2. MOTION IN A STRAIGHT LINE

1. INTRODUCTION

Mechanics is a branch of Physics which deals with motion of bodies and the cause behind it. Motion of a body that includes position and time can be determined with respect to other bodies. Branch of physics that deals with the motion of particles and rigid bodies irrespective of the forces responsible for their motion is known as Kinematics. When the size of a body is too small such that its motion can be described by a point mass moving along straight line, motion is known as rectilinear motion or one-dimensional motion.

1.1 Motion

A body is said to be in motion when it changes its position with respect to the observer while it is said to be at rest when there is no change in its position with respect to the observer. For instance, two passengers travelling in a moving train are at rest with respect to each other but in motion for a ground observer.

1.2 Particle

Physically, a particle is considered as analogues to a point. A body with a definite size is considered as a particle when all of its parts have same displacement, velocity and acceleration. The motion of any such body can be studied by the motion of any point on that body.

1.3 Basic Definitions

1.3.1 Position and Position Vectors

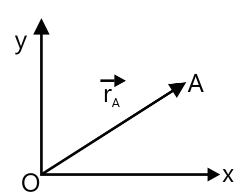


Figure 2.1

Position of any point can be represented by its coordinates with respect to an origin in Cartesian system. For example, point A can be represented by (x_A, y_A, z_A).

$$\vec{r}_A = \overline{OA} = x_A\hat{i} + y_A\hat{j} + z_A\hat{k}$$

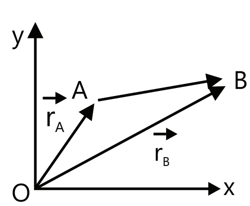


Figure 2.2

Given the co-ordinates of two points A & B, the position vector of B w.r.t. A can be determined as follows:

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A \Rightarrow \overrightarrow{AB} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$$

Illustration 1: Find the torque (τ) exerted by force $\hat{i} + \hat{j} + 2\hat{k}$ at point P (2, 3, 4) w.r.t origin. Given, $\vec{\tau} = \vec{r} x \vec{F}$

$\vec{r}$: is the position vector of point at which force is acting w.r.t to the given point (in this case origin) **(JEE MAIN)**

Sol: The force $\vec{F}$ is given in Cartesian coordinates. Express the position vector of point P in Cartesian coordinates and find the cross product $\vec{\tau} = \vec{r} x \vec{F}$

Here $\vec{r} = \overrightarrow{op} = \vec{P}$ [$\vec{P}$: Position vector of point P]

$\Rightarrow \vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{F} = \hat{i} + \hat{j} + 2\hat{k}$

$\therefore \vec{\tau} = \vec{r} \times \vec{F}$

$= \hat{i} + \hat{j} + 2\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix} = \hat{i}(2) - \hat{j}(0) + \hat{k}(-1) \qquad = \vec{\tau} = 2\hat{i} - \hat{k}$$

1.3.2. Distance and Displacement

A particle follows either a curve or a straight line when moving in the space. This curve or line is known as its trajectory. Distance is the length of the path or trajectory covered by the particle and displacement is the difference between the vectors of the first and the last position on this path.

Distance and displacement are illustrated in the Fig. 2.3 where AB (curve length) is the distance and vector $\Delta\vec{r}_{AB}$ is the displacement $\left(\vec{r}_B - \vec{r}_A\right)$. Following points should be considered about both the quantities:

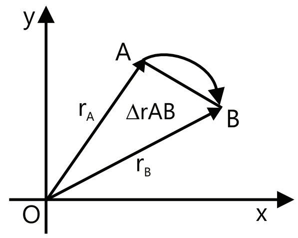


Figure 2.3

(a) Distance is a scalar quantity while displacement is vector.

(b) Displacement is always less than or equal to distance in magnitude.

(c) Displacement can be zero but distance cannot be zero for a moving body.

MASTERJEE CONCEPTS

For a particle moving in a straight line:

- the distance travelled is always equal to the displacement when there is no change in direction,

  i.e. distance travelled = |displacement|

- Else, distance travelled is always greater than displacement,

  i.e. distance travelled ≥ |displacement|

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 2: Find the distance and displacement of a particle travelling from one point to another, say from pt. A to B, in a given path. **(JEE MAIN)**

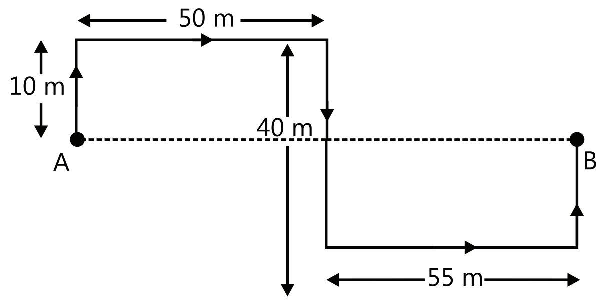


Figure 2.4

Sol: Distance is length of the path travelled. Displacement is the vector from initial point to final point.

Total distance travelled = 10+50+40+55+ (40 - 10) = 185 m

Total Displacement = 50 + 55 =105 m

Deduction of displacement is from A (initial position) to B (final position)

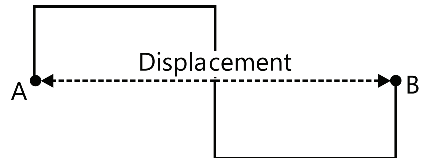


Figure 2.5

Illustration 3: If a particle travels a distance of 5 m in straight line and returns back to the initial point, then find

(i) Total distance travelled (ii) Total displacement **(JEE ADVANCED)**

Sol: Distance is length of the path travelled. Displacement is the vector from initial point to final point.

Total distance travelled = 5 + 5 = 10 m (since, initial and final position of the particle are same).

Displacement = 0 m (5-5=0, since the directions are opposite and cancel with each other)

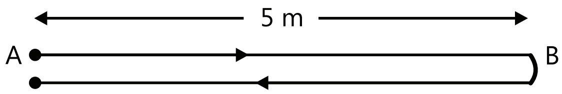


Figure 2.6

1.3.3 Average Speed and Average Velocity

The total distance travelled by a particle divided by the total time taken is known as its average speed.

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

While the average velocity is defined as $v_{av} = \frac{\text{displacement}}{\text{time elapsed}}$ or $v = \frac{r_B - r_A}{t_B - t_A} = \frac{\Delta r_{AB}}{\Delta t}$

Both average speed and average velocity are expressed in ms^{-1} or kmh^{-1}, the former is a scalar quantity while the latter is a vector.

MASTERJEE CONCEPTS

For a moving body:

- Average velocity can be zero but average speed cannot be zero.
- The magnitude of average velocity is always less than or equal to the average speed because,

  $|\text{displacement}| \le \text{distance}$

  $\left|\frac{\text{displacement}}{\text{time elapsed}}\right| \le \frac{\text{distance}}{\text{time elapsed}}$

  therefore,
- Average speed does not mean the magnitude of the average velocity vector.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 4: If a train moves from station A to B with a constant speed v =40 km/h and returns back to the initial point A with a constant speed V_2=30 km/h, then calculate the average speed and average velocity. **(JEE MAIN)**

Sol: Average speed is distance covered divided by time taken. Distance is length of the path travelled. Average velocity is displacement divided by time taken. Displacement is the vector from initial point to final point.

Let the distance AB = s, Time taken by train from A to B, $t_1 = \frac{s}{v_1}$

Time taken by train From B to A, $t_2 = \frac{s}{v_2}$; Average speed $= \frac{\text{Total distance}}{\text{Total time taken}} = \frac{s+s}{t_1+t_2} = \frac{s+s}{\frac{s}{v_1}+\frac{s}{v_2}}$

$V_{avg} = \frac{2v_1v_2}{v_1+v_2} = \frac{2x40x30}{40+30} = \frac{240}{7} = 34.3 km/h$; Average velocity $= \frac{\text{Net displacement}}{\text{Total time}} = \frac{0}{t_1+t_2} = 0$

Illustration 5: Consider a train moving from station A to B with a constant speed of 40 km/h for half the time and with constant speed of 30 km/h for the next half time of that journey. Calculate the average speed of the whole journey. **(JEE MAIN)**

Sol: Average speed is distance covered divided by time taken. Here we need to assume total time of journey as T. The speed in each half of the time T is constant. The distance covered in each half of the time can be easily written in terms of T. Average speed is distance covered divided by total time taken.

Let AB = s and T = Total time of journey.

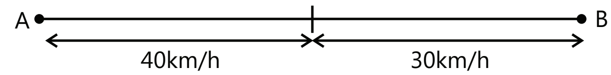


Figure 2.7

$\therefore$ Distance travelled in first half time $\frac{T}{2}$ is, $s_1 = v_1\frac{T}{2}$

Distance travelled in second half time $\frac{T}{2}$ is, $s_2 = v_2\frac{T}{2}$; Average speed $= \frac{\text{Total distance}}{\text{Total time}}$

$V_{avg} = \frac{v_1(T/2)+v_2(T/2)}{T}$; $V_{avg} = \frac{v_1+v_2}{2}$; $V_{avg} = \frac{40+30}{2} = 35 km/h$

Illustration 6: A particle travels half of the journey with speed 2 m/s. For second half of the journey, the particle travels with a speed of 3 m/s for half of remaining time, and for the other half it travel with a speed of 6 m/s. Find its average speed. **(JEE ADVANCED)**

Sol: Average speed is distance covered divided by total time taken. Here we need to assume total distance of journey. The speed in each part of the journey is constant. The time taken to cover each part of the journey can be calculated in terms of distance covered and speed in that part.

Let total distance = 4 s

Let the time taken in covering last half of journey = 2t

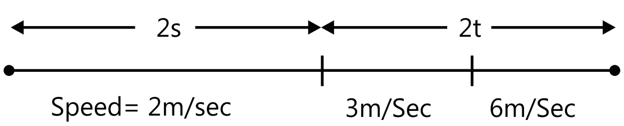


Figure 2.8

For the first half of the journey

speed (v_1) = 2 m/s and distance (d_1) = 2s

So, time taken $(t_1) = \frac{\text{distance}}{\text{speed}} = \frac{2s}{2} = s$

For the first part of second half of the journey

v_2 = 3 m/s ; t_2 = t s; $\therefore d_2 = v_2 x t_2 = 3t$

For the second part of second half of the journey

V_3 = 6 m/s ; T_3 = t s ; $d_3 = v_3 x t_3 = 6t$ m

We know $d_2 + d_3 = \frac{\text{Total distance}}{2} = \frac{4s}{2} = 2s$; $\Rightarrow$ 3t + 6t = 2s; $\Rightarrow t = \frac{2s}{9}$

Therefore, total time taken for journey = $t_1 + t_2 + t_3$ = s + t + t $= s + \frac{2s}{9} + \frac{2s}{9} = \frac{13s}{9}$

$\therefore V_{avg} = \frac{\text{Total distance}}{\text{Time interval}} = \frac{4s}{\frac{13s}{9}} = \frac{36}{13}$ m/s

Illustration 7: Consider the following:

Men running in a straight line with a speed of 15 m/s one behind the other at equal intervals of 20 m. Cyclists are also riding along the straight line with a speed of 25 m/s at equal intervals of 30 m.

