4. FORCES AND LAWS OF MOTION

1. INTRODUCTION

In this chapter, we study in detail the actual consequences due to motion, i.e., the concept of force, which we specifically define as a push or pull experienced by a particular body or system. As we are of aware of the fact that the equation(s) of motion is/are governed by the choice of reference frame made, we, therefore, also concentrate on the same by involving different types of reference frames.

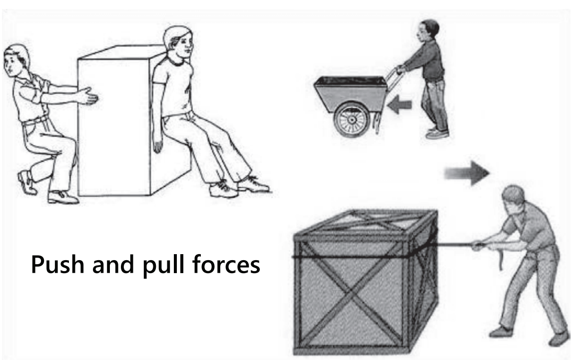


Figure: 4.1

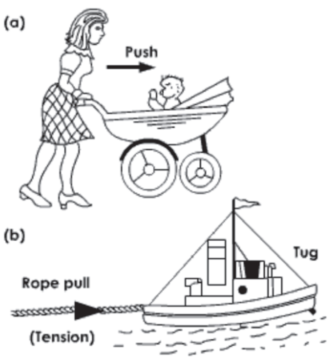


Figure: 4.2

2. FORCE

Force, by its nature, is better understood as any influence that causes an object to undergo a certain change, which maybe with respect to its movement, direction, or even geometrical construction. To be succinct, suffice it to say that a force can facilitate an object with mass to change its velocity, either to accelerate or deform a flexible object, or both. However, we can also define force using intuitive concepts such as a push or a pull. As mandated for a vector quantity, force has both magnitude and direction. We generally measure force based on the SI unit (of Newton) and represent the same using the symbol 'F.' It is imperative to understand, therefore, that in case if a body is subjected to more than one force, then the actual net force acting on that particular body is invariably a vector addition of all the forces in operation.

3. FREE BODY DIAGRAM

Suppose that we indicate all the operative external forces on an object, then the representation of the same is what we call as a free body diagram (FBD) of that particular object.

(a) Weight of a body/object: Weight of a body or an object is generally regarded as the force with which earth attracts that particular body/object toward its center. For example, if we consider 'M' as the mass of a body/

object and 'g' as its acceleration due to gravity, then we can conveniently express the weight of that particular body/object as Mg. However, we always consider that the weight of a body/object is in a direction that is vertically downward.

(b) Normal force: To understand the concept of normal force, let us consider a book resting on a table, as an example. The book has a specific weight, specifically in vertically downward direction and is at rest to begin with. Therefore, we understand that there is definitely one more force that is operative on the block but in an opposite direction, which helps to balance its weight. The source of this force is none other than the table and we hence call the same as a normal force. This signifies the fact that if in case two bodies are in contact with each other, then a contact force arises; further, if the contact surface is smooth, then the direction of the force is usually normal to the plane of contact. As stressed earlier, we always mean that its direction is towards the body under consideration.

Figure: 4.3

(c) Tension in a string: Let us assume that there is a block hanging from a fixed surface by a string. The weight of this block is acting vertically downward although it is not under motion; hence, its weight is adequately balanced by a force originating from the string. We call this force as 'tension in string.' Thus, we define 'tension' as a resisting force that is operative in a stretched string. Further we understand that its direction is along the string but away from the body/object under consideration.

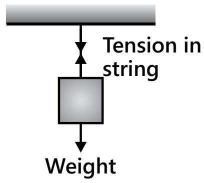


Figure: 4.4

Illustration 1: A cylinder of weight w is resting on groove V as shown in the Figure 4.5. Draw the FBD of the same. **(JEE MAIN)**

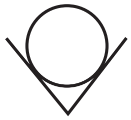


Figure: 4.5

Sol: Weight acts vertically downwards and contact force from the surface is normal to the surface at the point of contact.
The FBD of the cylinder is as shown in the Figure 4.6.

Here, w = weight of the cylinder and represent the normal reactions between the cylinder and the two inclined walls.

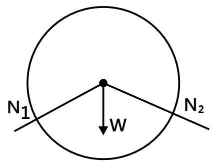


Figure: 4.6

Illustration 2: A block of mass m is attached with two strings as shown in the Figure 4.7. Draw the FBD of the same. **(JEE ADVANCED)**

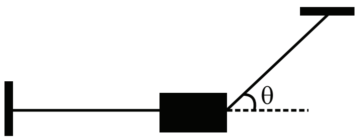


Figure: 4.7

Sol: Weight acts vertically downwards and tension forces along the length of the strings. The FBD of the block is as shown in the Figure 4.8.

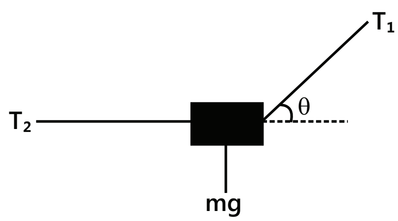


Figure: 4.8

4. NEWTON'S FIRST LAW OF MOTION

Suppose that if a body is observed from an inertial frame i.e., a frame which is at rest or moving with uniform velocity, then it will remain at rest or continue to move with uniform velocity unless an external force is applied on it. This property due to which a body remains at rest or continues its motion with uniform velocity is called as inertia. Force is push or pull, which disturbs or tends to disturb inertia of rest or inertia of uniform motion of a body. Thus, Newton's first law of motion defines inertia, force and intertial frame of reference. One example in this regard is the straight line motion of a body in the absence of the constraining force.

Illustration 3: A heavy particle of mass 0.50 kg is hanging from a string fixed with a roof. Find the force exerted by the string on the particle. (Take $g = 9.8\ m/s^2$) **(JEE MAIN)**

Sol: The weight of the particle is balanced by the force of tension in the string.

The forces acting on the heavy particle are

(a) Pull of the earth $0.50\ kg \times 9.8\ m/s^2 = 4.9\ N$, vertically downward

(b) Pull of the string, T, vertically upward.

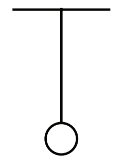


Figure: 4.9

The heavy particle is at rest with reference to position of the earth (which we assume to be an inertial frame). Hence, the sum of forces should be zero. Therefore, T is 4.9 N when acting vertically upward.

Illustration 4: The given diagram shows the forces in operation on a block. Determine whether the block is under acceleration or not. **(JEE MAIN)**

Sol: If the net force on the block is non-zero then the block accelerates. If the net force on the block is zero, then acceleration is zero.

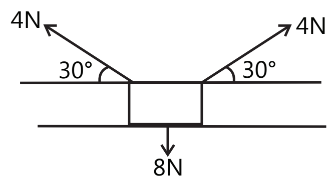


Figure: 4.10

To check whether the particle will have any acceleration or not, let us confirm if net force is zero or not by resolving the forces in both horizontal and vertical directions.

Net force in horizontal direction $= 4\cos 30° - 4\cos 30° = 0$

Net force in vertically downward direction $= 8 - 4\sin 30° - 4\sin 30° \neq 0$

The net force is not zero. Therefore, the particle will have downward acceleration.

5. INERTIA

Inertia is the resistance of any physical object to any change in its state of motion (including a change in direction). However, we need to understand that inertia is actually a passive property. Further, it does not permit a body to do anything but resists active agents such as torques and forces. In other words, it is tendency of objects to keep moving in a straight line at constant linear velocity.

5.1 Types of Inertia

There are basically three types of inertia.

5.1.1 Inertia of Rest

The inability of a body to change its state of rest by itself is known as inertia of rest. For example.

When we happen to shake the branch of a tree, we observe that the leaves or the fruits fall down. This is because the branches comes in motion, whereas the leaves or the fruits tend to remain at rest and hence fall down.

5.1.2 Inertia of Motion

The inability of a body to change its state of uniform motion by itself its state of uniform motion is known as inertia of motion.

Example: (i) When a moving car suddenly stops, we know that the person sitting in the car falls in the forward direction. This is because the lower portion of the person's body in contact with the car comes to rest, whereas the upper portion tends to remain in motion due to inertia of motion.

(ii) A person runs a certain distance before taking a long jump. This is mainly because the velocity acquired by running prior to attempting a long jump is added at the time of jump, so that he or she can cover a long distance.

5.1.3 Inertia of Direction

The inability of a body to change by itself its direction of motion is referred to as inertia of direction.

Example: (i) When a car moves around a curve, a person sitting inside it is thrown outward. This is to ensure his or her direction of motion.

5.2 Linear Momentum

The principle of linear momentum helps us to have a measure of an object's translational motion. The linear momentum p of a single particle is defined as the product of the mass m and velocity v of a particle in motion.

i.e., p = mv.

Linear momentum is a vector quantity. Its direction is in accordance with the direction of the velocity. The net momentum of a system of particles is the sum of momenta. In a system of two particles with masses m_1 and m_2 and, having velocities v_1 and v_2, respectively, the total momentum, $p = p_1 + p_2 = m_1v_1 + m_2v_2$

In the same manner, the momenta of more than two particles can be added.

6. NEWTON'S SECOND LAW OF MOTION

Newton's second law states that the net force on an object is equal to the rate of change of its linear momentum, p (i.e., the derivative) in an inertial reference frame: $F = \frac{dp}{dt} = \frac{d(mv)}{dt}$.

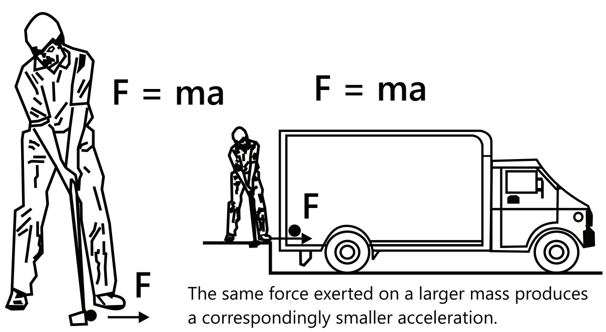


Figure: 4.11

However, the second law can also be stated in terms of an object's acceleration. As the law is valid only for constant-mass systems, the mass can be considered outside of the differentiation operator by the constant factor rule in differentiation. Thus,

$F = m\frac{dv}{dt} = ma,$ Where F is the net force applied, m is the mass of the body, and a is the body's acceleration.

Thus, we now know that the net force applied to a body results in a proportional acceleration. In other words, if a body is in an accelerating mode, then there is force acting on it. Both force and acceleration are vector quantities (as denoted by the bold type in the Figure 4.11). This shows that they have both a magnitude (size) and a direction relative to some reference frame.

Illustration 5: Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a 2 kg mass. If = 10 N and = 5 N, find the acceleration. **(JEE MAIN)**

Sol: Apply Newton's second law of motion.

Acceleration, as we already know, will be in the direction of the net force and hence will have magnitude as given by

$\sum \vec{F} = m\vec{a}$; $\vec{F} = \vec{F}_1 + \vec{F}_2 \Rightarrow |\vec{F}| = \sqrt{10^2 + 5^2 + 2.10.5\cos 120^o} + 5\sqrt{3}N$

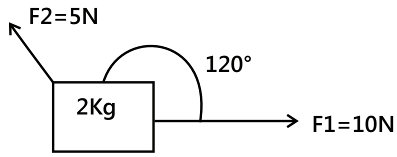


Figure: 4.12

$\Rightarrow |\vec{a}| = 2.5\sqrt{3} m / sec^2$

Further, if the resultant force is at angle α with $\vec{F}_1$

$$\tan\alpha = \frac{5\sin120°}{10+5\cos120°} \Rightarrow \alpha = 30°$$

Therefore acceleration is $2.5\sqrt{3} m / sec^2$ at an angle 30° with the direction of $\vec{F}_1$

Illustration 6: A block of mass M is pulled on a smooth horizontal table by a string making an angle with the horizontal as shown in the Figure 4.13. If the acceleration of the block is a, find the force applied by both the string and the table on the block. **(JEE ADVANCED)**

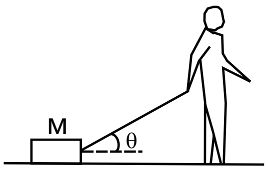


Figure: 4.13

Sol: List all the forces acting on the block. Take components of forces along the horizontal and the vertical. Apply Newton's second law along the horizontal and along the vertical.

Let us consider the block as the whole system. Therefore, the forces acting on the block are

(a) Pull of the earth, Mg, vertically downward,

(b) Contact force by the table, N, vertically upward, and

(c) Pull of the string, T, along the string.

Please observe the provided free body diagram for the block.

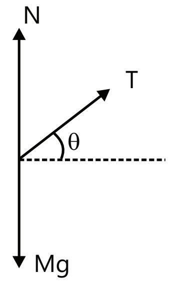


Figure: 4.14

The acceleration of the block is horizontal and toward the right. Now, take this direction as the x-axis and vertically upward direction as the y-axis. Therefore, we have

Component of Mg along the x-axis = 0; component of N along the x-axis = 0

Component of T along the Xx-axis = T cos

Hence, the total force along the x-axis = T cos.

Now, applying Newton's second law, T cos = Ma. ... (i)

Component of Mg along the y-axis =-Mg

Component of N along the y-axis = N

Component of T along the y-axis = T sin

The total force along the y-axis = N + T sin − Mg.

Again applying Newton's second law, N + T sin − Mg = 0; ... (ii)

From equation (i), $T = \frac{Ma}{\cos\theta}$. Substituting this in equation (ii) $N = Mg - Ma\tan\theta$.

7. BASIC FORCES IN NATURE

The various types of forces in nature can be grouped into four categories as listed hereunder:

(a) Gravitational, (b) Electromagnetic, (c) Nuclear, and (d) Weak.

7.1 Gravitational Force

Any two bodies attract each other by virtue of their masses. Now, the force of attraction between two masses is $F = G\frac{m_1 m_2}{r^2}$. where, m_1 and m_2 are the masses of the particles and r is the distance between the particles, and G is universal constant having the value 6.67×10^{-11} $N - m^2 / kg^2$

Similarly, the force of attraction exerted by the earth on any object is called gravity. The force exerted by the earth on a small body of mass m, kept near the earth's surface is mg in the vertically downward direction.

