9. FLUID MECHANICS

1. INTRODUCTION

Fluid is a collective term for liquid and gas. A fluid cannot sustain shear stress when at rest. We will study the dynamics of non-viscous, incompressible fluid. We will be learning about pressure variation, Archemides principle, equation of contինity, Bernoulli's Theorem and its applications and surface tension, Stoke's Law and Terminal velocity of a spherical body.

2. DEFINITION OF A FLUID

A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be.

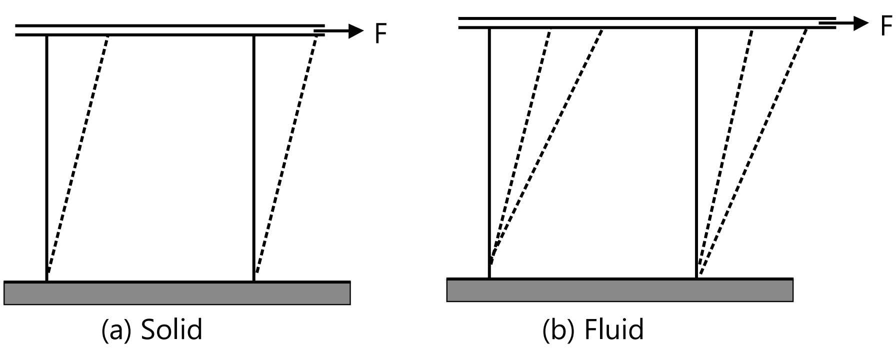

Figure 9.1: Behavior of a solid and a fluid, under the action of a constant shear force.

3. FLUID STATICS

It refers to the state when there is no relative velocity between fluid elements. In this section we will learn some of the properties of fluid statics.

3.1 Density

The density ρ of a substance is defined as the mass per unit volume of a sample of the substance. If a small mass element Δm occupies a volume ΔV, the density is given by $\rho = \frac{\Delta m}{\Delta V}$

In general, the density of an object depends on position, so that $\rho = f(x, y, z)$

If the object is homogeneous, its physical parameters do not change with position throughout its volume. Thus for a homogeneous object of mass M and volume V, the density is defined as $\rho = \frac{M}{V}$

Thus SI units of density are kg m^{-3}.

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Note: As pressure is increased, volume decreases and hence density will increase.

As the temperature of a liquid is increased, mass remains the same while the volume is increased.

Vaibhav Krishnan (JEE 2009, AIR 22)

3.2 Specific Gravity

The specific gravity of a substance is the ratio of its density to that of water at 4°C, which is 1000 kg/m^3. Specific gravity is a dimensionless quantity numerically equal to the density quoted in g/cm^3. For example, the specific gravity of mercury is 13.6, and the specific gravity of water at 100ºC is 0.999. $RD = \frac{\text{Density of substance}}{\text{Density of water at } 4°C}$

Illustration 1: Find the density and specific gravity of gasoline if 51 g occupies 75 cm^3? **(JEE MAIN)**

Sol: Density is mass per unit volume, and specific gravity is the ratio of density of substance and density of water.

$$\text{Density} = \frac{\text{mass}}{\text{volume}} = \frac{0.051\text{kg}}{75\times10^{-6}\text{m}^3} = 680 \text{ kg/m}^3$$

$$\text{Sp. gr} = \frac{\text{density of gasoline}}{\text{density of water}} = \frac{680\text{kg}/\text{m}^3}{1000\text{kg}/\text{m}^3} = 0.68 \text{ or Sp. gravity} = \frac{\text{mass of 75 cm}^3\text{gasoline}}{\text{mass of 75 cm}^3\text{water}}$$

$$= \frac{51\text{g}}{75\text{g}} = 0.68$$

Illustration 2: The mass of a liter of milk is 1.032 kg. The butterfat that it contains has a density of 865 kg/m^3 when pure, and it constitutes 4 percent of the milk by volume. What is the density of the fat-free skimmed milk? **(JEE MAIN)**

Sol: Find the mass of butterfat present in the milk. Subtract this from total mass to get mass of fat-free milk. The density of fat-free milk is equal to its mass divided by its volume.

Volume of fat in 1000 cm^3 of milk = 4% × 1000 cm^3 = 40 cm^3

Mass of 40 cm^3 fat = $V\rho$ = $(40 \times 10^{-6}$ $m^3)(865$ kg/$m^3)$ = 0.0346 kg

$$\text{Density of skimmed milk} = \frac{\text{mass}}{\text{volume}} = \frac{(1.032-0.0346)\text{kg}}{(1000-40)\times10^{-6}\text{m}^3}$$

3.3 Pressure

The pressure exerted by a fluid is defined as the force per unit area at a point within the fluid. Consider an element of area ΔA as shown in the figure and an external force ΔF is acting normal to the surface. The average pressure in the fluid at the position of the element is given by $P_{av} = \frac{\Delta F}{\Delta A}$ [A normal force ΔF acts on a small cylindrical element of cross-section area ΔA.]

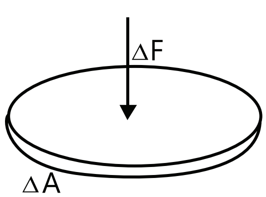


Figure 9.2

As $\Delta A \to 0$, the element reduces to a point, and thus, pressure at a point is defined as

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

When the force is constant over the surface A, the above equation reduces to $p = \frac{F}{A}$

The SI unit of pressure is Nm^{-2} and is also called Pascal (Pa). The other common pressure units are atmosphere and bar.

1 atm = 1.01325×10^5 Pa; 1 bar = 1.00000×10^5 Pa; 1 atm = 1.01325 bar

3.3.1 Pressure Is Isotropic

Imagine a static fluid and consider a small cubic element of the fluid deep within the fluid as shown in the figure. Since this fluid element is in equilibrium therefore, forces acting on each lateral face of this element must also be equal in magnitude. Because the areas of each face are equal, therefore, the pressure on each face is equal in magnitude. Therefore the pressure on each of the lateral faces must also be the same. In the limit as the cube element to a point, the forces on top and bottom surfaces also become equal. Thus, the pressure exerted by a fluid at a point is the same in all directions – pressure is isotropic.

Note: Since the fluid cannot support a shear stress, the force exerted by a fluid pressure must also be perpendicular to the surface of the container that holds it.

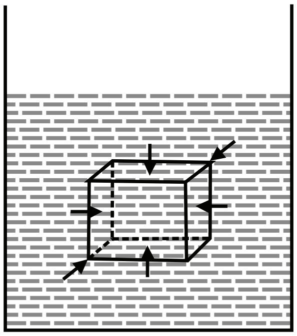


Figure 9.3: A small cubical element is in equilibrium inside a fluid

3.3.2 Atmospheric Pressure (P_0)

It is pressure of the earth's atmosphere. This changes with weather and elevation. Normal atmospheric pressure at sea level (an average value) is 1.013×10^5 Pa. Thus,

1 atm = 1.013×10^5 Pa=1.013 Bar

3.3.3 Absolute Pressure and Gauge Pressure

The excess pressure above atmospheric pressure is usually called gauge pressure and the total pressure is called absolute pressure. Thus, Gauge pressure = absolute pressure – atmospheric pressure. Aboslute pressure is always greater than or equal to zero. While gauge pressure can be negative also.

Illustration 3: Atmospheric pressure is about 1.01×10^5 Pa. How large a force does the atmosphere exert on a 2 cm^2 area on the top of your head? **(JEE MAIN)**

Sol: Force = Pressure × Area

Because p = F/A, where F is perpendicular to A, we have F = pA. Assuming that 2 cm^2 of your head is flat (nearly correct) and that the force due to the atmosphere is perpendicular to the surface (as it is), we have $F = pA = (1.01 \times 10^5\ N/m^2)(2 \times 10^{-4}\ m^2) \approx 20N$

3.3.4 Variation of Pressure with Depth

Weight of a fluid element of mass Δm, $\Delta W = (\Delta m)g$. The force acting on the lower face of the element is pA and that on the upper face is $(p + \Delta p)A$. The figure (b) shows the free body diagram of the element. Applying the condition of equilibrium, we get, $pA - (p + \Delta p)A - (\Delta m)g = 0$

if ρ is the density of the fluid at the position of the element, then $\Delta m = \rho A(\Delta y)$

and pA – (p + Δp) A – ρgA(Δy) = 0

or $\frac{\Delta p}{\Delta y} = -\rho g$

In the limit Δy approaches to zero, $\frac{\Delta p}{\Delta y}$ becomes $\frac{dp}{dy} = -\rho g$. The above equation indicates that the slope of p versus y is negative. That is, the pressure p decreases with height y from the bottom of the fluid. In other words, the pressure p increases with depth h, i.e., $\frac{dp}{dh} = \rho g$

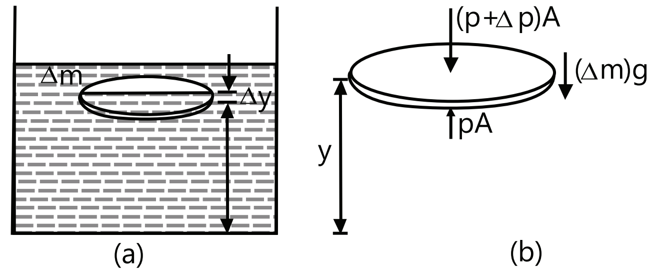


Figure 9.4

3.4 The Incompressible Fluid Model

For an incompressible fluid, the density ρ of the fluid remains constant throughout its volume. It is a good assumption for liquids. To find pressure at the point A in a fluid column as shown in the figure, is obtained by integrating the following equation:

$$dp = \rho g dh \text{ or } \int_{p_0}^{p} dp = \rho g \int_{0}^{h} dh \text{ or } p - p_0 = \rho g h \text{ or } p = p_0 + \rho g h \qquad \text{...(xvi)}$$

where ρ is the density of the fluid, and p_0 is the atmospheric pressure at the free surface of the liquid.

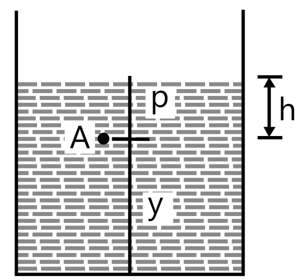


Figure 9.5: A point A is located in a fluid at a height from the bottom and at a deth h from the free surface

Note: Further, the pressure is the same at any two points at the same level in the fluid. The shape of the container does not matter.

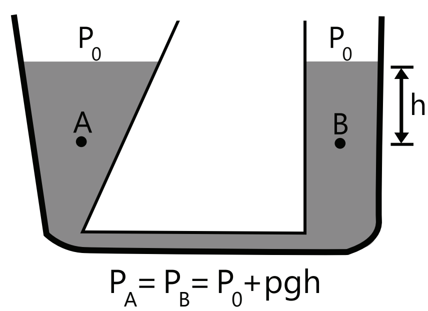


Illustration 4: Find the absolute pressure and gauge pressure at point A, B and C as shown in the Fig. 9.6 (1 atm = 10^5 Pa) **(JEE MAIN)**

Sol: Gauge Pressure = ρgh, Absolute Pressure is sum of gauge pressure and atmospheric pressure.

$P_{atm} = 10^5$ Pa.

Absolute Pressure **A** -> $P_A + P_{atm} = r_1 g h_A$ = (800)(10)1 = 8 kPa

$p'_A = p_A + p_{atm} = 108$ kPa

Gauge Pressure = 8 kPa.

B -> $p_B = \rho_1 g(2) + \rho_2 g(1.5)$

$p'_B = p_B + p_{atm}$ = 131 kPa = (800)(10)(2) + (10^3)(10)(1.5) = 131 kPa

Gauge Pressure = 31 kPa.

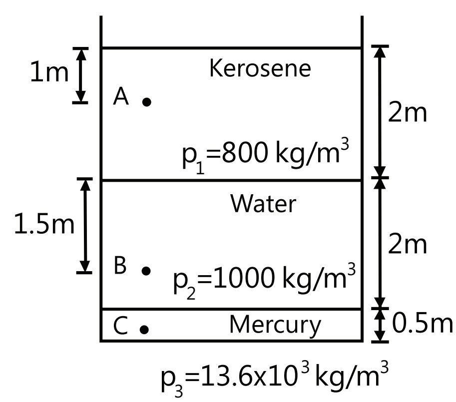


Figure 9.6

C-> $p_c = p_1 g(2) + \rho_2 g(2) + \rho_3 g(0.5)$

$\rho'_C = p_C + p_{atm} = 204\,\text{kPa}$

= (800)(10)(2) + $(10)^3$(10)(2) + 1(13.6 × 10^3)(10)(0.5) = 204 kPa

Gauge Pressure = 104 kPa.

Illustration 5: A glass full of water of a height of 10 cm has a bottom of area 10 cm^2, top of area 30 cm^2 and volume 1 litre. **(JEE ADVANCED)**

(a) Find the force exerted by the water on the bottom.

(b) Find the resultant force exerted by the side of the glass on the water.

(c) If the glass is covered by a jar and the air inside the jar is completely pumped out, what will be the answer to parts (a) and (b).

(d) If a glass of different shape is used provided the height, the bottom area and the volume are unchanged, will the answers to parts (a) and (b) change.

Take g = 10m/s^2, density of water = 10^3 kg/m^3 and atmospheric pressure = 1.01 × 10^5 N/m^2.

Sol: Pressure at the bottom depends on the height of water in the container. Force = Pressure × Area. The force on water surface due to atmospheric pressure plus the weight of water are balanced by the force on water by the container bottom and its walls.