An observer moving in the opposite direction with a speed of v such that comes across men and cyclists at the same time. Find the velocity v. **(JEE ADVANCED)**

Sol: The relative velocity between observer and men is (15 + v). The relative velocity between observer and cyclists is (25 + v). In the reference frame of observer, the time elapsed between the passing of two men is 20/(15 + v) and the time elapsed between the passing of two cyclists is 30/(25 + v). Both these timings should be equal so that the men and cyclists pass the observer together.

Let us assume that a man, a cyclist and the observer are in line. Now after t time, the observer again meets with the next man and cyclist

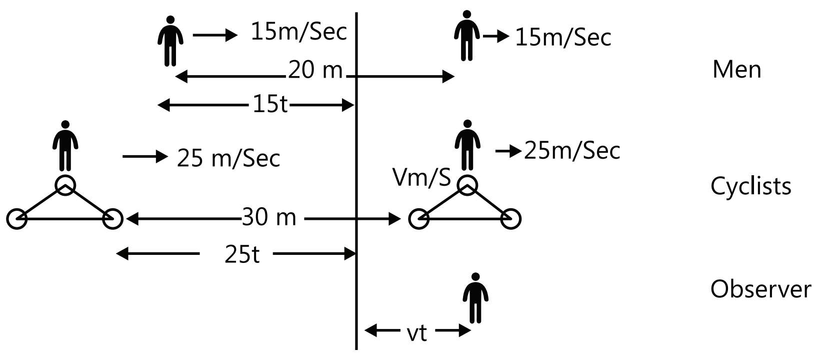


Figure 2.9

Then the distance travelled by the observer = vt

Distance travelled by the man = 15t = 20 –vt ... (i)

Distance travelled by the cyclist = 25t = 30 – vt ... (ii)

Simplifying (i) and (ii) we get

15t + vt = 20 ... (iii)

25t + vt = 30 ... (iv)

Dividing (iii) and (iv) we get $\frac{15t+vt}{25t+vt}=\frac{20}{30}$; $\Rightarrow \frac{15+v}{25+v}=\frac{2}{3}$; $\Rightarrow$ 45 + 3v = 50 + 2v; $\Rightarrow$ v = 5 m/sec

Illustration 8: 2 Cars A and B simultaneously start with speed 20 m/sec and 30 m/sec, respectively. Both cars have constant but different acceleration. On completing the race simultaneously, if the final velocity of A is 90 m/sec, then find the final velocity of B (i.e. v_B) **(JEE ADVANCED)**

Sol: Both cars travel equal distance in equal time. Initial velocities and accelerations of both the cars are different. Problem can be best solved by equating the average velocities.

Since both cars travel equal distance at equal intervals of time, both cars have equal average speeds, i.e. average speed of car A = average speed of car B and for

for constant acceleration, avg velocity $=\frac{u+v}{2}$; $\therefore \frac{20+90}{2}=\frac{30+v_B}{2}$; $\Rightarrow V_B$ = 80 m/s

1.3.4. INSTANTANEOUS SPEED AND INSTANTANEOUS VELOCITY

The average speed/velocity of a moving body with

infinitesimally small time interval (i.e. $\Delta t \to 0$) is known as instantaneous speed/velocity. Therefore

Instantaneous velocity $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$.

(a) As the time interval tends to zero (i.e. $\Delta t \to 0$), the displacement vector $\Delta \vec{r}$ is along the direction of motion of the particle i.e. tangential to the path of the particle at that instant. Thus, the instantaneous velocity direction is always tangential to the path of the the particle.

(b) Instantaneous speed and the magnitude of instantaneous velocity are always the same.

i.e. Instantaneous Speed= | Instantaneous Velocity |

A particle moving on a straight line, say along the x-axis, has an instantaneous velocity as follows

$$v(t) - \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} - \lim_{\Delta t \to 0} v_{av}$$

A particle is said to move in a uniform velocity when the velocity of a particle remains constant with respect to time. It is said to be accelerated when velocity changes with respect to time.

Illustration 9: The distance travelled by a particle in time t is given by $s(t) = (2.5\ m/s^2)t^2$.

Find (a) the average speed of the particle during time 0 to 5 s?

(b) The instantaneous speed at t = 5.0 sec **(JEE MAIN)**

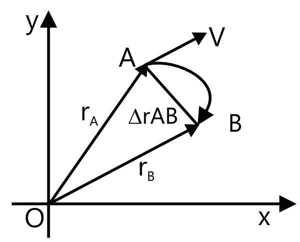


Figure 2.10

Sol: Average speed is distance covered divided by total time taken. Instantaneous speed is the rate of change of distance at a particular instant.

(a) The distance travelled during time 0 to 5.0 sec

$= s\ (5\ sec) = 2.5 \times (5)^2 = 62.5\ m$

$\therefore$ Average speed $= \frac{\text{distance travelled}}{\text{time interval}}$; $\Rightarrow V_{av} = \frac{62.5m}{5.0sec - 0.0sec} = 12.5m/s$

(b) $s(t) = 2.5t^2$; $\therefore \frac{ds(t)}{dt} = (2.5)x(2t) = 5t$; $\therefore$ instantaneous speed $(v) = \frac{ds}{dt}|_{t=5.0} = 5(5.0) = 25\ m/s$

1.3.5. Average acceleration and Instantaneous Acceleration

Rate of change of velocity is defined as acceleration. Velocity changes with change in magnitude or direction or both.

Suppose the velocity of a particle at time t_1 is $\overrightarrow{v_1}$ and at time t_2 it is $\overrightarrow{v_2}$. The change produced in time interval t_1 to t_2 is $\vec{v}_2 - \vec{v}_1$. Average acceleration $\overrightarrow{a_{av}}$ is defined as change in velocity with respect to change in time, i.e. $\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_1 - t_2}$

The average acceleration with infinitesimally small time interval Δt is known as instantaneous acceleration, i.e. $\vec{a} = \lim \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

(a) Acceleration is a vector quantity and its SI unit is ms^{-2}.

(b) The average acceleration vector and the change in velocity vector are in the same direction.

(c) The direction of velocity vector and the direction of acceleration vector are independent of each other.

(d) Acceleration is perpendicular to the velocity vector only when there is change in direction of velocity with time, with its magnitude being constant.

If a body moves with uniform acceleration along a straight line, then the average acceleration and instantaneous acceleration will always be the same.

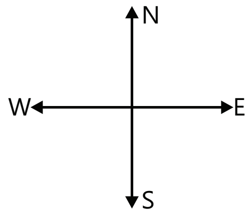
MASTERJEE CONCEPTS

Retardation:

- Negative acceleration does not imply retardation.
- Retardation refers to decrease in speed and not velocity.

Nivvedan JEE 2009 AIR 113

Illustration 10: Consider a particle moving with a speed of 5 m/s towards east. After 10 sec velocity of particle is 5 m/s towards north. Find the average acceleration and its direction. **(JEE MAIN)**



Sol: Average acceleration is change in velocity divided by total time taken.

$\vec{v}_i = 5\hat{i}$; $\vec{v}_f = 5\hat{j}$ $\quad \therefore \quad \overline{v_f} - \overline{f_i} = 5\hat{j} - 5\hat{i}$

Times interval = 10 sec

We know that , Average acceleration $= \dfrac{\vec{v}_f - \vec{v}_i}{\text{Time interval}}$

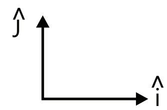
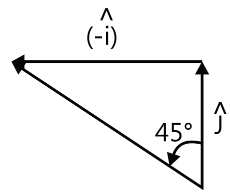


Figure 2.11

$\Rightarrow \quad acc^n_{avg} = \dfrac{5\hat{j} - 5i}{10} = \dfrac{1}{2}(\hat{j} - \hat{i}) m/sec^2$

$\therefore \quad |acc^n_{avg}| = \left|\dfrac{1}{2}(\hat{j} - \hat{i})\right| = \dfrac{1}{\sqrt{2}} \ m/sec^2$



Unit vector along that direction is $= \dfrac{1}{\sqrt{2}}(\hat{j} - \hat{i})$ [$45°$ due west of north]

Figure 2.12

MASTERJEE CONCEPTS

Motion of bodies in three dimensional space:

If a body has coordinates (x, y, z) in space, its position vector r at time is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

The velocity vector v is given by $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

where v_x, v_y v_z are magnitudes of components of the velocity along the x-, y-, and z-axes, respectively, at time t. $v_x = \dfrac{dx}{dt}, v_y = \dfrac{dy}{dt}, v_z = \dfrac{dz}{dt}$

The acceleration a is given by

$\vec{a} = a_x\hat{i} + a_y\hat{j} + a = \hat{k}$

where a_x, a_y and a_z are components of acceleration along x, y, and z directions respectively at time t.

Chinmay S Purandare (JEE 2012 AIR 698)

2. UNIFORMLY ACCELERATED MOTION FOR 1-D MOTION

2.1 Equations of Motion

Consider that the acceleration of a particle ‘a’ is constant.

First Equation: Acceleration is defined as the rate of change of velocity

$$a = \frac{dv}{dt} \Rightarrow dv = a.dt \qquad \text{... (i)}$$

The velocity at time 0 is u and at time t is v. Thus, at t = 0, v= u and

At t = t, v = v.

Integrating equation (i) for these limits: V = v to u = v and t = 0 to t = t

$$\int_u^v dv = \int_o^t adt\,; \Rightarrow [v]_u^v = a[t]_0^t \Rightarrow v-u = a(t-0)\;;\; \Rightarrow v-u = at\,; \Rightarrow \boxed{v-u=at}$$

Second Equation: Velocity is defined as the rate of change of displacement.

$$v = \frac{ds}{dt} \Rightarrow ds = vdt\,; \Rightarrow ds = (u+at)dt \qquad [\because v = u + at] \qquad \text{... (ii)}$$

Suppose the position of the particle is '0' at time '0' and 's' at time 't'. Hence,

at t = 0, s = 0 and

at t = t, s = s

Integrating equation (ii) for these limits: $\int_o^s ds = \int_o^t (u+at)$

$$\Rightarrow [s]_0^s = \int_0^t udt + \int_0^t at.dt \quad \Rightarrow \quad [s-0] = u[t]_0^t + \int_0^t tdt \qquad \Rightarrow s = ut + \frac{1}{2}at^2$$

Third Equation: From the definition of acceleration

$$a = \frac{dv}{dt} \Rightarrow a\frac{dv}{dx}.\frac{dx}{dt}; \quad \Rightarrow a = \frac{dv}{dt}.v\;; \quad \Rightarrow v.dv = a.dx \qquad \text{... (iii)}$$

If the velocity at position 0 is u and at position s is v,

at s=0, v=u and

at s=s, v=v

Integrating equation (iii) for these limits: V=u to v = v and x = 0 to x = s

$$\int_u^v vdv = \int_o^s adx\,; \Rightarrow \left[\frac{v^2}{2}\right] = a[x]_o^s \quad \Rightarrow \frac{v^2}{2} - \frac{u^2}{2} = a[s-0]$$

$$\frac{v^2-u^2}{2} = as\,; \Rightarrow \boxed{v^2-u^2 = 2as}$$

Displacement In n^th^ Second

Displacements S_n of a particle in n seconds and S_{n-1} in (n-1) seconds are given as: $s_n = un + \frac{1}{2}an^2$; $s_n = u(n-1) + \frac{1}{2}a(n-1)^2$

Displacement in n^th^ second: $s_n - s_{n-1} = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$; $x_n = u + \frac{a}{2}(2n-1)$

Illustration 11: Consider a particle moving in straight line with constant acceleration "a" traveling 50 m in 5th second and 100 m in 10th second. Find

(1) Initial velocity (u)

(2) Acceleration (a)

(3) Displacement till 7 s

(4) Velocity after 7 s

(5) Displacement between t = 6 s and t = 8 s **(JEE MAIN)**

Sol: We know the formula for displacement in nth second. For 5th second and 10th second we get two equations and two variables u and a. So we solve the equations to get the values of u and a.