7.2 Electromagnetic Force

Consider two particles having charges at rest with respect to the observer. Now, the force between them has magnitude $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ where $\varepsilon = 8.85419 \times 10^{-12}\ C^2 / N\text{–}m^2$ is a constant

The quantity $\frac{1}{4\pi\varepsilon_0}$ is $9.0 \times 10^9 \frac{N-m^2}{C^2}$

This is called Coulomb force and it acts along the line joining the particles. Electromagnetic force is realized in many forms in our day-to-day life. Some examples having practical importance in this regard are listed hereunder:

(1) Forces between two surfaces in contact, (2) Tension in a string or a rope, and (3) Force due to spring.

7.3 Nuclear Force

The nuclear force (or nucleon–nucleon interaction or residual strong force) is the actual force between two or more nucleons. However, its fundamental laws are unknown as of now unlike the laws of Coulomb and Newton. This force is responsible for binding protons and neutrons in an atomic nucleus.

7.4 Weak Force

Weak force is a fundamental force of nature that underlies some forms of radioactivity. This force controls the decay of unstable subatomic particles such as mesons, and initiates the nuclear fusion reaction that fuels the Sun. We should know that the weak force acts upon all known fermions—i.e., elementary particles with half-integer values of intrinsic angular momentum, or spin. Particles are known to interact through the weak force by exchanging force-carrier particles known as the W- and Z particles. These particles are generally heavy, with masses of about 100 times the mass of a proton. It is precisely their heavier nature that defines the extremely short-range nature of the weak force. Understandably, therefore, this makes the weak force appear weak at the low energies associated with radioactivity.

8. NEWTON'S THIRD LAW OF MOTION

According to this law, when two bodies interact, they apply forces to one another that are equal in magnitude but opposite in direction.

However, for simplicity we state this law as, "To every action there is an equal and opposite reaction". Therefore, the third law is known also as the law of action and reaction.

But what is the meaning of action and reaction? Further, which force is "action" and which force is "reaction"? We know that every force that acts on a body is due to the presence other bodies in environment. Suppose that a body A experiences a force due to other body B. Then, the body B will also experience a force due to A. As per Newton's third law, two forces are equal in magnitude and opposite in direction. Therefore, mathematically we represent it as $\vec{F}_{AB} = -\vec{F}_{BA}$ Here, in this case, we can take either $\vec{F}_{AB}$ or $\vec{F}_{BA}$ as action force and the other will be the reaction force. Another important thing is that these two forces always act on different bodies.

Practical examples of law of motion

(a) First Law: "Every object persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed on it."

Before release: Object is in state of rest, air speed is zero, and there is weight but no drag. When the object is released: Object accelerates – airspeed increases.

As drag depends on airspeed – drag increases.

When drag is equal to weight: Object no longer accelerates but holds a constant velocity – terminal velocity.

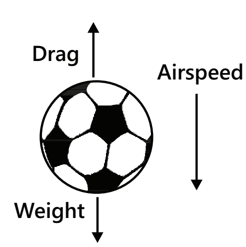


Figure: 4.15

(b) Second Law: Differential form: Force = change of momentum with change of time, i.e., $F = \frac{d(mv)}{dt}$ with mass constant: Force = mass × acceleration F=ma

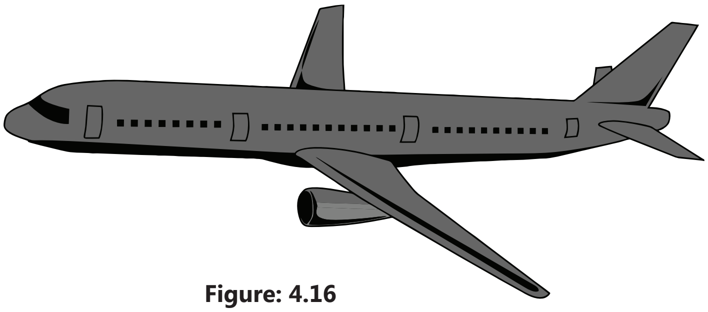


Figure: 4.16

Force = mass × change in velocity with time

$$F = \frac{m(V_1 - V_0)}{(t_1 - t_0)}$$

Hence, each has both magnitude and direction.

(c) Third Law: For every action, there is an equal and opposite reaction.

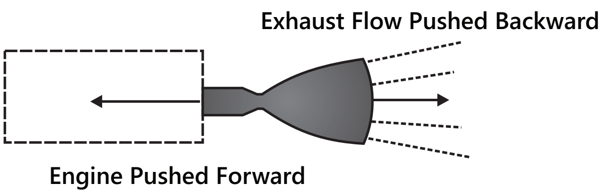


Figure 4.17: Rocket Engine Thrust

MASTERJEE CONCEPTS

Working with laws of motion

Step 1: Decide the system: The first step is to decide the system on which the laws of motion to be applied .The system may be a single particle, a block, a combination of two blocks one kept over the other, two blocks connected by a string, a piece of string, etc. The only restriction is that all parts of the system should have identical acceleration.

Step 2: Identify the forces: Once the system is decided, make a list of the forces acting on the system due to all the objects other than the system. Any force applied by the system should not be included in the list of forces.

Step 3: Make a Free Body Diagram (FBD): Now, represent the system by a point in a separate diagram and draw vectors representing the forces acting on the system with this point as the common origin.

Step 4: Choose the axes and Write Equations: Any three mutually perpendicular directions may be chosen as X-Y-Z axes.

Some suggestions are given below for choosing the axes to solve the problems

MASTERJEE CONCEPTS

If the forces are coplanar, only two axes say X and Y, taken in the plane of forces are needed. Choose the X-axis along the direction in which the system is known to have or is likely to have acceleration. A direction perpendicular to it may be chosen as the Y-axis. If the system is in equilibrium, any mutually perpendicular directions in the plane of the diagram may be chosen as the axes. Write the components of all the forces along the X-axis and equate their sum to the product of the mass of the system and its acceleration. This gives you an equation. Write the components of the forces along the Y-axis and equate the sum to zero. This gives you another equation. If the forces are collinear, this second equation is not needed.

If necessary you can go to step 1, choose another object as the system, repeat steps 2, 3 and 4 to get more equations. These are called equations of motion. Use mathematical techniques to get unknown quantities out of these equations. This completes the algorithm.

Note: (i) If the system is in equilibrium we will write the two equations as: $\sum F_x = 0$ and $\sum F_y = 0$ (ii) If the system is in collinear, the second equation, i.e. $\sum F_y = 0$ is not needed.

Nivvedan (JEE 2009, AIR 113)

9. IMPULSE

Definition: The impulse of a force is defined as the product of the average force $\vec{F}$ and the time interval Δt during which the force acts: $\text{Impulse} = \vec{F}\Delta t$.

Impulse, hence, is a vector quantity and has the same direction as the average force. The SI unit of impulse is Newton-second (Ns).

However, we can also define impulse as the change in the linear momentum of a body. Forces acting for a very short duration are called impulsive forces.

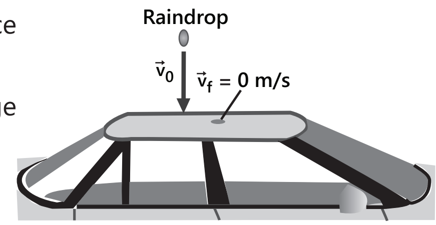


Figure: 4.18

9.1 Impulse Linear Momentum Theorem

When a net force is in operation on an object, then the impulse of the net force is equal to the change in momentum of the object: $\text{Impulse} = \text{Change in momentum } \bar{F}\Delta t = m\overrightarrow{v_f} - m\overrightarrow{v_0}$

Illustration 7: A truck of mass travelling at 4 m/s is brought to rest in 2 s when it strikes a wall. What force (assume constant) is exerted by the wall? **(JEE MAIN)**

Sol: Force on the truck is the change in momentum per unit time.

Using the relation, impulse = change in linear momentum

We have, $F.t = mv_f - mv_0 = m(v_f - v_0)$ or $F(2) = 2\times10^3\left[0-(-4)\right]$ or $2F = 8x10^3$ or $F = 4x10^3$

Illustration 8: Assume that on a certain day rain comes down at a velocity of –15 m/s and hits the roof of a car. The mass of rain per second that strikes the roof of the car is 0.060 kg/s. Assuming that rain comes to rest upon striking the car, find the average force exerted by the rain on the roof. **(JEE MAIN)**

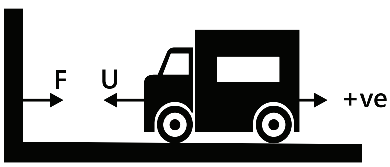


Figure: 4.19

Sol: Force on the roof of the car is equal to the momentum imparted to it per second by rain drops.

$$\left(\sum \vec{F}\right)\Delta t = m\overrightarrow{v_f} - m\overrightarrow{v_0}\,; \quad \vec{F}\Delta t = m\overrightarrow{v_f} - m\overrightarrow{v_0} \Rightarrow \vec{F} = -\left(\frac{m}{\Delta t}\right)\overrightarrow{v_0}$$

$\vec{F} = -(0.060\,kg/s)(-15\,m/s) = +0.90\ N$ [Hint: Third law of motion]

Illustration 9: A bullet of mass strikes an obstacle and moves at to its original direction. If its speed also changes from 20 m/s to 10 m/s, then find the magnitude of impulse acting on the bullet. **(JEE ADVANCED)**

Sol: Find the impulse along the initial line of motion and along the perpendicular to the initial line of motion.

Mass of the bullet, $m = 10^{-3}$kg

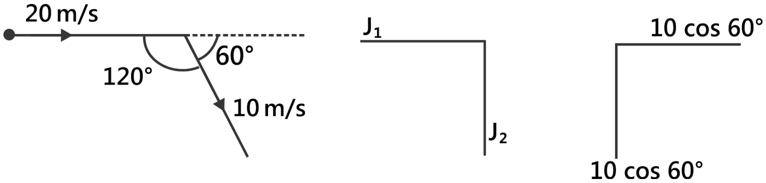


Figure: 4.20

Consider components parallel to J_1 $J_1 = 10^{-3}\left[-10\cos 60° - (-20)\right]$ $J_1 = 15 \times 10^{-3}$ N.s

Put 8 & 8.1 alter a and adjust the numbering

Similarly, parallel to J_2, we have $J_2 = 10^{-3}\left[10\sin 60° - 0\right] = 5\sqrt{3} \times 10^{-3}$ N.s

The magnitude of the resultant impulse is given by

$J = \sqrt{J_1^2 + J_2^2} = 10^{-3}\sqrt{(15)^2 + (5\sqrt{3})^2}$ or $J = \sqrt{3} \times 10^{-2}$N.s

10. APPLICATION OF LAWS OF MOTION

10.1 Two Blocks in Contact

If two blocks of masses m_1 and m_2 are in contact on a horizontal frictionless surface, so that a force F applied horizontally imparts an acceleration a and F_c is the contact force, which is equal and opposite for and. , then Newton's second law, when applied to free body diagram, gives the following equations:

$F - F_c = m_1 a\,; \quad F_c = m_2 a$

Adding $a = \dfrac{F}{m_1 + m_2}$

$\therefore \quad Fc = \dfrac{m_2 F}{m_1 + m_2}$

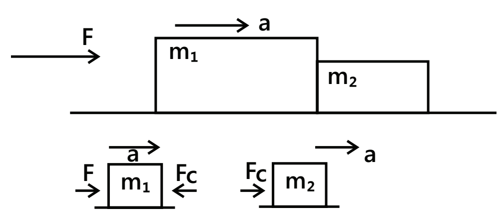


Figure: 4.21

10.2 Blocks Connected by Strings

If two blocks of masses and are connected by an inextensible string so that if force F is applied to and there is an equal and opposite tension T in the string and if a is acceleration of the masses, then Newton's law gives

$F - T = m_1 a\,;\ T = m_2 a$ Adding $F = (m_1 + m_2)a$ or $a = \dfrac{F}{m_1 + m_2}\,; \quad T = \dfrac{m_2 F}{m_1 + m_2}$

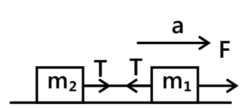


Figure: 4.22

If three blocks of masses, , and are connected by two strings with tension and when a force F applied to imparts an acceleration a to all the blocks, then Newton's law gives the following relations for these three blocks: $F - T_1 = m_1 a\,; \quad T_1 - T_2 = m_2 a \ T_2 = m_3 a$

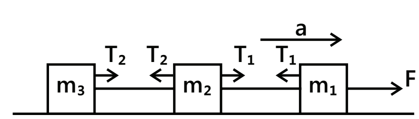


Figure: 4.23

Adding $F = (m_1 + m_2 + m_3)a$ or $a = \dfrac{F}{m_1 + m_2 + m_3}$

$$T_1 = \frac{(m_2 + m_3)F}{m_1 + m_2 + m_3};\quad T_2 = \frac{m_3 F}{m_1 + m_2 + m_3}$$

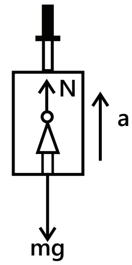


Figure: 4.24

10.3 Apparent Weight in a Lift

If a person is standing in a stationary lift, his or her weight mg acts downward and his or her normal reaction on the floor of the lift acts upward so that N = mg as per Newton's third law of motion. However, if the lift is moving with a constant velocity, then N is equal and opposite to mg as now the net force is zero.

$$N - mg = 0 \quad \text{or} \quad N = mg$$

Thus, the apparent weight is equal to true weight.

Further, if the lift is moving upward with acceleration a, $N - mg = ma$ or $N = m(g + a)$

Thus, the apparent weight is greater than the actual weight.

However, if the lift is accelerating downward with a′, $mg - N = ma'$ or $N = m(g - a')$

Therefore, in this case, the apparent weight is lesser than the actual weight.

10.4 Horse Cart Problem

To analyze this properly, it is probably best to individually consider the cart and the horse. The cardinal rules while dealing with introductory physics courses are first identify and isolate the body that you intend to apply Newton's second law to, and then identify all forces acting on that body and add them (as vectors) to get the net force, and finally, use the relation $F_{net} = ma$.