(a) Force exerted by the water on the bottom $F_1 = (P_0 + \rho g h)A_1$ ... (i)

Here, P_0 = atmospheric pressure = 1.01 × 10^5 N/m^2; ρ = density of water = 10^3 kg/m^3

g = 10 m/s^2, h = 10 cm = 0.1 m and A_1 = area of base 10 cm^2 = 10^{-3} m^2 . Substituting in Eq. (i), we get F_1= (1.01 × 10^5 + 10^3 × 10 × 0.1) ×10^{-3} or F_1 = 102 N (downwards)

(b) Force exerted by atmosphere on water $F_2 = (P_0)A_2$

Here, A_2 = area of top = 30 cm^2 = 3 × 10^{-3} m^2 ; F_2 = (1.01 × 10^5)(3 × 10^{-3}) = 303 N (downwards)

Force exerted by bottom on the water $F_3 = -F_1$ or F_3= 102 N (upwards)

Weight of water W = (volume)(density)(g) = (10^{-3})(10^3)(10) = 10 N (downwards)

Let F be the force exerted by side walls on the water (upwards). Then, from equilibrium of water

Net upward force = net downward force or $F + F_3 = F_2 + W$

$F - F_2 + W - F_3$ = 303 + 10 – 102 or F = 211 N (upwards)

(c) If the air inside of the Jar is completely pumped out,

$F_1 = (\rho g h)A_1$ (as P_0 = 0) = (10^3)(10)(0.1)(10^{-3}) = 1 N (downwards). In this case F_2 = 0 and F_3 = 1 N (upwards)
∴ $F = F_2 + W - F_3$ = 0 + 10 – 1 = 9 N (upwards)

(d) No, the answer will remain the same. Because the answers depend upon P_0, ρ, g, h , A_1 and A_2.

Illustration 6: Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill up to a particular common height. Is the force exerted by water on the base of the vessel the same in the two cases? If so, why do the vessels filled with water to the same height give different reading on a weighing scale? **(JEE MAIN)**

Sol: Force on the base of the vessel depends on the pressure on it, and pressure depends on the height of the liquid in the vessel. On the other hand the normal reaction from the surface on which the vessel is kept, depends on both the pressure at the base as well as the weight of the liquid in the vessel.

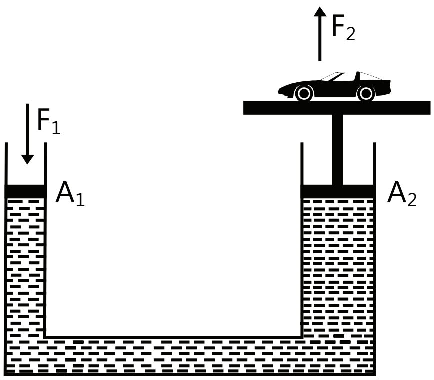
Pressure (and therefore force) on the two equal base areas are identical. But force is exerted by water on the sides of the vessels also, which has non-zero vertical component when the sides of the vessel are not perfectly normal to the base. This net vertical component of force by water on the side of the vessel is greater for the first vessel than the second. Hence, the vessels weigh different when the force on the base is the same in the two cases.

3.4.1 Pascal's Laws

According to the equation $p = p_0 + \rho gh$. Pressure at any depth h in a fluid may be increased by increasing the pressure p_0 at the surface. Pascal recognized a consequence of this fact that we now call Pascal's Law. A pressure applied to a confined fluid at rest is transmitted equally undiminished to every part of the fluid and the walls of the container.

This principle is used in a hydraulic jack or lift, as shown in the figure.

The pressure due to a small force F_1 applied to a piston of area A_1 is transmitted to the large piston of area A_2. The pressure at the two pistons is the same because they are at the same level.



A hydraulic jack

Figure 9.7

$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}$ Or $F_2 = \left(\frac{A_2}{A_1}\right)F_1$. Consequently, the force on the larger piston is large.

Thus, a small force F_1 acting on a small area A_1 results in a larger force F_2 acting on a larger area A_2.

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Since energy is always conserved, $F_1x_1 = F_2x_2$ where x_1 and x_2 are the distances moved by the pistons.

Nitin Chandrol (JEE 2012, AIR 134)

Illustration 7: Find the pressure in the air column at which the piston remains in equilibrium. Assume the pistons to be massless and frictionless.

(JEE MAIN)

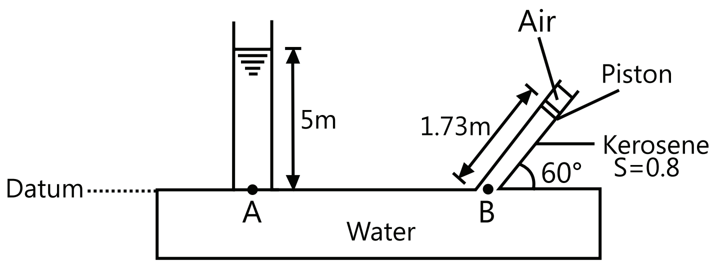


Figure 9.8

Sol: Apply Pascal's law at two points at equal height from a common datum.

Let p_a be the air pressure above the piston.

Applying Pascal's law at point A and B.

$P_{atm} + r_w g(5) = p_a + r_k g(1.73)\frac{\sqrt{3}}{2}$; $P_a = 138$ kPa

Illustration 8: A weighted piston confines a fluid of density ρ in a closed container, as shown in the figure. The combined weight of piston and container is W = 200 N, and the cross-sectional area of the piston is A = 8 cm^2. Find the total pressure at point B if the fluid is mercury and h = 25 cm (p_m = 13600 kgm^{-3}). What would be an ordinary pressure gauge reading at B? **(JEE ADVANCED)**

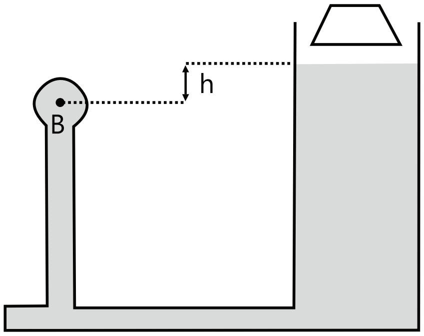


Figure 9.9

Sol: Pressure difference between two points at different heights is equal to ρgh, where h is difference in heights of two points. Apply Pascal's law at two points at different heights from a common datum.

Pascal's principle tells us about the pressure applied to the fluid by the piston and atmosphere. This added pressure is applied at all points within the fluid. Therefore the total pressure at B is composed of three parts: Pressure of atmosphere = 1.0×10^5 Pa

Pressure due to piston and weight = $\frac{W}{A} = \frac{200N}{8\times10^{-4}m^2} = 2.5 \times 10^5$ Pa

Pressure due to height h of fluid = $h\rho g = 0.33 \times 10^5$ Pa

In this case, the pressure of the fluid itself is relatively small. We have

Total pressure at B = 3.8×10^5 Pa = 383 kPa. The gauge pressure does not include atmospheric pressure. Therefore,

Gauge pressure at B = 280 kPa

Illustration 9: For the system shown in the figure, the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm². The piston on the right at S, has cross-sectional area 25 cm² and negligible weight. If the apparatus is filled with oil (ρ=0.78 g/cm³), find the force F required to hold the system in equilibrium as shown in figure. **(JEE ADVANCED)**

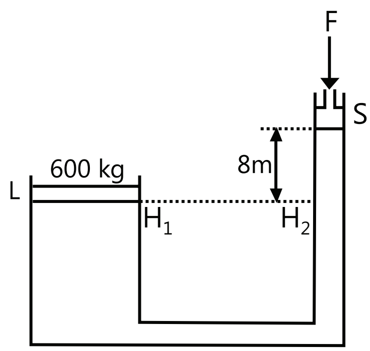


Figure 9.10

Sol: Apply Pascal's law at two points at different heights from a common datum.

The pressures at point H_1 and H_2 are equal because they are at the same level in the single connected fluid. Therefore, Pressure at H_1 = pressure at H_2 = (pressure due to F plus pressure due to liquid column above H_2)

$$\frac{(600)(9.8)N}{0.08m^2} = \frac{F}{25\times10^{-4}m^2} + (8m)(780\ kg/m^{-3})(9.8)$$

After solving, we get, F = 31 N

Illustration 10: As shown in the figure, as column of water 40 cm high supports 31 cm of an unknown fluid. What is the density of the unknown fluid? **(JEE MAIN)**

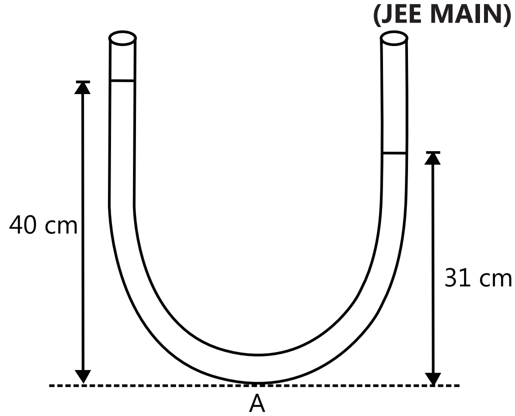


Figure 9.11

Sol: Find the hydrostatic pressure at the bottom most point A due to both the water column and the unknown fluid column.

The pressure at point A due to the two fluids must be equal (or the one with the higher pressure would push lower pressure fluid away). Therefore, pressure due to water = pressure due to known fluid; $h_1r_1g = h_2r_2g$, from which $r_2 = \frac{h_1}{h_2}p_1 = \frac{40}{31}$ (1000 kg/m²) = 1290 kg/m³

For gases, the constant density assumed in the compressible model is often not adequate. However, an alternative simplifying assumption can be made that the density is proportional to the pressure, i.e., $\rho = kp$. Let r_0 be the density of air at the earth's surface where the pressure is atmospheric p_o, then $r_0 = kp_0$; After eliminating k, we get $\rho = \frac{\rho_0}{p_0}p$

Putting the value of ρ in equation $dp = -\rho g dy$ or $dp = -\left(\frac{\rho_0}{p^0}p\right)gdy$

On rearranging, we get $\int_{p_0}^{p}\frac{dp}{p} = -\frac{\rho_0}{p_0}g\int_0^h dy$ where p is the pressure at a height y = h above the earth's surface.

After integrating, we get $\ln\left|\frac{p}{p_0}\right| = -\frac{\rho_0}{p_0}gh$ or $p = p_0 e^{\frac{-\rho_0}{p_0}gh}$

Note: Instead of a linear decrease in pressure with increasing height as in the case of an incompressible fluid, in this case pressure decreases exponentially.

4. PRESSURE MEASURING DEVICES

4.1 Manometer

A manometer is a tube open at both ends and bent into the shape of a "U" and is partially filled with mercury. When one end of the tube is subjected to an unknown pressure p, the mercury level drops on that side of the tube and rises on the other so that the difference in mercury level is h as shown in the figure.

When we move down in a fluid, pressure increases with depth and when we move up the pressure decreases with height. When we move horizontally in a fluid, pressure remains constant. Therefore, $p + r_0gh_0 - r_mgh = p_0$ where p_0 is atmospheric pressure, and r_m is the density of the fluid inside the vessel.

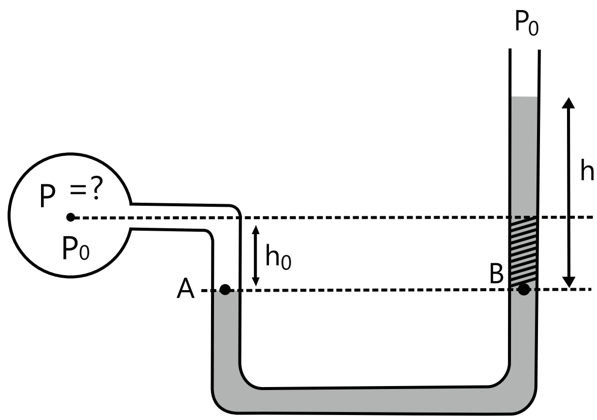


Figure 9.12: An U-shaped manometer tube connected to a vessel

4.2 The Mercury Barometer

It is a straight glass tube (closed at one end) completely filled with mercury and inserted into a dish which is also filled with mercury as shown in the figure. Atmospheric pressure supports the column of mercury in the tube to a height h. The pressure between the closed end of the tube and the column of mercury is zero, p = 0. Therefore, pressure at points A and B are equal and thus $p_0 = 0 + r_mgh$. Hence, $p_0 = (13.6 \times 10^3)(9.8)(0.76) = 1.01 \times 10^5\ Nm^{-2}$ for Pa.

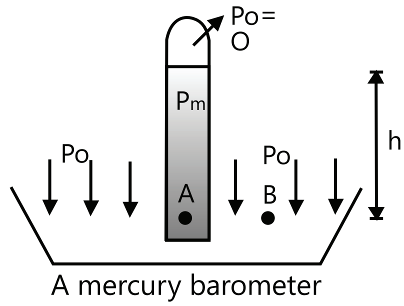


Figure 9.13

Illustration 11: What must be the length of a barometer tube used to measure atmospheric pressure if we are to use water instead of mercury? **(JEE MAIN)**

Sol: The length of the barometer tube will be inversely proportional to the density of fluid used in it.

We know that $p_0 = r_mgh_m = r_wgh_w$ where r_w and h_w are the density and height of the water column supporting the atmospheric pressure p_0.

$\therefore \quad h_w = \frac{\rho_m}{\rho_w}h_w$; Since $\frac{\rho_m}{\rho_w} = 13.6$; $h_w = 0.76$ m $= (13.6)(0.76) = 10.33$ m.

