \text{Mantissa} \end{array}$$

(b) The mantissa part of the log of a number is always kept non-negative, it ranges from [0, 1]

(c) If the characteristic of $\log_{10}N$ is C then the number of digits in N is (C + 1)

(d) If the characteristic of $\log_{10}N$ is (–C) then there exist (C – 1) number of zeros after decimal point of N.

Illustration 16: Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$. **(JEE ADVANCED)**

Sol: Simply by applying log on both sides and using various logarithm formulas we can solve the above illustration.

$$\log x = \log(0.15)^{20} = 20\log\left(\frac{15}{100}\right) = 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2] = 20[\log 3 + 1 - \log 2 - 2]\left(\because \log_{10}5 = \log_{10}\frac{10}{2}\right)$$

$$= 20[-1 + \log 3 - \log 2] = -20 \times 0.824 = -16.48 = \overline{17}.52$$

Hence, characteristic = – 17 and mantissa = 0.52

Illustration 17: Find the number of digits in the following: (i) 2^{100} (ii) 3^{10} **(JEE ADVANCED)**

Sol: By considering $x = 2^{100}$ and 3^{10} respectively and applying log on both sides we can solve the problems given above.

(i) Let, $x = 2^{100}$

$\log_{10}X = \log_{10}2^{100} = 100\log_{10}2 = 100 \times 0.3010 = 30.10$

Characteristic = 30, Mantissa = 0.10

Number of digits before decimal = C + 1 = 30 + 1 = 31

(ii) Let, $X = 3^{10}$

$\log_{10}x = 10 \log 3 = 10 \times 0.4771 = 4.771$

C = 4, M = 0.771

Number of digits before decimal = C + 1 = 4 + 1 = 5

Note: Let y = log (N) when 0 < N < 1

If N lies between 0 and 1, then the characteristic is negative

$N = 1/10$, $\log_{10}N = \log_{10}(1/10) = -1, C = -1, M = 0$

$N = 0.01$, $\log_{11}N = \log_{10}(10)^{-2} = -2$, $C = -2, M = 0$

$N = 0.001$, $\log_{10}N = \log_{10}10^{-3} = -3$, $C = -3, M = 0$

No. of zeros after decimal = |–3| –1 = 2

$N = 0.002$, $\log_{10}N = \log(2 \times 10^{-3}) = \log 2 + \log 10^{-3} = 0.03010 + (-3) = -0.3010 = -2.699$

C = – 3, M = 0.3010

Number of zeros after decimal = magnitude of the characteristic –1 = |C| –1 = |–3| –1 = 2

Illustration 18: Find the number of zeros after decimal before a significant figure in

(i) 3^{-50} (ii) 2^{-100} (iii) 7^{-100} **(JEE ADVANCED)**

Sol: Similar to the illustration above, we can solve these too.

(i) $N = 3^{-50}$

$\log_{10} N = \log_{10} 3^{-50} = -50 \log_{10}3 = -50 \times (0.4771) \Rightarrow \log_{10} N = -23.855$

Now to find the characteristic and mantissa many would say that (c = –23, m = –0.855) (which is wrong) because mantissa is always non-negative.

$\log_{10}N = -23.855 = -23 - 1 + 1 - 0.855 = -24 + 0.145$

C = –24, M = 0.145. Number of zeroes after decimal = |–24| –1 = 23 or |–24 + 1| = 23

(ii) $N = 2^{-100}$

$\log_{10} N = -100 \log 2 = -30.10 = -30 - 0.10 = -31 + 0.90$. Number of zeroes after decimal = |–31| –1 = 30 or |–31 + C| = 30

(iii) $N = 7^{-100}$

$\log_{10}N = -100 \log 7 = -100 \times 0.8451 = -84.51 = -84 - 1 + 0.49 = -85 + 0.49$

C = –85, M = 0.49. Number of zeroes after decimal = |–85| –1 = 84 or |–85 + 1| = 84

Illustration 19: Find the number of positive integers which have the characteristic 2, when base of log is 6. **(JEE ADVANCED)**

Sol: If any number x has the characteristic a, when base of log is b then $x = b^a$. By using the given condition we can solve the problem above.

$x = 6^2 = 36$; $\log_6 x = \log_6 6^2 = 2 \log_6 6 = 2$

The smallest natural number which has characteristic 3 with base 6 is 6^3

$x = 6^3 = 216$; $\log_6 x = \log_6 6^3 = 3$

Hence x = 215 will give characteristic 2.

Natural numbers ranging from 36 to 215 will give characteristic 2, when taken log with base 6.