We know that displacement in n^{th} second (x_{nth})

$$x_{nth} = u + \frac{a}{2}(2n - 1) \quad \text{... (i)}$$

Given $x_{5th} = 50$; $\Rightarrow u + \frac{a}{2}[2x5 - 1] = 50$

$$\Rightarrow u + \frac{9}{2}a = 50 \quad \text{... (ii)}$$

and X_{10th} = 100 ; $\Rightarrow u\frac{a}{2}[2x10 - 1] = 100$

$$\Rightarrow u + \frac{19}{2}a = 100 \quad \text{... (iii)}$$

Subtracting (ii) from (i), we get $\left(\frac{19}{2} - \frac{9}{2}\right)a = 100 - 50 \Rightarrow a = 10\,m/sec^2$

Substituting the value of a in (i) we get, $u + \frac{9}{2}(10) = 50 \Rightarrow u = 50 - 45 = 5\,m/s$

We know $x = ut + \frac{1}{2}at^2$

At t= 7 sec $x = (5)(7) + \frac{1}{2}(10)(7)^2 = 35 + 245 = 280$ m

We know v = u + at

$\therefore$ At t = 7sec $\Rightarrow v = (5) + (10)(7) = 75$ m/sec

$$X_{6\,sec\,to\,8\,sec} = x_8 - x_6 = \left(ut + \frac{1}{2}at^2\right)_{at\ t=8\,sec} - \left(ut + \frac{1}{2}at^2\right)_{at\ t=6\,sec} = u(8-6) + \frac{1}{2}a\left[8^2 - 6^2\right] = 5(2) + \frac{1}{2}x10(28) = 150m$$

Illustration 12: Consider a particle moving in a straight line with constant acceleration, has a velocity (v_p) = 7 m/s and V_Q = 17 m/s, when it crosses the point P and Q respectively. Find the speed of the particle at mid-point of PQ. **(JEE ADVANCED)**

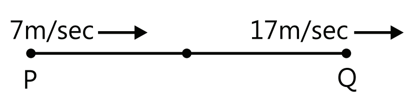


Figure 2.13

Sol: Initial and final velocity are known for constant acceleration and a particular displacement. The final velocity for half the displacement is to be calculated. This problem can be easily solved by using the third equation of motion with constant acceleration.

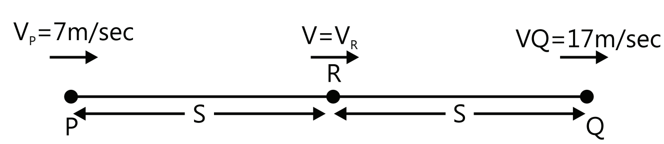


Figure 2.14

Let the mid-point be R

Then PR=RQ = s (say)

From PR	From RQ
$u = v_p = 7$ m/s	$u = V_R$
$V = V_R$	V=17 m/s

x=s x=s

Using formula $v^2 = u^2 + 2ax$

We get: $V_R^2 = 7^2 + 2as$ ... (i)

And $17^2 = V_R^2 + 2as$ ... (ii)

Subtracting (i) from (ii) we get $17^2 - V_R^2 = V_R^2 - 7^2 \Rightarrow 2V_R^2 = 17^2 + 7^2$

$\Rightarrow 2V_R^2 = 338 \therefore V_R = 13 m/sec$

Illustration 13: A body moving with uniform acceleration covers 24 m in the 4th second and 36 m in the 6th second. Calculate the acceleration and initial velocity. **(JEE MAIN)**

Sol: We know the formula for displacement in nth second. For 4th second and 6th second we get two equations and two variables u and a. So we solve the equations to get the values of u and a.

$$S_n = u + \frac{a}{2}(2n-1)$$

$$\therefore \quad 24 = u + \frac{a}{2}(2x4-1) \quad \text{... (i)}$$

$$36 = u + \frac{a}{2}(6x2-1) \quad \text{... (ii)}$$

From equation (i) and (ii) we get $12 = 2a \Rightarrow a = \frac{12}{2} = 6 m/s^2$

From equation (i) $24 = u + \frac{6}{2}(2x4-1) \Rightarrow u = 3\ ms^{-1}$

Observation: Motion is independent of the mass of the body and hence no equation of motion considers mass.

MASTERJEE CONCEPTS

- For uniformly accelerated motion i.e. constant acceleration:

  Average velocity $\{V_{avg}\} = (v + u)/2$

  Proof: V_{avg} =Displacement (s)/time interval (t) = s/t

  $= (ut + ½ at^2) / t$

  $= u + ½ at$

  $= (2u + at)/2$

  $= [u + (u+at)]/2$

  $= (u + v)/2$

- If initial vector of a particle is $\vec{r}_o$, then position vector at time t can be written as

  $$\vec{r} = \vec{r}_o + \vec{s} = \vec{r}_o + \left(\vec{u}t + \frac{1}{2}\vec{a}t^2\right)$$

- Difference between distance (d) and displacement (s)

  From equations of motion

  $$s = ut + \frac{1}{2}at^2 \text{ and } v^2 = u^2 + 2as$$

s is the displacement and not the distance of the particle. The values are different when u and a are of opposite sign or $u \uparrow\downarrow a$.

Case 1: When velocity u is either zero or parallel to a, then motion is simply accelerated and in this case distance is equal to displacement. So, we can write, $d = s = ut + \frac{1}{2}at^2$.

Case 2: When u is not parallel to a, the motion is first retarded and then accelerated in opposite direction. Hence distance is either greater than or equal to displacement ($d \geq |s|$).

Nitin Chandrol (JEE 2012 AIR 134)

Illustration 14: Consider an object moving with an initial velocity of 10 m/s and acceleration of 2 m/s^2. Find distance travelled from t = 0 to 6 s. **(JEE MAIN)**

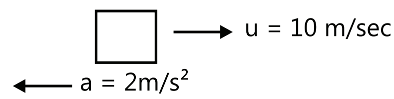


Figure 2.15

Sol: Distance covered is equal to displacement if the object moves in a straight line and there is no change in direction of motion. If direction of motion changes, distance should be calculated separately for different parts of the path.

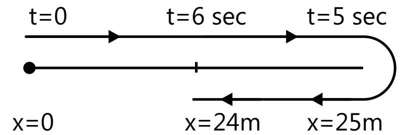


Figure 2.16

$\because V = u + at$; $\Rightarrow 0 = 10 + (-2)t$; $\Rightarrow t = 5\,sec$; $x_5 = ut + \frac{1}{2}at^2$

$= (10)(5) + \frac{1}{2}(-2)(5)^2$; $= 25m$ and $x_6 = (10)(6) + \frac{1}{2}(-2)(6)^2 = 24m$

$\therefore$ Total distance travelled = 25 + (25-24) = 26 m.

Illustration 15: Consider a body moving with velocity 9 m/s. It is subjected to acceleration of -2 m/s^2. Calculate the distance travelled by the body in fifth second. **(JEE ADVANCED)**

Sol: Distance covered is equal to displacement if the object moves in a straight line and there is no change in direction of motion. If direction of motion changes, distance should be calculated separately for different parts of the path.

Advice: Distance travelled in 5th sec need to be calculated and not the displacement.

Hence displacement formula cannot be used directly to calculate the distance in nth second.

According to equations of motion,

$S_n = u + \frac{a}{2}(2t-1)$ $= 9 - \frac{2}{2}(2x5-1)$ $= 9 - 9 = 0$

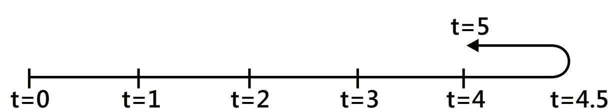


Figure 2.17

The value obtained is for displacement and not distance.

Hence distance S in 5th sec. can be calculated as S = 2

$(S_{4.5} - S_4) = 2\left[\left\{9x4.5 - \frac{1}{2}x2x(4.5)^2\right\} - \left\{9x4 - \frac{1}{2}x2(4)^2\right\}\right]$

= 2 [4.5-4.25] = 2 x 0.25 = 0.5 m

3. MOTION OF BODY UNDER GRAVITY (FREE FALL)

The force of attraction that the earth exerts on all the bodies is called force of gravity and the acceleration induced by gravity is called acceleration due to gravity, represented by g. All bodies irrespective of their size, weight or composition fall with the same acceleration near the surface of earth in the absence of air. Motion of a body falling towards the earth from a small altitude (h<<R) is known as free fall (R: Radius of Earth).

3.1 Body Projected Vertically Upwards

(a) Equation of motion: Considering point of projection as origin and direction of motion (i.e. vertically up) as positive

$a = -g$ [as acceleration is downwards]

If a body is projected with velocity u and after time *t* it reaches to a height *h* then

$v = u - gt$ ... (i)

$h = ut - ½ gt^2$ ... (ii)

$v^2 = u^2 - 2gh$ ... (iii)

(b) For maximum height (H): v =0

Using equation (ii) we get $0^2 = u^2 - 2gH \Rightarrow H = \frac{u^2}{2g}$ (c) Time taken to reach maximum height (t): v=0

Using equation (ii) we get: 0 = u-gt; T = u/g.

(c) Time of flight (T) is the time during which the object travels. In this case, it is the time between the the maximum height and the ground.

Thus, h = 0. Using equation (iii): $0 = uT - \frac{1}{2}gt^2;\ 0 = T\,(u - \frac{1}{2}gt)$

$\Rightarrow$ either T=0 or T=2u/g = 2 x. Time taken to reach maximum height (t)

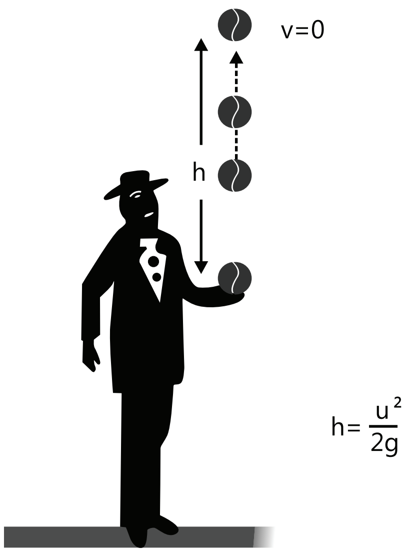


Figure 2.18

(d) The following graphs show the displacement, velocity and acceleration with respect to time (for maximum height) when body is thrown upwards:

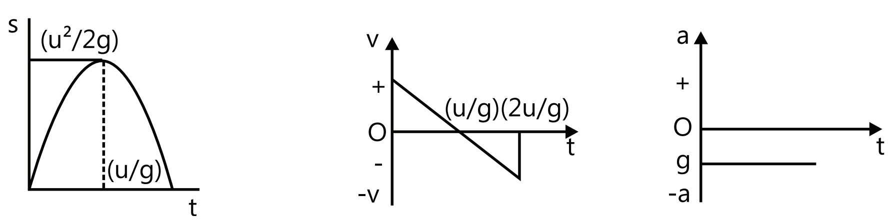


Figure 2.19

Observation:

(a) Time taken by the body to travel up is equal to the time taken by the body to fall down. Time of descent (t_2) = time of ascent (t_1) = u/g

(b) The speed with which a body is projected up is equal to the speed with which it comes down. The magnitude of velocity at any point is same whether the body is moving up or down.