5. PRESSURE DIFFERENCE IN ACCELERATING FLUIDS

Consider a beaker filled with some liquid of density p accelerating upwards with an acceleration a_y along positive y-direction. Let us draw the free body diagram of a small element of fluid of area A and length dy as shown in figure. Equation of motion for this fluid element is, PA – W – (P + dP)A = (mass)(a_y) or –W – (dP) A =(Aρ dy)(a_y)

or (Aρg dy) – (dP)A = (Aρ dy)(a_y) or $\frac{dP}{dy} = -\rho(g + a_y)$

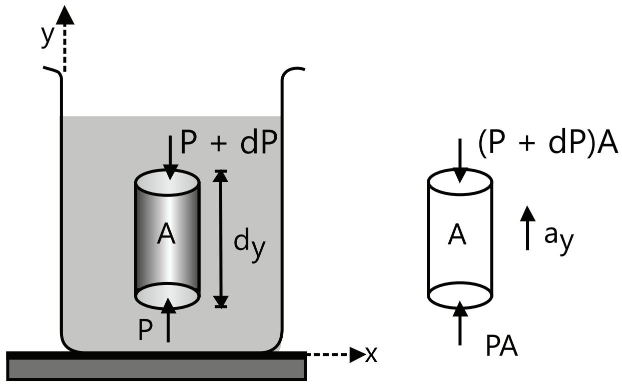


Figure 9.14

Similarly, if the beaker moves along positive x-direction with acceleration a_x, the equation of motion for the fluid element shown in the figure is, PA – (P + dP) A = (mass)(a_x)

or (dP)A = (Aρ dx)a_x Or $\frac{dP}{dx} = -\rho a_x$

But suppose the beaker is accelerated and it has components of acceleration a_x and a_y in x and y directions respectively, then the pressure decreases along both x and y directions. The above equation

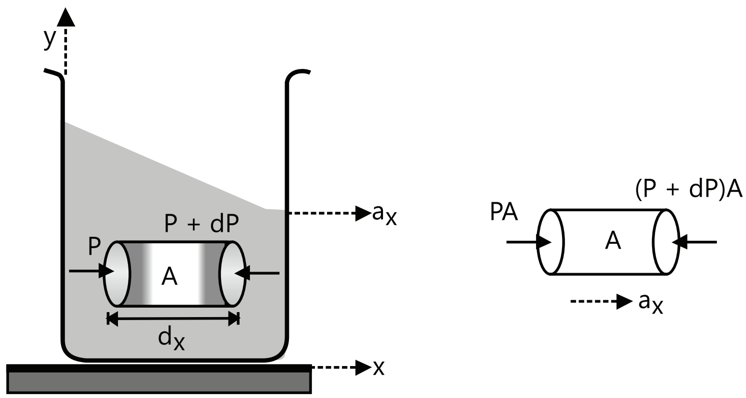


Figure 9.15

in that case reduces to,

$$\frac{dP}{dx} = -\rho a_x \quad \text{and} \quad \frac{dP}{dy} = -\rho(g + a_y) \quad \text{..... (i)}$$

For surface of a Liquid Accelerated in Horizontal Direction.

Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration 'a'. Let A and B be two points in the liquid at a separation x in the same horizontal line. As we have seen in this case.

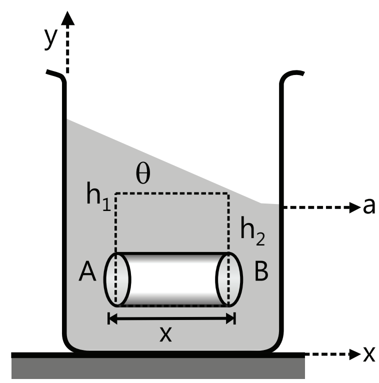


Figure 9.16

$\frac{dP}{dx} = -\rho a$ or dP = -ra dx. Integrating this with proper limits, we get

$P_A - P_B$ = pax ..... (ii)

Further, $P_A = P_0 + \rho g h_1$ And $P_B = P_0 + \rho g h_2$

Substituting in Eq. (ii), we get pg($h_1 - h_2$) = pax ∴ $\frac{h_1 - h_2}{x}$

$= \frac{a}{g} = \tan\theta$ ∴ $\boxed{\tan\theta = \frac{a}{g}}$

Note: When a_y is not equal to zero then the angle of inclination is given by

$$\tan\theta = \frac{dy}{dx} = \left(\frac{\overline{(dp)}}{\left(\frac{dp}{dy}\right)}\right) = \frac{a_x}{g + a_y}$$

Illustration 12: A liquid of density ρ is in a bucket that spins with angular velocity ω as shown is the figure. Show that the pressure at a radial distance r from the axis is

$P = P_0 + \frac{\rho\omega^2 r^2}{2}$ where P_0 is the atmospheric pressure. **(JEE ADVANCED)**

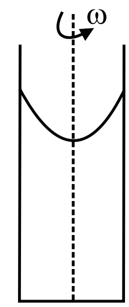


Figure 9.17

Sol: The net force on the liquid surface in equilibrium is always perpendicular to it as the liquid surface cannot sustain shear stress.

Consider a fluid particle P of mass m at coordinates (x, y). From a non-inertial rotating frame of reference, two forces are acting on it.

(i) Pseudo force $(mx\omega^2)$

(ii) Weight (mg) in the direction shown in figure.

Net force on it should be perpendicular to the free surface (in equilibrium). Hence,

$$\tan\theta = \frac{mx\omega^2}{mg} = \frac{x\omega^2}{g} \text{ or } \frac{dy}{dx} = \frac{x\omega^2}{g}$$

$$\therefore \int_0^y dy = \int_0^x \frac{x\omega^2}{g} \cdot dx \quad \therefore \quad y = \frac{x^2\omega^2}{2g}$$

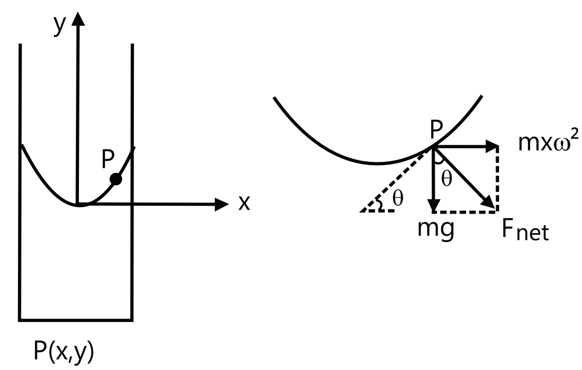


Figure 9.18

This is the equation of the free surface of the liquid, which is a parabola.

As x = r, $y = \frac{r^2\omega^2}{2g} \therefore P(r) = P_0 + \rho g y$ or $\quad P(r) = P_0 + \frac{\rho\omega^2 r^2}{2}$

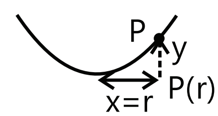


Figure 9.19

Hence proved.

Illustration 13: An open rectangular tank 5 m × 4 m × 3 m high containing water up to a height of 2 m is accelerated horizontally along the longer side.

(a) Determine the maximum acceleration that can be given without spilling the water.

(b) Calculate the percentage of water split over, if this acceleration is increased by 20%.

(c) If initially, the tank is closed at the top and is accelerated horizontally by 9 m/s², find the gauge pressure at the bottom of the front and rear walls of the tank. (Take g = 10 m/s²) **(JEE MAIN)**

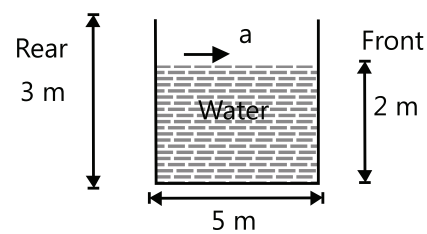


Figure 9.20

Sol: As the water column is accelerated towards right in horizontal direction, the free surface will not be horizontal but will be inclined at an angle with the θ horizontal, such that the left edge of the surface is at a higher level than the right edge. This is because the pressure at the left of water column will be more than the pressure at the right of it.

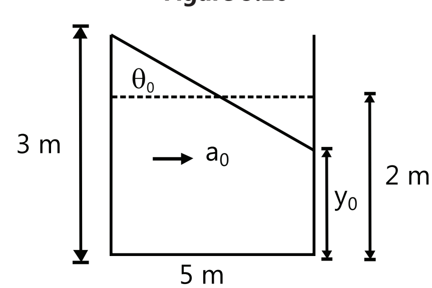


Figure 9.21

(a) Volume of water inside the tank remains constant

$\left(\frac{3+y_0}{2}\right) 5 \times 4 = 5 \times 2 \times 4$ or $y_0 = 1$m $\therefore \tan q_0 = \frac{3-1}{5} = 0.4$

Since, $\tan q_0 = \frac{a_0}{g}$, therefore $a_0 = 0.4\,g = 4$ m/s²

(b) When acceleration is increased by 20%

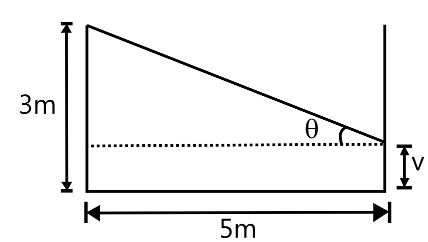


Figure 9.22

$a = 1.2\ a_0 = 0.48\ g \therefore \tan\theta = \frac{a}{g} = 0.48$

Now, $y = 3 - 5\tan\theta = 3 - 5\ (0.48) = 0.6$ m

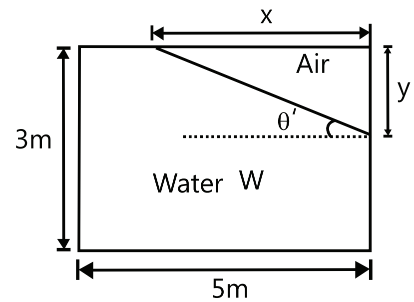


Figure 9.23

Fraction of water split over $= \dfrac{4\times2\times5-\dfrac{(3+0.6)}{2}\times5\times4}{2\times5\times4} = 0.1$

Percentage of water split over = 10%

(c) $a' = 0.9\ g$; $\tan\theta' = \frac{a'}{g} = 0.9$

Volume of air remains constant -> $4\times\frac{1}{2}yx = (5)(1)\times4 \Rightarrow$ Pressure does not change in the air.

Since $y = x\tan\theta' \therefore \frac{1}{2}x^2\tan\theta' = 5$ or $x = 3.33$ m; $y = 3.0$m

Gauge pressure at the bottom of the

(i) Front wall p_f = zero

(ii) Rear wall $p_r = (5\tan\theta')r w g = 5(0.9)(10^3)(10) = 4.5\times10^4$ Pa

Illustration 14: A vertical U-tube with the two limbs 0.75 m apart with water and rotated about a vertical axis 0.5 m from the left limb, as shown in the figure. Determine the difference in elevation of the water levels in the two limbs, when the speed of rotation is 60 rpm.

(JEE MAIN)

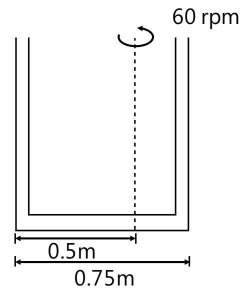


Figure 9.24

Sol: Each element of water in the tube is accelerated towards the axis. Along the horizontal part of the tube, the pressure will increase gradually as one moves radially away from the axis. The extra pressure provides the required centripetal acceleation.

Consider a small element of length dr at a distance r from the axis of rotation. Considering the equilibrium of this element.

$(p + dp) - p = r\omega^2\ r\ dr$ or $dp = r\omega^2\ r\ dr$

On integrating between 1 and 2

$$p_1 - p_2 = \rho\omega^2\int_{r_2}^{r_1} r\ dr = \frac{\rho\omega^2}{2}(r_1^2 - r_2^2)$$

or $h_1 - h_2 = \dfrac{\omega^2}{2g}[r_1^2 - r_2^2] = \dfrac{(2\pi)^2}{2(10)}[(0.5)^2 - (0.25)^2] = 0.37$ m.

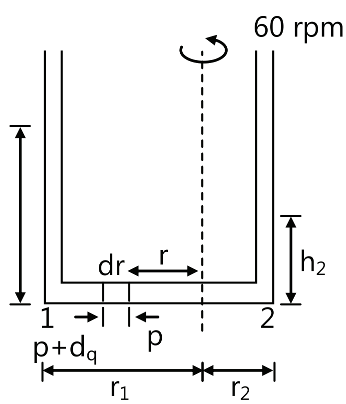


Figure 9.25

6. BUOYANCY

If a body is partially or wholly immersed in a fluid, it experiences an upward force due to the fluid surrounding it.

The phenomenon of force exerted by fluid on the body is called buoyancy and the force is called buoyant force. A body experiences buoyant force whether it floats or sinks, under its own weight or due to other forces applied on it.

Note: The buoyant force is due to the fact that the hydrostatic pressure at different depths is not the same. Buoyant force is independent of:

(a) Total volume and shape of the body.

(b) Density of the body.

6.1 Archimedes Principle

A body immersed in a fluid experiences an upward buoyant force equivalent to the weight of the fluid displaced by it. The proof of this principle is very simple. Imagine a body of arbitrary shape completely immersed in a liquid of density ρ. A body is being acted upon by the forces from all directions. Let us consider a vertical element of height h and cross-sectional area dA.The force acting on the upper surface of the element is F_1 (downward) and that on the lower surface is F_2 (upward). Since $F_2 > F_1$, therefore, the net upward force acting on the element is $dF = F_2 - F_1$. It can be easily seen that

$F_1 = (\rho g h_1)dA$ and $F_2 = (\rho g h_2)dA$. So $dF = \rho g(h)\,dA$

Also, $h_2 - h_1 = h$ and $h(dA) = dV$ $\therefore$ The net upward force is $F = \int \rho g dV = \rho V g$

Hence, for the entire body, the buoyant force is the weight of the volume of the fluid displaced.

Note: Buoyant force acts on the centre of gravity of the displacement liquid. This point is called centre of Buoyancy.