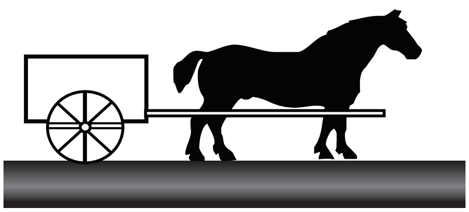
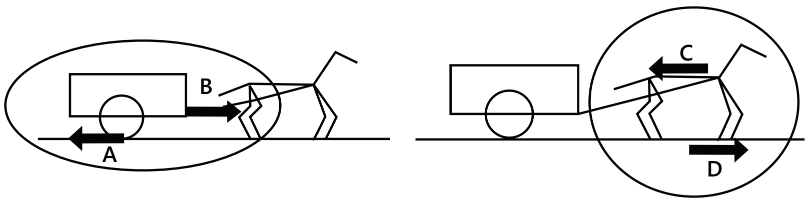


Figure: 4.25

In the diagrams provided, we have used an oval or a circle to enclose the subsystem being analyzed. The forces acting on the cart include the forward force that the horse exerts on the cart and the backward force due to friction at the ground, acting on the wheels. However, at rest, or at constant velocity, these two are equal in magnitude, because the acceleration of the cart is zero.

On the contrary, the forces that are acting on the horse include the backward force the cart exerts on the horse and the forward force of the ground on its hooves. However, at rest, or at constant velocity, these two are equal in magnitude, because the acceleration of the horse is zero. Therefore, $\vec{C} = -\vec{D}$. Similarly, for the cart, $\vec{A} = -\vec{B}$.

By Newton's third law, the force the horse exerts on the cart is of equal size and opposite in direction to the force the cart exerts on the horse. (These two forces are an action–reaction pair.) Therefore, $\vec{B} = -\vec{C}$, and this is true whether or not anything is accelerating.



Forces on the cart, no acceleration.

Forces on the horse, no acceleration.

Figure: 4.26

Since the horse is not accelerating, $\vec{C} = -\vec{D}$, by Newton's second law, and, finally we see that all the forces shown in the diagram are of the same size.

Illustration 10: Two blocks of masses and are placed in contact with each other on a frictionless horizontal surface as shown in the Figure 4.27. Constant forces and are applied on and as shown in the Figure 4.27. Find the magnitude of acceleration of the system. Also, calculate the contact force between the blocks. **(JEE MAIN)**

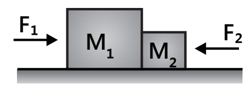


Figure: 4.27

Sol: Draw the FBD of each block. Apply Newton's first law along the vertical and Newton's second law along the horizontal.

In this problem, acceleration of both blocks will be the same as they are rigid and in contact with each other. As the surfaces are frictionless, contact force on any surface will be normal force only. Let us assume that the acceleration of blocks be a and contact forces, and N as shown in free body diagrams of blocks.

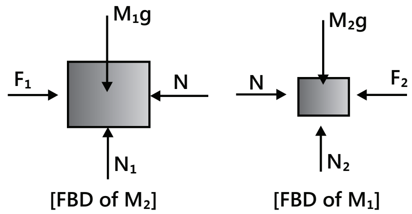


Figure: 4.28

Therefore, by applying, Newton's second law for

$F_1 - N = M_1 a$ ...... (i) and $M_1 g - N_1 = 0$ ....(ii)

Applying, Newton's second law for

$N - F_1 = M_2 a$ ...... (iii) and $M_2 g - N_2 = 0$ ....(iv)

By solving (i) and (iii) $a = \dfrac{F_1 - F_2}{M_1 + M_2}$ and $N = \dfrac{M_2 F_1 + M_1 F_2}{M_1 + M_2}$

Illustration 11: A rope of length L is pulled by a constant force F. What is the tension in the rope at a distance x from one end where the force is applied? **(JEEMAIN)**

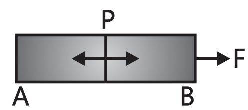


Figure: 4.29

Sol: Acceleration of all parts of the rope will be same. Net force on a part of rope is equal to acceleration multiplied by the mass of that part.

Let AB be a string of length L and F the constant force pulling the rope as shown in the Figure 4.29 provided.

Mass per unit length of rope $= \dfrac{M}{L}$

where M is the total mass. Let P be a point at a distance x from B. If T is the tension in the rope at P then for the part AP, the tension is toward right while for the part PB it is toward left. If a is the acceleration produced in the rope, then for part PB

$F - T = \text{mass of PB} \times a = \dfrac{Mx}{L} a \cdot$ Also for rope, $F = Ma$ $\quad \therefore\ T = \dfrac{F(L-x)}{L}$

Illustration 12: Two blocks each having mass of 20 kg rest on frictionless surfaces as are shown in the Figure 4.30. Assume that the pulleys to be light and frictionless. Now, find:

(a) The time required for the block A to move 1 m down the plane, starting from rest;

(b) The tension in the cord connecting the blocks. $\sin\theta = 3/5$ **(JEE ADVANCED)**

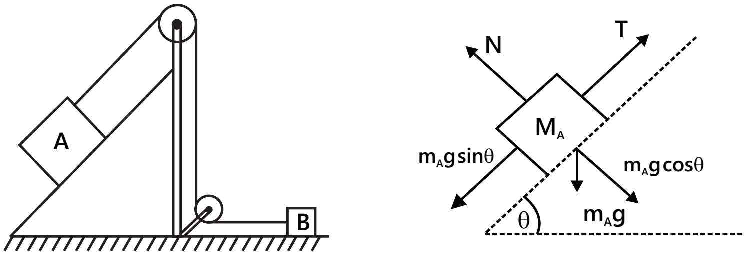


Figure: 4.30

Sol: Draw the FBD of each block. Apply Newton's second law along the direction of motion for each block. Solve the equations obtained to get the values for two variables T and a.

Blocks A and B are considered as two systems. The free body diagrams for the blocks A and B are shown in the Figure 4.31 where T is tension in the string.

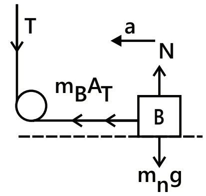


Figure: 4.31

$m_A g\sin\theta - T = m_A a$ ....(i)

$N = m_A g\cos\theta$ ....(ii)

$T = m_B a$ ....(iii)

Adding equation (i) and (iii), $m_A g\sin\theta = (m_A + m_B)a$

$$\Rightarrow a = \left(\frac{m_A}{m_A + m_B}\right) g\sin\theta = \left(\frac{20}{20+20}\right)(10)\left(\frac{3}{5}\right) = 3m/s^2$$

$$\text{(a) } s = \frac{1}{2}at^2; t = \left(\frac{2s}{a}\right)^{\frac{1}{2}} = \left(2\times\frac{1}{3}\right)^{\frac{1}{2}} = 0.82$$

(b) $T = m_B a = 20 \times 3 = 60N.$

11. LAW OF CONSERVATION OF LINEAR MOMENTUM

The law of conservation of linear momentum states that if no external forces act on the system of objects, then the vector sum of the linear momentum of each body remains constant and is not affected by their mutual interaction.

By applying the principle of conservation of linear momentum

(a) Decide which objects are included in the system.

(b) Identify the internal and external forces relative to the system.

(c) Verify that the system is isolated.

(d) Set the final momentum of the system equal to its initial momentum. Remember that momentum is a vector.

Illustration 13: From a gun (mass = M) a bullet (mass = m) is fired with speed relative to barrel of the gun which is inclined at an angle of with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of the gun. **(JEE ADVANCED)**

Sol: Apply the law of conservation of linear momentum along the horizontal direction.

Let the recoil speed of the gun is v. By taking gun + bullet as the system, the net external force on the system in horizontal direction is zero. Initially, the system was at rest. Therefore, applying the principle of the conservation of linear momentum in horizontal direction,

we get $Mv - m(v_r\cos 60° - v) = 0$

$$v = \frac{mv_r\cos 60°}{M+m} \text{ or } v = \frac{mv_r}{2(M+m)}$$

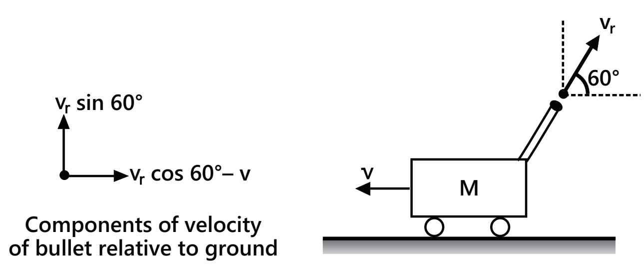


Figure: 4.32

Illustration 14: A man of mass is standing on a platform of mass kept on a smooth horizontal surface. Now, the man starts moving on the platform with velocity relative to the platform. Based on the above, find the recoil velocity of the platform. **(JEE MAIN)**

Sol: Apply the law of conservation of linear momentum along the horizontal direction.

Absolute velocity of the man = – v where v = recoil velocity of the platform. By considering together the platform and the man as a system, the net external force acting on the system in horizontal direction is zero. However, the linear momentum of the system remains

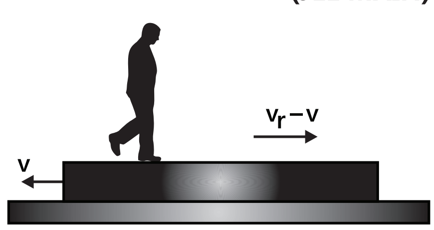


Figure: 4.33

constant. Initially, both the man and the platform were at rest.

Hence, $0 = m_1(v_r - v) - m_2 v \qquad \therefore v = \frac{m_1 v_r}{m_1 + m_2}$

12. VARIABLE MASS

Problems related to variable mass can be solved in the following three steps

(a) Make a list of all the forces acting on the main mass and then apply them on it.

(b) Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by $\vec{v}_r$, in case the mass is increasing otherwise the direction of $-\vec{v}_r$, if it is decreasing.

(c) Find the net force on the mass and then apply $\vec{F}_{net} = m\frac{d\vec{v}}{dt}$ (m = mass at that particular instant)

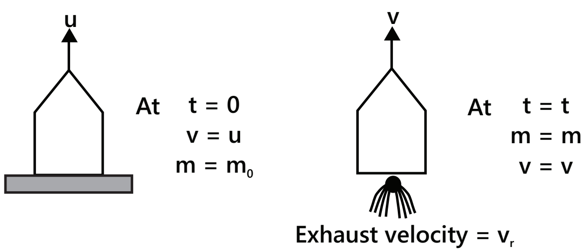


Figure: 4.34

Let m_0 be the mass of the rocket at time t = 0. Let m be its mass at any time t and v its velocity at that moment. Initially, let us suppose that velocity of the rocket is u.

Further, let $\left(-\frac{dm}{dt}\right)$ be the mass of gas ejected per unit time and, the exhaust velocity of the gases. Usually $\left(-\frac{dm}{dt}\right)$ and, are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time t = t,

(a) Thrust force on the rocket $\quad F_t = v_r\left(-\frac{dm}{dt}\right)$ (upwards)

(b) Weight of the rocket $\quad W = mg$ (downwards)

(c) Net force on the rocket $\quad F_{net} = F_t - W$ (upwards)

or $\quad F_{net} = v_r\left(-\frac{dm}{dt}\right) - mg$

(d) Net acceleration of the rocket $\quad a = \frac{F}{m}$ or $\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$ or $dv = v_r\left(\frac{-dm}{m}\right) - gdt$

or $\int_u^v dv = v_r\int_{m_0}^{m}\frac{-dm}{m} - g\int_0^t dt$ or $v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$

Thus, $\quad v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$ ... (i)

Note: $F_t = V_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.

If gravity is ignored and initial velocity of the rocket u = 0, eq(i) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$

Illustration 15: (a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2 km/s. What should be its minimum rate of fuel consumption?

(a) (i) To just lifting it off from the launching pad?

(ii) To give it an acceleration of 20 m/?

(b) What will be the speed of the rocket when the rate of consumption of the fuel is 10 kg/s after whole of the fuel is consumed? (take g = 9.8 m/). **(JEE ADVANCED)**

Sol: Use the equation of motion for variable mass.

(a) (i) to just lift it off from the launching pad

Weight = thrust force or $mg = v_r\left(\frac{-dm}{dt}\right)$ or $\left(\frac{-dm}{dt}\right) = \frac{mg}{v_r}$

Substituting the value, we get $\left(\frac{-dm}{dt}\right) = \frac{(450+50)(9.8)}{2\times10^3} = 2.45\ kg/s$

(ii) net acceleration $a = 20 m/s^2$

$ma = F_t - mg$ or $a = \frac{F_t}{m} - g$ or $a = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$

This gives $\left(\frac{-dm}{dt}\right) = \frac{m(g+a)}{v_r}$

Substituting the values, we get $\left(\frac{-dm}{dt}\right) = \frac{(450+50)(9.8+20)}{2\times10^3} = 7.45 kg/s.$

(b) The rate of fuel consumption is 10 kg/s. So, the time for the consumption of entire fuel is t = 450/10 = 45 s

Using Eq. (i), i.e., $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Here $u = 0, v_r = 2\times10^3 m/s, m_0 = 500 kg$ and $m = 50 kg$

Substituting the values, we get $v = 0 - (9.8)(45) + (2\times10^3)\ln\left(\frac{500}{50}\right)$

Other Example of Variable Mass System is Falling raindrop

Illustration 16: Suppose that a raindrop falls through a cloud and accumulates mass at a rate of kmv where k > 0 is a constant, m is the mass of the raindrop, and v its velocity. What is the speed of the raindrop at a given time if it starts from rest, and what is its mass? **(JEE ADVANCED)**

Sol: Use the equation of motion for variable mass.

Then, the external force is its weight mg and so we have

$$mg = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = m\frac{dv}{dt} + kmv^2$$

Since we know that dm/dt = kmv. Cancelling the mass and rearranging $\frac{dv}{dt} = g - kv^2$,

So that, $\int_0^v \frac{dv}{g-kv^2} = \int_0^t dt = t$

Now set $V^2 = g/k$ and use partial fractions to get

$$t = \int_0^v \frac{dv}{g-kv^2} = \frac{1}{2kV}\int_0^v \frac{1}{V+v} + \frac{1}{V-v}dv = \frac{1}{2kV}\log\left(\frac{V+v}{V-v}\right)$$

so, $V+v=(V-v)e^{2kVt}$, i.e. $v=V\left(\dfrac{e^{2kVt}-1}{e^{2kVt}+1}\right)=V\tanh(Vkt)$, so that $v=\sqrt{\dfrac{g}{k}}\tanh(\sqrt{kgt})$

Now we may find the mass : we have $\dfrac{dm}{dt}=kmv=km\sqrt{\dfrac{g}{k}}\tanh(\sqrt{kgt})=m\sqrt{kg}\tanh(\sqrt{kgt})$.