Illustration 16: Consider a ball being thrown upwards with an initial speed of u. Find out u if the ball is at a height of 80 m and the interval between two times is 6 s. ($g=10\ m/s^2$) **(JEE MAIN)**

Sol: Body thrown vertically upwards reaches the maximum height, stops momentarily and then starts falling vertically downwards. So for any point at height less than the maximum height, the body will reach the point twice during its travel, first time while ascending and the second time while descending.

u = u m/s, a=g= $-10\ m/s^2$ and s=80 m

Substituting the value, $s = ut + \frac{1}{2}at^2$, we have $80 = ut - st^2$ or $st^2 - ut + 80 = 0$

Or, $t = \frac{u + \sqrt{u^2 - 1600}}{10}$ and $\frac{u - \sqrt{u^2 - 1600}}{10}$

Given that $\frac{u + \sqrt{u^2 - 1600}}{10} - \frac{u - \sqrt{u^2 - 1600}}{10} = 6$

$\frac{\sqrt{u^2 - 1600}}{5} = 6$ or $\sqrt{u^2 - 1600} = 30$

Or $u^2 - 1600 = 900$; $\therefore$ $u^2 = 2500$; Or $u = \pm 50$ m/s

Ignoring the negative sign, u =50 m/s.

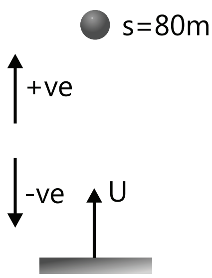


Figure 2.20

3.2 Body Dropped From Some Height (Initial Velocity Zero)

(a) Equations of motion: Let the initial position be the origin and direction of motion (i.e. downward direction) to be positive, then

u = 0 [As body starts from rest]

a = g [As acceleration is in the direction of motion]

Therefore, equations of motion are:

$v = u + gt$ ...(i)

$v = ut + \frac{1}{2} gt^2$ ...(ii)

$v^2 = u^2 + 2gh$ ...(iii)

(b) Velocity (v) of the particle just before hitting the ground: h = H

Therefore, using equation (iii): $V^2 = 0 + 2gH$; $v = \sqrt{2gH}$ (c) Time (t) taken by the object to reach the ground: h = H

Therefore, using equation (ii): $H = (0)T + \frac{1}{2}gT^2$; $T = \sqrt{\frac{2H}{g}}$

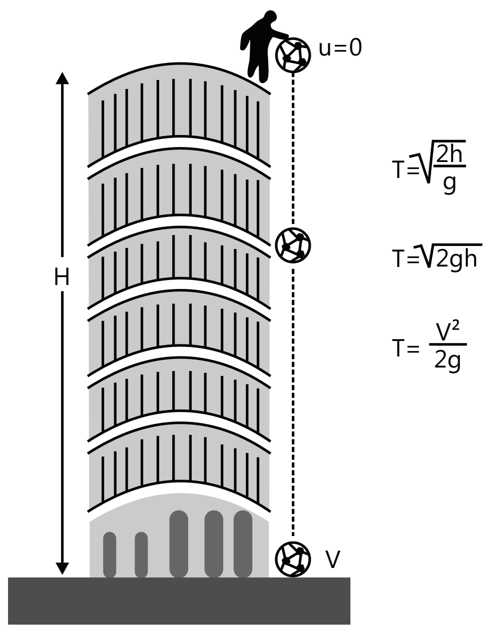


Figure 2.21

(c) The following graphs show the distance, velocity and acceleration with respect to time (for free fall):

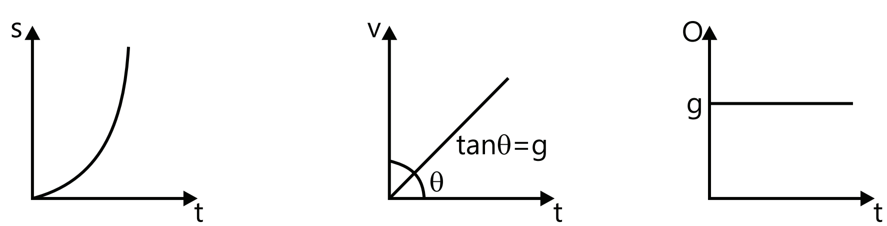


Figure 2.22

(d) The distance covered in the nth sec, $h_n = \frac{1}{2}g(2n - 1)$

Hence the ratio of the distance covered in 1st, 2nd, 3rd sec, etc. is 1:3:5 i.e. only odd integers.

These results obtained are the corollary of the Galileo's Theorem:

For a uniform accelerating body, the distance travelled is always odd ratio, i.e. 1:3:5:7, for regular time interval.

(e) As $h = (1/2)\ gt^2$, i.e. $h \propto t^2$, distance covered in time t, 2t, 3t, etc., will be in the ratio of $1^2:2^2:3^2$, i.e. square of integers.

3.3 Body Thrown Vertically from a Height

There are two possibilities when an object is thrown.

First, when an object is thrown in upward direction, velocity is upwards whereas acceleration acts downwards, i.e. they are in opposite directions. Hence initially the object undergoes retardation and rises through a certain height and then it undergoes free fall from that height.

Second, when an object is thrown from a certain height, both the velocity and acceleration are in the downward direction, i.e. velocity and acceleration are in the same direction. In this case, the object undergoes acceleration. It hits the ground with a speed greater than the speed if it had gone through free fall.

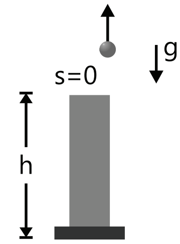


Figure 2.23

Equations of motion are used in both the cases.

Assume a sign convention for the direction.

Then, note the values of displacement, velocity and acceleration with appropriate signs.

Finally, use the appropriate equations of motion.

Illustration 17: A ball is thrown upwards from 40 m high tower with a velocity of 10 m/s. Calculate the time when it strikes the ground. ($g = 10\ m/s^2$) **(JEE MAIN)**

Sol: In the second equation of motion with constant acceleration, value of all the quantities need to be substituted with proper sign. If the displacement and acceleration are in the opposite direction of initial velocity (taken as positive) then substitute there values with negative sign.

$u = +10\ m/s$, $a = -10\ m/s^2$

$s = -40$ m (at the point where the ball strikes the ground)

Substituting in $S = ut + \frac{1}{2}at^2$, we have $-40 = 10t - 5t^2$

or $5t^2 - 10t - 40 = 0$ or $t^2 - 2t - 8 = 0$

Solving this, we get t = 4 s and -2 s. Considering the positive value, t=4 s.

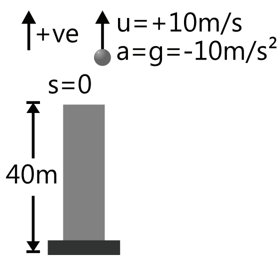


Figure 2.24

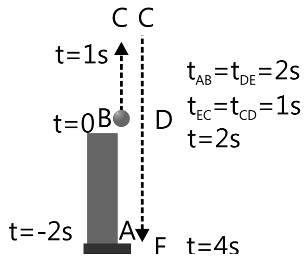


Figure 2.25

Note: The significance of t=-2 s can be understood by Fig. 2.25:

MASTERJEE CONCEPTS

We have studied the formula of maximum height (H) and time taken (T) to reach the point where the velocity of an object becomes zero under gravity.

The retardation of the object is 'g'. $H = u^2/2g$ and $T = u/g$.

This can be used for an object having an initial velocity 'u' and retardation 'a'.

Thus, distance at which velocity of the particle becomes zero $H = u^2/2a$;

Distance after which the particle changes its direction;

Time taken to reach this distance = u/a

Time taken to reach its initial position = 2u/a

Anand K (JEE 2011 AIR 47)

4. NON-UNIFORMLY ACCELERATED MOTION

Equations of motion cannot be considered for particles travelling with constant or uniform acceleration. While deriving equations of motion, we considered acceleration to be constant. So, for solving non-uniformly accelerated motion, we will follow the two basic equations: (i) $\vec{v} = \frac{d\vec{s}}{dt}$ or sometimes $\vec{v} = \frac{d\vec{r}}{dt}$ (ii) $\vec{a} = \frac{d\vec{v}}{dt}$

Vector quantity is not required for one-dimensional motion. Therefore the above equations can be re-written as: $(i)\ v = \frac{ds}{dt}\ \ (ii)\ a = \frac{dv}{dt} = v\frac{dv}{ds}$

MASTERJEE CONCEPTS

$a = v\frac{dv}{ds}$ is useful when acceleration displacement is known and velocity displacement is required.

Yashwanth Sandupatla (JEE 2012, AIR 821)

Illustration 18: For a particle moving along x-axis, displacement time equation is $x = 20 + t^3 - 12t$.

(a) Find the position and velocity of the particle at time t = 0

(b) Find out whether the motion is uniformly accelerated or not.

(c) Find out the position of particle when velocity is zero. **(JEE MAIN)**

Sol: Displacement is given as a function of time. Differentiating the equation of displacement w.r.t time we get the velocity as a function of time. Differentiating the equation of velocity w.r.t time we get the acceleration as a function of time.

(a) $x = 20 + t^3 - 12t$ ... (i)

At t = 0, x = 20 + 0 – 0 = 20 m

By differentiating Equation (i) w.r.t. time i.e. $v = \frac{dx}{dt} = 3t^2 - 12$ ... (ii)

Velocity of particle can be obtained at time t.

At t = 0, v = 0 -12 =- 12 m/s

(b) Differentiating equation (ii) w.r.t. time, we get the acceleration $a = \frac{dv}{dt} = 6t$

As acceleration is a function of time, the motion is non-uniformly accelerated.