MASTERJEE CONCEPTS

The fluid exerts force on the immersed part of the body from all directions.

The net force experienced by every vertical element of the body is in the upward direction.

A uniform body floats in a liquid if density of the body is less than or equal to the density of the liquid and sinks if density of the uniform body is greater than that of the liquid.

B Rajiv Reddy (JEE 2012, AIR 11)

6.1.1 Detailed Explanation

An object floats on water if it can displace a volume of water whose weight is greater than that of the object. If the density of the material is less than that of the liquid, it will float even if the material is a uniform solid, such as a block of wood floats on water surface. If the density of the material is greater than that of water, such as iron, the object can be made to float provided it is not a uniform solid. An iron built ship is an example to this case

Apparent weight of a body immersed in a liquid = $w - w_0$, where 'w' is the true weight of the body and w_0 is the apparent loss in weight of the body, when immersed in the liquid.

6.1.2 Buoyant Force in Accelerating Fluids

Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with acceleration $\vec{a}$. The buoyant force F in this case becomes, $F = V\rho_L g_{eff}$;

Here, $g_{eff} = |\vec{g} - \vec{a}|$

Illustration 15: An iceberg with a density of 920 kgm^{-3} floats on an ocean of density 1025 kgm^{-3}. What fraction of the iceberg is visible? **(JEE MAIN)**

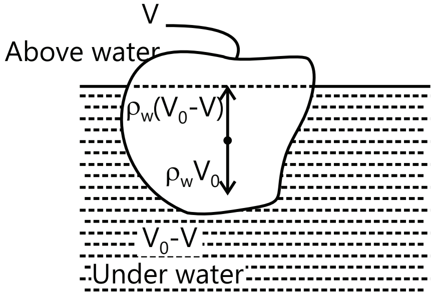


Figure 9.26

Sol: The buoyant force on the iceberg will be equal to its weight. The buoyant force is equal to the weight of water displaced by the iceberg, i.e. the weight of volume of water equal to the volume of iceberg immersed.

Let V be the volume of the iceberg above the water surface, then the volume under inside is $V_0 - V$. Under floating conditions, the weight $(\rho_I V_0 g)$ of the iceberg is balanced by the buoyant force $\rho_w(V_0 - V)g$.

Thus, $\rho_I V_0 g = \rho_w(V_0 - V)g$

or $\rho_w V = (\rho_w - \rho_I)V_0$

or $\frac{V}{V_0} = \frac{\rho_w - \rho_I}{\rho_w}$

Since, r_w = 1025 kg m^{-3} and r_i = 920 kg m^3, therefore, $\frac{V}{V_0} = \frac{1025 - 920}{1025} = 0.10$

Hence 10% of the total volume is visible.

Illustration 16: When a 2.5 kg crown is immersed in water, it has an apparent weight of 22 N. What is the density of the crown? **(JEE MAIN)**

Sol: Apply Archemides principle.

Let W = actual weight of the crown and W' = apparent weight of the crown

ρ = density of crown, ρ_0 = density of water. The buoyant force is given by $F_E = W - W'$ or

$\rho_0 Vg = W - W'$. Since $W = \rho Vg$, therefore, $V = \frac{W}{\rho g}$. Eliminating V from the above equation, we get

$\rho = \frac{\rho_0 W}{W - W'}$. Here W = 25 N; W' = 22 N; $\rho_0 = 10^3$ kg m^{-3} ; $\rho = \frac{(10)^3(25)}{25-22} = 9.3 \times 10^3$ kg m^{-3}.

Illustration 17: The tension in a string holding a solid block below the surface of a liquid (of density greater than that of solid) as shown in figure is T_0 when the system is at rest. What will be the tension in the string if the system has an upward acceleration a? **(JEE MAIN)**

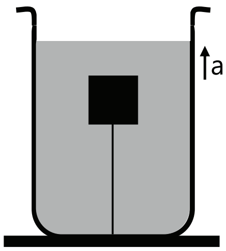


Figure 9.27

Sol: The weight and tension force on the block are balanced by the buoyant force on it. When the system is accelerated upwards, the effective value of g is increased.

Let m be the mass of block.

Initially for the equilibrium of block, $F = T_0 + mg$ ....(i)

Here, F is the up thrust on the block.

When the lift is accelerated upwards, g_{eff} becomes g + a instead of g.

Hence $F' = F\left(\frac{g+a}{g}\right)$ ...(ii)

From Newton's second law, $F' - T - mg = ma$ ...(iii)

Solving equations (i), (ii) and (iii), we get $T = T_0\left(1 + \frac{a}{g}\right)$

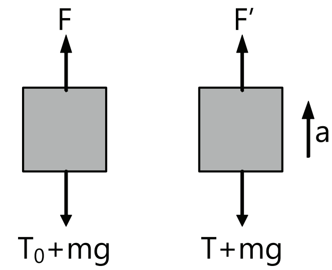


Figure 9.28

Illustration 18: An ice cube of side 1 cm is floating at the interface of kerosene and water in beaker of base area 10 cm^2. The level of kerosene is just covering the top surface of the ice cube.

(a) Find the depth of submergence in the kerosene and that in the water.

(b) Find the change in the total level of the liquid when the whole ice melts into water. **(JEE ADVANCED)**

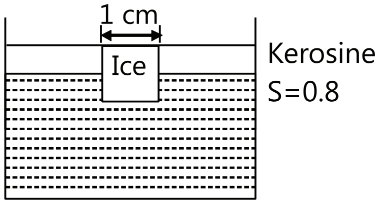


Figure 9.29

Sol: Apply Archemedes principle. Sum of the buoyant forces by kerosene and water will be equal to the weight of the ice cube.

(a) Condition of floating $0.8\ \rho_w g h_k + \rho_w g h_w = 0.9\ \rho_w g h$

or $0.8\ h_k + h_w = (0.9)h$ ... (i)

Where h_k and h_w are the submerged depths of the ice in the kerosene and water, respectively.

Also $h_k + h_w = h$ ... (ii)

Here it is given that h = 1 cm

Solving equations (i) and (ii), we get

$h_k = 0.5$ cm, $h_w = 0.5$ cm

(b) $\underset{\text{Ice}}{1\ cm^3} \xrightarrow{\text{m heat}} \underset{\text{(water)}}{0.9\ cm^3}$

Fall in the level of kerosene $\Delta h_k = \frac{0.5}{A}$; Rise in the level of water $\Delta h_w = \frac{0.9-0.5}{A} = \frac{0.4}{A}$

Net fall in the overall level $\Delta h = \frac{0.1}{A} = \frac{0.1}{10} = 0.01$ cm = 0.1 mm.

6.2 Stability of a Floating Body

The stability of a floating body depends on the effective point of application of the buoyant force. The weight of the body acts at its centre of gravity. The buoyant force acts at the centre of gravity of the displaced liquid. This is called the centre of buoyancy. Under equilibrium condition, the centre of gravity G and the centre of buoyancy B lie along the vertical axis of the body as shown in the figure(s).

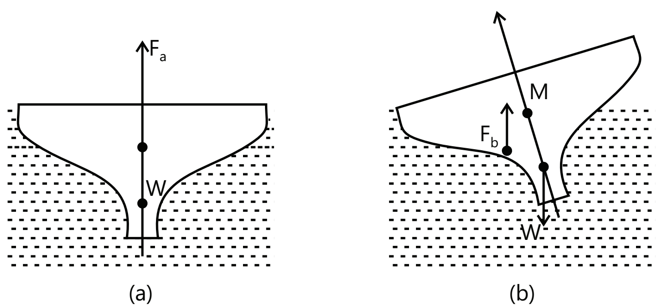


Figure 9.30

(a) The buoyant force acts at the centre of gravity of the displaced fluid.

(b) When the boat tilts, the line of action of the buoyant force intersects the axis of the boat at the metacentre M. In a stable boat, M is above the centre of gravity of the boat. When the body tilts to one side, the centre of buoyancy shifts relative to the centre of gravity as shown in the figure (b). The two forces act along different vertical lines. As a result, the buoyant force exerts a torque about the centre of gravity. The line of action of the buoyant force crosses the axis of the body at the point M, called metacentre. If G is below M, the torque will tend to restore the body to its equilibrium position. If G is above M, the torque will tend to rotate the body away from its equilibrium position and the body will be unstable.

Illustration 19: A wooden plank of length 1 m and uniform cross section is hinged at one end to the bottom of a tank as shown in the figure. The tank is filled with water up to a height of 0.5 m. The specific gravity of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position. (Exclude the case θ = 0) **(JEE ADVANCED)**

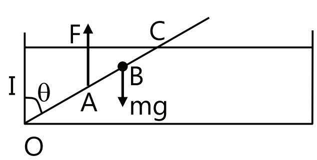


Figure 9.31

Sol: The net torque about the hinge due the weight of the plank and due to the buoyant force acting on the plank should be zero.

The forces acting on the plank are shown in the figure. The height of water level is 0.5m. The length of the plank is 1.0 = 2ℓ. We have OB = ℓ. The buoyant force F acts through the mid-point of the dipped part OC of the plank.

We have $OA = \frac{OC}{2} = \frac{\ell}{2\cos\theta}$; Let the mass per unit length of the plank be ρ.

Its weight mg = 2 ℓ ρg; The mass of the part OC of the plank = $\left(\frac{\ell}{\cos\theta}\right)\rho$.

The mass of water displaced = $\frac{1}{0.5}\frac{1}{\cos\theta}\rho = \frac{2\ell\rho}{\cos\theta}$; The buoyant force F is, therefore, F = $\frac{2\ell\rho g}{\cos\theta}$.

Now, for equilibrium, the torque of mg about O should balance the torque of F about O.

So, mg (OB) sin θ = F(OA) sin θ or (2 ℓ ρ) ℓ = $\left(\frac{2\ell\rho}{\cos\theta}\right)\left(\frac{\ell}{2\cos\theta}\right)$ or $\cos^2\theta = \frac{1}{2}$ or $\cos\theta = \frac{1}{\sqrt{2}}$, or $\theta = 45°$

6.3 Forces on Fluid Boundaries

Whenever a fluid comes in contact with solid boundaries, it exerts a force on it. Consider a rectangular vessel of base size l × b filled with water to a height H as shown in figure The force acting at the base of the container is given by F_b = p × (area of the base)

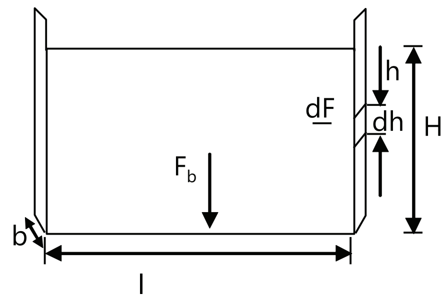


Figure 9.32

Pressure is same everywhere at the base and is equal to ρgH. Therefore, $F_b = \rho gH(lb) = \rho$ glb H Since, lbH = V (volume of the liquid) .Thus, $F_b = \rho gV$ = weight of the liquid inside the vessel.

A fluid contained in a vessel exerts forces on the boundaries. Unlike the base, the pressure on the vertical wall of the vessel is not uniform but increases linearly with depth from the free surface. Therefore, we have to perform the integration to calculate the total force on the wall. Consider a small rectangular element of width b and thickness dh at depth h from the free surface. The liquid pressure at this position is given by p = ρgh. The force at the element is dF = p(dbh) = ρgbh dh;

The total force is F = $\rho gb\int_O^H h\ dh = \frac{1}{2}\rho gbH^2$. The total force acting per unit width of the critical walls is $\frac{F}{b} = \frac{1}{2}\rho gH^2$

The point of application (the centre of force) of the total force from the free surface is given by $h_c = \frac{1}{F}\int_0^H h\,dF$

Where $\int_0^H h\,dF$ is the moment of force about the free surface.

Here $\int_0^H h\,dF = \int_0^H h(\rho\ gbh\ dh) = \rho gb\int_0^H h^2 dh = \frac{1}{3}\rho gH^3$;

Since F = $\frac{1}{2}\rho gbH^2$, therefore, $h_c = \frac{2}{3}H$

Illustration 20: Find the force acting per unit width on a plane wall inclined at an angle θ with the horizontal as shown in the figure. **(JEE MAIN)**

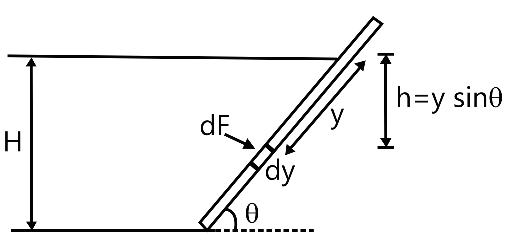


Figure 9.33

Sol: The pressure at each point on the wall will be different, depending on the height. Find pressure on a small element, and use the method of integration.

Consider a small element of thickness dy at a distance y measured along the wall from the free surface. There pressure at the position of the element is p = $\rho gh = \rho gy\sin\theta$. The force given by dF = p(b dy) = $\rho gb(y\,dy)\sin\theta$

The total force per unit width b is given by $\frac{F}{b} = \rho g\sin\theta . \int_0^{H/\sin\theta} y\ dy = \rho g\sin\theta\left[\frac{y^2}{2}\right]_0^{H/\sin\theta}$

Or $\frac{F}{b} = \frac{1}{2}\rho g \frac{H^2}{\sin\theta}$

Note: That the above formula reduces to $\frac{1}{2}\rho g H^2$ for a vertical wall (θ = 90°)

6.4 Oscillations of a Fluid Column

The initial level of liquid in both the columns is the same. The area of cross-section of the tube is uniform. If the liquid is depressed by x in one limb, it will rise by x along the length of the tube is the other limb. Here, the restoring force is provided by the hydrostatic pressure difference.