Thus, $\displaystyle\int_0^t \frac{1}{m}\frac{dm}{dt}dt=\int_0^t\sqrt{kg}\tanh(\sqrt{kgt})dt$

$$\int_{m_0}^{m}\frac{dm}{m}=\int_0^t\sqrt{kg}\tanh(\sqrt{kgt})dt$$

$$\log m-\log m_0=\log\cosh(\sqrt{kgt})$$

which gives $m=m_0\cosh(\sqrt{kgt})$

13. EQUILIBRIUM

Equilibrium is the condition of a system, when net external farce is zero.

13.1 Equilibrium of Concurrent Forces

A simple mechanical body is said to be in equilibrium, if it does not experience any linear acceleration; however, unless it is disturbed by an outside force, it will continue in that condition indefinitely. For a body facing concurrent forces, equilibrium arises if the vector sum of all forces acting upon the body is zero. There are two types of equilibrium as listed hereunder:

(a) Static equilibrium: When a body is at rest under the influence of external forces acting on the it.

(b) Dynamic equilibrium: If net external force is zero but the velocity of a body is not zero, i.e., body moves with a constant velocity.

13.2 Constrained Motion

When a motion of a body can be controlled, then the motion is said to be a constrained motion. For example, when a body tied with a string is lowered under the effect of gravity, then its motion is a constrained motion. Also, motion of masses and can be controlled by choosing an appropriate value for and.

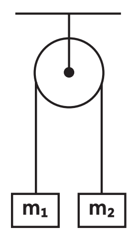


Figure: 4.35

13.2.1 Masses Connected by Pulley and Constraint Relation

Let us consider blocks of masses and connected by a string and passing over the pulley as shown in the Figure 4.36. Let be the acceleration of downward and be the acceleration of upward. Let T is the tension in the string, so that the pulley moves clockwise. For block,

As there are three unknown parameters, we take the following steps for writing the constraint relation and hence find the parameters:

(a) Assume direction of acceleration of each body.

(b) Locate position of each block from any fixed point like, for example, center of the pulley.

(c) Identify the constraint and write the equation of constraint in terms of distance.

(d) Write the equation of constraint and hence differentiate twice to find one of the parameters.

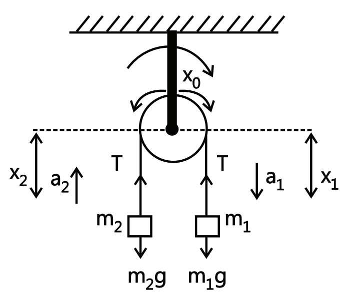


Figure: 4.36

In this case, the string is inextensible; therefore, the constraint the length of string remains constant. If is the length of the string passing over the pulley, and lengths of string from the pulley to and respectively, then the total length L of the string remains constant.

$\therefore x_1 + x_2 + x_0 = L = \text{constant}$

Differentiate, $\frac{dx_1}{dt} + \frac{dx_2}{dt} = 0$; Differentiate, $\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = 0$

As and have opposite directions,

$a_1 - a_2 = 0$; $a_1 = a_2 = a$ $\therefore m_1g - T = m_1a$; $T - m_2g = m_2a$

Adding $a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$; $T = m_2a + m_2g = \frac{m_2(m_1 - m_2)g}{m_1 + m_2} + m_2g$; $T = \left(\frac{2m_1m_2}{m_1 + m_2}\right)g$

If the pulley is pulled in upward direction with an acceleration a, then $T = \left(\frac{2m_1m_2}{m_1 + m2}\right)(g + a)$

Illustration 17: Find the relationship between accelerations of blocks A and B based on the Figure 4.37. **(JEE MAIN)**

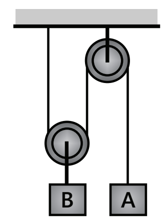


Figure: 4.37

Sol: Measure all distances of pulley and block from a fixed point (stationary point).

The physical property that we can use here is the inextensibility of string,

i.e., ab + bc + cd + de + ef = constant.

Let at any moment A and B are distances and from the support as shown in the Fig. 4.37.

Let us take gh= and ik= and hence express the length of string in equation (i) in terms of, l_1 and l_2.

We hence obtain $X_B - l_1 + bc + (X_B - l_1 - l_2) + de + (X_A - l_2) = \text{constant}$ Here,, bc and de are constants.

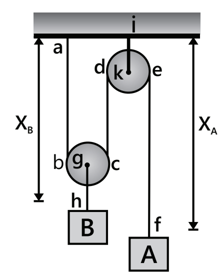


Figure: 4.38

$\therefore 2X_B + X_A = \text{constant}$ ... (i)

Let at time Δt, changes to + Δ and changes to - Δ

[therefore, B is assumed to move downward]

Then, $2(X_B + \Delta X_B) + (X_A - \Delta X_B) = \text{constant}$ ... (ii)

From (i) and (ii) $2\Delta X_B - \Delta X_A = 0$

Also, $\left(\frac{2\Delta X_B}{\Delta t}\right) - \left(\frac{\Delta X_A}{\Delta t}\right) = 0$; $2V_B - V_A = 0$ also, $2\Delta V_B - \Delta V_A = 0$

$\frac{2\Delta V_B}{\Delta t} - \frac{\Delta V_A}{\Delta t} = 0$ $\therefore 2a_B = a_A$

Hence, we prove that magnitude of acceleration of A is twice the magnitude of acceleration of B.

Let us assume that B moves by a distance x during an interval of time, and this will cause movement of pulley g by x. Now, an extra length of 2x of string will come to the left of pulley k. This must be coming from the right side of the pulleys. Hence, displacement of A will be 2x. On the basis of this discussion, we can say that if acceleration of block B is a, then the acceleration of A will be 2a.

14 PSEUDO FORCE

4.1 Inertial and Non-inertial Frames of References

Non-accelerated frames of reference are called inertial frames, whereas accelerated frames are called non-inertial frames. If one is travelling in a train which is accelerating forward, the body in the train is pushed backward and he or she is pushed forward when the brakes are applied. This is due to inertia of the body. Such an accelerated frame is called a non-inertial frame.

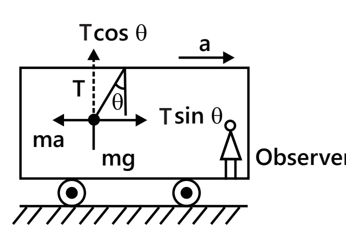


Figure: 4.39

In order to make Newton's laws applicable to such a frame, a fictitious force or pseudo force is applied on the body. Based on the above discussion, we now understand that the magnitude of this pseudo force is equal to the product of the mass m of the body and acceleration a of the reference frame and its direction is opposite to the acceleration of the frame.

$\therefore$ pseudo force, $F = -m \times a$. Thus in a non-inertial frame trolley moving with an acceleration a hanging bob of mass m will be deflected through an angle due to a pseudo force acting in backward direction. In the non-inertial frame of reference, this bob is in equilibrium under the action of force due to tension T, weight mg and the pseudo force ma in a direction making an angle with the vertical.

$T\sin\theta = ma$; $T\cos\theta = mg$; $\tan\theta = \frac{a}{g}$ or $\theta = \tan^{-1}\left(\frac{a}{g}\right)$

Example: Motion of a block on an inclined plane is an example of accelerated frame of motion.

Motion of a Block on a Smooth Inclined Plane: Let us consider a block of mass m placed on a frictionless inclined plane, inclined at an angle to the horizontal. We observe that the normal reaction N acts perpendicular to the plane and its weight is resolved into component mg sin along the plane which slides the block downward with acceleration a and component mg cos perpendicular to the plane downward, which is equal and opposite to the normal reaction.

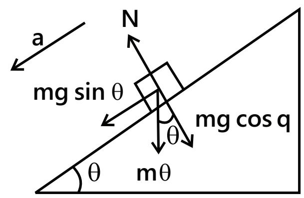


Figure: 4.40

$\therefore mg\sin\theta = ma$ $\quad mg\cos\theta = N$ $\quad$ or $a = g\sin\theta$ However, if the plane is provided with a horizontal acceleration a′ in the horizontal direction as shown in the Figure 4.44, then the body lies in an accelerating frame of reference and a pseudo force ma′ acts horizontally in a direction opposite to that of a′ because an inertial force ma′ acts on it in the direction of a′. Thus ma′ can be resolved into a component ma′cos up the plane and ma′sinperpendicular to the plane in the downward direction as shown in the Figure 4.44. From Newton's second law of motion, we know that: $mg\sin\theta - ma'\cos\theta = ma$ or $a = g\sin\theta - a'\cos\theta$

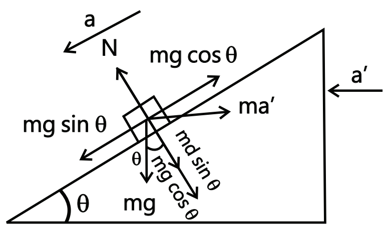


Figure: 4.41

$N = m(g\cos\theta + a'\sin\theta)$. If the body is at rest relative to the inclined plane, then a=0 $\quad$ or $g\sin\theta = a'\cos\theta$ or $a' = g\tan\theta$

Illustration 18: A frictionless block 3 carries two other frictionless blocks 1 and 2 connected by a light string passing over a weightless and frictionless pulley as shown in the Figure 4.42. What horizontal force must continuously be applied to block 3 so that 1 and 2 do not move relative to 3? **(JEE ADVANCED)**

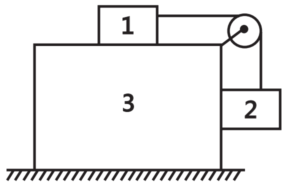


Figure: 4.42

Sol: Analyze the motion of blocks 1 and 2 in the reference frame of block 3. As block 3 is accelerated, blocks 1 and 2 experience pseudo forces in the frame of block 3.

Let a_1, a_2 and a_3 be the accelerations of 1, 2, and 3, respectively. Let a_{1x}, a_{2x} and a_{3x} be the absolute horizontal acceleration of 1, 2 and 3 to the right and a_{1y}, a_{2y} and a_{3y} be their downward accelerations. According to the constraints of the problem $a_{1y} = 0$, $a_{3y} = 0$

Let a_{13x} = relative acceleration of 1 w.r.t. 3. $= a_{1x} - a_{3x} = 0 \Rightarrow a_{3x} = a_{1x}$

a_{23y} = relative acceleration of 2 w.r.t 3. $= a_{2y} - a_{3y} = a_{2y} - 0 \Rightarrow a_{2y} = 0$

a_{23x} = relative acceleration of 2 w.r.t. 3. $= a_{2x} - a_{3x} = 0 \Rightarrow a_{2x} = a_{3x}$

$\therefore a_{1x} = a_{2x} = a_{3x}$ and $a_{1y} = a_{2y} = a_{3y} = 0$

Consider the free-body diagrams of blocks 1 to 3.

From the FBD, we obtain the following equations.

$$\left.\begin{aligned} T &= m_1 a_{1x} \quad \text{........(i)} \\ N - m_1 g &= 0 \quad \text{.......(ii)} \end{aligned}\right\} \text{For Block I}$$

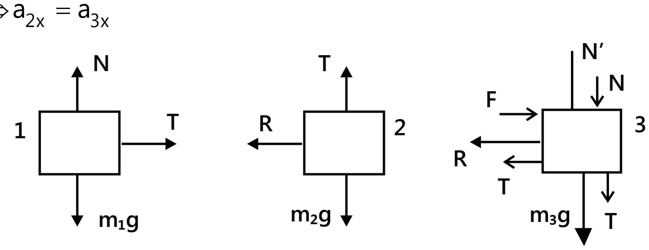


Figure: 4.43

$$\left.\begin{array}{ll} R = m_2 a_{2x} & \text{........(iii)} \\ T - m_2 g = 0 & \text{.......(iv)} \end{array}\right\} \text{For Block II}$$

$$\left.\begin{array}{ll} F - R - T = m_3 a_{3x} & \text{.......(v)} \\ N' - N - T - m_3 g = 0 & \text{........(vi)} \end{array}\right\} \text{For Block III}$$

Adding (i), (iii) and (v)

$$F = m_1 a_{1x} + m_2 a_{2x} + m_3 a_{3x} = a_{1x}(m_1 + m_2 + m_3) \quad (\because a_{1x} = a_{2x} = a_{3x})$$

From (i) and (iv), $a_{1x} = \frac{m_2}{m_1}.g$; $F = (m_1 + m_2 + m_3).\frac{m_2}{m_1}.g$

15. FRICTION

If there are two bodies in contact with each other, then the force which opposes the relative motion between two bodies in contact is called force of friction. Further, the magnitude of the frictional force depends upon the nature of two surfaces in contact. This is primarily due to surface irregularities at molecular levels, with the result that even a highly polished surface has irregularities. This results in producing interlocking of uneven surfaces. Once there is smooth motion of the body, the friction is less than the maximum force of static friction or limiting friction.The variation of force of friction with the applied force is shown in the graph when any block is moving over another surface. However, when any block is at rest, the resultant force of static friction is equal to the force applied. Then, it reaches to a maximum value at A, the limiting friction. Once the motion resumes, a lesser force is required for maintaining uniform motion.

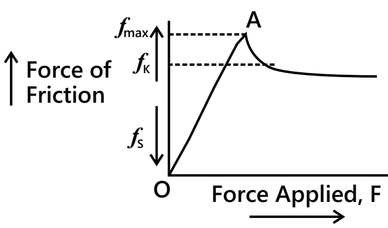


Figure: 4.44

5.1 Static Friction

Suppose that if the force applied in the horizontal direction on a surface is less, then the body does not have any motion because an equal and opposite frictional force is present. Hence, it is clear that static friction is in operation only between surfaces that are at rest with respect to each other. As F is increased the frictional force too increases continuously until a stage is reached when the body is just at the point of sliding. The force of friction at this stage is called a limiting friction or maximum static friction.