Number of positive integers = 215 – 35 = 180

6.7 Algebraic Inequalities

(a) If $a < b$ and $b < c \Rightarrow a < c$

(b) If $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc$, if b and d are of same sign. $\Rightarrow ad > bc$ if b and d are of opposite sign.

(c) If $a > b$ then, $a\lambda > b\lambda$ if $\lambda > 0$; $a\lambda < b\lambda$ if $\lambda < 0$

6.8 Logarithmic Inequalities

If the base is less than one, then the inequality will change. If base is greater than one, then inequality will remain the same.

$$\left.\begin{aligned}\log_a x < \alpha &\Rightarrow 0 < x < a^{\alpha}\\ \log_a x < \log_a y &\Rightarrow 0 < x < y\end{aligned}\right\}\text{ if } a > 1$$

$$\left.\begin{aligned}\log_a x < \alpha &\Rightarrow x > a^{\alpha}\\ \log_a x < \log_a y &\Rightarrow x > y > 0\end{aligned}\right\}\text{ if } 0 < a < 1$$

Illustration 20: Solve $(1/2)^{x^2-2x} < 1/4$ **(JEE MAIN)**

Sol: Here we can write the given equation as $(1/2)^{x^2-2x} < (1/2)^2$ and then by comparing powers on both side we can solve this.

We have $(1/2)^{x^2-2x} < (1/2)^2$. It means $x^2 - 2x > 2$

$\Rightarrow \left(x-\left(1+\sqrt{3}\right)\right)\left(x-\left(1-\sqrt{3}\right)\right) > 0 \Rightarrow x > 1+\sqrt{3}$ or $x < 1-\sqrt{3} \Rightarrow x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

Illustration 21: Solve $\frac{1+5^x}{7^{-x}+97} \geq 0$. **(JEE MAIN)**

Sol: Simply by multiplying $\left(7^{-x}-7^2\right)$ on both sides and solving we will get the result.

$g(x) = \frac{1-5^x}{7^{-x}-7} \leq 0$. Now $\left(1-5^x\right)\left(7^{-x}-7\right) \leq 0$; $5^x - 1 = 0 \Rightarrow x = 0$; $7^{-x} - 7 = 0 \Rightarrow x = -1$

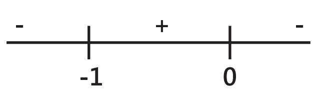


Figure 1.8

g(x) behavior on the number line. Hence, from above, $x \in (-\infty, -1) \cup [0, \infty)$

6.9 Modulus Function

Definition: Modulus of a number. The modulus of a number is denoted by |a|

$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$ Also, $\sqrt{a^2} = |a|$; Eg: $y = |x|$

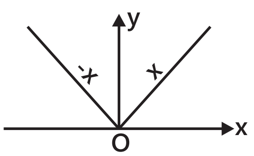


Figure 1.9

Basic properties of modulus

(A) $|ab| = |a|\,|b|$

(B) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $b \neq 0$

(C) $|a + b| \leq |a| + |b|$

(D) $|a - b| \geq |a| - |b|$ equality holds if $ab \geq 0$

Using triangle inequality

If $a > 0$

(i) $|x| = a \Rightarrow x = \pm a$

(ii) $|x| = -a \Rightarrow$ No solution

(iii) $|x| > a \Rightarrow x < -a$ or $x > a$

(iv) $|x| < a \Rightarrow -a < x < a$

(v) $|x| > -a \Rightarrow x \in R$

(vi) $|x| < -a \Rightarrow$ No solution

(vii) $a < |x| < b \Rightarrow x \in (-b, -a) \cup (a, b)$ where $a, b \in R^+$

Illustration 22: Solve for x, $|x - 2| = 3$ **(JEE MAIN)**

Sol: The above illustration can be solved by taking two cases; the first one is by taking $x - 2$ as greater than 0 and second one is by taking x - 2.

Case-I: When $x - 2 \geq 0 \Rightarrow x \geq 2$ ... (i)

Since $x - 2$ is non negative, the modulus can simply be removed. $x - 2 = 3$; $x = 5$

We had taken $x \geq 2$ and we got $x = 5$ hence this result satisfy the initial condition $\Rightarrow x = 5$

Case-II: When $x - 2 < 0 \Rightarrow x < 2$; Since $x - 2$ is negative, the modulus will open with a –ve sign.