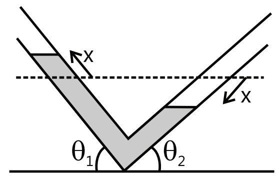


Figure 9.34

$\therefore \quad F = -(\Delta P)A = -(h_1 + h_2)\rho g A = -\rho g A(\sin\theta_1 + \sin\theta_2)x$

suppose, m is the mass of the liquid in the tube. Then, $ma = -\rho g A(\sin\theta_1 + \sin\theta_2)x$

Since, F or a is proportional to –x, the motion of the liquid column is simple harmonic in nature, time period of which is given by,

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{\rho g A(\sin\theta_1 + \sin\theta_2)}}$$

6.5 Oscillations of a Floating Cylinder

Consider a wooden cylinder of mass m and cross-sectional area A floating in a liquid of density ρ. At equilibrium, the cylinder is floating with a depth h submerged [See Fig. 8.35]. If the cylinder is pushed downwards by a small distance y and then released, it will move up and down with SHM. It is desired to find the time period and the frequency of oscillations.

According to the principle of flotation, the weight of the liquid displaced by the immersed part of the body is equal to the weight of the body. Therefore, at equilibrium,

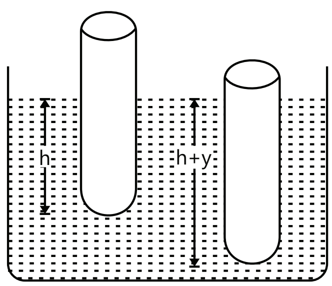


Figure 9.35

Weight of cylinder = Weight of liquid displaced by the immersed part of cylinder

or $\quad mg = (\rho A h)g \;\therefore\;$ Mass of cylinder, $m = \rho A h$

When the cylinder is pushed down to an additional distance y, the restoring force F (upward) equal to the weight of additional liquid displaced acts on the cylinder.

$\therefore$ Restoring force, F= - (weight of additional liquid displaced) or $F = -(\rho A y)g$

The negative sign indicates that the restoring force acts opposite to the direction of the displacement.

Acceleration a of the cylinder is given by $a = \frac{F}{m} = \frac{-(\rho A y)g}{\rho A h} = -\left(\frac{g}{h}\right)y$ ... (i)

Since g/h is constant, $a \alpha - y$ Thus the acceleration a of the body (wooden cylinder) is directly proportional to the displacement y and its direction is opposite to the displacement. Therefore, motion of the cylinder is simple harmonic.

$\therefore$ Time period $T = 2\pi\sqrt{\frac{h}{g}}$ ... (ii)

$\therefore$ Frequency $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$ ... (iii)

These very interesting results show that time period and frequency have the same form as that of simple pendulum. The submerged depth at equilibrium takes the place of the length of the pendulum.

7. FLUID DYNAMICS

In the order to describe the motion of a fluid, in principle, one might apply Newton's laws to a particle (a small volume element of fluid) and follow its progress in time. This is a difficult approach. Instead, we consider the properties of the fluid, such as velocity, pressure, at fixed points in space. In order to simplify the discussion we take several assumptions:

(i) The fluid is non viscous (ii) The flow is steady

(iii) The flow is non rotational (iv) The fluid is incompressible.

7.1 Equation of Continuity

It states that for streamlined motion of the liquid, the volume of liquid flowing per unit time is constant through different cross-sections of the container of the liquid. Thus, if v_1 and v_2 are velocities of fluid at respective points A and B of areas of cross-sections a_1 and a_2 and r_1 and r_2 be the densities respectively. Then the equation of continuity is given by $r_1a_1v_1 = r_2a_2v_2$ ... (i)

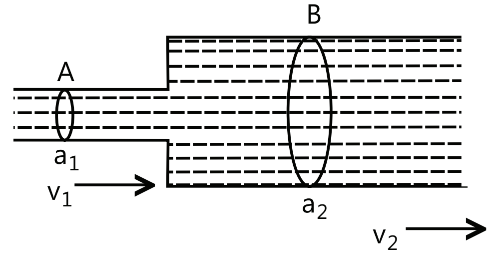


Figure 9.36

If the same liquid is flowing, then $\rho_1 = \rho_2$; then the equation (i) can be written

As $a_1v_1 = a_2v_2$ ...(ii)

$\Rightarrow$ av = constant $\Rightarrow$ $v \propto 1/a$

MASTERJEE CONCEPTS

Equation of continunity repersents the law of conservation of mass of moving fluids.

$a_1v_1r_1 = a_2v_2r_2$ (General equation of continuity)

This equation is applicable to actual liquids or to other fluids which are not incompressible.

Yashwanth Sandupatla (JEE 2012, AIR 821)

Illustration 21: Water is flowing through a horizontal tube of non-uniform cross-section. At a place, the radius of the tube is 1.0 cm and the velocity of water is 2 m/s. What will be the velocity of water, where the radius of the pipe is 2.0 cm? **(JEE MAIN)**

Sol: Apply the equation of continuity. Where area of cross-section is larger, the velocity of water is lesser and vice-versa.

Using equation of continuity, $A_1v_1 = A_2v_2$; $v_2 = \left(\frac{A_1}{A_2}\right)v_1$ or $v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right)v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$

Substituting the value, we get $v_2 = \left(\frac{1.0\times10^{-2}}{2.0\times10^{-2}}\right)$ or v_2 = 0.5 m/s

Illustration 22: Figure shows a liquid being pushed out of a tube by pressing a piston. The area of cross-section of the piston is 1.0 cm^2 and that of the tube at the outlet is 20 mm^2 If the piston is pushed at a speed of 2 $cm\text{-}s^{-1}$, what is the speed of the outgoing liquid?

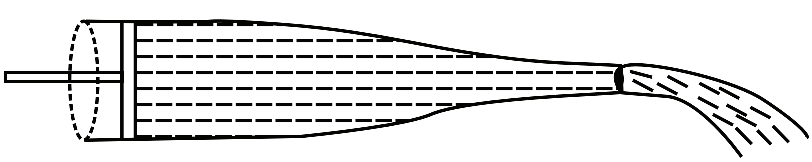


Figure 9.37

Sol: Apply the equation of continuity. Where area of cross-section is larger, the velocity of liquid is lesser and vice-versa.

From the equation of continuity $A_1v_1 = A_2v_2$

or $(1.0\ cm^2)(2\ cm\ s^{-1}) = (20\ mm^2)\ v_2$

or $v_2 = \frac{1.0\ cm^2}{20\ mm^2} \times 2\ cm\ s^{-1}$

$= \frac{100\ mm^2}{20\ mm^2} \times 2\ cm\ s^{-1} = 10\ cm\ s^{-1}$

SHM of fluids in tubes:

Tubes form angles θ_1 and θ_2 with the horizontal.

$$T = 2\pi\sqrt{\frac{m}{\rho gA(\sin\theta_1 + \sin\theta_2)}}$$

m is total mass of fluid in tubes, A is area of cross – section ρ is density of fluid.

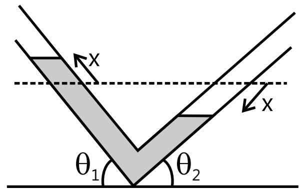


Figure 9.38

8. BERNOULLI'S THEOREM

When a non-viscous and an incompressible fluid flows in a streamlined motion from one place to another in a container, then the total energy of the fluid per unit volume is constant at every point of its path. Total energy = pressure energy + Kinetic energy + Potential energy

$= PV + \frac{1}{2}Mv^2 + Mgh$

Where P is pressure, V is volume, M is mass and h is height from a reference level.

$\therefore$ The total energy per unit volume = $P + \frac{1}{2}\rho v^2 + \rho gh$

Where ρ is density. Thus if a liquid of density ρ, pressure P_1 at a height h_1 which flows with velocity v_1 to another point in streamline motion where the liquid has pressure P_2, at height h_2 which flows with velocity v_2,

then $P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$

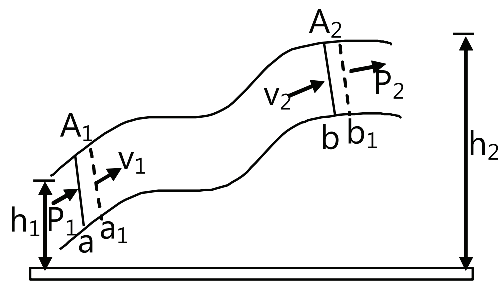


Figure 9.39

8.1 Derivations

8.1.1 Pressure Energy

If P is the pressure on the area A of a fluid, and the liquid moves through a distance due to this pressure, then Pressure energy of liquid = work done = force × displacement = PAl

The volume of the liquid is Al.

$\therefore$ Pressure energy per unit volume of liquid = $\frac{PAl}{Al} = P$

8.1.2 Kinetic Energy

If a liquid of mass m and volume V is flowing with velocity v, then the kinetic energy is $\frac{1}{2}mv^2$.

$\therefore$ Kinetic energy per unit volume of liquid. = $\frac{1}{2}\left(\frac{m}{V}\right)v^2 = \frac{1}{2}\rho v^2$. Here, ρ is the density of liquid.

8.1.3 Potential energy

If a liquid of mass m is at a height h from the reference line (h = 0), then its potential energy is mgh. $\therefore$ Potential energy per unit volume of the liquid = $\left(\frac{m}{v}\right)gh = \rho gh$

Thus, the Bernoulli's equation $P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$

This can also be written as: Sum of total energy per unit volume (pressure + kinetic + potential) is constant for an ideal fluid.

MASTERJEE CONCEPTS

$\frac{P}{\rho g}$ is called the 'pressure head', $\frac{v^2}{2g}$ the velocity head and h the gravitational head.

GV Abhinav JEE 2012, AIR 329

Intresting takeaway is the SI unit of each of these is meter (m).

Illustration 23: Calculate the rate of flow of glycerin of density 1.25×10^3 kg/m^3 through the conical section of a pipe, if the radii of its ends are 0.1 m and 0.04 m and the pressure drop across its length is 10 N/m^2. **(JEE MAIN)**

Sol: Apply the equation of continuity. Where area of cross-section is larger, the velocity of fluid is lesser and vice-versa.

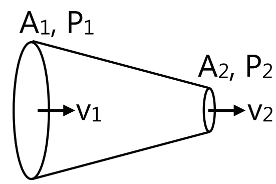


Figure 9.40

From continuity equation, $A_1v_1 = A_2v_2$

or $\frac{v_1}{v_2} = \frac{A_2}{A_1} = \frac{\pi r_2^2}{\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2 = \left(\frac{0.04}{0.1}\right) = \frac{4}{25}$ ... (i)

From Bernoulli's equation , $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

or $v_2^2 - v_1^2 = \frac{2\times10}{1.25\times10^3} = 1.6\times10^{-2}m^2/s^2$ ... (ii)

Solving equations (i) and (ii), we get $v_2 = 0.128$ m/s

$\therefore$ Rate of volume flow through the tube

$Q = A_2v_2 = (\pi r_2^2)\,v_2 = \pi\,(0.04)^2(0.128) = 6.43\times10^{-4}$ m^3/s

Illustration 24: Figure shows a liquid of density 1200 kg m^{-3} flowing steadily in a tube of varying cross section. The cross section at a point A is 1.0 cm^2 and that at B is 20 mm^2, the points A and B are in the same horizontal plane. The speed of the liquid at A is 10 cm s^{-1}. Calculate the difference in pressure at A and B. **(JEE ADVANCED)**

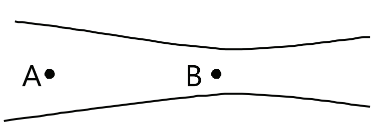


Figure 9.41

Sol: Apply the equation of continuity. Where area of cross-section is larger, the velocity of fluid is lesser and vice-versa.

From equation of continuity. The speed v_2 at B is given by, $A_1v_1 = A_2v_2$

or (1.0 cm^2) (10 cm s^{-1}) = (20 mm^2)v_2 or $v_2 = \frac{1.0cm^2}{20mm^2} \times 10cm\,s^{-1} = 50\,cm\,s^{-1}$

By Bernoulli equation, $P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$

Here $h_1 = h_2$. Thus $P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = \frac{1}{2} \times (1200\,kg\,m^{-2})(2500\,cm^2\,s^{-2} - 100\,cm^2\,s^{-2})$

$= 600\,kg\,m^{-3} \times 2400\,cm^2\,s^{-2} = 144\,Pa$

8.2 Application Based on Bernoulli's Equation

8.2.1 Venturimeter

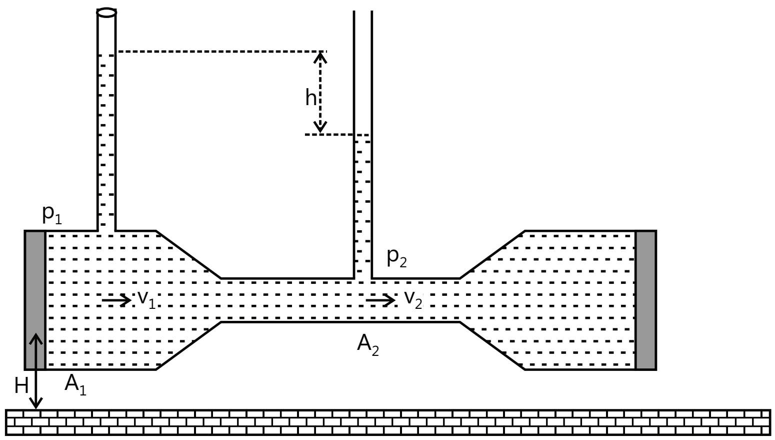
Figure shows a venturimeter used to measure flow speed in a pipe of non-uniform cross-section. We apply Bernoulli's equation to the wide (point 1) and narrow (point 2) parts of the pipe, with $h_1 = h_2$

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

From the continuity equation $v_2 = \frac{A_1v_1}{A_2}$

Substituting and rearranging,

we get $P_1 - P_2 = \frac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2} - 1\right)$ ...(i)



Venturimeter

Figure 9.42

The pressure difference is also equal to ρgh, where h is the difference in liquid level in the two tubes.