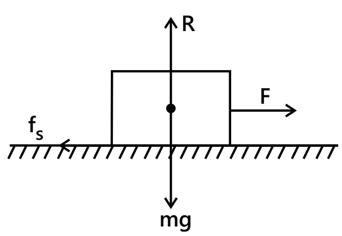


Figure: 4.45

15.2 Coefficient of Friction

The coefficient of friction, generally specified by the Greek letter mu (μ), is the ratio of limiting or maximum value of force of friction to reaction R between surfaces when the body is just about to move. $\mu = \frac{f_{max}}{R}$; $f_{max} = f_{limiting} = \mu_s R$

Thus if the body is not in motion, the static frictional force and external applied force parallel to the surface are equal in magnitude but opposite in direction and hence F is directly proportional to . However, if external force F exceeds, then the body slides on the surface and magnitude of frictional force decreases than, the frictional force for sliding = = R where is the coefficient of kinetic friction. As, is less than, the coefficient of kinetic friction is less than the coefficient of static friction.

15.3 Angle of Friction

The angle between the normal reaction R and the resultant of limiting friction with normal reaction is called the angle of friction and is denoted by λ.

$\tan\lambda = \frac{f_{max}}{R} = \mu$ or $\lambda = \tan^{-1}(\mu)$ Suppose that if a body of mass m is placed on an inclined

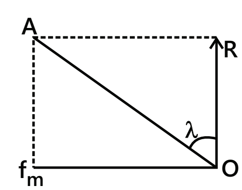


Figure: 4.46

plane whose inclination is gradually increasing. Then the body just starts sliding down at a certain angle of inclination θ. Now, the weight mg can be resolved into a component mg sin due to which the body is about to slide down against maximum or limiting value of friction and therefore the second component mg cos balances the normal reaction R perpendicular to the inclined plane.

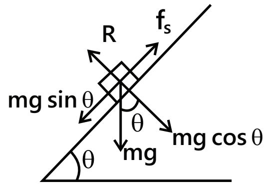


Figure: 4.47

$\therefore f_s = mg\sin\theta$; $R = mg\cos\theta$; $\therefore \frac{f_s}{R} = \tan\theta = \mu$ as $\mu = \tan\lambda$; $\theta = \lambda = \tan^{-1}(\mu)$

Here, the angle θ is called angle of inclination. We now know that the angle of friction λ is that minimum angle of inclination of the inclined plane at which a body placed at rest on the inclined plane just starts sliding down.

However, when θ < λ, then the body is in equilibrium and does not slide. On the contrary, when θ > λ, then the body starts sliding down with an acceleration.

15.4 Motion of a Block on a Rough Inclined Plane

Let us assume that a block of mass m is moving down an inclined plane with an acceleration a. Now, the coefficient of friction between the block and inclined plane equal to μ, the force of friction μN will be acting along the plane upward as shown in the Figure 4.48. Thus, the weight mg of the block is resolved into component mg cosθ opposite of normal reaction and component mg sin downward opposite to μN.

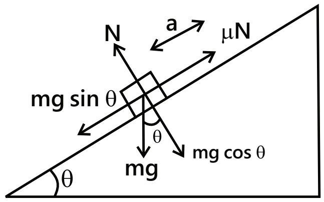


Figure: 4.48

Thus, from Newton's second law of motion, mg sin θ – μN = ma

N = mg cosθ or mg sinθ - μ mg cosθ = ma $\therefore$ a = g(sinθ – μcos θ)

However, if the block is moving upward and its retardation is a, where the frictional force acts downward, then

$ma = mg\sin\theta + \mu mg\cos\theta$ $\therefore a = g\sin\theta + \mu g\cos\theta$.

MASTERJEE CONCEPTS

Value of friction is not always equal to μN. Further, μN is the maximum value of friction. Friction does not oppose motion; rather, it opposes relative motion between two surfaces.

Anand K (JEE 2011, AIR 47)

Illustration 19: A heavy box of mass 20 kg is pulled on a horizontal surface by applying a horizontal force. If the coefficient of kinetic friction between the box and the horizontal surface is 0.25, then find the force of friction exerted by the horizontal surface on the box. **(JEE MAIN)**

Sol: Force of friction on a body sliding on a surface is equal to the normal reaction multiplied by the coefficient of kinetic friction between the pair of surfaces.

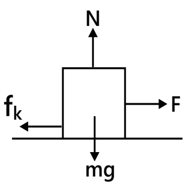


Figure: 4.49

The situation is shown in the Figure 4.49. In the vertical direction, there is no acceleration; therefore, N=mg.

As the box slides on the horizontal surface, the surface exerts kinetic friction on the box. Therefore, the magnitude of the kinetic friction is $f_k = \mu_k N = \mu_k Mg$

$= 0.25 \times (20\text{kg}) \times (9.8\text{m/s}^2) = 49\text{N}$. This force thus acts in the direction opposite to that of the pull.

Illustration 20: Two blocks, M_1 and M_2, connected by a massless string slide down an inclined plane, having an angle of inclination of. The masses of the two blocks are = 4 kg and = 2 kg, respectively and the coefficients of friction of and with inclined plane are 0.75 and 0.25, respectively. Assuming the string to be taut, find

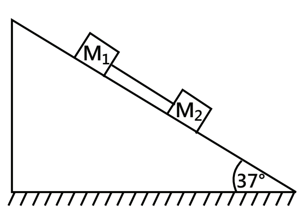


Figure: 4.50

(a) The common acceleration of the two masses and

(b) The tension in the string. (note: sin =0.6, cos =0.8) **(JEE ADVANCED)**

Sol: Let each block is having acceleration a down the incline plane. Draw the FBD of each block and apply the Newton's second law of motion along the direction of motion. Solve the equations obtained to get the value of two variables a and T.

Let a be the common acceleration of the system and T be the tension in the string $\mu_1 = 3/4,\ \mu_2 = 1/4$

Equation of motion for M_1 and M_2 are

$M_1a = M_1g\sin37° + T - \mu_1M_1g\cos37°$ ......(i)

$M_2a = M_2g\sin37° - T - \mu_2M_2g\cos37°$ .....(ii)

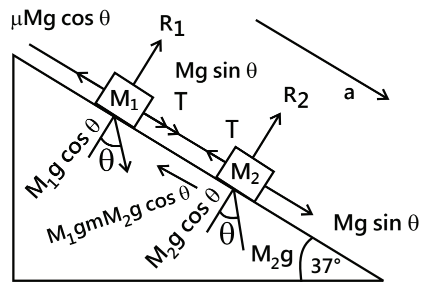


Figure: 4.51

Now, by adding, (i) and (ii)

$(M_1+M_2)a = (M_1+M_2)g\sin37° - (\mu_1 M_1 + \mu_2 M_2)g\cos37°$

$\therefore (4+2)a = (4+2)g\times(0.6) - \left(4\times\frac{3}{4}+\frac{1}{4}\times2\right)g\times0.8$

$6a = g[3.6-2.8]$ or $a = \frac{9.8\times0.8}{6} = \frac{7.64}{6} = 1.27\text{m}/\sec^2$

From (ii) $T = M_2g\sin37° - \mu_2M_2g\cos37° - M_2a = 2\times10\times\frac{6}{10} - \frac{1}{4}\times2\times10\times\frac{8}{10} - 2\times1.27 = 12-4-2.54 = 5.46$ Newtons

16. CIRCULAR DYNAMICS

The motion of a particle particularly along a circular path is called its circular motion and it can be uniform, with constant angular rate (and constant speed), or non-uniform with a changing rate. In uniform circular motion, a resultant non-zero force is in operation on the particle. This is because a particle moving in a circular path is accelerated even if speed of the particle remains constant. This acceleration is due to change in direction of the velocity vector. As we have already seen that in uniform circular motion tangential acceleration (a_t) is zero, the acceleration of the particle is toward the center and its magnitude is $\frac{v^2}{r}$. Here, v is the speed of the particle and r is the radius of the circle. The direction of the resultant force F is, therefore, toward the center and its magnitude is

$F = ma$ or $F = \frac{mv^2}{r}$ Or $F = mr\omega^2$ (as $v = r\omega$)

Here, ω denotes the angular speed of the particle. The force F is called the centripetal force. Thus, a centripetal force of magnitude $\frac{mv^2}{r}$ is required to keep the particle moving in a circular path with constant speed. This force is generally provided by some external sources such as friction, magnetic force, Coulomb force, gravitation, tension, etc.

Illustration 21: Assume that a small block of mass 100 g moves with uniform speed in a horizontal circular groove, with vertical side walls, of radius 25 cm. If the block takes 2 s to complete one round, then find the normal contact force by the side wall of the groove. **(JEE MAIN)**

Sol: The normal contact force provides the necessary centripetal force to the block to move in a circle.

The speed of the block is $v = \frac{2\pi\times(25\text{ cm})}{2.0\text{s}} = 0.785\text{m}/\text{s}.$

The acceleration of the block is $a = \frac{v^2}{r} = \frac{(0.785\text{m}/\text{s})^2}{0.25\text{m}} = 2.5\text{m}/\text{s}^2$

However, toward the center, the only force in this direction is the normal contact force due to the side walls. Thus, from Newton's second law, this force is $N = ma = (0.100\text{kg})(2.5\text{m}/\text{s}^2) = 0.25\text{N}$

MASTERJEE CONCEPTS

I have found students often confused over the concept of centripetal force. They think that this force acts on a particle moving in a circle. This force does not act but required for moving in a circle which is being provided by the other forces acting on the particle. Let us take an example, Suppose a particle of mass 'm' is moving in a vertical circle with the help of a string of length l fixed at point O.

Let v be the speed of the particle at its lowest position. When I ask the students what forces are acting on the particle in this position, they immediately say, three forces are acting on the particle: (i) tension, T (ii) weight, mg and (iii) centripetal force, $\frac{mv^2}{I}$ (r=I). However, they are wrong. Only the first two forces T and mg are acting on the particle.

The third force $\frac{mv^2}{I}$ is required for circular motion which is being provided by T and mg. Thus, the resultant of these two forces is $v^2 = \mu rg$ or $v = \sqrt{\mu rg}$ toward O. Or we can write $T - mg = \frac{mv^2}{I}$

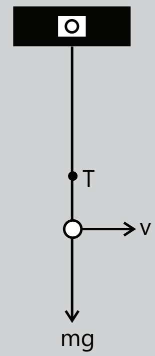


Figure: 4.52

Ankit Rathore (JEE Advanced 2013, AIR 158)

17. UNIFORM CIRCULAR MOTION

If a particle moves on a circular path with constant speed, its motion is called as a uniform circular motion. In this type of motion, angular speed of the particle is also constant. Further, linear acceleration in such motion will not have any tangential component; therefore, the particle possesses only radial or centripetal acceleration. Therefore in case of uniform circular motion the particle will have acceleration toward the center only and is called as centripetal acceleration having magnitude $\frac{v^2}{R}$ or $\omega^2 R$. However, the magnitude of acceleration remains constant but its direction changes with time. If a particle moving on circular path is observed from an inertial frame, then we know that it has an acceleration $\omega^2 R$ or $\frac{v^2}{R}$ acting toward the center. Therefore, from Newton's second law of motion, there must be a force acting on the particle toward the center of magnitude $m\omega^2 R$ or $\frac{mv^2}{R}$. This required force

for a particle to move on a circular path is called as centripetal force. $\therefore$ Centripetal force $= \frac{mv^2}{R}$

The term "centripetal force" merely signifies a force toward the center; however, it tells nothing about its nature or origin. Further, the centripetal force may be a single force due to a rope, a string, the force of gravity, friction, and so forth or it may be resultant of several forces.

Illustration 22: A ball of mass 0.5 kg is attached to the end of a cord whose length is 1.50 m. The ball is whirled in a horizontal circular path. If the cord can withstand a maximum tension of 50.0 N, what is the maximum speed the ball can have before the cord breaks? **(JEE MAIN)**

Sol: The tension force in the cord provides the necessary centripetal force to the ball to move in a circular path.

Because the centripetal force in this case is the force T exerted by the cord on the ball, we have $T = m\frac{v^2}{r}$; therefore, solving for v, we have $v = \sqrt{\frac{Tr}{m}}$

The maximum speed that the ball can have corresponds to the maximum tension. Hence, we find

$$v_{max} = \sqrt{\frac{T_{max} r}{m}} = \sqrt{\frac{(50.0N)(1.50m)}{0.500kg}} = 12.2 m/s$$

18. NON-UNIFORM CIRCULAR MOTION

If the speed of a particle moving in a circle is not constant, then the acceleration has both radial and tangential components. These radial and tangential accelerations are given as: $a_r = \omega^2 r = \frac{v^2}{r}$; $a_t = \frac{dv}{dt}$

Then, the magnitude of the resultant acceleration will be: $a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$

However, if the direction of the resultant acceleration makes an angle β with the radius, where $\tan\beta = \frac{dv/dt}{v^2/r}$

then, in the direction $\tan^{-1}\left(\frac{dv/dt}{v^2/r}\right)$ with radius of circle, acceleration needs a resultant force of $m\sqrt{\left(\frac{v^2}{R}\right)^2 + \left(\frac{dv}{dt}\right)^2}$

Illustration 23: A car moves on a horizontal circular road of radius R. The speed of the car is increasing at a rate $\frac{dv}{dt} = a$. The frictional coefficient between the road and tire is μ. Find the speed at which the car will skid.

(JEE ADVANCED)

Sol: The net acceleration of the car is the vector sum of the centripetal acceleration and the tangential acceleration. By Newton's second law the friction force on the car is (mass)$\times$(net acceleration).

Here, at any time t, the speed of the car becomes V; therefore, the net acceleration in the plane of the road is $\sqrt{\left(\frac{v^2}{R}\right)^2 + (a^2)}$. This acceleration is provided by the frictional force. At the moment, the car will slide if it reaches the speed as given by $M\sqrt{\left(\frac{v^2}{R}\right)^2 + (a^2)} = \mu Mg \Rightarrow v = \left[R^2(\mu^2 g^2 - a^2)\right]^{1/4}$

Illustration 24: A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in the Figure 4.53. The small mass m, which lies in the horizontal plane, moves around in a circular path. The length of the string from the mass m to the top of the tube is l and θ is the angle this length makes with the vertical. What should be the frequency of rotation of the small mass m so that the large mass M remains stationary? **(JEE MAIN)**

Sol: For the mass M to be stationary the tension in the string should balance the weight of M. For mass m the horizontal component of tension in the string provides the centripetal force. The vertical component of tension balances the weight of m.