(c) Substituting v=0 in equation (ii) $0 = 3t^2 - 12$

From the above equation, t = 2 sec. Substituting it in equation (i) we have $x = 20 + (2)^3 - 12(2)$ or x=4 m

Illustration 19: Acceleration of an object moving in straight line is $a=v^2$ and initial velocity of that object is u m/sec Find. (i) v(x) i.e. velocity as a function of displacement (ii) v(t) i.e. velocity as a function of time **(JEE ADVANCED)**

Sol: Acceleration is the differentiation of velocity with respect to time. We can use the following transformation:

$$a=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$$

(i) $a=v\frac{dv}{dx}=v^2;\ \Rightarrow\frac{dv}{v}=dx$ Integration $\Rightarrow\int_u^v\frac{dv}{v}=\int_o^x dx\ \Rightarrow\ \ell nv\Big|_u^v=x\Big|_o^x;\ \ell nv-\ell nu=x;\ \Rightarrow\ell n\frac{v}{u}=x;\ \therefore\ v=u\,e^{x}$

(ii) $a=\frac{dv}{dt}=v^2;\ \frac{dv}{v^2}=dt$ Integration $\Rightarrow\int_u^v\frac{dv}{v^2}=\int_o^t dt;\ \Rightarrow-\frac{1}{v}\Big|_u^v=t\Big|_o^t\Rightarrow-\left[\frac{1}{v}-\frac{1}{u}\right]=t;\ \Rightarrow\frac{1}{v}=-t+\frac{1}{u}=\frac{1-ut}{u};\ \therefore\ v=\frac{u}{1-ut}$

5. ANALYSIS OF MOTION THROUGH GRAPHS

Let us see some basics points of graphs before analyzing the motion of an object through graphs.

Basic Graphs: (a) A linear relationship between x and y represents a straight line.

E.g., y=4x-2, y=5x + 3, 3x = y-2

(b) A proportionality relationship between x and y (i.e. $x\propto y$ or $y=kx$) represents a straight line passing through origin.

(c) Inverse proportionality $\left(x\propto\frac{1}{y}\right)$ or xy=k represents a rectangular hyperbola.

Shape of a rectangular hyperbola is given in the graph:

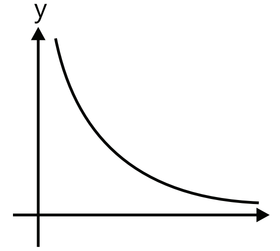


Figure 2.26

(d) A quadratic equation in x and y represents a parabola in x-y graph. E.g., $y=3x^2+2$, $y^2=4x$, $x^2=y-2$

Analysis of Graphs: (a) If $z=\frac{dy}{dx}$, the value of z at any point on x-y graph can be obtained by the slope of the graph at that point.

(b) If z = y(dx) or x(dy), the value of z between x_1 and x_2 or y_1 and y_2 is obtained by the area of graph between x_1 and x_2 or y_1 and y_2.

5.1 Displacement–Time Graph

With displacement of a body plotted on y-axis against time on x-axis, displacement–time curve is obtained.

(a) The slope of the tangent at any point of time gives the instantaneous velocity at any given instant.

(b) At a uniform motion the displacement–time graph is a straight line.

(i) If the graph obtained is parallel to time axis, the velocity is zero.

(ii) If the graph is an oblique line, the velocity is constant (OC and EF in Fig. 2.28).

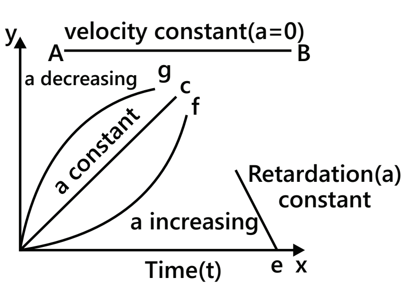


Figure 2.27

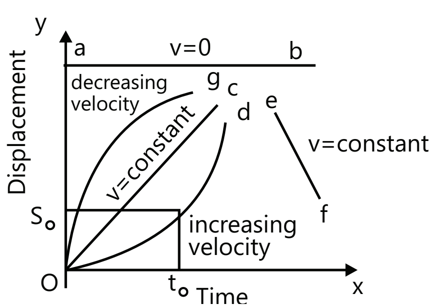


Figure 2.28

Illustration 20: Displacement–time graph of a particle moving in a straight line is shown in the Fig. 2.30. State whether the motion is accelerated or not. Describe the motion of the particle in detail. Given s_0=20 m and t_0=4 s. **(JEE MAIN)**

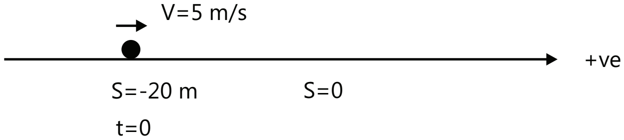


Figure 2.29

Sol: The velocity of the particle at any instant is the slope of the displacement time graph at that instant. If the slope is constant, velocity is constant.

Slope s–t is a straight line. Hence, velocity of particle is constant. At time t = 0, displacement of the particle from its mean position is $-s_0$ i.e. -20 m. Velocity of particle,

$$V = \text{slope} = \frac{s_0}{t_0} = \frac{20}{4} = 5\text{m/s}$$

At t = 0 particle is at –20 m and has a constant velocity of 5 m/s. At t_0 = 4 sec, particle will pass through its mean position.

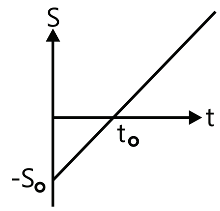


Figure 2.30

5.2 Velocity–Time Graph

Displacement, velocity and acceleration, specifying the entire motion, can be determined by the velocity–time curve.

(a) Instantaneous acceleration can be obtained by the slope of the tangent at any point corresponding to a particular time on the curve.

(b) Displacement during a time interval is obtained from the area enclosed by velocity–time graph and time axis for a time interval.

(c) For a uniformly accelerated motion, velocity–time graph is a straight line.

(d) For constant velocity (i.e. acceleration is zero), the graph obtained is a straight line AB parallel to x-axis (time).

(e) For constant acceleration, the graph obtained is oblique.

5.3 Acceleration–Time Graph

(a) Change in velocity for a given time interval is the area enclosed between acceleration–time graph and time axis.

(b) For constant acceleration, the obtained graph is a straight line parallel to x-axis, i.e. time (t).

(c) If the acceleration is non-uniform, then the graph is oblique.

Inference: Displacement–time graph for uniformly accelerated or retarded motion is a parabola. Since, for constant acceleration, then relation between displacement and time is: $s = ut \pm \frac{1}{2}at^2$ which is quadratic in nature. Thus, displacement–time graph will be parabolic in nature.

Illustration 21: Acceleration–time graph of a particle moving in a straight line is shown in Fig. 2.31. At time t=0, velocity of the particle is 2 m/s. Find velocity at the end of the 4th second. **(JEE MAIN)**

Sol: The area enclosed by the acceleration-time graph between t = 0 and t = 4s will give the change in velocity in this time interval.

dv = a dt

or change in velocity = area under a-t graph

Hence $v_f - v_i = \frac{1}{2}(4)(4) = 8\text{m/s}$; $\therefore$ $v_f = v_i + 8 = (2+8)\text{m/s} = 10\text{m/s}$

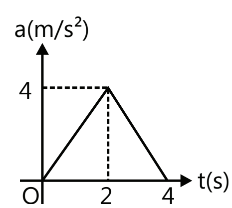
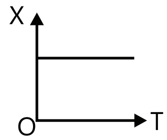
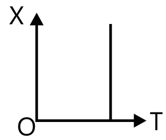
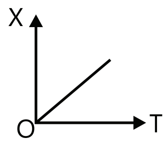
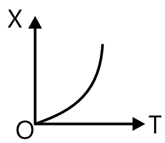
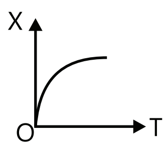
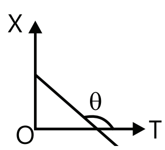
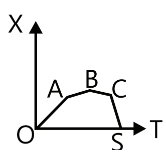
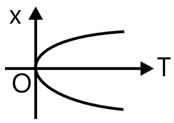
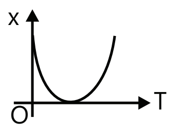
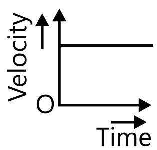
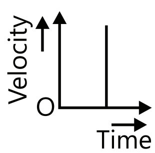
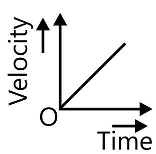
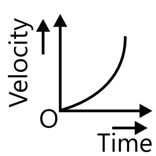
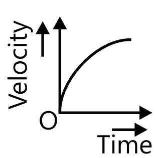
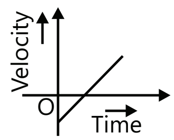
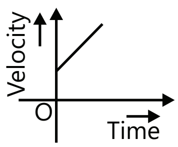
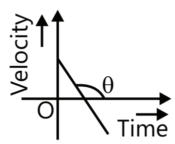
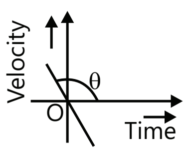
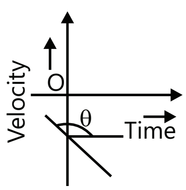


Figure 2.31

Various Position–time graphs and their interpretation	
 **Figure 2.32**	$\theta = 0^\circ$, so v = 0 i.e. line parallel to time axis represents that the particle is at rest
 **Figure 2.33**	$\theta = 90^\circ$,so $v = \infty$ i.e. line perpendicular to time axis represents that particle is changing its position with constant time. Hence, particle possesses infinite velocity (which is not possible practically).
 **Figure 2.34**	θ = constant, so v = constant, a=0 i.e. line with constant slope represents uniform velocity of the particle.
 **Figure 2.35**	θ is increasing, so v is increasing and a is positive. i.e. line bending towards position axis x represents increase in velocity of particle. Hence, the particle possesses acceleration.
 **Figure 2.36**	θ is decreasing, so v is decreasing and a is negative i.e. line bending towards time axis t represents decreasing velocity of the particle. Hence, the particle possesses retardation.
 **Figure 2.37**	θ is constant but $>90^\circ$, so v will be constant but negative i.e. line with negative slope represents that particle returns to the point of reference (i.e. negative displacement).
 **Figure 2.38**	Straight line segments of different slopes represent that velocity of the body is different for different intervals of time.

 **Figure 2.39**	At one point the particle has two positions, which is not possible.
 **Figure 2.40**	Particle moves towards origin initially and after that moves away from origin.
Various Velocity–time graphs and their interpretation	
 **Figure 2.41**	θ = 0°, a = 0, v = constant i.e. line parallel to time axis represents that the particle is moving with constant velocity.
 **Figure 2.42**	θ = 90°, a = ∞, v is increasing i.e. line perpendicular to time axis represents that particle is increasing in velocity, but there is no change in time. Hence the particle possesses infinite acceleration (which is not possible practically).
 **Figure 2.43**	θ =constant, so a = constant and v is increasing uniformly with time i.e. the slope represents uniform acceleration of the particle.
 **Figure 2.44**	θ is increasing, so acceleration is increasing i.e. line bending towards velocity axis represents increasing acceleration of the body.
 **Figure 2.45**	θ is decreasing, so acceleration is decreasing i.e. line bending towards time axis represents the decreasing acceleration in the body.

 **Figure 2.46**	θ is constant and <90° i.e. acceleration is constant and positive but initial velocity of the particle is negative.
 **Figure 2.47**	θ is constant and <90° i.e. acceleration is constant and positive but initial velocity of the particle is positive.
 **Figure 2.48**	θ is constant and >90° i.e. acceleration is constant and negative but initial velocity of the particle is positive.
 **Figure 2.49**	θ is constant and >90° i.e. acceleration is constant and negative but initial velocity of the particle is zero.
 **Figure 2.50**	θ is constant and >90° i.e. acceleration is constant and negative but initial velocity of the particle is negative.