Substituting in equation (i), we get $v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$

MASTERJEE CONCEPTS

Because A_1 is greater than A_2, v_2 is greater than v_1 and hence the pressure P_2 is less than P_1.

The discharge or volume flow rate can be obtained as, $\frac{dV}{dt} = A_1v_1 = A_1\sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$

Anurag Saraf (JEE 2011, AIR 226)

9. TORRICELLI'S THEOREM

It states that the velocity of efflux of a liquid through an orifice is equal to that velocity which a body would attain in falling from a height from the free surface of a liquid to the orifice. If h is the height of the orifice below the free surface of a liquid and g is acceleration due to gravity, the velocity of efflux of liquid = v= $\sqrt{2gh}$. Total energy per unit volume of the liquid at the surface = KE + PE + Pressure energy = 0 + ρgh + P_0 ...(i)

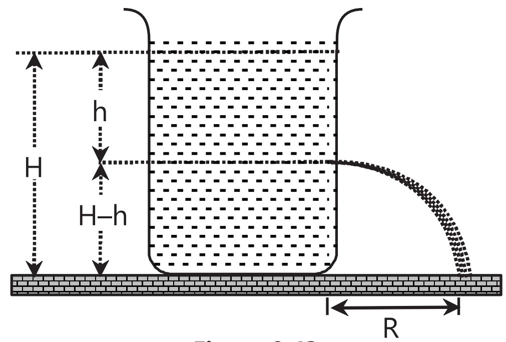


Figure 9.43

and total energy per unit volume at the orifice = KE + PE + Pressure energy = $\frac{1}{2}\rho v^2 + 0 + P_0$

Since total energy of the liquid must remain constant in steady flow, in accordance with Bernoulli's equation,

we have $\rho gh + P_0 = \frac{1}{2}\rho v^2 + P_0$ or $v = \sqrt{2gh}$

Range = velocity × time ; R = V_x × time = $\sqrt{2gh} \times 1$

Now, $H - h = \frac{1}{2}gt^2 \Rightarrow t = \frac{\sqrt{2(H-h)}}{g}$. From equation (i),

$$R = \sqrt{2gh} \times \frac{\sqrt{2(H-h)}}{g} = \sqrt{2h \times 2(H-h)} \times \sqrt{h(H-h)}2$$

∴ $\boxed{R = 2\sqrt{h(H-h)}}$

Range is max. if $\frac{dR}{dh} = 0 \Rightarrow 2 \times \frac{H-2h}{2\sqrt{h(H-h)}} = 0 \Rightarrow H - 2h = 0 \Rightarrow \boxed{h = \frac{H}{2}}$

MASTERJEE CONCEPTS

$R_h = R_{H-h}$

$R_h = 2\sqrt{h(H-h)}$

$R_{H-h} = 2\sqrt{h(H-h)}$

i.e. Range would be the same when the hole is at a height h

or at a height H – h from the ground or from the top of the beaker.

R is maximum at h = $\frac{H}{2}$ and R_{max} =H.

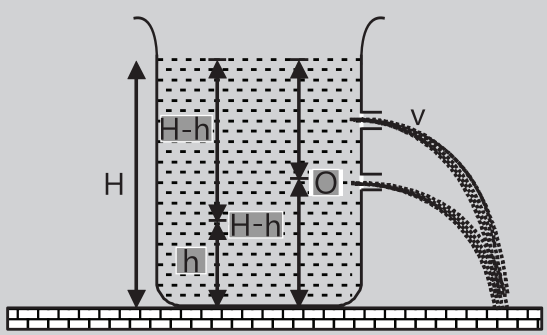


Figure 9.44

Vijay Senapathi (JEE 2011, AIR 71)

9.1 An Expression for the Force Experienced by the Vessel

The force experienced by the vessel from which liquid is coming out.

$F = \frac{dp}{dt}$ (Rate of change of momentum) $= \frac{d}{dt}(mv) = \frac{d}{dt}(\rho Avtv)$

$\boxed{F = \rho Av^2}$ Where ρ = It is the density of the liquid.

A = It is the area of hole through which liquid is coming out.

9.2 Time taken to Empty a Tank

Consider a tank filled with a liquid of density ρ up to a height H. A small hole of area of cross section a is made at the bottom of the tank. The area of cross-section of the tank is A.

Let at some instant of time the level of liquid in the tank be y. Velocity of efflux at this instant of time would be, $v = \sqrt{2gy}$.

At this instant volume of liquid coming out of the hole per second is $\left(\frac{dV_1}{dt}\right)$.

Volume of liquid coming down in the tank per second is $\left(\frac{dV_2}{dt}\right)$.

$$\frac{dV_1}{dt} = \frac{dV_2}{dt};\ \therefore\ av = A\left(-\frac{dy}{dt}\right)\ \ \therefore a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right) \text{Or}\ \ \int_0^t dt = -\frac{A}{a\sqrt{2g}}\int_H^0 y^{-1/2}dy$$

$$\therefore t = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_0^H = \frac{A}{a}\sqrt{\frac{2H}{g}}$$

Illustration 25: A tank is filled with a liquid up to a height H. A small hole is made at the bottom of this tank. Let t_1 be the time taken to empty first half of the tank and t_2 the time taken to empty rest half of the tank.

Then find $\frac{t_1}{t_2}$. **(JEE MAIN)**

Sol: This problem needs to be solved by method of integration.

Substituting the proper limit in equation (i), derived in the theory, we have

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}}\int_H^{H/2} y^{-1/2}dy\ \text{ Or }\ t_1 = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_{H/2}^{H}\ \text{Or}\ = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$

$$\text{Or}\ t_1 = \frac{A}{a}\sqrt{\frac{H}{g}}(\sqrt{2}-1) \quad \text{...(ii)}$$

$$\text{Similarly}\ \int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}}\int_{H/2}^{0} y^{-1/2}dy \quad \text{Or} \quad t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \quad \text{... (iii)}$$

From equations (ii) and (iii), we get $\frac{t_1}{t_2} = \sqrt{2}-1$ Or $\frac{t_1}{t_2} = 0.414$

MASTERJEE CONCEPTS

From here we see that $t_1 < t_2$. This is because inititally the pressure is high and the liquid comes out with greater speed.

Ankit Rathore (JEE Advanced 2013, AIR 158)

10. VISCOSITY

When a liquid moves slowly and steadily on a horizontal surface, its layer in contact with the fixed surface is stationary and the velocity of the layers increase with the distance from the fixed surface.

Consider two layers CD and MN of a liquid at distances x and x + dx from the fixed surface AB having velocities v and v + dv respectively as shown in the figure. Here $\left(\frac{dv}{dx}\right)$ denotes the rate of change of velocity with distance and is known as velocity gradient. The tendency of the upper layer is to accelerate the motion and the lower layer tries to retard the motion of upper layer. The two layers together tend to destroy their relative motion as if there is some backward dragging force acting tangentially on the layers. To maintain the motion, an external force is applied to overcome this backward drag.

Hence the property of a liquid virtue of which it opposes the relative motion between its different layers is known as viscosity.

The viscous force is given by $F = -\eta A \frac{dv}{dx}$

Where η is a constant, called the coefficient of viscosity.

The SI unit of η is N-s/m^2. It is also called decapoise or Pascal second. Thus,

1 decapoise = N-s/m^2 = 1 Pa-s = 10 poise.

Dimensions of h are $[ML^{-1}T^{-1}]$

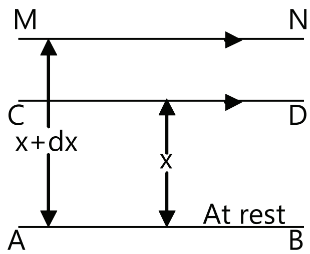


Figure 9.45

MASTERJEE CONCEPTS

The negative sign in the above equation shows that the direction of viscous force F is opposite to the direction of relative velocity of the layer.

Viscous force depends upon the velocity gradient whereas the mechanical frictional force is independent of the velocity gradient.

Vaibhav Gupta (JEE 2009, AIR 54)

10.1 Effect of Temperature

In case of liquids, coefficient of viscosity decreases with increase of temperature as the cohesive forces decrease with increase of temperature.

Illustration 26: A plate of area 2 m^2 is made to move horizontally with a speed of 2 m/s by applying a horizontal tangential force over the free surface of a liquid. The depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate. **(JEE MAIN)**

Sol: Apply the Newton's formula for the frictional force between two layers of a liquid.

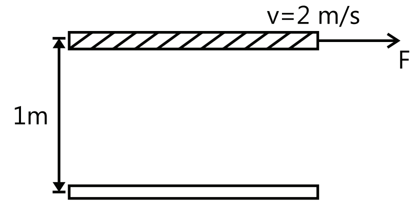


Figure 9.46

Velocity gradient = $\frac{\Delta v}{\Delta y} = \frac{2-0}{1-0} = 2\frac{m/s}{m}$

From Newton's law of viscous force,

$|F| = \eta A\frac{\Delta v}{\Delta y}$ = $(0.01 \times 10^{-1})(2)(2) = 4 \times 10^{-3}$ N.

So, to keep the plate moving, a force of 4×10^{-3} N must be applied.

10.2 Stokes' Law and Terminal Velocity

Stokes established that the resistive force or F, due to the viscous drag, for a spherical body of radius r, moving with velocity V, in a medium of coefficient of viscosity η is given by

F = 6 pη rV

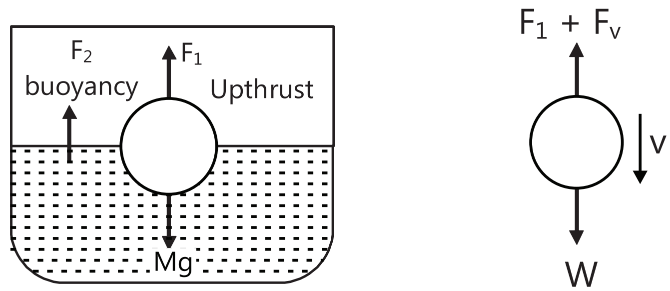


Figure 9.47

10.3.1 An Experiment for Terminal Velocity

Consider an established spherical body of radius r and density ρ falling freely from rest under gravity through a fluid of density σ and coefficient of viscosity η. When the body acquires the terminal velocity V

$W = F_t + 6\pi\eta rV$;

$$6\pi\eta rV = \frac{4}{3}\pi pr^3(\rho-\sigma)g \Rightarrow \boxed{V = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}}$$

Note: From the above expression we can see that terminal velocity of a spherical body is directly proportional to the densities of the body and the fluid (ρ – σ). If the density of the fluid is greater than that of the body (.i.e. σ>ρ), the terminal velocity is negative. This means that the body instead of falling, moves upward. This is why air bubbles rise up in water.

Illustration 28: Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 1 m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?

(JEE MAIN)

Sol: Use the formula for terminal velocity for spherical body.

$v_T \propto r^2$. Let r be the radius of small rain drops and R the radius of large drop.

Equating the volume, we have $\frac{4}{3}\pi R^2 = 2\left(\frac{4}{3}\pi r^3\right)$

$\therefore \quad R = (2)^{1/3}.\ r \quad$ or $\quad \frac{R}{r} = (2)^{1/3} \backslash \quad \frac{v_T'}{v_T} = \left(\frac{R}{r}\right)^2 = (2)^{2/3}$

$\therefore \quad v_T' = (2)^{2/3}\ v_T = (2)^{2/3}\ (1.0)$ m/s = 1.587 m/s.

Illustration 29: An air bubble of diameter 2 mm rises steadily through a solution of density 1750 kg m^{-3} at the rate of 0.35 cm s^{-1}. Calculate the coefficient of viscosity of the solution. The density of air is negligible. **(JEE MAIN)**

Sol: As the air bubble rises with constant velocity, the net force on it is zero.

The force of buoyancy B is equal to the weight of the displaced liquid. Thus B = $\frac{4}{3}pr^3\sigma g$.

This force is upward. The viscous force acting downward is F = $6\pi\eta rv$.

The weight of the air bubble may be neglected as the density of air is small. For uniform velocity

$$F = B \text{ or, } 6\pi\eta rv = \frac{4}{3}\pi r^3\sigma g \text{ or, } \eta = \frac{2r^2\sigma g}{9v} = \frac{2\times(1\times10^{-3}\text{m})^2\times(1750\text{ kg m}^{-3})(9.8\text{ ms}^{-2})}{9\times(0.35\times10^{-2}\text{ ms}^{-1})} \approx 11\text{ poise}.$$

This appears to be a highly viscous liquid.

10.3 Stream Line Flow

When liquid flows in such a way that the velocity at a particular point is the same in magnitude as well as in direction. As shown in figure every molecule should have the same velocity at A, if it crossed from that point. Notice that the velocity at the point B will be different from that of A. But every molecule which reaches at the point B, gets the velocity of the point B.

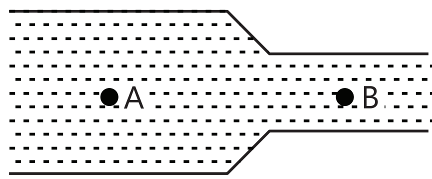


Figure 9.48

10.4 Turbulent Flow

When the motion of a particle at any point varies rapidly in magnitude and direction, the flow is said to be turbulent or beyond critical velocity. If the paths and velocities of particles change continuously and haphazardly, then the flow is called turbulent flow.