$-(x - 2) = 3; -x + 2 = 3 \Rightarrow x = -1$ Since $x < 2$ Hence $x = -1, 5$

Illustration 23: Solve for x, $|x + 3| + |x - 2| = 11$ **(JEE ADVANCED)**

Sol: As $x + 3 = 0 \Rightarrow x = -3$ and $x - 2 = 0 \Rightarrow x = 2$. Therefore we can solve it by using the modulus inequality.

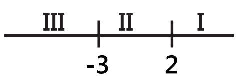


Figure 1.10

Case-I: For $x \geq 2$, $x + 3 > 0$, $x - 2 > 0$; $x + 3 + x - 2 = 11 \Rightarrow 2x = 10 \Rightarrow x = 5$

Case-II: For $-3 \leq x < 2$, $x + 3 \geq 0$, $x - 2 < 0$; $|x + 3| + |x - 2| = 11$

$\Rightarrow x + 3 - x + 2 = 11 \Rightarrow 5 = 11$ is impossible $\Rightarrow$ Hence, No value of x

Case-III: For $x < -3$

$x + 3 < 0, x - 2 < 0$; $|x + 3| + |x - 2| = 11 \Rightarrow -(x + 3) - (x - 2) = 11$

$\Rightarrow -x - 3 - x + 2 = 11 \Rightarrow -2x = 12 \Rightarrow x = -6$, since $x < -3$

Hence, to satisfy the initial condition, combining all we get $x = -6, 5$

Illustration 24: Solve for x, $x|x| = 4$ **(JEE MAIN)**

Sol: Here we can solve this problem by using two case, first one for $x > 0$ and the other one is for $x < 0$.

Case-I: For $x > 0$; $x.x = 4$

$x^2 = 4 \Rightarrow x = \pm 2$ but $x > 0$, hence $x = 2$ (–2 rejected)

Case-II: For $x < 0$; $x(-x) = 4$

$x^2 = -4$ no solution ; Hence, the only solution is $x = 2$

Illustration 25: Solve for x, $|x - 3| + 2|x + 1| = 4$ **(JEE ADVANCED)**

Sol: As $x - 3 = 0 \Rightarrow x = 3$; and $x + 1 = 0 \Rightarrow x = -1$.

Therefore by applying the cases $X \ge 3$, $-1 \le x < 3$ and $x < -1$ we can solve this.

Mark the points on number line

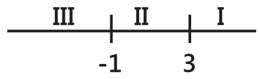


Figure 1.11

Case-I: For $x \ge 3$

$(x - 3)$ is non-negative $(x + 1)$ is also positive

$\Rightarrow (x - 3) + 2(x + 1) = 4 \Rightarrow 3x = 5 \Rightarrow x = 5/3 \Rightarrow x = 5/3$ is discarded, since x should be > 3

Case-II: For $-1 \le x < 3$; $x - 3$ is –ve, $x + 1$ is positive

$\Rightarrow -(x - 3) + 2(x + 1) = 4 \Rightarrow -x + 3 + 2x + 2 = 4 \Rightarrow x = -1$ satisfies the initial condition

Case-III: $x < -1$

$\Rightarrow -(x - 3) - 2(x + 1) = 4$; $-3x = 3 \Rightarrow x = -1$ $\Rightarrow$ Does not satisfy $x < 1$ Hence, solution is $x = -1$ from case-II.

6.10 Exponential and Logarithm Series

6.10.1 The Number 'e'

The sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ .... $+ \infty$ is denoted by the number e

i.e. $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(i) The number e lies between 2 and 3. Approximate value of e = 2.718281828.

(ii) e is an irrational number.

6.10.2 Some Standard Deduction from Exponential Series

(i) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......\frac{x^n}{n!}+.......\infty$

(ii) $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+.......\frac{(-1)^n}{n!}x^n+.......\infty$ (Replace x by –x)

(iii) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.......\infty$ (Substituting x = 1 in (i))

(iv) $e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+.......\infty$ (Substituting x = – 1 in (i))

(v) $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+.......\infty$

(vi) $\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+.......\infty$

(vii) $a^x = 1+x(\ln a)+\frac{x^2}{2!}(\ln a)^2+\frac{x^3}{3!}(\ln a)^3+$ ....; $(a > 0)$, where $\ln a = \log_e(a)$

6.10.3 Logarithmic Series

If $-1 < x \le 1$

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

(ii) $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

(iii) $\ln(x + 1) - \ln(1 - x) = \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$

(iv) $\ln(1 + x) + \ln(1 - x) = \ln(1 - x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...\right)$

6.11 Antilogarithm

The positive number n is called the antilogarithm of a number m if m = log n. If n is the antilogarithm of m, we write n = antilog m. For example