MASTERJEE CONCEPTS

The following are the lists of motions that are not possible practically:

- Slopes of v-t or s-t graphs can never be infinite at any point, because infinite slope of v-t graph means infinite acceleration. Similarly, infinite slope of s-t graph means infinite velocity. Hence, the graphs shown here are not possible.
- At a particular time, two values of velocities v_1 and v_2 or displacements S_1 and S_2 are not possible. Hence, the following graphs shown here are not possible.

GV Abhinav (JEE 2012 AIR 329)

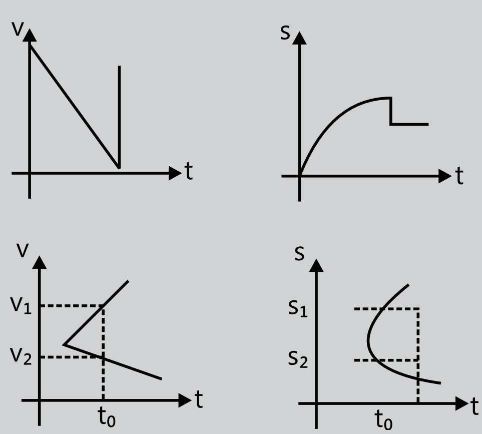


Figure 2.51

Illustration 22: At t = 0, a particle is at rest at origin. For the first 3 s the acceleration is 2 ms^{-2} and for the next 3 s acceleration is -2 ms^{-2}. Find the acceleration versus time, velocity versus time and position versus time graphs.

(JEE ADVANCED)

Sol: The area enclosed by the acceleration-time graph and the time-axis between t = 0 and t = t gives the change in velocity in this time interval. Similarly the area enclosed by the velocity-time graph and the time-axis between t = 0 and t = t will give the change in displacement in this time interval.

Given for the first 3 s acceleration is 2 ms^{-2} and for next 3 s acceleration is -2 ms^{-2}. Hence acceleration–time graph is as shown in the Fig. 2.52.

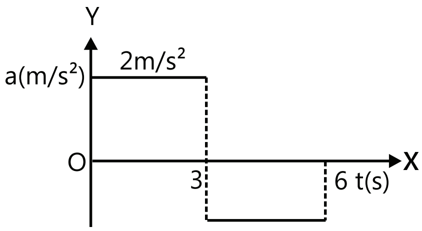


Figure 2.52

The area enclosed between a–t curve and t-axis gives change in velocity for the corresponding interval. Also at t=0, v=0, hence final velocity at t=3 s will increase to 6 ms^{-1}. In next 3 s the velocity will decrease to zero. Hence the velocity–time graph is as shown in figure.

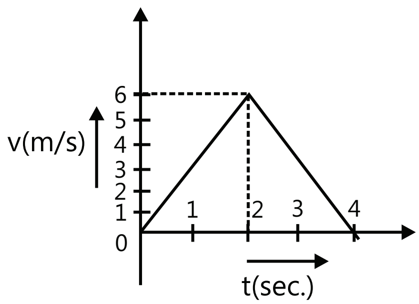


Figure 2.53

Note that due to constant acceleration v-t curves are taken as straight line.

Now for x-t curve, we will use the fact that area enclosed between v-t curve and time axis gives displacement for the corresponding interval. Hence displacement in the first 3 s is 4.5 m and in next 3 s is 4.5 m. Also the x–t curve will be of parabolic nature as the motion has a constant acceleration. Therefore, x–t curve is as shown in figure.

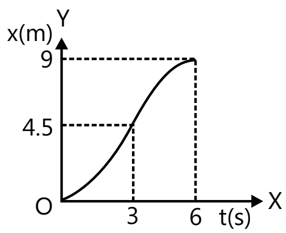


Figure 2.54

Illustration 23: The graph in the Fig. 2.55 shows the velocity of a body plotted as a function of time.

(a) Find the instantaneous acceleration at t = 3 s, 7 s, 10 s, and 13 s.

(b) Find the distance travelled by the body in the first 5 s, 9 s, and 14 s

(c) Find the total distance covered by the body during motion.

(d) Find the average velocity of the body during motion.

(JEE ADVANCED)

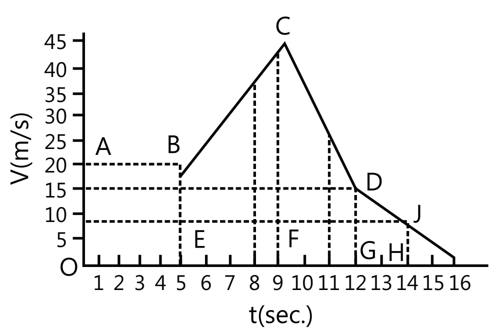


Figure 2.55

Sol: The area enclosed by the acceleration-time graph and the time-axis between t = 0 and t = t gives the change in velocity in this time interval. Similarly the area enclosed by the velocity-time graph and the time-axis between t = 0 and t = t will give the change in displacement in this time interval.

(a) Acceleration at t = 3 s

When the particle travels from point A to B for the first 5 s, the body moves with a constant velocity. Hence, the acceleration is zero.

Acceleration at t = 7 s.

When the particle travels from point B to C for the interval 5 to 9 s, the acceleration is uniform.

$$a = \frac{45-20}{(9-5)} = 6.25\,m/s^2$$

Hence acceleration at t=7 s is 6.25 m/s^2. The acceleration at t = 10 s and 13 s are respectively -10 m/s^2 and -3.75 $m.s^2$.

(b) The distance covered in t seconds is the area enclosed by the curve in t seconds on velocity–time graph. The distance covered by the body in 5 s = Area of rectangle ABEO = 20 x 5 = 100 m

The distance covered in first 9 s = The area of the figure ABCFO = Area ABEO + Area EBCF

$$=100+\frac{1}{2}(20+45)\times 4 = 100 + 130 = 230\text{ m.}$$

The distance covered by the body in first 14 s = Area [(ABCFO) + (CDGF) + DJHG)]

$$=230+\frac{1}{2}(45+15)\times 3+\frac{1}{2}(15+7.5)\times 2 = 230 + 90 + 22.5 = 342.5\text{ m.}$$

(c) The distance covered by the body during the entire motion $=342.5+\frac{1}{2}7.5\times 2 = 350\ m/s.$

(d) Average velocity for the motion $= \frac{350}{16} m/s = 21.9\,m/s.$

6. RELATIVE MOTION

(a) Motion of an object is dependent on observation.

(b) Motion is a relative term.

(c) An observation of motion is always with respect to frame of reference.

6.1 Types of Frames of Reference

(a) **Inertial frames of reference:** It is defined as the frame of reference with uniform velocity (both in magnitude and direction). Thus, acceleration is also zero, i.e. $\vec{v} = \text{constant}$ and $\vec{a} = 0$.

(b) **Non-Inertial frames of reference**: It is defined as the frames of reference with non-uniform velocity (either magnitude or direction is not constant). Thus, acceleration is also non-zero, i.e. $\vec{a} \neq 0$.

6.2 Relative Velocity (Introduction to motion in 2D)

Consider two bodies A and B travelling with velocities V_{AO} and V_{BO}, respectively, with respect to origin O, then the relative velocity of B with respect to an observer A, V_{BA}, is given as follows:

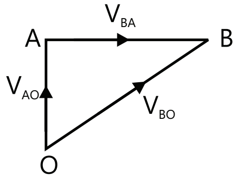


Figure 2.56

$V_{BA} = V_{BO} - V_{AO}$

Similarly, $V_{AB} = V_{AO} - V_{BO}$

Thus the relative velocity of any two bodies moving from the same origin is equal to the vector difference of their velocities.

The relative rate of change of V_{BA} gives relative acceleration of B with respect to A and is given by $a_{BA} = a_{BO} - a_{AO}$ and $a_{AB} = a_{AO} - a_{BA}$

Fact: Distance between two objects with respect to is independent of the reference frame.

If 'x' is the minimum distance between the two objects at time 't' then in any frame of reference the minimum distance of the objects remains constant at time 't'.

MASTERJEE CONCEPTS

To find the relative velocity of an object A (say) w.r.t to object B (say), inverse (change the direction of) the velocity vector of object B and then add it to velocity of A.

Anurag Saraf (JEE 2011 AIR 226)

Illustration 24: A man whose velocity in still water is 5 m/s swims from point A to B (100 m down-stream of A) and back to point A. Velocity of the river is 3 m/s. Find the time taken in going down-stream and upstream and the average speed of the man during the motion. **(JEE MAIN)**

Sol: The velocity of man in ground frame is the vector sum of the velocity of river and the velocity of man in river frame. In going down-stream, the magnitude of velocity of river and the magnitude of velocity of man in river frame are added to get the magnitude of velocity of man in ground frame. In going up-stream, the magnitude of velocity of river is subtracted from the magnitude of velocity of man in river frame to get the magnitude of velocity of man in ground frame.

During down-stream, velocity of the man $= \overrightarrow{V_m} = \overrightarrow{V_{mw}} + \overrightarrow{V_w} = 3 + 5 = 8$ m / s Time taken during down-stream = 100/8 =12.5 s

During upstream, velocity of the man $= \overrightarrow{V_m}' = \overrightarrow{V_{mw}} + \overrightarrow{V_w}$ = - 5 + 3 = - 2 m/s.

Time taken during upstream = 100/2 = 50 s

Average speed = $\dfrac{200}{62.5} = 3.2$ m/ s

Illustration 25: Yashwant started moving with constant speed 10 m/s to catch the bus. When he was 40 m away from the bus, it started moving away from him with acceleration of 2 m/s^2. Find whether Yashwant catches the bus or not. If yes, at what time he catches the bus. If no, then find the minimum distance between the bus and him. **(JEE MAIN)**

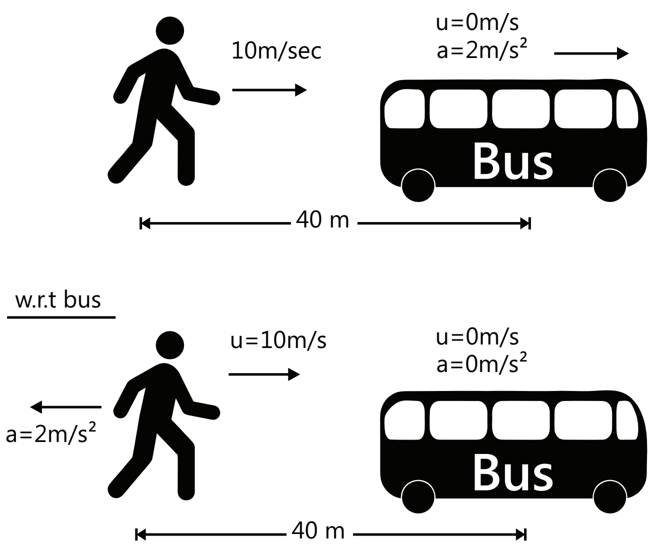


Figure 2.57

Sol: This problem is best solved in the reference frame of bus. In this frame the initial velocity of Yashwant is towards the bus (assumed positive) and the acceleration of Yashwant is in the opposite direction i.e. away from the bus (assumed negative).

Yashwant moves with the initial velocity of 10 m/s and acceleration of 2 m/s^2.