10.5 Critical Velocity and Reynolds Number

When a fluid flows in a tube with small velocity, the flow is steady. As the velocity is gradually increased, at one stage the flow becomes turbulent. The largest velocity which allows a steady flow is called the critical velocity.

Whether the flow will be steady or turbulent mainly depends on the density, velocity and the coefficient of viscosity of the fluid as well as the diameter of the tube through which the fluid is flowing. The quantity N = $\frac{\rho vD}{\eta}$ is called the Reynolds number and plays a key role in determining the nature of flow. It is found that if the Reynolds number is less than 2000, the flow is steady. If it is greater than 3000, the flow is turbulent. If it is between 2000 and 3000, the flow is unstable.

11. SURFACE TENSION

The properties of a surface are quite often marked different from the properties of the bulk material. A molecule well inside a body is surrounded by similar particles from all sides. But a molecule on the surface has particles of one type on one side and of a different type on the other side. Figure shows an example: A molecule of water well inside the bulk experiences force from water molecules from all sides, but a molecule at the surface interacts with air molecules from above and water molecules from below. This asymmetric force distribution is responsible for surface tension.

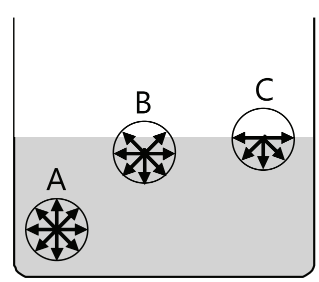


Figure 9.49

A surface layer is approximately 10-15 molecular diameters. The force between two molecules decreases as the separation between them increases. The force becomes negligible if the separation exceeds 10-15 molecular diameters. Thus, if we go 10-15 molecular diameters deep, a molecule finds equal forces from all directions.

Imagine a line AB drawn on the surface of a liquid (figure). The line divides the surface in two parts, surface on one side and the surface on the other side of the line. Let us call them surface to the left of the line and surface to the right of the line. It is found that the two parts of the surface pull each other with a force proportional to the length of the line AB. These forces of pull are perpendicular to the line separating the two parts and are tangential to the surface. In this respect the surface of the liquid behave like a stretched rubber sheet. The rubber sheet which is stretched from all sides is in the state of tension. Any part of the sheet pulls the adjacent part towards itself.

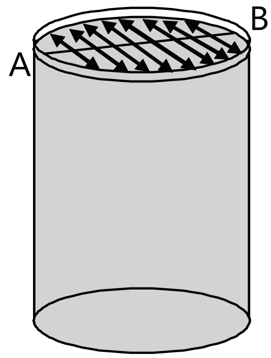


Figure 9.50

Let F be the common magnitude of the forces exerted on each other by the two parts of the surface across a line of length ℓ. We define the surface tension T of the liquid as $T = F/\ell$

The SI unit of surface tension is N/m.

Note: The surface tension of a particular liquid usually decreases as temperature increases. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers. This requires increasing the surface area of the water, which is difficult to do because of surface tension. Hence, hot water and soapy water is better for washing.

MASTERJEE CONCEPTS

Surface tension acts over the free surface of a liquid only and not within the interior of the liquid.

Due to surface tension the insects can walk on liquid surface.

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 30: Calculate the force required to take away a flat circular plate of radius 4 cm from the surface of water, surface tension of water being 75 dyne cm^{-1}. **(JEE MAIN)**

Sol: Force = Surface tension×length of the surface

Length of the surface = circumference of the circular plate = $2\pi r = (8\pi)$ cm

Required force = $T \times L = 72 \times 8\pi = 1810$ dyne.

12. SURFACE ENERGY

When the surface area of a liquid is increased, the molecules from the interior rise to the surface. This requires work against force of attraction of the molecules just below the surface. This work is stored in the form of potential energy. Thus, the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called 'surface energy'. The surface energy is related to the surface tension as discussed below:

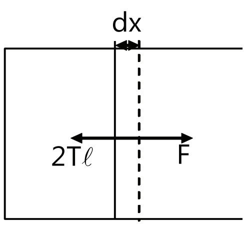


Figure 9.51

Let a liquid film be formed on a wire frame and a straight wire of length ℓ can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solution, each surface will pull the wire parallel to itself with a force $T\ell$. Thus, net force on the wire due to both the surfaces is $2T\ell$. One has to apply an external force F equal and opposite to it to keep the wire in equilibrium. Thus, $F = 2T\ell$

Now, suppose the wire is moved through a small distance dx, the work done by the force is,

$dW = F\,dx = (2T\ell)dx$

But $(2\ell)(dx)$ is the total increase in the area of both the surfaces of the film. Let it be dA. Then,

$dW = T\,da$ or $T = \frac{dW}{dA}$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by unity. Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface.

$\therefore T = \frac{dU}{dA}$ (as dW = dU)

Thus, the surface tension of a liquid is equal to the surface energy per unit surface area.

Illustration 31: How much work will be done in increasing the diameter of a soap bubble from 2 cm to 5 cm? Surface tension of soap solution is 3.0×10^{-2} N/m. **(JEE MAIN)**

Sol: Work done will be equal to the increase in the surface porential energy, which is surface tension multiplied by increase in area of surface of liquid.

Soap bubble has two surfaces. Hence, $W = T\,\Delta A$

Here, $\Delta A = 2[4\pi\{(2.5\times10^{-2})^2 - (1.0\times10^{-2})^2\}] = 1.32 \times 10^{-2}\ m^2$

$W = (3.0\times10^{-2})(1.32\times10^{-2})J = 3.96\times10^{-4}J$

Illustration 32: Calculate the energy released when 1000 small water drops each of same radius 10^{-7}m coalesce to form one large drop. The surface tension of water is 7.0×10^{-2} N/m. **(JEE MAIN)**

Sol: Energy released will be equal to the loss in surface potential energy.

Let r be the radius of smaller drops and R of bigger one.

Equating the initial and final volumes, we have $\frac{4}{3}\pi R^3 = (1000)\left(\frac{4}{3}\pi r^3\right)$

$R = 10r = (10)(10^{-7})\ m = 10^{-6}$ m. Further, the water drops have only one free surface. Therefore,

$\Delta A = 4\pi R^2 - (1000)(4\pi r^2) = 4\pi[(10^{-6})^2 - (10^3)(10^{-7})^2] = -36\pi(10^{-12})m^2$

Here, negative sign implies that surface area is decreasing. Hence, energy is released in the process.

$U = T[\Delta A] = (7\times10^{-2})(36\pi\times10^{-12})J = 7.9\times10^{-12}J$

13. EXCESS PRESSURE

The pressure inside a liquid drop or a soap bubble must be in excess of the pressure outside the bubble drop because without such pressure difference, a drop or a bubble cannot be in stable equilibrium. Due to the surface tension, the drop or bubble has got the tendency to contract and disappear altogether. To balance this, there must be excess of pressure inside the bubble.

13.1 Excess Pressure Inside a Drop

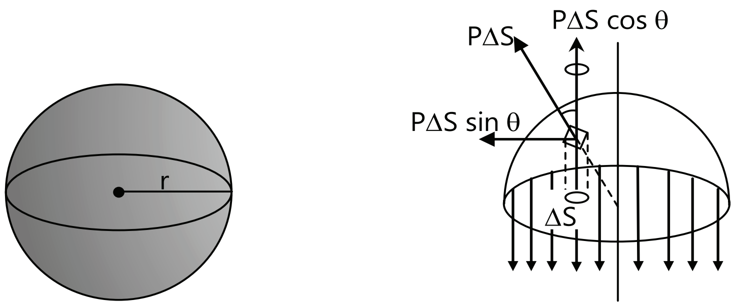


Figure 9.52

To obtain a relation between the excess of pressure and the surface tension, consider a water drop of radius r and surface tension T. Divide the drop into two halves by a horizontal passing through its centre as shown in figure and consider the equilibrium of one-half, say, the upper half. The force acting on it are:

(a) Force due to surface tension distributed along the circumference of the section.

(b) Outward thrust on elementary areas of it due to excess pressure.

Obviously, both the types of forces are distributed. The first type of distributed forces combine into a force of magnitude $2\pi r \times T$. To find the resultant of the other type of distributed forces, consider an elementary area ΔS of the surface. The outward thrust on $\Delta S = p\Delta S$ where p is the excess of the pressure inside the bubble. If this thrust makes an angle θ with the vertical, then it is equivalent to $\Delta Sp \cos\theta$ along the vertical and $\Delta Sp \sin\theta$ along the horizontal. The resolved component $\Delta Sp \sin\theta$ is infective as it is perpendicular to the resultant force due to surface tension. The resolved component $\Delta Sp \cos\theta$ is equal to balancing the force due to surface tension

The resultant outward thrust = $\Sigma\Delta Sp\cos\theta = p\Sigma\Delta S\cos\theta = p\Sigma\Delta S\cos\theta \quad = p\Sigma\Delta S'$

where $\Delta S' = \Delta S\cos\theta$ = area of the projection of ΔS on the horizontal dividing plane

$= p \times \pi r^2$ $(\ell\,\Delta S' = \pi r^2)$

For equilibrium of the bubble we have $\pi r^2\, p = 2\pi r\, T$ or $p = \dfrac{2T}{r}$

MASTERJEE CONCEPTS

If we have an air bubble inside a liquid, a single surface is formed.

There is air on the concave side and liquid on the convex side.

The pressure in the concave side (that is in the air) is greater than

the pressure in the convex side (that is in the liquid) by an amount $\dfrac{2T}{R}$.

$\therefore \boxed{P_2 - P_1 = \dfrac{2T}{R}}$

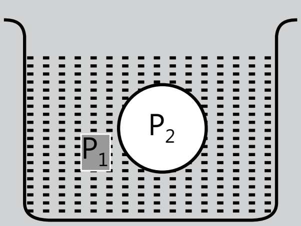


Figure 9.53

Nivvedan (JEE 2009, AIR 113)

13.2 Excess Pressure Inside Soap Bubble

A soap bubble consists of two spherical surface films with a thin layer of liquid between them. $P' - P_1 = 2S/R$ where R is the radius of the bubble.

As the thickness of the bubble is small on a macroscopic scale, the difference in the radii of the two surfaces will be negligible.

Similarly, looking at the inner surface, the air is on the concave side of the surface, hence $P_2 - P' = 2S/R$. Adding the two equations, $P_2 - P_1 = 4S/R$

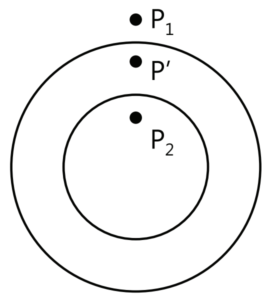


Figure 9.54

Illustration 33: What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface? Surface tension of water = 7.2×10^{-2} N/m and atmospheric pressure = 1.013×10^5 N/m^2.

(JEE MAIN)

Sol: Pressure inside the air bubble is larger than that outside it by amount 2T/R, where T is surface tension and R is its radius.

Surface tension of water T = 7.2×10^{-2} N/m; Radius of air bubble R = 0.1 mm = 10^{-4} m

The excess pressure inside the air bubble is given by, $P_2 - P_1 = \frac{2T}{R}$

$\therefore$ Pressure inside the air bubble, $P_2 = P_1 + \frac{2T}{R}$; Substituting the values, we have,

$$P_2 = (1.013\times10^5) + \frac{(2\times7.2\times10^{-2})}{10^{-4}} = 1.027\times10^5\ \text{N/m}^2$$

Illustration 34: A 0.02 cm liquid column balances the excess pressure inside a soap bubble of radius 7.5 mm. Determine the density of the liquid. Surface tension of soap solution = 0.03 Nm^{-1}. **(JEE MAIN)**

Sol: Pressure inside the soap bubble is larger than that outside it by amount 4T/R, where T is surface tension and R is its radius. Gauge pressure of liquid column is ρgh where symbols have the usual meaning.

The excess pressure inside a soap bubble is DP = 4S/R = $\frac{4\times0.03\text{Nm}^{-1}}{7.5\times10^{-3}\text{m}}$ = 16 Nm^{-2}

The pressure due to 0.02 cm of the liquid column is P = $h\rho g$ = (0.02×10^{-2} m) ρ (9.8 ms^{-2})

Thus, 16 N m^{-2} = (0.02×10^{-2} m) ρ (9.8 ms^{-2}); $\rho = 9.2\times10^3$ kg m^{-3}.

14. CAPILLARY ACTION

When a glass tube of very fine bore called a capillary tube is dipped in a liquid (like water), the liquid immediately rises into it due to the surface tension. The phenomenon of rise of a liquid in a narrow tube is known as capillarity.

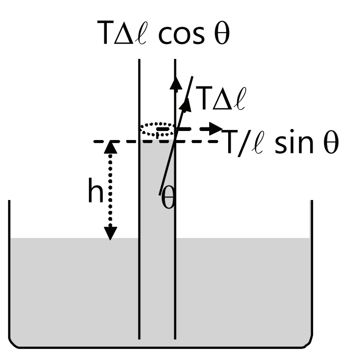


Figure 9.55

Suppose that a capillary tube of radius r is dipped vertically in a liquid. The liquid surface meets the wall of the tube at some inclination θ called the angle of contact. Due to surface tension, a force, $\Delta\ell T$ acts on an element $\Delta\ell$ of the circle of contact along which the liquid surface meets the solid surface and it is tangential to the liquid surface at inclination θ to the wall of the tube. (The liquid on the wall of the tube exerts this force. The tube also exerts the same force on the liquid in the opposite direction.) Resolving this latter force along and perpendicular to the wall of the tube, we have $\Delta\ell T\cos\theta$ along the tube vertically upwards and $\Delta\ell T\sin\theta$ perpendicular to the wall. The latter component is ineffective. It simply comes the liquid against the wall of the tube. The vertical component $\Delta\ell T\cos\theta$ pulls the liquid up the tube.