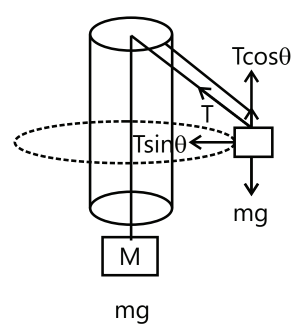


Figure: 4.53

The forces acting on the small mass m and the large mass M are shown in the Figure 4.56. When mass M is stationary, then T = Mg ... (i)

where T is tension in the string.

For the smaller mass, the vertical component of tension T cos θ balances mg and the horizontal component T sin θ supplies the necessary centripetal force.

$T\cos\theta = mg$ ... (ii)

$T\sin\theta = mr\omega^2$ ... (iii)

ω being the angular velocity and r is the radius of horizontal circular path.

From (i) and (iii), Mg sinθ = $mr\omega^2 \Rightarrow \omega = \sqrt{\frac{Mg\sin\theta}{mr}} = \sqrt{\frac{Mg\sin\theta}{ml\sin\theta}} = \sqrt{\frac{Mg}{ml}}$

Frequency of rotation $= \frac{1}{T'} = \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}$ $\quad\therefore$ Frequency $= \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$

19. CENTRIPETAL FORCE

An observer in a rotating system is another example of a non-inertial observer category. Suppose that a block of mass m lying on a horizontal frictionless turntable is connected to a string as shown in the Figure 4.54. Then, according to an inertial observer, if the block rotates uniformly it hence undergoes an acceleration of magnitude $\frac{v^2}{r}$ where v is the tangential speed. The inertial observer hence concludes that this centripetal acceleration is provided by the force exerted by the string T and writes as per Newton's second law $T = \frac{mv^2}{r}$.

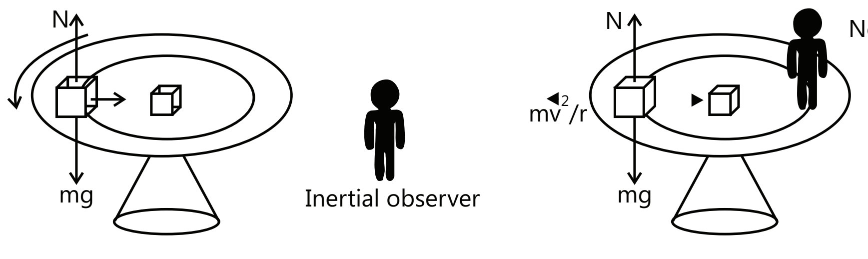


Figure: 4.54

However, according to a non-inertial observer attached to the turntable, the block is at rest. Therefore, by applying Newton's second law, this observer introduces a fictitious outward force of magnitude $\frac{mv^2}{r}$. According to the non-inertial observer, this outward force balances the force exerted by the string and therefore $T - \frac{mv^2}{r} = 0$. In fact, the centrifugal force is sufficient pseudo force only if we were analyzing the particles at rest in a uniformly rotating frame. In contrast, if we analyze the motion of a particle that moves in the rotating frame then we may have to assume other pseudo forces together with the centrifugal force, such forces are called Coriolis forces. The Coriolis force, named after the 19th century French engineer-mathematician, is perpendicular not only to the velocity of the particle but also to the axis of rotation of the frame. Once again we should be remembering that all these pseudo forces, centrifugal or Coriolis, are needed only if the reference frame is rotating. We must know that if we work from an inertial frame, then there is no need to apply any pseudo force. However, we should be aware of the fact that there should not be a misconception that centrifugal force acts on a particle because the particle describes a circle. Therefore, when we are working from a frame of reference that is rotating at a constant angular velocity ω w.r.t. an inertial frame, then we have to obviously assume that a force $m\omega^2 r$ acts radially outward on a particle of mass m kept at the distance r from the axis of rotation. Then only we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called centrifugal force. One should be careful when using fictitious forces to describe such physical phenomena. Remember that fictitious forces are used only in non-inertial frames of references. Therefore, when solving problems of this nature, it is often best to use an inertial frame.

Illustration 25: A table with smooth horizontal surface is fixed in a cabin that rotates with angular speed ω in a circular path of radius R. A smooth groove AB of length L(<<R) is made on the surface of table as shown in the Figure 4.55. A small particle is kept at the point A in the groove and is released to move. Find the time taken by the particle to reach the point B. **(JEE MAIN)**

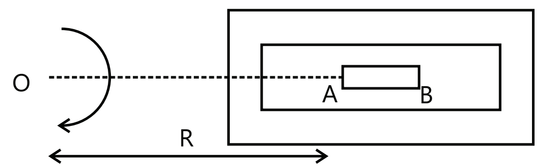


Figure: 4.55

Sol: In the reference frame of cabin the particle experiences centrifugal force in the radial direction. This force can be assumed constant for L<<R.

Let us analyze the motion of particle with respect to the table which is moving with cabin at an angular speed of ω. Now, along the smooth groove AB centrifugal force of magnitude $m\omega^2 R$ will act at A on the particle which can be treated as constant from A to B as L<<R.

$\therefore$ Acceleration of the particle along AB with respect to the cabin a = $\omega^2 R$ (constant)

Therefore, required time "t" is given by $s = ut + \frac{1}{2}at^2 \Rightarrow L = 0 + \frac{1}{2} \times \omega^2 R t^2 \Rightarrow t = \sqrt{\frac{2L}{\omega^2 R}}$

19.1 Applications of Centripetal Force

19.1.1 Circular Turning of Roads

When vehicles go through turnings, we observe that they travel along a nearly circular arc. Naturally, there must be some force which will produce the required centripetal acceleration. However, if the vehicles travel in a horizontal circular path, then this resultant force is also horizontal. The necessary centripetal force, which we are discussing about, is being provided to the vehicles by any one or combination of the following three ways:

(a) By friction only, **(b)** By banking of roads only, or **(c)** By both friction and banking of roads

In real life, the necessary centripetal force is provided by both friction and banking of roads. Now, let us write equations of motion in each of these three cases separately and find out what the constraints in each case are.

(a) **By Friction only:** Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, therefore, the necessary centripetal force to the car will be provided by force of friction, f, acting toward the center. Thus, $f = \frac{mv^2}{r}$ Further, limiting value of f is μN Or $f_L = \mu$; $N = \mu mg$ (N=mg)

Therefore, for a safe turn without sliding $\frac{mv^2}{r} \le f_L$ or $\frac{mv^2}{r} \le \mu mg$ or $\mu \ge \frac{v^2}{rg}$ or $v \le \sqrt{\mu rg}$

Here, however, two situations may arise. If μ and r are known to us, then the speed of the vehicle should not exceed and if v and r are known to us, then the coefficient of friction should be greater than $\frac{v^2}{rg}$.

MASTERJEE CONCEPTS

You might have seen that if the speed of a car is too high, the car starts skidding outward with the radius of the circle increased of the necessary centripetal force is reduced.

centripetal force $\propto \frac{1}{r}$

Anurag Saraf (JEE 2011, AIR 71)

(b) **By Banking of Roads only**: It is a common fact that friction is not always reliable particularly at circular turns when in high speeds and where sharp turns are also involved. To avoid dependence on friction, the roads are banked at the turn in such a way that the outer part of the road is somewhat lifted compared to the inner part.

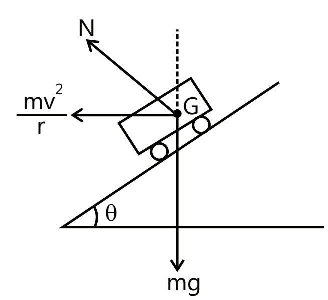


Figure: 4.56

Now, by applying Newton's second law along the radius and the first law in the vertical direction, we obtain $N \sin\theta = \frac{mv^2}{r}$ and $N \cos\theta = mg$

Thus, from the above two equations, we obtain

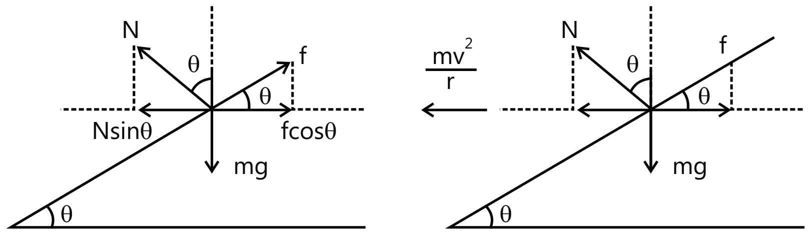
$\tan\theta = \frac{v^2}{rg}$ ... (i) or $v = \sqrt{rg\tan\theta}$ ... (ii)

MASTERJEE CONCEPTS

This is the speed at which a car does not slide down even if a track is smooth. If the track is smooth and speed is less than $\sqrt{rg\tan\theta}$, then the vehicle will move down so that r gets decreased and if speed is more than this, then the vehicle will move up.

Vijay Senapathi (JEE 2011, AIR 71)

(c) By Both Friction and Banking of Road: If a vehicle is moving on a circular road which is both rough and banked, then three forces may act on the vehicle, and out of these the first force, i.e., weight (mg) is fixed both in magnitude and direction. The direction of the second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force, i.e., friction f can be either inward or outward while its magnitude can be varied up to a maximum limit ($f_L = \mu N$). Therefore, the magnitude of normal reaction, N and direction plus magnitude of friction, f, are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the center.



(a) **Figure: 4.57** (b)

Therefore, magnitude of N and direction plus magnitude of friction mainly depends on the speed of vehicle, v. Thus, the situation varies on a case-to-case basis even though we can observe the following scenarios:

(i) Friction f is outward if the vehicle is at rest or v = 0. Because, in this case, the component of weight mg sin θ is balanced by f.

(ii) Friction f is inward if $v > \sqrt{rg\tan\theta}$

(iii) Friction f is outward if $v < \sqrt{rg\tan\theta}$

(iv) Friction f is zero if $v = \sqrt{rg\tan\theta}$

Let us now observe how the force of friction and normal reaction changes as speed is gradually increased.

In Figure 4.57 (a): When the car is at rest, then the force of friction is upward. However, we can resolve the forces in any two mutually perpendicular directions. Let us resolve them in both horizontal and vertical directions.

$\sum F_H = 0 \; \therefore N\sin\theta - f\cos\theta = 0$ ... (i)

$\sum F_V = 0 \; \therefore N\cos\theta + f\sin\theta = mg$ ... (ii)

In Figure 4.57 (b): Now the car is given a small speed v, so that a centripetal force $\frac{mv^2}{r}$ is now required in horizontal
direction toward the center. Therefore, Eq. (i) will now become $N\sin\theta - f\cos\theta = \frac{mv^2}{r}$

Or we can say, in case (a) $N\sin\theta$ and $f\cos\theta$ were equal while in case (b) their difference is $\frac{mv^2}{r}$. This can occur in any of the following three ways:

(i) N increases while f remains same

(ii) N remains same while f decreases

(iii) N increases and f decreases

But only the third case is possible, i.e., N will increase but f will decrease. This is because Eq. (ii), $N\cos\theta + f\sin\theta = mg$ is still has to be valid.

Therefore, to keep $N\cos\theta + f\sin\theta$ to be constant (=mg), N should increase but f should decrease (as θ =constant). Now, as speed goes on increasing, force of friction first decreases But becomes zero at v= $\sqrt{rg\tan\theta}$ and then reverses it direction. Let us show an example which illustrates this theory.

Illustration 26: A turn of radius 20 m is banked for the vehicle of mass 200 kg moving at a speed of 10 m/s. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed (a) 5 m/s and (b) 15 m/s. Assume that friction is sufficient to prevent slipping. ($g = 10\ m/s^2$) **(JEE ADVANCED)**

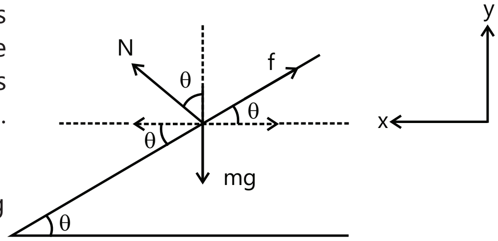


Figure: 4.58

Sol: At the correct speed for which the road is banked, the self-adjusting static friction acting on the vehicle is zero. When the speed decreases, the vehicle has a tendency to slip downwards, thus the static friction acts upwards. When the speed increases, the vehicle has a tendency to slip upwards, thus static friction acts downwards.

(a) the turn is banked for speed v = 10 m/s

Therefore, $\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$ Now, as the speed is decreased, force of friction f acts upward.

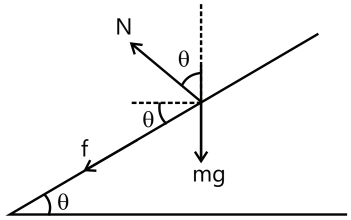


Figure: 4.59

Using the equations $\sum F_X = \frac{mv^2}{r}$ and $\sum F_y = 0$ we obtain

$$N\sin\theta - f\cos\theta = \frac{mv^2}{r} \qquad \text{... (i)}$$

and $N\cos\theta + f\sin\theta = mg$ ... (ii)

Now, by substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, v = 5 m/s, m = 200 kg and r = 20 m, in the above equations,

we obtain $f = 300\sqrt{5}N$ (outward)

(b) In the second case, force of friction f will act downward.

Using $\sum F_X = \frac{mv^2}{r}$ and $\sum F_y = 0$ we obtain

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{... (iii)}$$

$$N\cos\theta - f\sin\theta = mg \qquad \text{... (iv)}$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, v = 15 m/s, m = 200 kg and r = 20 m, in the above equations, we obtain $f = 500\sqrt{5}$ N (downward).