Assuming Yashwant never catches the bus, let us find the distance at which his velocity becomes zero.

We know $v^2 = u^2 + 2as$

Here v = 0; u = 10 m/s; a = -2 m/s^2; $\therefore\ 0^2 = (10)^2 + 2(-2)\,s \Rightarrow s = \frac{100}{4} = 25\text{m}$

Our assumption was right, since Yashwant travels only 25 m in the positive direction in bus reference frame.

So, minimum distance = 40 – 25 = 15 m.

6.3 Applications of relative velocity

Relative motion is widely used in two- and three-dimensional motions. The four types of problems arising based on relative motion are as follows:

(a) Problems on minimum distance between two bodies in motion

(b) River–boat problems

(c) Aircraft–wind problems

(d) Rain problems

(a) Minimum distance between two bodies in motion: Minimum distance between two moving bodies or the time taken when one body overtakes the other can be solved easily by the principle of relative motion. Here we consider one body to be at rest and other body to be in relative motion of the other body. By combining two problems into one, the solution becomes easy. Following examples will illustrate the statement.

Illustration 26: Car A and car B start moving simultaneously in the same direction along the line joining them. Car A moves with a constant acceleration a = 4 m/s^2, while car B moves with a constant velocity v = 1 m/s. At time t = 0, car A is 10 m behind car B. Find the time when car A overtakes car B. **(JEE MAIN)**

Sol: This problem is best solved in the reference frame of any one of the two cars (say car B). In this frame the initial velocity of car A is in the direction away from car B (assumed negative) and the acceleration of car A is in the direction towards the car B (assumed positive).

Given: u_A=0, u_B=1m/s, a_A = 4 m/s^2 and a_B = 0

Assuming car B to be at rest, we have

$u_{AB} = u_A - u_B = 0 - 1 = -1\text{m/s}$

$a_{AB} = a_A - a_B = 4 - 0 = 4\text{m/s}^2$

Now, the problem can be solved easily as follows:

Substituting the proper values in equation

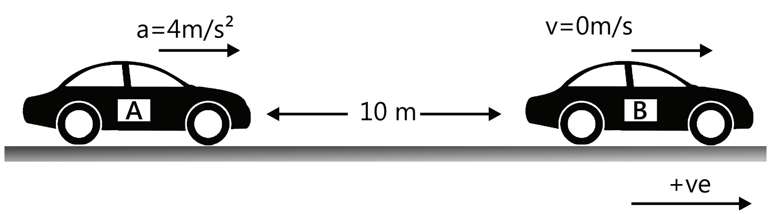


Figure 2.58

$s = ut + \frac{1}{2}at^2$, we get $10 = -t + \frac{1}{2}(4)(t^2)$ or, $2t^2 - t - 10 = 0$ or, $t = \frac{1 \pm \sqrt{1+80}}{4} = \frac{1 \pm \sqrt{81}}{4}$

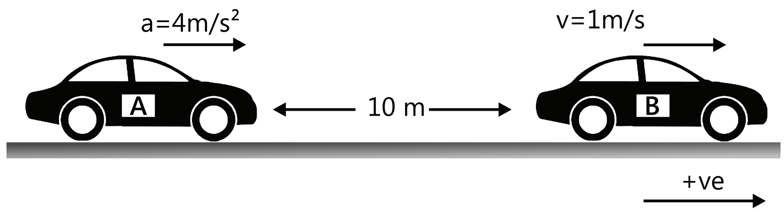


Figure 2.59

$=\frac{1\pm 9}{4}$ or $t=2.5\text{ s}$ and -2 s Ignoring the negative value, the desired time is 2.5 s.

(b) River–boat problem

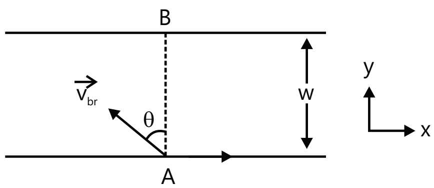


Figure 2.60

We come across the following three terms:

$\vec{v_r}$ = absolute velocity of river

$\vec{v_{br}}$ = velocity of boatman with respect to river or velocity of boatman in still water and $\vec{v_b}$ = absolute velocity of boatman

Here, it is important to note that $\vec{v_{br}}$ is the velocity of boatman with which he steers and $\vec{v_b}$ is the actual velocity of boatman relative to ground.

Further, $\vec{v_b}=\vec{v_{br}}+\vec{v_r}$ Now, let us derive some standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\vec{v_{br}}$ in the direction shown in Fig. 2.60.

River is flowing along positive x-direction with velocity $\vec{v_r}$.

Width of the river is w, then

Therefore, $v_{bx} = v_{rx} + v_{brx} = v_r - v_{br}\sin\theta$ and $v_{by} = v_{ry} + v_{bry\,=\,0} + v_{br}\cos\theta$

Now, time taken by the boatman to cross the river is:

$$t=\frac{w}{v_{br}\cos\theta} \quad \text{... (i)}$$

Further, displacement along x-axis when he reaches on the other bank (also called drift) is:

$$x = v_{bx}t = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta}$$

$$x = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta} \quad \text{... (ii)}$$

Or the three special cases are:

(i) Condition when the boatman crosses the river in shortest interval of time

From Equation (i) we can see that time (t) will be minimum when $\theta = 0^\circ$, i.e. the boatman should steer his boat perpendicular to the river current.

Also, $t_{mim}=\frac{w}{v_{br}}$ as $\cos\theta=1$

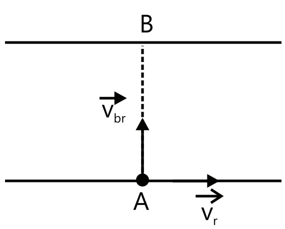


Figure 2.61

(ii) Condition when the boatman wants to reach point B, i.e. at a point just opposite from where he started

In this case, the drift (x) should be zero. Therefore x = 0

Or $(v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta}=0$ or $v_r = v_{br}\sin\theta$

$\left(\frac{v_r}{v_{br}}\right)$ or $\sin\theta = \frac{v_r}{v_{br}}$ or $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$

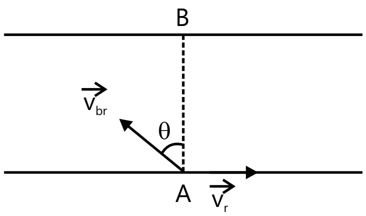


Figure 2.62

Hence, to reach point B, the boatman should row at an angle $\theta = \sin^{-1}$ upstream from AB. Further, since $\sin\theta \le 1$.

So, if $v_r \ge v_{br}$, the boatman can never reach point B. If $v_r = v_{br}$, $\sin\theta = 1$ or 90° it is quite impossible to reach B if $\theta = 90°$. Moreover it can be seen that $v_{by} = 0$ if $v_r = v_{br}$ and $\theta = 90°$. Similarly, if $v_r \ge v_{br}$, $\sin\theta > 1$ i.e. no such angle exists. Practically it is not possible to reach B if river velocity (v_r) is too high.

(iii) Shortest path

Distance travelled by the boatman when he reaches the opposite shore is $s = \sqrt{w^2 + x^2}$

Here, w = width of river which is constant. For s to be minimum, modulus of x (drift) should be minimum. Now two cases are possible

When $v_r < v_{br}$: In this case x = 0, when $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$ or $s_{mim} = w$ at $\theta \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$

When $v_r > v_{br}$: In this case x is minimum, where $\left(\frac{dx}{d\theta}\right) = 0$; $\frac{d}{d\theta}\left\{\frac{w}{v_{br}\cos\theta}(v_r - v_{br}\sin\theta)\right\} = 0$

or $-v_{br} + v_r\sin\theta = 0$ or $\theta = \sin^{-1}\left(\frac{v_{br}}{v_r}\right)$

Now, at this angle we can find x_{min} and then s_{min}

$$s_{min} = w\left(\frac{v_r}{v_{br}}\right) \text{at } \theta = \sin^{-1}\left(\frac{v_{br}}{v_r}\right)$$

Illustration 27: A man rows a boat at 4 km/h in still water. If he is crossing a river with a 2 km/h current

(a) What will be the direction of the boat, if he wants to reach a point directly opposite the starting point on the other bank?

(b) With these conditions, how much time it will take for him to cross the river, given the width of river is 4 km?

(c) What will be the minimum time and what direction should he head to cross the river in shortest time?

(d) If he wants to row 2 km up the stream and back to the origin, what will be the time required? **(JEE MAIN)**

Sol: The velocity of boat in ground frame is the vector sum of the velocity of river and the velocity of boat in river frame. If the boat heads in the direction perpendicular to the direction of river flow in the frame of the river, it will cross the river in shortest time. But in doing so it will get drifted in the direction of the river flow as well, and thus will not reach the other bank directly opposite to the starting point.

(a) Given, that $v_{br} = 4$ km/h and $v_r = 2$ km/h ;

$$\therefore \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right) = \sin^{-1}\left(\frac{2}{4}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30°$$

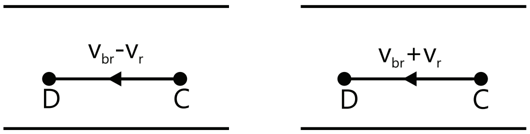


Figure 2.63(a)

To reach the point directly opposite to starting point, the boat should head at an angle of 30° with AB or 90° + 30° = 120° with the river flow.

(b) Time taken to cross the river w = width of river = 4 km

v_{br} = 4 km/h and θ = 30°; $t = \frac{4}{4\cos 30^o} = \frac{2}{\sqrt{3}} h$

(c) For shortest time θ = 0° and $t_{min} = \frac{w}{v_{br}\cos 30^o} = \frac{4}{4} = 1h$

Hence, he should incline his boat perpendicular to the current for crossing the river in shortest time of 1 h.

(d) t = t_{CD} + t_{DC} or $t = \frac{CD}{v_{br-v_r}} + \frac{DC}{v_{br}+v_r} = \frac{2}{4-2} + \frac{2}{4+2} = 1 + \frac{1}{3} = \frac{4}{3} h$

(c) Aircraft wind problem: The only difference between this and the river boat is that $\overrightarrow{v_{br}}$ is replaced by $\overrightarrow{v_{aw}}$ (velocity of aircraft with respect to wind or velocity of aircraft in still air), $\overrightarrow{v_r}$ is replaced by $\overrightarrow{v_w}$ (velocity of wind) and $\overrightarrow{v_b}$ is replaced $\overrightarrow{v_a}$ (absolute velocity of aircraft) . Further, $\overrightarrow{v_a} = v_{aw} + v_{w.}$ Following example will illustrate it.

Illustration 28: An aircraft flies 400 km/h in still air. If 200√2 k/h wind is blowing from the south and the pilot wants to travel from point A to a point B, north east of A. Find the direction in which the aircraft is to be steered and time of journey if AB = 100 km. (**JEE MAIN)**

Sol: The velocity of aircraft in the ground frame is the vector sum of its velocity in the wind frame and the velocity of the wind. This velocity in ground frame is along the known direction A to B which is 45° east of north. The direction of wind is towards north. The direction of velocity in wind frame is un-known which can be found using triangle law of vector addition.