(i) log (100) = 2 $\Rightarrow$ antilog 2 = 100

(ii) log (431.5) = 2.6350 $\Rightarrow$ antilog (2.6350) = 431.5

(iii) log (0.1257) = $\bar{1}.0993$ $\Rightarrow$ antilog ($\bar{1}.0993$) = 0.1257

6.12 To find the Antilog of a Number

Step I: Determine whether the decimal part of the given number is positive or negative. If it is negative make it positive by adding 1 to the decimal part and by subtracting 1 from the integral part. For, example, in – 2.5983

$-2.5983 = -2 - 0.5983 = -2 - 1 + 1 - 0.5983 = -3 + 0.4017 = \bar{3}.4017$

Step II: In the antilogarithm, look into the row containing the first two digits in the decimal part of the given number.

Step III: In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

Step IV: In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number obtained in step III.

Step V: Obtain the integral part (characteristic) of the given number.

If the characteristic is positive and is equal to n, then insert decimal point after (n + 1) digits in the number obtained in step IV.

Illustration 26: Find the antilogarithm of each of the following: **(JEE MAIN)**

(i) 2.7523 (ii) 0.7523 (iii) $\bar{2}.7523$ (iv) $\bar{3}.7523$

Sol: By using log table and following the above mentioned steps we can find the algorithms of above values.

(i) The mantissa of 2.7523 is positive and is equal to 0.7523.

Now, look into the row starting 0.75. In this row, look at the number in the column headed by 2. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 3. The number there is 4. Add this number to 5649 to get 5653. The characteristic is 2. So, the decimal point is put after 3 digits to get 565.3

(ii) Proceeding as above, we have antilog (0.7523) = 5.653.

(iii) In this case, the characteristic is $\bar{2}$, i.e., – 2. So, we write one zero on the digit side of the decimal point. Hence, antilog ($\bar{2}$.7523) = 0.05653

(iv) Proceeding as above, antilog ($\bar{3}$.7523) = 0.005653

PROBLEM-SOLVING TACTICS

(a) The main thing to remember about surds and working them out is that it is about manipulation. Changing and manipulating the equation so that you get the desired result. Rationalizing the denominator is all about manipulating the algebra expression.

(b) Strategy for Solving Equations containing Logarithmic and Non-Logarithmic Expressions:

- **(i)** Collect all logarithmic expressions on one side of the equation and all constants on the other side.
- **(ii)** Use the Rules of Logarithms to rewrite the logarithmic expressions as the logarithm of a single quantity with coefficient of 1.
- **(iii)** Rewrite the logarithmic equation as an equivalent exponential equation.
- **(iv)** Solve for the variable.
- **(v)** Check each solution in the original equation, rejecting apparent solutions that produce any logarithm of a negative number or the logarithm of 0. Usually, a visual check suffices!

Note: The logarithm of 0 is undefined

(c) Logarithmic series

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$
(ii) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$
(iii) $\ln(x+1) - \ln(1-x) = \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$
(iv) $\ln(1+x) + \ln(1-x) = \ln(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...\right)$

FORMULAE SHEET

(a) Laws of indices

(i) $a^0 = 1$, $(a \neq 0)$	**(ii)** $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$
(iii) $a^{m+n} = a^m.a^n$, where m and n are real numbers	**(iv)** $a^{m-n} = \frac{a^m}{a^n}$

(v) $\left(a^m\right)^n = a^{mn}$	**(vi)** $a^{p/q} = \sqrt[q]{a^p}$
(vii) $(ab)^n = a^n b^n$	**(viii)** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(b) Some Important Identities

(i) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
(ii) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
(iii) $a^2 - b^2 = (a + b)(a - b)$
(iv) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
(v) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
(vi) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$
(vii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$
(viii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$
(ix) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$
(x) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
(xi) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$
(xii) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

(c) Laws of Surds

(i) $\sqrt[n]{a} = a^{\frac{1}{n}}$	**(ii)** $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$
(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	**(iv)** $\left(\sqrt[n]{a}\right)^n = a$
(v) $\left(\sqrt[m]{\sqrt[n]{a}}\right) = \sqrt[mn]{a}$	**(vi)** $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

(d) Logarithm formulas

(i) $\log_a(M.N) = \log_a M + \log_a N$	**(ii)** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
(iii) $y = \log_a x \Leftrightarrow a^y = x(a, x > 0, a \neq 1)$	**(iv)** $\log_a M^x = x \log_a.M$