19.1.2 Conical Pendulum

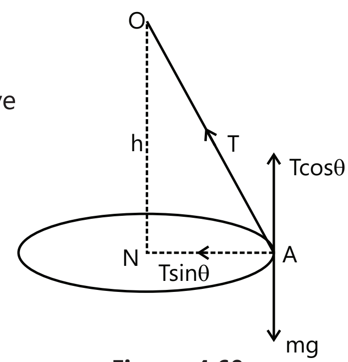


Figure: 4.60

A conical pendulum consists of a string OA whose upper end O is fixed and a bob is tied at the free end. When a horizontal push is given to the bob by drawing aside and let it describe a horizontal circle with uniform angular velocity ω in such way that the string makes an angle θ with the vertical, then the string traces the surface of a cone of semi-vertical angle θ. It is called a conical pendulum. Let us assume that T be the tension in string, l be the length and r be the radius of the horizontal circle described. Now, the vertical component of tension balances the weight, whereas the horizontal component supplies the centripetal force.

$$T\cos\theta = mg; \quad T\sin\theta = mr\omega^2 \quad \therefore \tan\theta = \frac{r\omega^2}{g}$$

$$\omega = \sqrt{\frac{g\tan\theta}{r}}; \quad r = l\sin\theta \quad \therefore \omega = \frac{2\pi}{T}$$

T being the period, i.e., time for one revolution $\therefore \frac{2\pi}{T} = \sqrt{\frac{g\tan\theta}{l\sin\theta}}$; $T = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$ where $h = l\cos\theta$

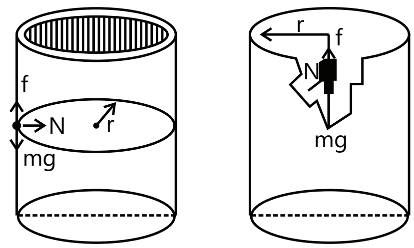
MASTERJEE CONCEPTS

This is similar to the case, when necessary centripetal force to vehicles is provided by the property of banking. The only difference here is that the normal reaction is being replaced by the tension.

Yashwanth Sandupatla (JEE 2012, AIR 821)

19.1.3 Death Well or Rotor

In the case of a 'death well', a person drives a bicycle on a vertical surface of a large wooden well, while in the case of rotor at a certain angular speed of rotor a person hangs resting against the wall without any support from the bottom. In the death well, all walls are at rest and the person revolves while in case of the rotor the person is at rest but the walls rotate.



(A)
Death well

(B)
Rotor

Figure: 4.61

In both the cases, friction balances the weight of the person while reaction provides centripetal force for circular motion, i.e., f = mg and $N = \frac{mv^2}{r} = mr\omega^2$

A cyclist on the bend of a road: In the Figure 4.62, $F = \sqrt{N^2 + f^2}$

When the cyclist is inclined to the center of the rounding along its path, the resultant N, f and mg are directed horizontally to the center of the circular path of the cycle. This resultant force naturally imparts a centripetal acceleration to the cyclist. The resultants of N and f, i.e., F should pass through G, the center of gravity of the cyclist (for complete equilibrium, rotational as well as translational). Hence, $\tan\theta = \frac{f}{N}$ where $f = \frac{mv^2}{r}$ and $N = mg$

$\therefore \tan\theta = \frac{v^2}{rg}$

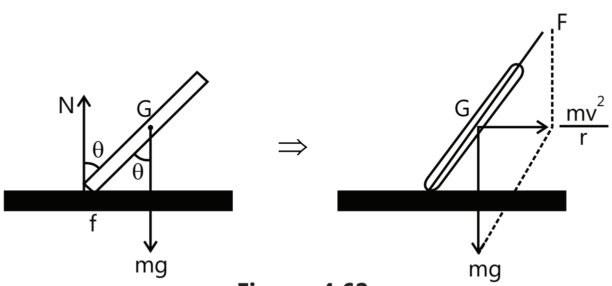


Figure: 4.62

20. CENTRIFUGAL FORCE

We know that Newton's laws are valid only in inertial frames. This is because in non-inertial frames a pseudoforce $-m\vec{a}$ has to be applied on a particle of mass m ($\vec{a}$ =acceleration of frame of reference). After applying the pseudo force, then one can apply Newton's laws in their usual form. Now, suppose that a frame of reference is rotating with constant angular velocity ω in a circle of radius 'r'. Then, it will become a non-inertial frame of acceleration θ toward the center. Now, if we observe an object of mass 'm' from this frame then obviously a pseudo force of magnitude $mr\omega^2$ will have to be applied to this object in a direction away from the center. This pseudo force is what we call as centrifugal force. After applying this force, we can apply Newton's laws in their usual form. The following example will illustrate this concept clearly.

Illustration 27: A particle of mass m is placed over a horizontal circular table rotating with an angular velocity ω about a vertical axis passing through its center. The distance of the object from the axis is r. Based on the above, find the force of friction f between the particle and the table. **(JEE ADVANCED)**

Sol: The particle is stationary in the rotating reference frame rigidly fixed to the rotating table. In the list of all the forces acting on the particle, include the centrifugal force (pseudo force) acting on the particle radially outwards.

Let us solve this problem from both the frames. The one is a frame fixed on the ground while the other is a frame fixed on the table itself.

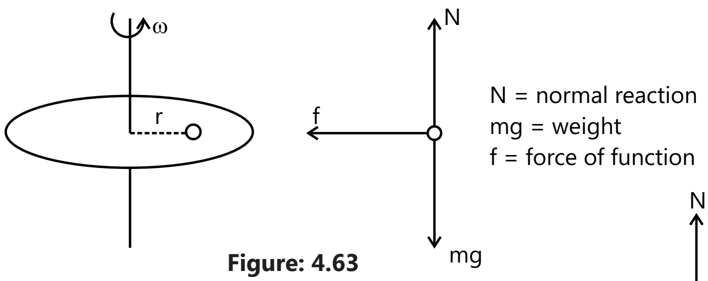


Figure: 4.63

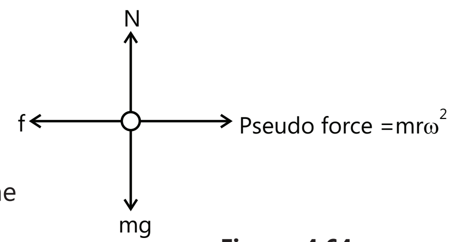


Figure: 4.64

From frame of reference fixed on ground (inertial)

Here, N will balance its weight and force of friction, whereas f will provide the necessary centripetal force.

Thus, $f = mr\omega^2$

From frame of reference fixed on table itself (non-inertial)

In the FBD of particle with respect to table, in addition to the above three forces (N, mg and f) a pseudo force of magnitude $mr\omega^2$ will have to be applied in a direction away from the center. But one significant point here is that in this frame the particle is in equilibrium, i.e., N will balance its weight in vertical direction, whereas f will balance the pseudo force in horizontal direction. $f = mr\omega^2$

Thus, we observe that 'f' equals $mr\omega^2$ from both the frames. Now, let us work up few more examples of circular motion.

Illustration 28: A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at the upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Based on the above, find the tension in the string and the magnitude of net force on the bob at that instant.

(JEE MAIN)

Sol: Apply Newton's second law on the bob in two perpendicular directions. One along the string and the other along the tangent to its circular path, i.e. along the perpendicular to the string.

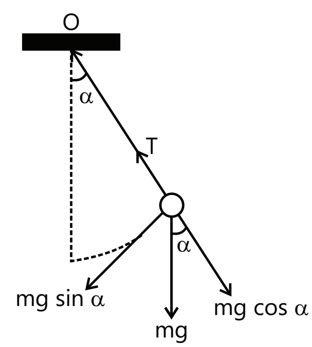


Figure: 4.65

The forces acting on the bob are:

(a) Tension, T and

(b) Weight, mg

As observe that the bob moves in a circle of radius L with center at O, it is imperative that a centripetal force of magnitude $\frac{mv^2}{L}$ is required toward O.

However, this force will be provided by the resultant of T and mg cos α. Thus,

(i) $T - mg\cos\alpha = \frac{mv^2}{L}$ or $T = m\left(g\cos\alpha + \frac{v^2}{L}\right)$

(ii) $\left|\vec{F}_{net}\right| = \sqrt{(mg\sin\alpha)^2 + \left(\frac{mv^2}{L}\right)^2} = m\sqrt{g^2\sin^2\alpha + \frac{v^4}{L^2}}$

Illustration 29: Suppose that a hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. Further, a small ball kept in the bowl rotates with the bowl but without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is $\sqrt{5gR}$, then find the angular speed at which the bowl is rotating. **(JEE MAIN)**

Sol: The horizontal component of normal contact force acting on the ball will provide the necessary centripetal acceleration to move in a circular path. The vertical component of normal contact force acting on the ball will balance its weight.

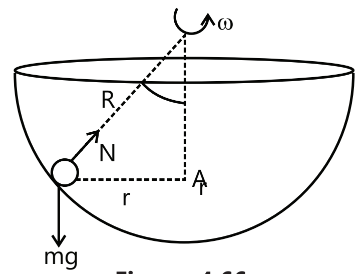


Figure: 4.66

Let us assume that ω be the angular speed of rotation of the bowl. Now, the two forces acting on the ball are

(a) Normal reaction, N and

(b) Weight, mg

We know that the ball is rotating in a circle of radius r (=$R\sin\alpha$) with center A at an angular speed ω.

Thus, $N\sin\alpha = mr\omega^2 = mR\omega^2\sin\alpha$ ... (i)

and $N\cos\alpha = mg$ ... (ii)

Thus, by dividing Eq. (i) by (ii), we obtain $\dfrac{1}{\cos\alpha} = \dfrac{\omega^2 R}{g}$ $\quad \therefore\ \omega = \sqrt{\dfrac{g}{R\cos\alpha}}$

21. EFFECT OF EARTH'S ROTATION ON CIRCULAR WEIGHT

We are very familiar with the fact that the earth rotates about its own axis at an angular speed of one revolution for every 24 hours. We also know that the line joining the north and the south poles is the axis of its rotation, and every point on the earth moves in a circular path. Further, a point at the equator moves in a circle of radius equal to that of the radius of the earth and the center of the circle is the same as the center of the earth. However, for any other point on the earth, the circle of rotation is smaller than this. Now, consider a place P on the earth.

If we drop a perpendicular PC from P to the axis SN, then the place P rotates in a circle with the center at C, and the radius of this circle is CP. Now, the angle between the axis SN and the radius OP through P is called the colatitude of the place P. We have $CP = OP\sin\theta$ or, $r = R\sin\theta$ where R is the radius of the earth.

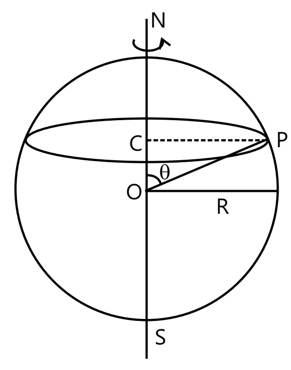


Figure: 4.67

However, if we work from the frame of reference of the earth, then we shall have to assume the existence of the pseudoforces. In particular, a centrifugal force, $m\omega^2 r$ has to be assumed on any particle of mass m placed at P. Here ω is the angular speed of the earth. In contrast, if we discuss the equilibrium of bodies at rest in the earth's frame, then no other pseudo force is needed. Let us now consider a heavy particle of mass m that is suspended through a string from the ceiling of a laboratory at colatitude θ. Observing from the earth's frame it appears that the particle is in equilibrium and hence the forces acting on it are

(a) Gravitational attraction, mg, toward the center of the earth, i.e., vertically downward,

(b) Centrifugal force, $m\omega^2 r$, toward CP, and

(c) The tension in the string, T, along the string.

As the particle is in equilibrium (in the frame of earth), the three forces on the particle therefore should add up to zero.

The resultant of mg and

$$m\omega^2 r \quad = \sqrt{(mg)^2 + (m\omega^2 r)^2 + 2(mg)(m\omega^2 r)\cos(90^\circ + \theta)}$$

$$= m\sqrt{g^2 + \omega^4 R^2 \sin^2\theta - 2g\omega^2 R\sin^2\theta} = mg'$$

where $g' = \sqrt{g^2 + \omega^4 R^2 \sin^2\theta - 2g\omega^2 R\sin^2\theta}$

Also, it is obvious that the direction of the resultant makes an angle α with the vertical OP, where

$$\tan\alpha = \frac{m\omega^2 r\sin(90° + \theta)}{mg + m\omega^2 r\cos(90° + \theta)} = \frac{\omega^2 R\sin\theta\cos\theta}{g - \omega^2 R\sin^2\theta}$$

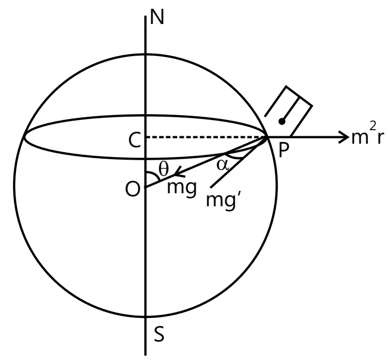


Figure: 4.68

From the above, as the three forces acting on the particle must add up to zero, we understand that the force of tension must be both equal and opposite to the resultant of the rest two. Therefore, the magnitude of the tension in the string must be mg′ and the direction of the string hence should make an angle α with the true vertical.

The direction of g′ is in the apparent vertical direction. This is because a plumb line stays in this direction only. However, the walls of the buildings are constructed by making them parallel to g′ and not to g. Further, the water surface placed at rest is perpendicular to g′.

We observe further that the magnitude of g′ is also different from g. As 2g > $\omega^2 R$, it is clear from the above equation that g′ < g. Here, one way of measuring the weight of the body is to suspend it by a string and hence find the tension in the string. This is because the tension itself is taken as a representative measure of the weight. As T = mg′, the weight so observed is clearly less than the true weight, mg. We call this quantity as the apparent weight. Similarly, if a person stands on the platform of a weighing machine, the platform then exerts a normal force N which is equal to mg′. The reading of the weighing machine responds to the force exerted on it and hence the weight recorded is the apparent weight, mg′ of the person. At the equator, we know that $\theta = 90°$ and hence Eq. (7.14) gives

$$g' = \sqrt{g^2 - 2g\omega^2 R + \omega^4 R^2} = g - \omega^2 R \quad \text{Or,}\ mg' = mg - m\omega^2 R.$$

This value, however, can be obtained in a more straightforward way. At the equator, we know that $m\omega^2 R$ is directly opposite to mg and the resultant is simply mg − $m\omega^2 R$. Also, it is clear that this resultant is towards the center of the earth so that at the equator the plumb line is along the true vertical.