The total vertical upward force = $\Sigma\Delta\ell T\cos\theta = T\cos\theta\Sigma\Delta\ell = T\cos\theta.2\pi r$ (ℓ $\Sigma\Delta\ell$ = 2pr). Because of this upward pull liquid rises up in the capillary tube till it is balanced by the downward gravitational pull. If h is the height of the liquid column in the tube up to the bottom, the gravitational pull, i.e. weight of the liquid inside the tube is $(\pi r^2 h + V)\rho g$, where V is the volume of the liquid in meniscus. For equilibrium of the liquid column in the tube $2\pi rT\cos\theta = (\pi r^2 h + V)\rho g$

If value of the liquid in meniscus is negligible then, $2\pi rT\cos\theta = (\pi r^2h)\rho g$; $h = \dfrac{2T\cos\theta}{r\rho g}$

The small volume of the liquid above the horizontal plane through the lowest point of the meniscus can be calculated if θ is given or known. For pure water and glass $\theta = 0°$ and hence the meniscus is hemispherical.

∴ V = volume of the cylinder of height r – volume of hemisphere.

$$= \pi r^3 - \frac{1}{2}\frac{4\pi}{3}r^3 = \pi r^3 - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

∴ For water and glass $2\pi rT = \left(\pi r^2 h + \dfrac{\pi r}{3}\right)^3 \rho g$

$$2T = r\left(h + \frac{r}{3}\right)\rho g \quad \Rightarrow \quad h = \frac{2T}{r\rho g} - \frac{r}{3}$$

For a given liquid and solid at a given place as ρ, T, θ and g are constant, ∴ hr = constant

i.e. lesser t the radius of capillary greater will be the rise and vice-versa.

Illustration 36: A capillary tube of radius 0.20 mm is dipped vertically in water. Find the height of the water column raised in the tube. Surface tension of water = 0.075 N m^{-1} and density of water = 1000 kg m^{-3}. Take g = 10 m s^{-2}. **(JEE MAIN)**

Sol: Use the formula for height of the liquid in the capillary.

We have, $h = \dfrac{2S\cos\theta}{r\rho g} = \dfrac{2\times 0.075\,\text{N m}^{-1}\times 1}{(0.20\times 10^{-3}\text{m})\times(1000\text{ kg m}^{-3})(10\text{ m s}^{-2})}$ = 0.075 m = 7.5 cm.

PROBLEM SOLVING TACTICS

(a) Suppose two liquids of densities r_1 and r_2 having masses m_1 and m_2 are mixed together.

Then the density of the mixture will be = $\dfrac{(m_1+m_2)}{\left(\dfrac{m_1}{\rho_1}+\dfrac{m_2}{\rho_2}\right)}$

If two liquids of densities r_1 and r_2 having volume V_1 and V_2 are mixed, then the density of the mixture will be

$\dfrac{\rho_1V_1+\rho_2V_2}{V_1+V_2}$.

(b) When solving questions on Bernoulli's always assume a reference level and calculate the heights from the reference level.

FORMULAE SHEET

Fluid Statics:

1. Density = $\frac{\text{mass}}{\text{volume}}$, S.I. units: kg/m^3

2. Specific gravity / Relative density / Specific density = $\frac{\text{Ratio of its density}}{\text{Ratio of density of water at } 4^\circ\text{C}}$,

S.I. units: No units

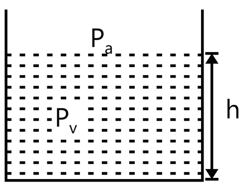
3. If two liquids of volume V_1 and V_2 and densities d_1 and d_2 respectively are mixed then the density d of the mixture is $d = \frac{V_1 d_1 + V_2 d_2}{V_1 + V_2}$; If $V_1 = V_2$ then $d = \frac{d_1 + d_2}{2}$

4. If two liquids of densities d_1 and d_2 and masses m_1 and m_2 respectively are mixed together, then the density d of the mixture is $d = \frac{m_1 + m_2}{\frac{m_1}{d_1} + \frac{m_2}{d_2}}$; if $m_1 = m_2$ then $d = \frac{2d_1 d_2}{d_1 + d_2}$

5. Pressure = $\frac{\text{Normal component of force}}{\text{Area on which force acts}} = \frac{f}{A}$, S.I. units: N/m^2, Pa

6. Pressure P acting at the bottom of an open fluid column of height h and density d is

= 1.013×10^5 Pa = 1.013×10^5 Pa = 1.013×10^6 dynes/cm^2= 76 cm of Hg = 760 torr = 1.013 bars.



$P = P_a + hdg$ (Absolute pressure = P; Atmospheric pressure = P_a)

$\frac{dp}{dh} = sg$ (As h increases P increases)

Figure 9.56

$P - P_a = hdg$

gauge pressure = absolute – atmospheric pressure.

7.

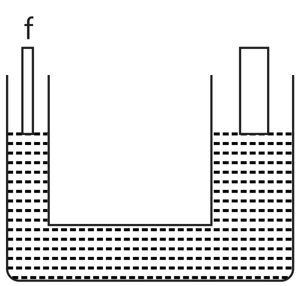


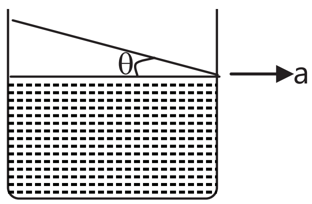
Figure 9.57

Area of smaller piston, a; area of larger piston, A, f is applied on the smaller piston

Force F developed on the larger piston $\frac{F}{A} = \frac{f}{a}$

$\therefore \quad F = \frac{fA}{a}$

8. Beaker is accelerated in horizontal direction



$\tan\theta = \frac{a}{g}$

a is the acceleration of the beaker in horizontal direction.

Figure 9.58

9. Beaker is accelerated and it has components of acceleration a_x, and a_y in x and y directions respectively.

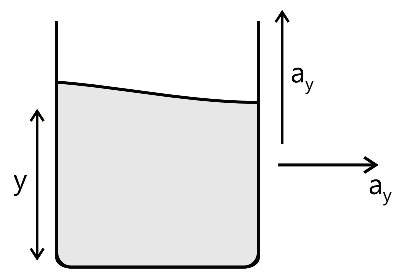


Figure 9.59

P increases with depth $\rightarrow \frac{dP}{dy} = p(g + ay)$

P is the density of the fluid.

ρ is the density of the fluid. $\frac{dP}{dx} = -pax$

10. Buoyant force $F = V_1\rho_1(\vec{g} - \vec{a})$

V_1 = immersed volume of liquid

ρ_1 = density of liquid

g = acceleration due to gravity

a = acceleration of body dipped inside liquid.

11. Body floats when Buoyant force balances the weight of the body.

$$\underset{\text{(Buoyant force)}}{V_i\rho_2 g} = \underset{\text{(Weight of body)}}{V_b\rho_b g}$$

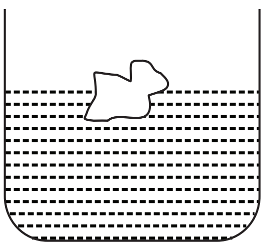


Figure 9.60

V_b, ρ_b = volume and density of body.

V_i = Volume of the immersed part of body.

ρ_2 = density of liquid.

Fraction of volume immersed $\frac{V_i}{V_b} = \frac{\rho_b}{\rho_2}$

% of volume immersed $\frac{V_i}{V_b} \times 100 = \frac{\rho_b}{\rho_2} \times 100.$

12. Apparent weight of a body inside a fluid is $W_{app} = W_{act}$ – Upthrust

$$W_{app} = V_b g\,(\rho_b - \rho_2)$$

V_b, δ_b = volume and density of body.

V_i = Volume of the immersed part of body.

ρ_2 = density of liquid.

13. General equation of continuity

$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ Generally $\rho_1 = \rho_2$ i.e., density is uniform.

A_1 & A_2 are area of cross-section at point P and Q.

V_1 & V_2 are velocities of the fluid at point P and Q.

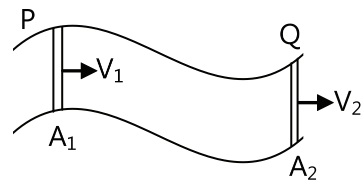


Figure 9.61

14. Bernoulli's Equation

$$P_1 + \rho g h_1 + \frac{1}{2}\rho V_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho V_2^2$$

i.e.,

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{constant}$$

Pressure Height Velocity at the point

at that point from the reference level

$$\frac{P}{\rho g} + \frac{V^2}{2g} + h = \text{constant}$$

Pressure head Velocity head gravitational head

15. Volumetric flow $Q = A\nu = \frac{dV}{dt}$ A – Area of cross section; ν – Velocity; V– Volume

S.I. unit = $\frac{m^3}{S}$

16. Torricelli Theorem:

$V = \sqrt{2gh}$ → height

V → velocity of efflux

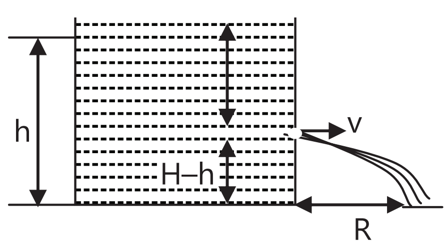


Figure 9.62

Range $R = 2\sqrt{h(H-h)}$

Range is maximum at $h = \frac{H}{2}$ and $R_{max} = H$

A_b – Area of orifice

A – Area of cross-section of the container.

Time taken to fall from H_1 to H_2 = $t \times \frac{A}{A_0}\sqrt{\frac{2}{g}}$

17. Viscous Force $F = \eta A\frac{dv}{dy}$

↓

coefficient of viscosity

L – Length of pipe

P_1 and P_2 are pressure at two ends of pipe.

R – Radius of pipe.

When liquid is flowing through a tube, velocity of flow of a liquid at distance from the axis.

$V = \frac{P}{4\eta L}\left(r^2 - x^2\right)$. Velocity distribution curve is a parabola.

18. Stoke's Law: Formula for the viscous force on a sphere

$F = 6\pi\eta rv$ (η – coefficient of viscosity)

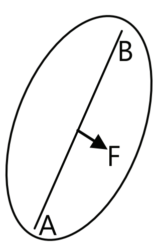
(r – radius of sphere)

(v – velocity of sphere)

$V_T = \frac{2}{9}r^2\frac{(\rho-\sigma)g}{\eta}$ (ρ – density of sphere)

($\propto$ – density of fluid)

19. Surface Tension



$T = \frac{F}{L}$

F is the total force acting on either side of AB.

L is length of AB.

Figure 9.63

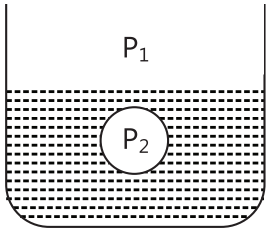
20. Surface Energy: dW = TdA

Surface Tension $T = \frac{dV}{dA} = \frac{\text{Surface energy}}{\text{Area}}$

21. Pressure inside the soap bubble is P, then

$P - P_0 = \frac{4T}{R}$

22. Air Bubble Inside a Liquid



$$P_2 - P_1 = \frac{2T}{R}$$

Figure 9.64

R – radius of bubble

T – surface tension force

23. Capillary Rise

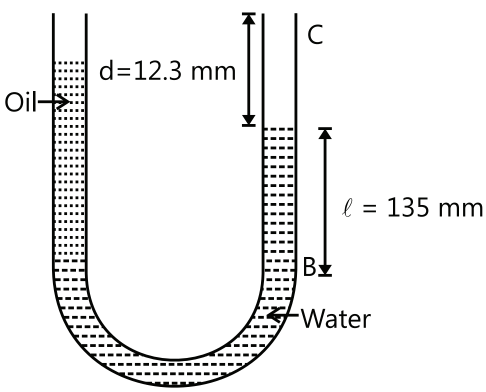
$h = \frac{2T\cos\theta}{r\rho g}$ r = is the radius of capillary tube

θ = angle of contact

Solved Examples

JEE Main/Boards

Example 1: For the arrangement shown in the figure. What is the density of oil?



Sol: Pressure will be same at all points at the same height in the same liquid.

$P_0 + \rho_w gl = P_0 + \rho_{oil}(\ell + d)g$

$$\Rightarrow \rho_{oil} = \frac{\rho_w \ell}{\ell + d} = \frac{1000.(135)}{(135+12.3)} = 916\ kg/m^3$$

Example 2: A solid floats in a liquid of different material. Carry out an analysis to see whether the level of liquid in the container will rise or fall when the solid melts.

Sol: Level of liquid will rise or fall depending on the density of the solid.

Let M = Mass of the floating solid.

ρ_1 = density of liquid formed by the melting of the solid.

ρ_2= density of the liquid in which the solid is floating. The mass of liquid displaced by the solid is M. Hence, the volume of liquid displaced is $\frac{M}{\rho_2}$. When the solid melts, the volume occupied by it is $\frac{M}{\rho_1}$. Hence, the level of liquid in container will rise or fall according as

$\frac{M}{\rho_2} - \frac{M}{\rho_1}$ is less than or greater than zero.

$\Rightarrow$ rises for $\rho_1 < \rho_2$

$\Rightarrow$ falls for $\rho_1 > \rho_2$

There will be no change in the level if the level if $\rho_1 = \rho_2$. In case of ice floating in water $\rho_1 = \rho_2$ and hence, the level of water remains unchanged when ice melts.

Example 3: An iron casting containing a number of cavities weighs 6000 N in air and 4000 N in water. What