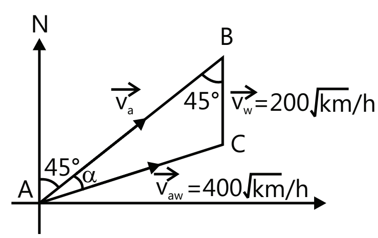


Figure 2.64

Give that $v_w = 200\sqrt{2km/h}$ v_{aw} = 400 km/h. $\vec{v}_a$ should be along AB or in north-east direction.

The direction of $\vec{v}_{aw}$ should be such that the resultant of $\vec{v}_w$ and $\vec{v}_{aw}$ is along AB or in north-east direction.

If $\vec{v}_{aw}$ makes an angle α with AB as shown in Fig. 2.64, then by applying sin law in triangle ABC,

$$\frac{AC}{\sin 45^o} = \frac{BC}{\sin\alpha} \quad \text{or} \quad \sin\alpha = \left(\frac{BC}{AC}\right)\sin 45^o = \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}} = \frac{1}{2}; \ \therefore \alpha = 30^o$$

Therefore, the pilot should steer in a direction at an angle of (45° + α) or 75° from north towards east.

$$\text{Further, } \frac{[\vec{v}_a]}{\sin(180° - 45° - 30°)} = \frac{400}{\sin 45°}; \ |\vec{v}_0| = \frac{\sin 105^o}{\sin 45^o} \times (400)\frac{km}{h}$$

$$= \left(\frac{\cos 15^o}{\sin 45^o}\right)(400)\frac{km}{h} = \left(\frac{0.9659}{0.707}\right)(400)\frac{km}{h} = 546.47 \text{ km/h}$$

$\therefore$ The time for journey from A to B is $t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47} h$; t=1.83 h

(d) Rain problem: In these type of problems, we again come across three terms $\vec{v}_r, \vec{v}_m$ and $\vec{v}_{rm}$, Here

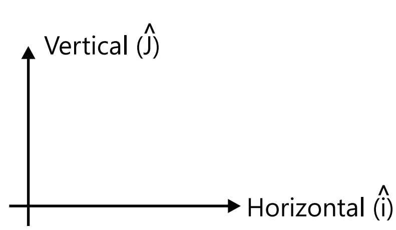


Figure 2.65

$\vec{v}_r$ = velocity of rain

$\vec{v}_m$ = velocity of man (it may be velocity of cyclist or velocity of motorist also)

and $\vec{v}_{rm}$ = velocity of rain with respect to man

Here, $\vec{v}_{rm}$ is the velocity of rain which appears to the man. Now, let us take one example of this

Illustration 29: Rain appears to fall vertically to a man walking at a rate of 3km/h. At a speed of 6 km/h, it appears to meet him at an angle of 45^0 of vertical. Find out the speed of rain. **(JEE MAIN)**

Sol: This problem is best solved by using Cartesian coordinates. Take x-axis along the horizontal and y-axis vertically upwards. The velocity of man is along positive x-axis. The velocity of rain has both horizontal and vertical components. Express the velocity of man and rain in terms of unit vectors $\hat{i}$ and $\hat{j}$.

Let $\hat{i}$ and $\hat{j}$ be the unit vectors in horizontal and vertical directions, respectively.

Velocity of rain

$$\vec{v}_r = a\hat{i} + b\hat{j} \qquad \text{... (i)}$$

Then the speed of rain will be

$$|\vec{v}_r| = \sqrt{a^2 + b^2} \qquad \text{... (ii)}$$

In the first case, $\vec{v}_m = \text{velocity of man} = 3\hat{i}$

$\therefore\ \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (a-3)\hat{i} + b\hat{j}$ It seems to be in vertical direction. Hence, $a - 3 = 0$ or $a = 3$

In the second case $\vec{v}_m = 6\hat{i}$

$\therefore\ \vec{v}_m = (a-6)\hat{i} + b\hat{j} = -3\hat{i} + b\hat{j}$

This seems to be at 45^0 of vertical

Hence, $|b| = 3$

Therefore, from Eq. (ii) speed of rain is $|\vec{v}_r| = \sqrt{3^2 + 3^2} = 3\sqrt{2}\,\text{km/h}$

Alternative Solution:

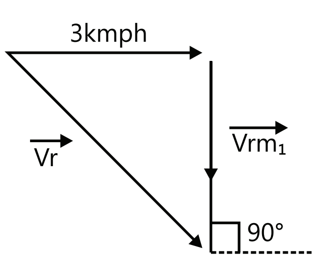


Figure 2.66

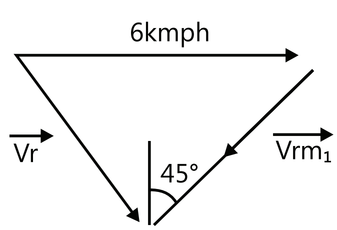


Figure 2.67

Combining these two we get

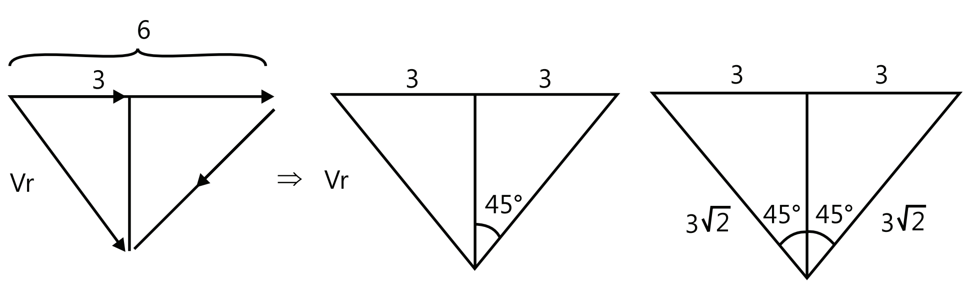


Figure 2.68

$|\vec{V}_r| = 3\sqrt{2}$ km/h

PROBLEM-SOLVING TACTICS

To avoid confusion while using signs in equations of motion, the following points need to be considered:

(a) Assuming any one direction to be positive, the other automatically becomes negative.

Generally, vertically up is considered as positive and right side is taken as positive.

(b) Write down the values of velocity, displacement and acceleration according to the sign convention.

(c) On completion of sign convention, then simply use the equations of motions.

FORMULAE SHEET

Position vector of point A with respect to O:

$\vec{r}_A = \overrightarrow{OA} = x_A\hat{i} + y_A\hat{j} + z_A\hat{k}$

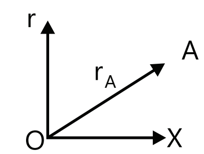


Figure 2.69

$\overrightarrow{AB} = \vec{r}_s - \vec{r}_A$

$\overrightarrow{AB} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$

For un-accelerated Motion: Distance = Speed x Time

Displacement = Velocity x Time

v_{av} = Average speed $= \dfrac{\Delta s}{\Delta t}$; $\vec{v}_{av}$ = Average Velocity $= \dfrac{\Delta \vec{r}}{\Delta t}$

$v = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$ and $\vec{v} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta r}{\Delta t} = \dfrac{d\vec{r}}{dt}$

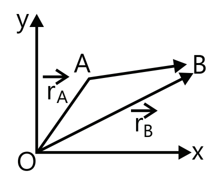


Figure 2.70

(a) Uniform motion is due to constant Velocity

Average acceleration has the same direction as of change in velocity

$\vec{a}_{av} = \dfrac{\Delta \vec{v}}{\Delta t}$

(b) One-dimensional Uniformly Accelerated Motion

- $v = u + at,\ s = ut + \dfrac{1}{2}at^2$ and $v^2 = u^2 + 2as$ and $v_{avg} = (v+u)/2$

(i) Maximum height attained by a particle, thrown upwards from ground with initial velocity u is $h = \dfrac{u^2}{2g}$

(c) Displacement of particle in tth second of its motion , $s_t = u + \dfrac{a}{2}(2t-1)$

- Time taken to reach maximum height = u/g

(i) Velocity of a particle (u=0) when it touches the ground when dropped from a height h is,

$v = \sqrt{2gh}$

(ii) In (b) time of collision with ground $t = \sqrt{2h/g}$

- If acceleration of particle is not constant, basic equations of velocity and acceleration are used, i.e.

(i) $\vec{v}=\frac{d\vec{s}}{dt}$ or $\vec{v}=\frac{d\vec{r}}{dt}$ (ii) $\vec{a}=\frac{d\vec{v}}{dt}$ (iii) $d\vec{s}=\vec{v}dt$ (iv) $d\vec{v}=\vec{v}dt$

- For one dimensional motion,

(i) $v=\frac{ds}{dt}$ (ii) $a=\frac{dv}{dt}=v\frac{dv}{ds}$

(iii) $ds=v\,dt$ and (iv) $dv=adt$ or $vdv=ads$

- If $z=\frac{dy}{dx}$ or $\frac{y}{x}$, the value of z at any point on x-y graph can be obtained by the slope of the graph at that point.

(i) slope of displacement–time graph gives velocity $\left(\text{as } v=\frac{ds}{dt}\right)$

(ii) slope of velocity–time graph gives acceleration $\left(\text{as } a=\frac{dv}{dt}\right)$

If z = yx, y (dx), or x (dy), the value of z between x_1 and x_2 or y_1 and y_2 can be obtained by the area of graph between x_1 and x_2 or y_1 and y_2

(iii) velocity–time graph gives displacement (as ds=v dt)

(iv) acceleration–time graph gives change in velocity (as dv= adt).

relative velocity of A with respect to B (written as $\vec{v}_{AB}$) is $\vec{v}_{AB}=\vec{v}_A-\vec{v}_B$

Similarly, relative acceleration of A with respect to B is $\vec{a}_{AB}=\vec{a}_A-\vec{a}_B$

$\vec{v}_{AB}=-\vec{v}_{BA}$ or $\vec{a}_{BA}=-\vec{a}_{AB}$

- In case of a one dimensional motion when we can treat the vectors as scalars by assigning positive to one direction and negative to another, the above equations can be written as $v_{AB}=v_A-v_B$ and $a_{AB}=a_A-a_B$

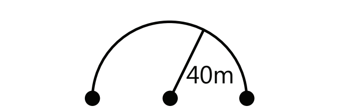
Solved Examples

JEE Main/Boards

Example 1: A person in his morning walk moves on a semicircular track of radius 40 m. Find the distance travelled and the displacement, when he starts from one end of the track and reaches the other end.

Sol: Distance is length of the path travelled. Displacement is the vector from initial point to final point.

The distance covered = length of the semicircular track



= π R = 3.14 x 40 m = 125.6 m

Displacement = Final position - initial position

= diameter of semicircular track

= 2 R = 2 x 40 = 80 m

Initial point to final point gives the direction of displacement.

Example 2: A man walks 2.5 km from his house to the market on a straight road with a speed of 5 km/h. He instantly turns back home with a speed of 7.5 km/h finding the market closed. Calculate the

(a) magnitude of average velocity and (b) the average speed of the man over the interval of time.

(i) 0 to 30 min (ii) 0 to 50 min

(iii) 0 to 40 min

Sol: Average speed is distance covered divided by time taken. Distance is length of the path travelled. Average velocity is displacement divided by time taken. Displacement is the vector from initial point to final point.