At poles, invariably θ =0 and the first equation gives g′=g and by the next equation we know that α = 0. Thus, at the poles there is no apparent change in g. This is basically because of the fact that the poles do not rotate and hence the effect of earth's rotation is negligible there.

Illustration 30: A body weighs 98 N on a spring balance at the North Pole. What will be its weight recorded on the same scale if it is shifted to the equator? Use $g=GM/R^2=9.8\ m/s^2$ and the radius of the earth R=6400 km.

(JEE MAIN)

Sol: At the equator due to the rotation of earth the body experiences centrifugal force, directed away from the center of earth. So the apparent weight at equator is less than the actual weight at the poles.

At poles, we know that the apparent weight is the same as the true weight. Thus, 98 N = mg = m (9.8 m/s^2)

or, m = 10 kg. At the equator, however, the apparent weight is mg′ = mg − $m\omega^2 R$.

It is given that the radius of the earth is 6400 km and hence the angular speed is $\omega = \frac{2\pi\ \text{rad}}{24\times 60\times 60\ \text{s}} = 7.27\times 10^{-5}\ \text{rad/s}$

Thus, mg′= 98N – (10 kg) $(7.27\times 10^{-5} s^{-1})^2$ (6400 km) = 97.66 N.

PROBLEM SOLVING TACTICS

Step 1: First, construct a big schematic diagram of the physical situation. Then, while reading and rereading the problem statement construct your diagram accordingly including every available information from the statement on the diagram. Thereafter, if applicable, attach appropriate symbols to each important parameter in the problem irrespective of the fact that whether the value of the parameter is known or not. Eventually, make straight lines straight, parallel lines parallel, perpendicular lines perpendicular, etc., to the best of your ability in order to avoid confusion later on.

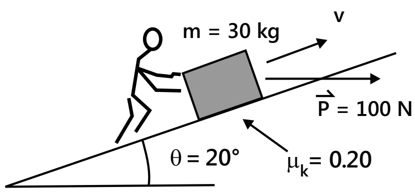


Figure: 4.69

Example at right. (Note how the following items recorded from the statement of the problem into specific items on the drawing: "... 30 kg suitcase ... moves upward ... 20 degree incline ... applied horizontal force of 100 N ... kinetic coefficient of friction ... is 0.20. ")

Step 2: Select a «system» to which you intend to apply Newton›s second law. In some problems, however, there may be more than one candidate for the "system." You may not choose the best one always the first time. That should not be a case for worry; just choose another one and do it again.

Example: We will select the "suitcase" as our system because it is the thing to which many obvious forces are being applied and it is the thing whose acceleration we want to find.

Step 3: Identify all the forces acting on «the system.» You can do this by drawing a dotted line around the system chosen in step 2 and identifying all physical objects that come in contact with the system. Each of these will exert a definite force on the system. Then, look for «field» forces—those forces that act without touching through the intermediary of a field of some sort. We know that in introductory mechanics the only "field" force is the force of gravity. It is a force exerted by the earth (or some other very massive body) on the system through the intermediary of the gravitational field. **Important!** It should be understood that every force on a system is exerted by some physical object exterior to the system. If you cannot identify that object and the method of interaction (contact or field), then the force DOES NOT EXIST! Listed here are the some commonly encountered forces and some tips on dealing with them:

(a) Ropes or strings: These exert «tension» forces on the system in question. They are always directed away from the system and along the direction of extension of the rope or string used.

(b) Contacts with surfaces: We generally split up the force due to contact with a surface into two components called the "normal"—meaning "perpendicular"—force and the "frictional" force. The normal force is generally a "push" type of force directed toward the system, unless the surface is sticky enabling it to exert a «pull» type of force. In contrast, the frictional force is parallel to the surface, opposes motion or potential motion (i.e., a system on the verge of «slipping») and is often assumed to be related to the normal force through a «coefficient of friction.» Readers may kindly refer to the discussion on the topic, "coefficient of friction" in this regard.

(c) Hinges or Pins: These exert forces of arbitrary magnitude and direction as required so as to ensure that the point of attachment remains stationary.

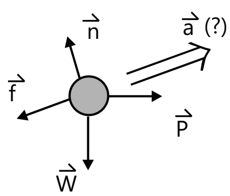


Figure: 4.70

(d) General pushes or pulls: If a working problem specifies that some object is being pushed or pulled in some direction, then you may have to assume that the force specified is being exerted by some physical object. Therefore, it is very important that you do not forget to include the same.

(e) Air resistance: Air may is not visible, but it is very likely that it does establish a physical contact with your system. Quite often we neglect air resistance because its effects are deemed negligible. However, if a problem specifies a certain amount of air resistance is involved or tells you that the air resistance depends in some way on velocity or other parameters, then do not forget to include it.

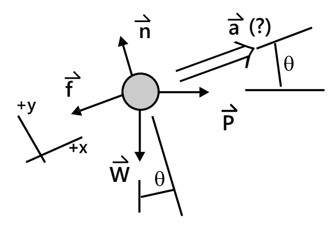


Figure: 4.71

(f) Gravitational force: We are aware of the fact that the gravitational force—commonly called the "weight" of the system—is the only force that acts without being in physical contact with the system (at least until you learn about electric and magnetic forces later on.) Generally, it acts in the downward direction (by definition!) and is equal to the mass of the system times the local gravitational field strength g—commonly, but misleadingly called "the acceleration due to gravity."

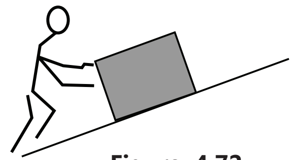


Figure: 4.72

The example at right. Here, we find two objects in contact with the system—one being the "surface" and other one "pusher." Thus, we find a total of four forces—the normal force, the frictional force (from the surface), the push (from the pusher), and the weight (due to the only force—so far—that acts without needing to touch—gravity.)

Step 4: Draw an "FBD." Now, the system may be represented by a simple circle or square; however, we want to focus our attention on the forces on and the resulting acceleration of the system. Now, draw each force with its tail at the surface of the system extending in the proper direction. Further, include the acceleration vector as well, but distinguish it from the force vectors by drawing it in a different way.

Example at the right. In this example, note that the normal force is directed perpendicular to the surface (not shown in the FBD), the frictional force is directed opposite to the direction that the system slips with respect to the surface, the push is in its given direction, and the weight is directed "down." We also show the acceleration as a different-looking vector that is directed upward along the plane, but we do not know this for certain; it may be directed downward along the incline. To keep us remind ourselves of this fact, we put a "(?)" next to the acceleration vector.

Step 5: Now, pick a coordinate system and hence determine the angles that the forces and accelerations make with the coordinate axes. It is usually "clever" and preferable to pick a coordinate system that minimizes the number of unknown vectors that will have to be broken down into components. The answers you obtain thereafter must and will be independent of your choice of coordinate system, but clever choices will help us to arrive at equations that are more easy to solve. However, you may need to do some geometrical work on another sheet of paper to figure out how the angles are related to those given in the problem statement.

Example at right. In this example, we have chosen a coordinate system that requires us to determine the components of only the weight and the push—the two forces about which we know a lot. These two forces lie at the angle theta (given in the problem statement as 20 degrees) from one of the axis directions.

Step 6: Now, write Newton's second law. This law is the basic physical principle you are applying; i.e., the "starting point" for your calculations. Just proceed to do it! Example: $\sum \vec{F} = m\vec{a}$

Step 7: Thence, apply the basic equation to this problem. Now, simply write what the "sum of forces" is in this case. If the acceleration is zero, then use that fact to simplify the equation too. Example: $\vec{n} + \vec{f} + \vec{P} + \vec{W} = m\vec{a}$

Step 8: Now, continue by writing the component equations. This is simply a matter of recognizing that every vector equation is shorthand for two (or, more generally, three) scalar equations. Then, simply rewrite the vector equation for each component direction with each vector quantity rewritten as the corresponding component. Examples: $x: n_x + f_x + P_x + W_x = ma_x$ and $n_y + f_y + P_y + W_y = ma_y$

Step 9: Now, determine what each component is in terms of the vector magnitude and trigonometric functions of the associated angles. In this step, it is imperative that we explicitly indicate the signs of the vector components. This is also a good time to explicitly substitute "mg" for "W" if you really happen to know the mass of the system.

Example: Notice that the normal force is purely in the +y direction, the frictional force is purely in the –x direction, the push has a positive x-component and a negative y-component, the weight has negative x- and y-components, and the assumed acceleration is purely in the +x direction. Thus, we have: $x:\ 0 + (-f) + (+P\cos\theta) + (-mg\sin\theta) = m(+a)$
$y:\ (+n) + 0 + (-P\sin\theta) + (-mg\cos\theta) = m(0)$

Step 10: To conclude this procedure, as a final step, simplify the resulting equations and figure out where to go from here. This is the end of "the method." Example: $P\cos\theta - f - mg\sin\theta = ma$ $n - P\sin\theta - mg\cos\theta = 0$

FORMULAE SHEET

(a) $F_1 = F\cos\theta =$ component of $\vec{F}$ along AC $\quad F_2 = F\sin\theta =$ component of $\vec{F}$ perpendicular to AC

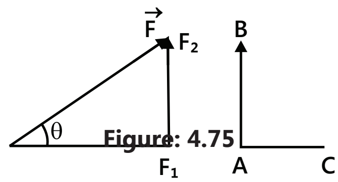


Figure: 4.73

(b) $F = \dfrac{dp}{dt} = \dfrac{d(mv)}{dt}$

(c) $\sum \vec{F} = \vec{F}_{net} = m\vec{a}$ or $\vec{a} = \dfrac{\vec{F}_{net}}{m}$

$\sum F_x = 0 \Rightarrow T\sin\theta = ma$

$\sum F_y = 0 \Rightarrow T\cos\theta = mg$

$\theta = \tan^{-1}\left(\dfrac{a}{g}\right)$

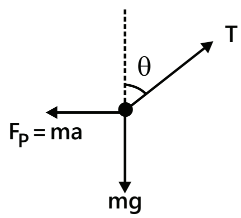
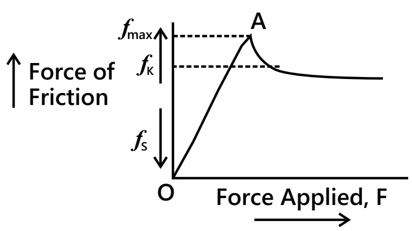


Figure: 4.74

(d) Impulse = change in momentum $\quad \vec{F}\Delta t = m\vec{v}_f - m\vec{V}_o$

(e) $\mu = \dfrac{f_{max}}{R}$



(f) $f_{max} = f_{limiting} = \mu_s R$

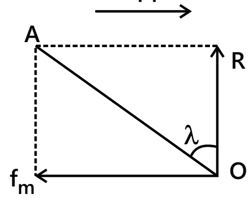


Figure: 4.76

(g) Angle of Friction: $\quad \tan\lambda = \dfrac{f_{max}}{R} = \mu \quad$ or $\lambda = \tan^{-1}(\mu)$

(h) Pseudo force: F = –ma; where m = mass of the object, a = acceleration of the reference frame

(i) A particle in circular motion may have two types of velocities as listed hereunder.

(i) Linear velocity v and

(ii) Angular velocity ω.These two are related by the equation $v = R\omega$ (R = radius of circular path)

(j) Acceleration of a particle in a circular motion may have two components as listed hereunder.

(i) Tangential component (a_t) and

(ii) Normal or radial component (a_n).

As the name suggests, the tangential component is tangential to the circular path, given by a_t = rate of change of speed

(k) $\frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = R\alpha$ where α = angular acceleration = rate of change of angular velocity $= \frac{d\omega}{dt}$

The normal or radial component, also known as centripetal acceleration is toward the center and is given by $a_n = R\omega^2 = \frac{v^2}{R}$

(l) Net acceleration of a particle is the resultant of two perpendicular components, a_n and a_t. Hence, $a = \sqrt{a_n^2 + a_t^2}$

(m) Tangential component a_t is responsible for change of speed of a particle. This can be positive, negative or zero, depending upon the situation whether the speed of the particle is increasing, decreasing or remains constant.

(n) In general, in any curvilinear motion, direction of instantaneous velocity is tangential to the path, while acceleration may assume any direction. If we resolve the acceleration in two normal directions, one parallel to velocity and another perpendicular to velocity, then the first component is a_t while the other is a_n.

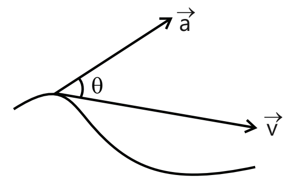


Figure: 4.77

Thus, a_t = component of $\vec{a}$ along $\vec{v}$ $= a\cos\theta = \frac{\vec{a}.\vec{v}}{v} = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt}$ = rate of change of speed.

Further, a_n = component of $\therefore a_{1x} = a_{2x} = a_{3x}$ perpendicular to $\vec{v}$ $= \sqrt{a^2 - a_t^2} = \frac{v^2}{R}$

Here, v is the speed of the particle at that instant and R is called the radius of curvature to the curvilinear path at that point.

(o) In $a_t = a\cos\theta$, if θ is acute, a_t will be positive and speed increases. However, if θ is obtuse a_t will be negative and speed will decrease. If θ is $90°$, a_t is zero and speed will remain constant.

(p) Now, depending upon the value of a_t, circular motion may be of three types as listed hereunder.

(i) Uniform circular motion in which speed remains constant or $a_t = 0$.

(ii) Circular motion of increasing speed, in which a_t is positive.

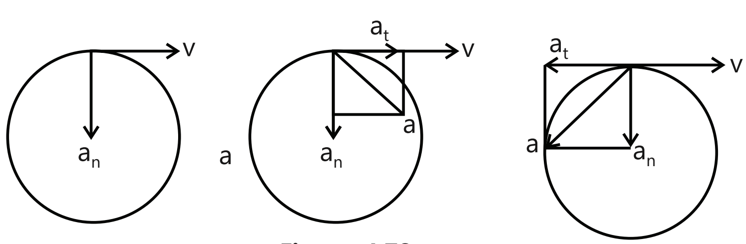


Figure: 4.78

(iii) Circular motion of decreasing speed, in which a_t is negative.